

# ON SINGULAR EQUIVALENCES OF MORITA TYPE

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**ABSTRACT.** Stable equivalences of Morita type preserve many interesting properties and is proved to be the appropriate concept to study for equivalences between stable categories. Recently the singularity category attained much attraction and Xiao-Wu Chen and Long-Gang Sun gave an appropriate definition of singular equivalence of Morita type. We shall show that under some conditions singular equivalences of Morita type have some biadjoint functor properties and preserve positive degree Hochschild homology.

## INTRODUCTION

For a Noetherian algebra  $A$  over a commutative ring its singularity category  $D_{sg}(A)$  is defined to be the Verdier quotient of the bounded derived category of finitely generated modules over  $A$  by the full subcategory of perfect complexes. This notion was introduced in an unpublished manuscript [5] by Ragnar-Olaf Buchweitz under the name of stable derived category. He related this category to maximal Cohen-Macaulay modules. Later Dmitri Orlov [22] rediscovered this notion independently in the context of algebraic geometry and mathematical physics, under the name of singularity category. The derived category of an algebra is replaced there by the derived category of coherent sheaves over a scheme. Orlov's notation for this object seems now to become the standard one, also in the case of the derived category of an algebra, and we shall concentrate here on this case.

If  $A$  is a selfinjective algebra, then  $D_{sg}(A)$  is equivalent to the stable category of  $A$  (cf [15, 25]). By definition  $D_{sg}(A)$  is always triangulated and it is easy to see that  $D_{sg}(A)$  is trivial if and only if  $A$  has finite global dimension. From this point of view  $D_{sg}(A)$  seems to have advantages with respect to the stable category of an algebra, in case the algebra is not selfinjective, and may be an appropriate replacement. Recently much work was undertaken to understand the structure of  $D_{sg}(A)$  under various conditions on  $A$ . We mention in particular Xiao-Wu Chen's work here [7, 8, 9, 10], but also Bernhard Keller, Daniel Murfet and Michel Van den Bergh [16] as well as Osamu Iyama, Kiriko Kato and Jun-Ichi Miyachi [14].

Abstract equivalences between stable categories of algebras are very ill-behaved, even in case the algebras are selfinjective. Very few properties of the algebras are preserved. However, if the equivalence is induced by an exact functor of the module categories, much more can be said and a rich structure is available. The concept developed for this purpose is Broué's concept of stable equivalence of Morita type [4]. Since the singularity category generalises the stable category, we cannot expect better properties in the singularity case than we have in the stable case.

Very recently analogous to the notion of stable equivalences of Morita type, Xiao-Wu Chen and Long-Gang Sun defined in [11] the concept of singular equivalences of Morita type. The purpose of the present note is to study this new concept of singular equivalences of Morita type. We obtain two main results. First, we shall prove in Theorem 3.1 that under mild conditions a singular equivalence of Morita type gives rise to a bi-adjoint pair. This section is inspired by an analogous approach by Alex Dugas and Roberto Martinez-Villa [12]. Then we shall investigate Hochschild homology and show in Theorem 4.1 that Hochschild homology of a finite dimensional algebra over a field and in strictly positive degrees is invariant under a singular equivalence of Morita type.

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The main tool here is Serge Bouc's generalisation [3] of the Hattori-Stallings trace to Hochschild homology.

The paper is organised as follows. We recall the notion and some properties of singularity categories in Section 1. Section 2 is devoted to the definition and some of the results of Chen and Sun on singular equivalences of Morita type. We prove the biadjoint property in Section 3 and we study Hochschild homology in Section 4.

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## 1. SINGULARITY CATEGORIES AND SINGULARLY STABLE CATEGORIES

Let  $A$  be a right Noetherian ring. We denote by  $\text{mod}(A)$  the category of finitely generated right  $A$ -modules, by  $D^b(\text{mod}(A))$  the bounded derived category of  $\text{mod}(A)$ , by  $\mathcal{P}^{<\infty}(A)$  the full subcategory of  $\text{mod}(A)$  consisting of modules of finite projective dimension, and by  $K^b(\text{proj}(A))$  the homotopy category of bounded complexes of finitely generated projective  $A$ -modules.

**Definition 1.1** ([5]). *Let  $A$  be a right Noetherian ring. Then the Verdier quotient category*

$$D_{sg}(A) := D^b(\text{mod}(A))/K^b(\text{proj}(A))$$

*is called the singularity category of  $A$ .*

It is well-known that  $K^b(\text{proj}(A))$  is a full triangulated subcategory of  $D^b(A)$ . We briefly recall the construction of the Verdier quotient. We refer to Gabriel and Zisman's book [13, Chapter 1] for more ample details, and give only the basic construction here for the convenience of the reader.

The objects of  $D_{sg}(A)$  are the same as those of  $D^b(A)$ . Let  $X$  and  $Y$  be objects of  $D_{sg}(A)$ . Then a morphism in  $\text{Hom}_{D_{sg}(A)}(X, Y)$  is represented by triples  $(\nu, Z, \alpha)$  where  $Z$  is an object in  $D^b(A)$ , where  $\alpha \in \text{Hom}_{D^b(A)}(Z, Y)$  and where  $\nu \in \text{Hom}_{D^b(A)}(Z, X)$  so that the mapping cone of  $\nu$  is isomorphic to an object in  $K^b(\text{proj}(A))$ . A triple  $(\nu, Z, \alpha)$  is covered by a triple  $(\nu', Z', \alpha')$  if there is a morphism  $\psi \in \text{Hom}_{D^b(A)}(Z', Z)$  so that  $\nu' = \nu \circ \psi$  and  $\alpha' = \alpha \circ \psi$ . Two triples  $(\nu, Z, \alpha)$  and  $(\nu'', Z'', \alpha'')$  are equivalent if both are covered by some triple  $(\nu', Z', \alpha')$ . This way the category of triples is directed, and the morphisms from  $X$  to  $Y$  is the limit of this category.

The construction of the singularity category as Verdier quotient implies that  $D_{sg}(A)$  is always triangulated.

Let  $A$  be any right Noetherian ring. Denote by  $\underline{\text{mod}}(A)$  the stable category of (finitely generated right)  $A$ -modules, with objects being the same as  $\text{mod}(A)$  and morphisms  $\underline{\text{Hom}}_A(M, N)$  being the equivalence classes of morphisms of  $A$ -modules modulo those factoring through a projective module. Recall that the category  $\underline{\text{mod}}(A)$  admits an endo-functor  $\Omega$ , the syzygy functor, defined as  $\ker(\pi_X)$ , where for every object  $X$  in  $\underline{\text{mod}}(A)$  we choose a projective object  $P_X$  in  $\text{mod}(A)$  and an epimorphism  $P_X \xrightarrow{\pi_X} X$  in  $\text{mod}(A)$ .

By the very construction there are natural functors

$$\begin{array}{ccc} \text{mod}(A) & \xrightarrow{F} & \underline{\text{mod}}(A) \\ D^b(A) & \xrightarrow{G} & D_{sg}(A) \\ \underline{\text{mod}}(A) & \xrightarrow{H} & D_{sg}(A) \end{array}$$

so that the diagram

$$\begin{array}{ccc} \text{mod}(A) & \longrightarrow & D^b(A) \\ \downarrow F & & \downarrow G \\ \underline{\text{mod}}(A) & \xrightarrow{H} & D_{sg}(A) \end{array}$$

commutes. Moreover,  $H(M) = 0$  if and only if  $M$  is of finite projective dimension. Finally  $H$  commutes with syzygies in the sense that

$$H \circ \Omega \simeq [-1] \circ H.$$

A consequence of this relation is an important observation, namely that the singularity category is in general not Hom-finite [9] and is in general not a Krull-Schmidt category. An example is given by the 3-dimensional local algebra  $A = K[X, Y]/(X^2, Y^2, XY)$  over a field  $K$ . Indeed, for the simple  $A$ -module  $S$  one gets  $\Omega(S) \simeq S \oplus S$  and this implies isomorphisms

$$H(S) \simeq H(\Omega(S))[1] \simeq H(S)[1] \oplus H(S)[1] \simeq (H(S)[2])^4 \simeq \cdots \simeq (H(S)[n])^{2^n}$$

in  $D_{sg}(A)$ . Moreover,  $H(S) \neq 0$  since  $S$  is of infinite projective dimension. Therefore,

$$\dim_K \text{End}_{D_{sg}(A)}(H(S)) = \dim_K \text{End}_{D_{sg}(A)}((H(S)[n])^{2^n}) \geq 2^n$$

for all  $n \geq 0$  and this implies that  $\dim_K \text{End}_{D_{sg}(A)}(H(S)) = +\infty$ .

As we have seen,  $M$  is an  $A$ -module of finite projective dimension if and only if  $H(M) = 0$ . Hence, it is natural to consider the following category  $\underline{\text{mod}}_{\mathcal{P}^{<\infty}}(A)$ .

**Definition 1.2.** Let  $A$  be a finite dimensional algebra. The singularly stable category is by definition the quotient category of  $\text{mod}(A)$  by  $\mathcal{P}^{<\infty}(A)$ , denoted by  $\underline{\text{mod}}_{\mathcal{P}^{<\infty}}(A) := \text{mod}(A)/\mathcal{P}^{<\infty}(A)$ .

More precisely, the objects of  $\underline{\text{mod}}_{\mathcal{P}^{<\infty}}(A)$  are the same objects as those in  $\text{mod}(A)$  and for two  $A$ -modules  $X$  and  $Y$  define  $\text{Hom}_{\underline{\text{mod}}_{\mathcal{P}^{<\infty}}(A)}(X, Y)$  to be the equivalence classes of  $A$ -module homomorphisms  $X \rightarrow Y$  modulo those factoring through an object in  $\mathcal{P}^{<\infty}(A)$ .

It is clear that  $H$  factors through the natural functor

$$\underline{\text{mod}}(A) \xrightarrow{\Pi} \underline{\text{mod}}_{\mathcal{P}^{<\infty}}(A)$$

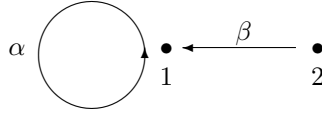
in the sense that there is a natural functor

$$\underline{\text{mod}}_{\mathcal{P}^{<\infty}}(A) \xrightarrow{L} D_{sg}(A)$$

so that

$$H = L \circ \Pi.$$

**Remark 1.3.** Observe that  $L$  is not an embedding in general. Let  $Q$  be the quiver



and let  $A = KQ/\langle \alpha^2, \beta\alpha \rangle$ . Let  $S_1$  and  $S_2$  be the two simple  $A$ -modules. Then  $H(S_1) \simeq H(S_2)$  since  $\Omega^2(S_2) \simeq S_1 \simeq \Omega^2(S_1)$ , but  $S_1 \not\simeq S_2$  in  $\underline{\text{mod}}_{\mathcal{P}^{<\infty}}(A)$  since there is no non zero homomorphism of  $A$ -modules between these objects.

**Remark 1.4.** We could consider modules of finite projective dimension as “smooth” objects. Then the singularly stable category measures the singularity of  $A$ . Clearly the algebra  $A$  has finite global dimension if and only if the singularly stable category has only one object with only one endomorphism. However, the singularly stable category is only an additive category, and in general it is not triangulated. If  $A$  is selfinjective,  $H$  is an equivalence (cf [15, 25]) and an  $A$ -module of finite projective dimension is actually projective. Hence also  $\Pi$  is an equivalence in this case.

**Remark 1.5.** Let  $A$  be the algebra introduced in Remark 1.3. Then it is easy to see that  $D_{sg}(A) \simeq D_{sg}(K[X]/X^2)$ . However, these two algebras are not stably equivalent. In fact, if they were stably equivalent, then there would be a one to one correspondence between the isomorphism classes of non-projective indecomposable modules. However, up to isomorphisms,  $A$  has more than two non-projective indecomposable modules and  $K[X]/(X^2)$  has only one such module.

We are grateful to one of the referees who suggested the above proof which is simpler than our original argument.

## 2. SINGULAR EQUIVALENCES OF MORITA TYPE

As mentioned in the introduction, general stable equivalences have very poor properties, even for selfinjective algebras. A richer concept is given by Broué [4]. Broué defined stable equivalences of Morita type as equivalences between stable module categories induced by tensor product with bimodules. This concept was highly successful in the understanding of equivalence between stable categories of self-injective algebras and was a subject of numerous studies.

We consider the question when the singularly stable categories of two algebras are equivalent. Since equivalences between singular categories of selfinjective algebras coincide with stable equivalences, we need a richer concept than just an equivalence between triangulated categories. Recently Xiao-Wu Chen and Long-Gang Sun introduced singular equivalences of Morita type [11] on the model of Broué's concept of stable equivalences of Morita type.

Let  $K$  be a commutative ring. For a  $K$ -algebra  $A$ , we denote by  $A^e = A^{op} \otimes_K A$  its enveloping algebra.

**Definition 2.1.** (cf [11]) *Let  $A$  and  $B$  be two  $K$ -algebras for a commutative ring  $K$ . Let  ${}_A M_B$  and  ${}_B N_A$  be two bimodules so that*

- $M$  is finitely generated and projective as  $A^{op}$ -module and as  $B$ -module;
- $N$  is finitely generated and projective as  $A$ -module and as  $B^{op}$ -module;
- ${}_A M \otimes_B {}_B N_A \simeq {}_A A_A \oplus {}_A X_A$  for a module  $X \in \mathcal{P}^{<\infty}(A^e)$ ;
- ${}_B N \otimes_A {}_A M_B \simeq {}_B B_B \oplus {}_B Y_B$  for a module  $Y \in \mathcal{P}^{<\infty}(B^e)$ .

*We then say that the pair  $({}_A M_B, {}_B N_A)$  induces a singular equivalence of Morita type.*

*We say that  $A$  and  $B$  are singularly equivalent of Morita type if there is a pair of bimodules  $({}_A M_B, {}_B N_A)$  which induces a singular equivalence of Morita type.*

**Remark 2.2.** • It is immediate from the definition that a pair of bimodules inducing a stable equivalence of Morita type induces a singular equivalence of Morita type as well.

However, a singular equivalence of Morita type will not be a stable equivalence of Morita type in general since the property of  $X$  to be in  $\mathcal{P}^{<\infty}(A^e)$  is in general much weaker than the condition to be projective as bimodule.

Nevertheless, if  $A$  is selfinjective (and thus so is any algebra singularly equivalent of Morita type to  $A$ , as is remarked in [11]), any module with finite projective resolution is actually projective, and hence a singular equivalence of Morita type is actually a stable equivalence of Morita type. The concept of a singular equivalence of Morita type and of a stable equivalence of Morita type coincide for selfinjective algebras.

- Let  $({}_A M_B, {}_B N_A)$  induce a singular equivalence of Morita type and let  $M \otimes_B N \simeq A \oplus X$  and  $N \otimes_A M \simeq B \oplus Y$ . Then  $X$  is projective as  $A$ -left module and as  $A$ -right module. Indeed,  $M$  is projective as  $B$ -right module, hence a direct factor of some  $B^n$ . Hence  $M \otimes_B N$  is a direct factor of  $B^n \otimes_B N \simeq N^n$ . Now,  $X$  is by definition a direct factor of  $N^n$  and since  $N$  is projective as  $A$ -right module,  $X$  is projective as  $A$ -right module. Similarly  $X$  is projective on the left. Likewise  $Y$  is projective as  $B$ -left module and as  $B$ -right module.

From now on to the end of the present section and in Section 3 fix a field  $K$  and  $K$ -algebras will always be supposed to be finite dimensional and modules will be always finitely generated.

The following result is a direct consequence of Definition 2.1.

**Proposition 2.3.** *Let  $({}_A M_B, {}_B N_A)$  be a pair of bimodules inducing a singular equivalence of Morita type between two  $K$ -algebras  $A$  and  $B$ . Then*

$$- \otimes_A M_B : D_{sg}(A) \longrightarrow D_{sg}(B)$$

*is an equivalence of triangulated categories with quasi-inverse*

$$- \otimes_B N_A : D_{sg}(B) \longrightarrow D_{sg}(A).$$

*Moreover, the same functors establish an equivalence of additive categories between  $\underline{\text{mod}}_{\mathcal{P}^{<\infty}}(A)$  and  $\underline{\text{mod}}_{\mathcal{P}^{<\infty}}(B)$ .*

The following result is an adaptation to the singular situation of a proof of Yu-Ming Liu for stable equivalences of Morita type (cf [18, Lemma 2.2]). The proof carries over verbatim.

**Proposition 2.4.** (cf [11]) *Let  $A$  and  $B$  be  $K$ -algebras. Suppose  $({}_A M_B, {}_B N_A)$  induces a singular equivalence of Morita type. Then  ${}_A M$  is a progenerator in  $\text{mod}(A^{\text{op}})$ , and likewise for  $M_B$ ,  ${}_B N$  and  $N_A$ .*

The following fact is proved in [11] analogous to [19, Proposition 2.1 and Theorem 2.2].

**Proposition 2.5.** (cf [11]) *Let  $A$  and  $B$  be two  $K$ -algebras without direct summands which have finite projective dimension as bimodules. Assume that two bimodules  ${}_A M_B$  and  ${}_B N_A$  induce a singular equivalence of Morita type between  $A$  and  $B$ .*

- (1) *Then  $A$  and  $B$  have the same number of indecomposable summands. In particular,  $A$  is indecomposable if and only if  $B$  is indecomposable.*
- (2) *Suppose that  $A = A_1 \times A_2 \times \cdots \times A_s$  and  $B = B_1 \times B_2 \times \cdots \times B_s$ , where all  $A_i$  and all  $B_i$  are indecomposable algebras. Then, there is a permutation  $\sigma$  of the set  $\{1, \dots, s\}$  so that  $A_i$  and  $B_{\sigma(i)}$  are singularly equivalent of Morita type for all  $i \in \{1, \dots, s\}$ .*

In analogy of what is known to hold for stable equivalences of Morita type, Chen and Sun also show the following lemma.

**Lemma 2.6.** (cf [11]) *Let  $K$  be a field and let  $A$  and  $B$  be finite dimensional  $K$ -algebras. Assume that bimodules  ${}_A M_B$  and  ${}_B N_A$  define a singular equivalence of Morita type between  $A$  and  $B$ , and suppose that  $A$  or  $B$  is indecomposable as an algebra. Then  $M$  and  $N$  each have a unique indecomposable bimodule summand of infinite projective dimension. If we denote these summands as  $M_1$  and  $N_1$  respectively, then  $(M_1, N_1)$  also induces a singular equivalence of Morita type between  $A$  and  $B$ .*

Let  $K$  be a field, and let  $A$  and  $B$  be finite dimensional  $K$ -algebras without direct summands having finite projective dimension as bimodules. Proposition 2.5 and Lemma 2.6 imply that for a singular equivalence of Morita type induced by  $({}_A M_B, {}_B N_A)$  we can always suppose that  $A$  and  $B$  are indecomposable algebras and that  ${}_A M_B$  and  ${}_B N_A$  are indecomposable bimodules.

**Remark 2.7.** During the ICRA 2012 in Bielefeld, Chang-Chang Xi raised the question whether there are algebras which are singularly equivalent of Morita type, but which are *not* stably equivalent of Morita type. This remark answers this question.

For any algebra  $A$  denote by  $T_2(A) := \begin{pmatrix} A & A \\ 0 & A \end{pmatrix}$  the algebra of upper  $2 \times 2$  triangular matrices over  $A$ . In a forthcoming paper, Yu-Ming Liu and the first author give two indecomposable  $K$ -algebras  $A$  and  $B$  which are stably equivalent but not Morita equivalent, but for which  $T_2(A)$  is not stably equivalent to  $T_2(B)$ . However, Chen and Sun ([11]) show that if  $A$  and  $B$  are singularly equivalent of Morita type, then also  $T_2(A)$  and  $T_2(B)$  are singularly equivalent of Morita type.

### 3. SINGULAR EQUIVALENCES OF MORITA TYPE GIVE ADJOINT PAIRS

Our aim is to prove analogous result of Dugas and Martinez-Villa [12, Theorem 2.7] for singular equivalences of Morita type. For a  $K$ -algebra  $A$ , denote by  $J(A)$  its Jacobson radical.

**Theorem 3.1.** *Let  $K$  be a field and let  $A$  and  $B$  be finite dimensional indecomposable  $K$ -algebras. Suppose  $A$  and  $B$  are not of finite projective dimension as bimodules and suppose that  $A/J(A)$  and  $B/J(B)$  are separable over  $K$ . Let  $({}_A M_B, {}_B N_A)$  be a pair of bimodules inducing a singular equivalence of Morita type between  $A$  and  $B$ . Suppose that  ${}_A M_B$  is indecomposable as a bimodule, and suppose that  $\text{Hom}_{A^{\text{op}}}({}_A M_B, {}_A A_A)$  is projective as a  $B^{\text{op}}$ -module.*

*Then*

$${}_B N_A \simeq \text{Hom}_{A^{\text{op}}}({}_A M_B, {}_A A_A)$$

*as  $B^{\text{op}} \otimes_K A$ -modules, and  $(- \otimes_B N, - \otimes_A M)$  is a pair of adjoint functors between the module categories  $\text{mod}(B)$  and  $\text{mod}(A)$ .*

**Remark 3.2.** Since a singular equivalence of Morita type induces an equivalence  $D_{sg}(A) \simeq D_{sg}(B)$  and  $\underline{\text{mod}}_{\mathcal{P}^{<\infty}}(A) \simeq \underline{\text{mod}}_{\mathcal{P}^{<\infty}}(B)$  it is clear that  $(-\otimes_A M, -\otimes_B N)$  is a pair of adjoint functors between  $D_{sg}(A)$  and  $D_{sg}(B)$ , as well as between  $\underline{\text{mod}}_{\mathcal{P}^{<\infty}}(A)$  and  $\underline{\text{mod}}_{\mathcal{P}^{<\infty}}(B)$ . Theorem 3.1 states that the functors form an adjoint pair *between the module categories*.

In order to prove Theorem 3.1, we shall use the following technical notion, motivated by Dugas and Martinez-Villa [12].

**Definition 3.3.** An  $A^{op} \otimes_K B$ -module  ${}_A U_B$  is called *strongly right nonsingular*, if for each  $A$ -module  $T_A$ , the  $B$ -module  $\Omega_A^n(T) \otimes_A U_B$  is projective for  $n \gg 0$ .

**Lemma 3.4.** Let  $K$  be a field and let  $A$  and  $B$  be finite dimensional  $K$ -algebras.

- (i) Let  ${}_A U_B$  be a bimodule which is projective as a left and as a right module. Then  ${}_A U_B$  is strongly right nonsingular if and only if for each  $A$ -module  $T_A$ ,  $T \otimes_A U_B$  has finite projective dimension.
- (ii) Objects in  $\mathcal{P}^{<\infty}(A^{op} \otimes_K B)$  which are projective as a left and as a right modules are strongly right nonsingular. In particular, for a singular equivalence of Morita type induced by the pair of bimodules  $({}_A M_B, {}_B N_A)$ , so that  $M \otimes_B N \simeq A \oplus X$  in  $\text{mod}(A \otimes_K A^{op})$  and  $N \otimes_A M \simeq B \oplus Y$  in  $\text{mod}(B \otimes_K B^{op})$ , the two bimodules  ${}_A X_A$  and  ${}_B Y_B$  are strongly right nonsingular bimodules.
- (iii) Let  $A$  be an algebra such that  $A/J(A)$  is separable over  $K$ . If the  $A^e$ -module  $A$  is not in  $\mathcal{P}^{<\infty}(A^e)$ , then the bimodule  ${}_A A_A$  is not strongly right nonsingular.
- (iv) Let  $A$  and  $B$  be finite dimensional indecomposable  $K$ -algebras which are not of finite projective dimension as bimodules and such that  $A/J(A)$  and  $B/J(B)$  are separable over  $K$ . Let  $({}_A M_B, {}_B N_A)$  be a pair of bimodules inducing a singular equivalence of Morita type between  $A$  and  $B$ . Then the two bimodules  ${}_A M_B$  and  ${}_B N_A$  are not strongly right nonsingular.
- (v) A direct summand of a strongly right nonsingular bimodule is also strongly right nonsingular. The direct sum of two right strongly non singular bimodules is also strongly right nonsingular.

**Proof** (i). Suppose that for each  $A$ -module  $T_A$ ,  $T \otimes_A U_B$  has finite projective dimension. Then for an  $A$ -module  $T_A$ , take a minimal projective resolution

$$\cdots \rightarrow P_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow T_A \rightarrow 0$$

and apply  $-\otimes_A U_B$ . The result is a complex of  $B$ -modules

$$\cdots \rightarrow P_n \otimes_A U_B \rightarrow P_{n-1} \otimes_A U_B \rightarrow \cdots \rightarrow P_1 \otimes_A U_B \rightarrow P_0 \otimes_A U_B \rightarrow T \otimes_A U_B \rightarrow 0.$$

This complex is actually exact, as  ${}_A U$  is projective. For  $n \geq 1$ , we have an exact sequence

$$0 \rightarrow \Omega_A^n(T_A) \otimes_A U_B \rightarrow P_{n-1} \otimes_A U_B \rightarrow \cdots \rightarrow P_1 \otimes_A U_B \rightarrow P_0 \otimes_A U_B \rightarrow T \otimes_A U_B \rightarrow 0.$$

Note that for  $0 \leq i \leq n-1$ ,  $P_i \otimes_A U_B$  is projective, as  $U_B$  is projective. Since  $T \otimes_A U_B$  has finite projective dimension, by Schanuel's Lemma, for  $n \gg 0$  we get that  $\Omega_A^n(T) \otimes_A U_B$  is projective as a  $B$ -module. This proves that  ${}_A U_B$  is strongly right nonsingular.

Conversely, suppose that  ${}_A U_B$  is strongly right nonsingular. Take a minimal projective resolution

$$\cdots \rightarrow P_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow T_A \rightarrow 0$$

and apply  $-\otimes_A U_B$  to get a complex

$$0 \rightarrow \Omega_A^n(T) \otimes_A U_B \rightarrow P_{n-1} \otimes_A U_B \rightarrow \cdots \rightarrow P_1 \otimes_A U_B \rightarrow P_0 \otimes_A U_B \rightarrow T \otimes_A U_B \rightarrow 0$$

of  $B$ -modules. This complex is exact, as  ${}_A U$  is projective. As for  $n \gg 0$ , we have that  $\Omega_A^n(T) \otimes_A U_B$  is projective,  $T \otimes_A U_B$  has finite projective dimension.

We shall use (i) in the proof of (ii)-(iv).

(ii). Let  ${}_A U_B$  be a bimodule of finite projective dimension which is projective as left and as right module. Then there exists an exact sequence of  $A^{op} \otimes_K B$ -modules

$$0 \rightarrow P_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_0 \rightarrow U \rightarrow 0,$$

where for any  $0 \leq i \leq n$ ,  $P_i$  is a projective  $A^{op} \otimes_K B$ -module. As  ${}_A U$  is projective, the above sequence splits as exact sequence of left modules. So if we apply  $T_A \otimes_A -$ , it remains exact. Observe that all the  $B$ -modules  $(T \otimes_A P_i)_B$  are projective and thus the  $B$ -module  $(T \otimes_A U)_B$  has finite projective dimension. We have proved that  ${}_A U_B$  is strongly right nonsingular. The second statement follows from the first one by observing that the two bimodules  ${}_A X_A$  and  ${}_B Y_B$  are projective as left and as right modules.

(iii). Suppose  ${}_A A_A$  is strongly right nonsingular. Then by (i) for each right  $A$ -module  $T_A$ , the module  $T_A \simeq T \otimes_A A_A$  is of finite projective dimension. Therefore  $A$  has finite global dimension and by [6, Section 1], we have  $A \in \mathcal{P}^{<\infty}(A^e)$ . This proves the statement. Note that the relevant conclusion from [6, Section 1] is shown only under the hypothesis that  $A/J(A)$  is separable.

(iv). For each right  $B$ -module  $T_B$  we get isomorphisms of  $B$ -modules

$$T \otimes_B N \otimes_A M_B \simeq T \otimes_B ({}_B B_B \oplus {}_B Y_B) \simeq T_B \oplus (T \otimes_B Y_B).$$

If  ${}_A M_B$  is strongly right nonsingular, by (i)  $T \otimes_B N \otimes_A M_B$  has finite projective dimension as a right  $B$ -module, and thus  $T_B$  has finite projective dimension. As in (iii), this implies that the  $B^e$ -module  $B$  is an object of  $\mathcal{P}^{<\infty}(B^e)$ , which is a contradiction to the hypothesis on  $B$ .

The case of  ${}_B N_A$  is similar.

(v) is trivial. □

**Remark 3.5.** In [6, Section 1] an example is given showing that we do need the hypothesis in (ii) and (iii) that  $A/J(A)$  and  $B/J(B)$  are separable over  $K$ .

**Proof of Theorem 3.1** Denote  ${}_B \check{M}_A := \text{Hom}_{A^{op}}({}_A M_B, {}_A A_A)$  to simplify the notation. Then  $(- \otimes_B \check{M}_A, - \otimes_A M_B)$  is an adjoint pair of functors between  $\text{mod}(B)$  and  $\text{mod}(A)$ , because

$$\text{Hom}_A({}_B \check{M}_A, {}_A A_A) = \text{Hom}_A(\text{Hom}_A({}_A M_B, {}_A A_A), {}_A A_A) \simeq {}_A M_B,$$

as  ${}_A M$  is finitely generated projective. This pair of adjoint functors can be defined on  $D^b(\text{mod}(B))$  and  $D^b(\text{mod}(A))$ , as  ${}_B \check{M}$  and  ${}_A M$  are finitely generated projective modules. They induce functors between  $D_{sg}(A)$  and  $D_{sg}(B)$  since  $- \otimes_A M_B$  maps  $K^b(\text{proj}(A))$  to  $K^b(\text{proj}(B))$ , and since  $- \otimes_B \check{M}_A$  maps  $K^b(\text{proj}(B))$  to  $K^b(\text{proj}(A))$ .

Moreover,  $- \otimes_A M_B$  and  $- \otimes_B \check{M}_A$  induce functors between  $\underline{\text{mod}}_{\mathcal{P}^{<\infty}}(A)$  and  $\underline{\text{mod}}_{\mathcal{P}^{<\infty}}(B)$  since  $\mathcal{P}^{<\infty}(A) \otimes_A M_B$  belongs to  $\mathcal{P}^{<\infty}(B)$  and likewise for  $- \otimes_B \check{M}_A$ .

Let  $\eta : id_{\text{mod}(B)} \rightarrow - \otimes_B \check{M}_A \otimes_A M_B$  be the unit of the adjoint pair  $(- \otimes_B \check{M}_A, - \otimes_A M_B)$  between  $\text{mod}(B)$  and  $\text{mod}(A)$  and let  $\eta_B : B \rightarrow {}_B \check{M} \otimes_A M_B$  be its evaluation on  $B$ . As  ${}_B \check{M} \otimes_A M_B \simeq \text{End}_A({}_A M_B)$  as  $B^e$ -modules,  $\eta_B$  identifies with the structure map of the right  $B$ -module structure of  $M$ . By Lemma 2.4,  $\eta_B$  is injective and we can form a short exact sequence as follows:

$$0 \rightarrow {}_B B_B \xrightarrow{\eta_B} {}_B \check{M} \otimes_A M_B \rightarrow {}_B U_B \rightarrow 0 \quad (*).$$

Applying  ${}_A M_B \otimes_B -$  to the exact sequence (\*) gives the exact sequence

$$0 \rightarrow {}_A M_B \xrightarrow{Id_M \otimes \eta_B} {}_A M \otimes_B \check{M} \otimes_A M_B \rightarrow {}_A M \otimes_B U_B \rightarrow 0.$$

Now it is easy to see that the monomorphism  $Id_M \otimes \eta_B$  is split by the bimodule map

$${}_A M \otimes_B \check{M} \otimes_A M \simeq {}_A M \otimes_B \text{End}_A({}_A M_B) \rightarrow {}_A M_B$$

where the second map is the evaluation map. Hence

$${}_A M \otimes_B \check{M} \otimes_A M_B \simeq {}_A M_B \oplus ({}_A M \otimes_B U_B).$$

**Claim 1:**  $U_B$  is projective and  ${}_B U_B$  is strongly right nonsingular.

We shall use this claim for the moment and give the proof of Claim 1 just after having finished the proof of Theorem 3.1.

Applying  $-\otimes_B N$  to the isomorphism

$${}_A M \otimes_B \check{M} \otimes_A M_B \simeq {}_A M_B \oplus ({}_A M \otimes_B U_B)$$

gives

$$\begin{aligned} {}_A M \otimes_B \check{M} \otimes_A M \otimes_B N_A &\simeq ({}_A M_B \oplus ({}_A M \otimes_B U_B)) \otimes_B N_A \\ &\simeq ({}_A M \otimes_B N_A) \oplus ({}_A M \otimes_B U \otimes_B N_A) \\ &\simeq {}_A A_A \oplus {}_A X_A \oplus ({}_A M \otimes_B U \otimes_B N_A). \end{aligned}$$

But we also get

$$\begin{aligned} {}_A M \otimes_B \check{M} \otimes_A M \otimes_B N_A &\simeq {}_A M \otimes_B \check{M} \otimes_A (M \otimes_B N_A) \\ &\simeq {}_A M \otimes_B \check{M} \otimes_A ({}_A A_A \oplus {}_A X_A) \\ &\simeq ({}_A M \otimes_B \check{M}_A) \oplus ({}_A M \otimes_B \check{M} \otimes_A X_A). \end{aligned}$$

**Claim 2:**  ${}_A M \otimes_B U \otimes_B N_A$  and  ${}_A M \otimes_B \check{M} \otimes_A X_A$  are strongly right nonsingular.

Again we shall use this claim for the moment and give the proof of Claim 2 just after having finished the proof of Theorem 3.1.

The indecomposable  $A^e$ -module  $A$  is not strongly right nonsingular by Lemma 3.4 part (iii). The Krull-Schmidt theorem shows that the  $A^e$ -module  $A$  is a direct factor of  $M \otimes_B \check{M}$  or of  ${}_A M \otimes_B \check{M} \otimes_A X_A$ . Claim 2 shows that  ${}_A M \otimes_B \check{M} \otimes_A X_A$  is strongly right nonsingular, and hence all of its direct factors. Hence the  $A^e$ -module  $A$  is a direct factor of  $M \otimes_B \check{M}$ . This shows that there is an  $A^e$ -module  $\tilde{X}$  such that

$${}_A M \otimes_B \check{M}_A \simeq {}_A A_A \oplus {}_A \tilde{X}_A \quad (**).$$

The bimodule  ${}_A \tilde{X}_A$  is strongly right nonsingular by Lemma 3.4 (ii) and (v), as  ${}_A \tilde{X}_A$  is a direct summand of  ${}_A X_A \oplus ({}_A M \otimes_B U \otimes_B N_A)$ .

Now we apply  $N \otimes_A -$  to  $(**)$  and get

$${}_B N \otimes_A M \otimes_B \check{M}_A \simeq {}_B N_A \oplus ({}_B N \otimes_A \tilde{X}_A),$$

but

$${}_B N \otimes_A M \otimes_B \check{M}_A \simeq ({}_B B_B \oplus {}_B Y_B) \otimes_B \check{M}_A \simeq {}_B \check{M}_A \oplus ({}_B Y \otimes_B \check{M}_A).$$

So

$${}_B N_A \oplus ({}_B N \otimes_A \tilde{X}_A) \simeq {}_B \check{M}_A \oplus ({}_B Y \otimes_B \check{M}_A).$$

**Claim 3:**  ${}_B N \otimes_A \tilde{X}_A$  and  ${}_B Y \otimes_B \check{M}_A$  are strongly right nonsingular; the  $B^{op} \otimes_K A$ -module  ${}_B \check{M}_A$  is indecomposable.

Again we shall use this claim for the moment and give the proof of Claim 3 just after having finished the proof of Theorem 3.1.

As in Lemma 3.4 (iv) the module  ${}_B N_A$  is not strongly right nonsingular. We hence obtain that the two indecomposable bimodules  ${}_B N_A$  and  ${}_B \check{M}_A$  are isomorphic. □

**Proof of Claim 1** As in the paragraph preceding the statement of Claim 1, we have an isomorphism of bimodules

$${}_A M \otimes_B \check{M} \otimes_A M_B \simeq {}_A M_B \oplus ({}_A M \otimes_B U_B).$$

Since  $M_B$  and  $\check{M}_A$  are projective,  $M \otimes_B U_B$  is projective as a right  $B$ -module and since  $M_B$  is a progenerator by Proposition 2.4, we see that  $U_B$  is projective.

Given a right  $B$ -module  $T_B$ , we apply  $T \otimes_B -$  to  $(*)$  and we get an exact sequence

$$T_B \xrightarrow{\eta_T} T \otimes_B \check{M} \otimes_A M_B \rightarrow T \otimes_B U_B \rightarrow 0,$$

where  $\eta_T = id_T \otimes_B \eta_B$ .

As  $\eta_T$  is an isomorphism in  $D_{sg}(B)$ , there exists  $n \gg 0$  such that  $\Omega^n(\eta_T)$  is an isomorphism in  $\underline{\text{mod}}(B)$ . In fact, by [15, Example 2.3] or [2, Corollary 3.9(1)], given two  $B$ -modules  $V$  and  $W$ , we have

$$\text{Hom}_{D_{sg}(B)}(V, W) = \varinjlim \underline{\text{Hom}}_B(\Omega^i V, \Omega^i W).$$



Suppose that a module homomorphism  $f : V \rightarrow W$  is invertible in the singularity category. Then its inverse is induced from a module homomorphism  $g : \Omega^i(W) \rightarrow \Omega^i(V)$ . We see that  $\Omega^i(f) \circ g$  coincides with  $Id_W$  (resp.  $g \circ \Omega^i(f)$  coincides with  $Id_V$ ) in the singularity category, so  $\Omega^{n-i}(\Omega^i(f) \circ g) = \Omega^n(f) \circ \Omega^{n-i}(g)$  coincides with  $Id_{\Omega^n(N)}$  in  $\underline{\text{Hom}}_B(\Omega^n(V), \Omega^n(W))$  for big enough  $n$ .

Let  $P_*$  be the minimal projective resolution of  $T_B$  and let  $Q_*$  be the minimal projective resolution of  $T \otimes_B \check{M} \otimes_A M_B$ . As  $P_* \otimes_B \check{M} \otimes_A M_B$  is also a projective resolution of  $T \otimes_B \check{M} \otimes_A M_B$ , the Comparison Lemma gives a chain map  $f_* : P_* \otimes_B \check{M} \otimes_A M_B \rightarrow Q_*$ . Therefore, we have a commutative diagram

$$\begin{array}{ccc} P_* & \xrightarrow{\quad} & T_B \\ \eta_{P_*} \downarrow & & \downarrow \eta_T \\ P_* \otimes_B \check{M} \otimes_A M_B & \xrightarrow{\quad} & T \otimes_B \check{M} \otimes_A M_B \\ f_* \downarrow & & \downarrow = \\ Q_* & \xrightarrow{\quad} & T \otimes_B \check{M} \otimes_A M_B \end{array}$$

Note that the induced map

$$\Omega_B^n(T_B) \xrightarrow{\eta_{\Omega_B^n(T_B)}} \Omega_B^n(T_B) \otimes_B \check{M} \otimes_A M_B \xrightarrow{f_n} \Omega_B^n(T \otimes_B \check{M} \otimes_A M_B)$$

is just  $\Omega^n(\eta_T)$ , which is an isomorphism as  $n$  is supposed to be large enough, as we have seen.

As  $f_n$  induces an isomorphism between  $\Omega_B^n(T \otimes_B \check{M} \otimes_A M_B)$  and  $\Omega_B^n(T) \otimes_B \check{M} \otimes_A M_B$  in  $\underline{\text{mod}}(B)$ , we obtain that  $\eta_{\Omega_B^n(T_B)} : \Omega_B^n(T_B) \rightarrow \Omega_B^n(T) \otimes_B \check{M} \otimes_A M_B$  is an isomorphism in  $\underline{\text{mod}}(B)$ .

As we have an exact sequence of  $B$ -modules

$$\Omega_B^n(T_B) \xrightarrow{\eta_{\Omega_B^n(T_B)}} \Omega_B^n(T_B) \otimes_B \check{M} \otimes_A M_B \rightarrow \Omega_B^n(T_B) \otimes_B U_B \rightarrow 0,$$

we deduce that  $\eta_{\Omega_B^n(T_B)}$  has projective cokernel. In fact, let  $S_B$  be an indecomposable direct summand of  $\Omega_B^n(T_B)$ . Then  $\eta_{\Omega_B^n(T_B)}$  is the direct sum of such  $\eta_S$  and  $\eta_S$  is an isomorphism in  $\underline{\text{mod}}(B)$ . If  $S_B$  is projective,  $\eta_S$  is injective and  $\eta_S$  has projective cokernel, since  $\eta_B$  is injective with projective cokernel  $U_B$  by Claim 1. If  $S_B$  is not projective, then the fact that  $\eta_S$  is an isomorphism in  $\underline{\text{mod}}(B)$  implies that  $\eta_S$  has projective cokernel.

Since  $\eta_{\Omega_B^n(T_B)}$  has projective cokernel,  $\Omega_B^n(T_B) \otimes_B U_B$  is projective and the module  ${}_B U_B$  is strongly right nonsingular.

□

**Proof of Claim 2** Let  $T_A$  be an  $A$ -module. For  $n \geq 1$ ,  $\Omega_A^n(T_A) \otimes_A M_B \simeq \Omega_B^n(T \otimes_A M_B) \oplus P_B$  with  $P_B$  projective. Then

$$\Omega_A^n(T) \otimes_A M \otimes_B U \otimes_B N_A \simeq (\Omega_B^n(T \otimes_A M_B) \otimes_B U \otimes_B N_A) \oplus (P \otimes_B U \otimes_B N_A).$$

The  $A$ -module  $\Omega_B^n(T \otimes_A M_B) \otimes_B U \otimes_B N_A$  is projective for  $n$  big enough, as  ${}_B U_B$  is strongly right nonsingular and that  ${}_B N_A$  is projective as a left and right module; the module  $P \otimes_B U \otimes_B N_A$  is projective since  $U_B$  is projective. We have proved that  $\Omega_A^n(T) \otimes_A M \otimes_B U \otimes_B N_A$  is projective for  $n \gg 0$  and that  ${}_A M \otimes_B U \otimes_B N_A$  is strongly right nonsingular.

The fact that  ${}_A M \otimes_B \check{M} \otimes_A X_A$  is strongly right nonsingular follows from the fact that  ${}_A X_A$  is in  $\mathcal{P}^{<\infty}(A^e)$  and that  ${}_A M \otimes_B \check{M} \otimes_A X_A$  is projective as a left and right module.

□

**Proof of Claim 3** The fact that  ${}_B N \otimes_A \check{X}_A$  is strongly right nonsingular follows from that  ${}_B N_A$  is projective as a left and right module and that  ${}_A \check{X}_A$  is strongly right nonsingular.

The fact that  ${}_B Y \otimes_B \check{M}_A$  is strongly right nonsingular follows from that  ${}_B Y_B$  is in  $\mathcal{P}^{<\infty}(B^e)$  and that  ${}_B \check{M}_A$  is projective as a left and right module.

Suppose  $\check{M} = \check{M}_1 \oplus \check{M}_2$  as  $B^{op} \otimes_K A$ -modules. Then  $\text{Hom}_A({}_B \check{M}_A, {}_A A_A) \simeq {}_A M_B$  is indecomposable as  $B^{op} \otimes_K A$ -module implies that  $\text{Hom}_A(\check{M}_1, {}_A A_A) = 0$  or  $\text{Hom}_A(\check{M}_2, {}_A A_A) = 0$ . But  $M_B$  is projective, and so this happens only if  $M_1 = 0$  or  $M_2 = 0$ . Therefore  ${}_B \check{M}_A$  is indecomposable.  $\square$

We obtain the analogous result to [12, Corollary 3.1].

**Corollary 3.6.** *Under the same assumption of Theorem 3.1, suppose further that  $\text{Hom}_B({}_A M_B, {}_B B_B)$  is projective as an  $A$ -module, or  $\text{Hom}_{B^{op}}({}_B N_A, {}_B B_B)$  is projective as a left  $A$ -module. Then*

$${}_B N_A \simeq \text{Hom}_{A^{op}}({}_A M_B, {}_A A_A) \simeq \text{Hom}_B({}_A M_B, {}_B B_B)$$

and

$${}_A M_B \simeq \text{Hom}_A({}_B N_A, {}_A A_A) \simeq \text{Hom}_{B^{op}}({}_B N_A, {}_B B_B).$$

Moreover  $(M \otimes_B -, N \otimes_A -)$  and  $(N \otimes_A -, M \otimes_B -)$  are adjoint functors between  $\text{mod}(A^{op})$  and  $\text{mod}(B^{op})$ , which induce pairs of equivalences of the corresponding singularity categories.

Finally  $(- \otimes_A M_B, - \otimes_B N_A)$  and  $(- \otimes_B N_A, - \otimes_A M_B)$  are adjoint functors between  $\text{mod}(A)$  and  $\text{mod}(B)$ , which induce pairs of equivalences of the corresponding singularity categories.

**Proof** As a left (resp. right) adjoint to a given functor is unique up to isomorphisms, Theorem 3.1 shows that  $\text{Hom}_A({}_B N_A, {}_A A_A) \simeq {}_A M_B$  and in particular,  $\text{Hom}_A({}_B N_A, {}_A A_A)$  is projective as a right  $B$ -module.

On the other hand, if we suppose in Theorem 3.1 that  $\text{Hom}_A({}_B N_A, {}_A A_A)$  is projective as a right  $B$ -module instead of being projective for  $\text{Hom}_{A^{op}}({}_A M_B, {}_A A_A)$  as a left  $B$ -module, a dual proof as that of Theorem 3.1, by considering the functors  $(\text{Hom}_A({}_B N_A, {}_A A_A) \otimes_B -, N \otimes_A -)$  between left module categories  $\text{mod}(B^{op})$  and  $\text{mod}(A^{op})$ , gives that

$${}_A M_B \simeq \text{Hom}_A({}_B N_A, {}_A A_A)$$

as  $A^{op} \otimes_K B$ -modules, and  $(M \otimes_B -, N \otimes_A -)$  is a pair of adjoint functors between the module categories  $\text{mod}(A^{op})$  and  $\text{mod}(B^{op})$ . As in the first paragraph, we see that  $\text{Hom}_{A^{op}}({}_A M_B, {}_A A_A) \simeq {}_B N_A$  and in particular,  $\text{Hom}_{A^{op}}({}_A M_B, {}_A A_A)$  is projective as a left  $B$ -module.

This shows that the condition that  $\text{Hom}_{A^{op}}({}_A M_B, {}_A A_A)$  is projective as a left  $B$ -module and the condition that  $\text{Hom}_A({}_B N_A, {}_A A_A)$  is projective as a right  $B$ -module are equivalent. Furthermore, under these two equivalent conditions, we know that

- (i)  ${}_B N_A \simeq \text{Hom}_{A^{op}}({}_A M_B, {}_A A_A)$  and  ${}_A M_B \simeq \text{Hom}_A({}_B N_A, {}_A A_A)$ .
- (ii)  $(M \otimes_B -, N \otimes_A -)$  is a pair of adjoint functors between  $\text{mod}(B^{op})$  and  $\text{mod}(A^{op})$ , which induce pairs of equivalences of the corresponding singularity categories.
- (iii)  $(- \otimes_B N_A, - \otimes_A M_B)$  is a pair of adjoint functors between  $\text{mod}(B)$  and  $\text{mod}(A)$ , which induce pairs of equivalences of the corresponding singularity categories.

A dual proof of the above argument shows that the condition that  $\text{Hom}_B({}_A M_B, {}_B B_B)$  is projective as an  $A$ -module, and the condition that  $\text{Hom}_{B^{op}}({}_B N_A, {}_B B_B)$  is projective as an  $A$ -module, are equivalent; under these two conditions, we have

- (i)  ${}_B N_A \simeq \text{Hom}_B({}_A M_B, {}_B B_B)$  and  ${}_A M_B \simeq \text{Hom}_{B^{op}}({}_B N_A, {}_B B_B)$ .
- (ii)  $({}_B N \otimes_A -, {}_A M \otimes_B -)$  is a pair of adjoint functors between  $\text{mod}(A^{op})$  and  $\text{mod}(B^{op})$ , which induce pairs of equivalences of the corresponding singularity categories.
- (iii)  $(- \otimes_A M_B, - \otimes_B N_A)$  is a pair of adjoint functors between  $\text{mod}(A)$  and  $\text{mod}(B)$ , which induce pairs of equivalences of the corresponding singularity categories.

$\square$

Let  $\nu_A := \text{Hom}_K(\text{Hom}_A(-, A), K)$  be the Nakayama functor on  $\text{mod}(A)$ . If  $Q_A$  is a projective  $A$ -module, then  $\nu(Q_A)$  is an injective  $A$ -module, and if  $I_A$  is an injective  $A$ -module, then  $\nu(I_A)$  is a projective  $A$ -module.

**Lemma 3.7.** *Under the same assumption of Theorem 3.1, if  $I$  is injective as a  $B$ -module, then  $M \otimes_B I$  is injective as an  $A$ -module. Moreover  $(M \otimes_B -) \circ \nu_B \simeq \nu_A \circ (M \otimes_B -)$ .*

**Proof** We know that  $N \simeq \text{Hom}_A(M, A)$  and that  $N \otimes_A -$  is (left and) right adjoint to  $M \otimes_B -$ . Hence for an injective  $A$ -module  $I$  we get

$$\text{Hom}_B(-, N \otimes_A I) \simeq \text{Hom}_A(M \otimes_B -, I)$$

by Corollary 3.6. Moreover  $M \otimes_B -$  is exact since  $M$  is projective as a  $B$ -module.  $\text{Hom}_A(-, I)$  is exact since  $I$  is injective as an  $A$ -module. Therefore  $\text{Hom}_B(-, N \otimes_A I)$  is exact as a functor  $B\text{-mod} \rightarrow (A\text{-mod})^{\text{op}}$ , and we get therefore that  $N \otimes_A I$  is injective.

We have

$$\begin{aligned} \text{Hom}_A(M \otimes_B -, A) &\simeq \text{Hom}_B(-, \text{Hom}_A(M, A)) \\ &\simeq \text{Hom}_B(-, N) \\ &\simeq \text{Hom}_B(-, B) \otimes_B N \end{aligned}$$

as right  $A$  modules, since  $N$  is projective as  $B$ -module. Hence,

$$\begin{aligned} \nu_A(M \otimes_B -) &= \text{Hom}_K(\text{Hom}_A(M \otimes_B -, A), K) \\ &\simeq \text{Hom}_K(\text{Hom}_B(-, B) \otimes_B N, K) \\ &\simeq \text{Hom}_B(N, \text{Hom}_K(\text{Hom}_B(-, B), K)) \\ &\simeq \text{Hom}_B(N, B) \otimes_B \text{Hom}_K(\text{Hom}_B(-, B), K) \\ &\simeq M \otimes_B \nu_B(-) \end{aligned}$$

This shows the lemma. □

**Corollary 3.8.** *Under the same assumption of Theorem 3.1, the functor  $- \otimes_A M_B$  sends projective injective  $A$ -modules to projective injective  $B$ -modules.*

#### 4. SINGULAR EQUIVALENCES OF MORITA TYPE AND HOCHSCHILD HOMOLOGY

In this section, we consider invariant property of Hochschild homology under singular equivalences of Morita type. For stable equivalences of Morita type, in [20], Yu-Ming Liu and Chang-Chang Xi proved that a stable equivalence of Morita type preserves Hochschild homology groups of positive degrees. Remark that by [21, Theorem 1.1] the invariance of degree zero Hochschild homology group under a stable equivalence of Morita type is equivalent to the famous Auslander-Reiten conjecture on the invariance of the number of non projective simple modules under stable equivalence.

We shall now prove that a singular equivalence of Morita type induces an isomorphism of Hochschild homology in positive degrees.

**Theorem 4.1.** *Let  $K$  be a Noetherian commutative ring and let  $A$  and  $B$  be Noetherian  $K$ -algebras which are projective as  $K$ -modules. Suppose that  $({}_A M_B, {}_B N_A)$  induce a singular equivalence of Morita type.*

- (1) *Then there is  $n_0 \in \mathbb{N}$  so that  $HH_n(A) \simeq HH_n(B)$  for each  $n > n_0$ .*
- (2) *If  $K$  is a field, and if  $A$  and  $B$  are finite dimensional, then  $HH_n(A) \simeq HH_n(B)$  for each  $n > 0$ .*

Our proof of the first statement, inspired by [27, Section 1.2], is similar to that of [20, Theorem 4.4], which uses a change-of-rings argument. Notice that our argument is simpler than the proof in [20] and in fact works also for stable equivalences of Morita type. Our proof of the second statement makes use of transfer maps and is similar to that of [21, Remark 3.3].

**Proof of Theorem 4.1.(1).** Let  $\mathbb{B}A$  be the bar resolution of  $A$ , that is

$$\mathbb{B}A : \dots \rightarrow A^{\otimes 5} \rightarrow A^{\otimes 4} \rightarrow A^{\otimes 3} \rightarrow A^{\otimes 2} (\rightarrow A \rightarrow 0).$$

Then, we may apply  $N \otimes_A - \otimes_A M$  and obtain an exact sequence  $N \otimes_A \mathbb{B}A \otimes_A M$  of  $B^e$ -modules:

$$\dots \rightarrow N \otimes_A A^{\otimes 4} \otimes_A M \rightarrow N \otimes_A A^{\otimes 3} \otimes_A M \rightarrow N \otimes_A A^{\otimes 2} \otimes_A M (\rightarrow N \otimes_A M \rightarrow 0).$$

of  $B^e$ -modules, since  $M$  and  $N$  are projective on the right, resp. on the left.

The key observation is the following isomorphism of complexes

$$\begin{aligned} ({}_A M \otimes_B N_A) \otimes_{A^e} \mathbb{B}A &\simeq B \otimes_{B^e} ({}_B N \otimes_A \mathbb{B}A \otimes_A M_B) \\ (m \otimes n) \otimes u &\mapsto 1 \otimes (n \otimes u \otimes m). \end{aligned}$$

which is easily verified. Taking homology groups gives

$$HH_n(A) \oplus \text{Tor}_n^{A^e}(X, A) \simeq \text{Tor}_n^{A^e}(M \otimes_B N, A) \simeq \text{Tor}_n^{B^e}(B, N \otimes_A M) \simeq HH_n(B) \oplus \text{Tor}_n^{B^e}(B, Y)$$

for each  $n \geq 0$ . When  $n$  is large,  $\text{Tor}_n^{A^e}(X, A) \simeq 0 \simeq \text{Tor}_n^{B^e}(B, Y)$ , as  $X \in \mathcal{P}^{<\infty}(A^e)$  and  $Y \in \mathcal{P}^{<\infty}(B^e)$ , we obtain that  $HH_n(A) \simeq HH_n(B)$  for  $n \gg 0$ .  $\square$

For the proof of Theorem 4.1.(2), let us recall some properties of transfer maps in Hochschild homology.

Let  $A$  and  $B$  be two algebras over a commutative ring  $k$ . Let  $M$  be an  $A$ - $B$ -bimodule such that  $M_B$  is finitely generated and projective. Then we can define a transfer map  $t_M : HH_n(A) \rightarrow HH_n(B)$  for each  $n \geq 0$ . As we don't need the construction of this map, we refer the reader to Bouc [3] (see also [21, 17] for a summary of Bouc's results).

**Proposition 4.2.** [3, Section 3] *Let  $A$ ,  $B$  and  $C$  be  $k$ -algebras over a commutative ring  $k$ .*

- (1) *If  $M$  is an  $A$ - $B$ -bimodule and  $N$  is a  $B$ - $C$ -bimodule such that  $M_B$  and  $N_C$  are finitely generated and projective, then we have  $t_N \circ t_M = t_{M \otimes_B N} : HH_n(A) \rightarrow HH_n(C)$ , for each  $n \geq 0$ .*
- (2) *Let*

$$0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$$

*be a short exact sequence of  $A$ - $B$ -bimodules which are finitely generated and projective as right  $B$ -modules. Then  $t_M = t_L + t_N : HH_n(A) \rightarrow HH_n(B)$ , for each  $n \geq 0$ .*

- (3) *Suppose that  $k$  is an algebraically closed field and that  $A$  and  $B$  are finite dimensional  $k$ -algebras. Then for a finitely generated projective  $A$ - $B$ -bimodule  $P$ , the transfer map  $t_P : HH_n(A) \rightarrow HH_n(B)$  is zero for each  $n > 0$ .*
- (4) *Consider  $A$  as an  $A$ - $A$ -bimodule by left and right multiplications, then  $t_A : HH_n(A) \rightarrow HH_n(A)$  is the identity map for any  $n \geq 0$ .*

**Proof of Theorem 4.1.(2).** For  $n \geq 0$ , we have transfer maps  $t_M : HH_n(A) \rightarrow HH_n(B)$  and  $t_N : HH_n(B) \rightarrow HH_n(A)$ . By the above result,

$$t_N \circ t_M = t_{M \otimes_B N} = t_A + t_X = \text{Id} + t_X$$

as maps from  $HH_n(A)$  to itself.

Let  $\bar{K}$  be the algebraic closure of  $K$  and write

$$\begin{aligned} \bar{A} &= A \otimes_K \bar{K}, \\ \bar{B} &= B \otimes_K \bar{K}, \\ \bar{M} &= M \otimes_K \bar{K}, \\ \bar{N} &= N \otimes_K \bar{K}, \\ \bar{X} &= X \otimes_K \bar{K}, \\ \bar{Y} &= Y \otimes_K \bar{K}. \end{aligned}$$

Then one verifies easily that  $(\bar{A} \bar{M}_{\bar{B}}, \bar{B} \bar{N}_{\bar{A}})$  induces a singular equivalence of Morita type between  $\bar{A}$  and  $\bar{B}$ , because

$$\bar{A} \bar{M} \otimes_{\bar{B}} \bar{N}_{\bar{A}} \simeq \bar{A} \bar{A}_{\bar{A}} \oplus \bar{A} \bar{X}_{\bar{A}}$$

with  $\bar{X} \in \mathcal{P}^{<\infty}(\bar{A}^e)$ ;

$$\bar{B} \bar{N} \otimes_{\bar{A}} \bar{M}_{\bar{B}} \simeq \bar{B} \bar{B}_{\bar{B}} \oplus \bar{B} \bar{Y}_{\bar{B}}$$

with  $\bar{Y} \in \mathcal{P}^{<\infty}(\bar{B}^e)$ . We also have  $t_{\bar{M}} = t_M \otimes_K \text{id}_{\bar{K}}$ .

Since  $\overline{X} \in \mathcal{P}^{<\infty}(\overline{A}^e)$ , there is an exact sequence of  $\overline{A}^e$ -modules

$$0 \rightarrow \overline{P}_n \rightarrow \cdots \rightarrow \overline{P}_0 \rightarrow \overline{X} \rightarrow 0$$

with  $\overline{P}_0, \dots, \overline{P}_n$  projective. By the point (2)(3) of Proposition 4.2, for  $n > 0$ , we have  $t_{\overline{X}} = \sum_{i=0}^n (-1)^i t_{\overline{P}_i} = 0$  as a homomorphism from  $HH_n(\overline{A}) \rightarrow HH_n(\overline{A})$ , and thus  $t_X = 0 : HH_n(A) \rightarrow HH_n(A)$  for  $n > 0$ . This shows that

$$t_N \circ t_M : HH_n(A) \rightarrow HH_n(A)$$

and

$$t_M \circ t_N : HH_n(B) \rightarrow HH_n(B)$$

are isomorphisms for  $n > 0$ . We deduce that

$$t_M : HH_n(A) \rightarrow HH_n(B)$$

is an isomorphism for  $n > 0$ . □

**Remark 4.3.** Finally we briefly mention what is known in this context about invariance of Hochschild cohomology under stable equivalence of Morita type and under singular equivalence of Morita type.

Chang-Chang Xi proved in [26, Theorem 4.2] that a stable equivalence of Morita type between Artin algebras preserves the Hochschild cohomology groups of positive degrees, generalising a previous result of Zygumt Pogorzały [24, Theorem 1.1] for selfinjective algebras. Sheng-Yong Pan and the first author further showed in [23] the invariance of stable Hochschild cohomology rings under stable equivalences of Morita type.

Chen and Sun prove in [11] that Tate-Hochschild cohomology rings of Gorenstein algebras are preserved under singular equivalences of Morita type. A careful study of the proof of [26, Theorem 4.2] shows that the proof of [26, Theorem 4.2] works for singular equivalences of Morita type. We obtain from this study that a singular equivalence of Morita type preserves Hochschild cohomology groups of large degrees. However, we don't know the algebra structure.

## REFERENCES

- [1] Maurice Auslander and Idun Reiten, *Stable equivalence of Artin algebras*. Proceedings of the Conference on Orders, Group Rings and Related Topics (Ohio State Univ., Columbus, Ohio, 1972), 8-71. Springer Lecture Notes in Mathematics 353 (1973).
- [2] Apostolos Beligiannis, *The homological theory of contravariantly finite subcategories: Auslander-Buchweitz contexts, Gorenstein categories and (co-)stabilization*. Communications in Algebra **28** (2000), no. 10, 4547-4596.
- [3] Serge Bouc, *Bimodules, trace généralisée, et transferts en homologie de Hochschild*. Preprint, 1997.
- [4] Michel Broué, *Equivalences of blocks of group algebras*. Finite-dimensional algebras and related topics (Ottawa, ON, 1992), 126, NATO Advanced Science Institute Series C Mathematics Physics Sci., 424, Kluwer Academic Publisher, Dordrecht, 1994.
- [5] Ragnar-Olaf Buchweitz, *Maximal CohenMacaulay modules and Tate-cohomology over Gorenstein rings*. Unpublished manuscript, 1987, 155 pp.
- [6] Ragnar-Olaf Buchweitz, Edward Green, Dag Madsen and Oyvind Solberg, *Finite Hochschild cohomology without finite global dimension*. Mathematical Research Letters **12** (2005) 805-816.
- [7] Xiao-Wu Chen, *Singularity categories, Schur functors and triangular matrix rings*. Algebras and Representation Theory **12** (2009), no. 2-5, 181-191.
- [8] Xiao-Wu Chen, *Unifying two results of Orlov on singularity categories*. Abhandlungen des Mathematischen Seminars der Universität Hamburg **80** (2010), no. 2, 207-212.
- [9] Xiao-Wu Chen, *The singular category of an algebra with radical square zero*. Documenta Mathematica **16** (2011) 921-936.
- [10] Xiao-Wu Chen, *Relative singularity categories and Gorenstein-projective modules*. Mathematische Nachrichten **284** (2011), no. 2-3, 199-212.
- [11] Xiao-Wu Chen and Long-Gang Sun, *Singular equivalences of Morita type*. Preprint 2012.
- [12] Alex Dugas and Roberto Martinez-Villa, *A note on stable equivalence of Morita type*. Journal of Pure and Applied Algebra **208** (2007) 421-433.

- [13] Pierre Gabriel and Michel Zisman, *CALCULUS OF FRACTIONS AND HOMOTOPY THEORY*. Springer Verlag Heidelberg 1967.
- [14] Osamu Iyama, Kiriko Kato and Jun-Ichi Miyachi, *Recollement of homotopy categories and Cohen-Macaulay modules*. Journal of K-Theory **8** (2011), no. 3, 507-542.
- [15] Bernhard Keller and Dieter Vossieck, *Sous les catégories dérivées*. Comptes Rendus de l'Académie des Sciences Paris, Série I Mathématique **305** (6) (1987) 225-228.
- [16] Bernhard Keller, Daniel Murfet and Michel van den Bergh, *On two examples by Iyama and Yoshino*. Compositio Mathematica **147** (2011), no. 2, 591-612.
- [17] Steffen König, Yu-Ming Liu and Guodong Zhou, *Transfer maps in Hochschild (co)homology and applications to stable and derived invariants and to the Auslander-Reiten conjecture*. Transactions of the American Mathematical Society **364** (2012) 195-232.
- [18] Yu-Ming Liu, *On stable equivalences of Morita type for finite dimensional algebras*. Proceedings of the American Mathematical Society **131** (2003), 2657-2662.
- [19] Yu-Ming Liu, *Summands of stable equivalences of Morita type*. Communications in Algebra **36**(10) (2008), 3778-3782.
- [20] Yu-Ming Liu and Chang-Chang Xi, *Constructions of stable equivalences of Morita type for finite dimensional algebras, II*. Mathematische Zeitschrift **251** (2005) 21-39.
- [21] Yu-Ming Liu, Guodong Zhou and Alexander Zimmermann *Higman ideal, stable Hochschild homology and Auslander-Reiten conjecture*. Mathematische Zeitschrift **270** (2012) 759-781.
- [22] Dmitri Orlov, *Derived categories of coherent sheaves and triangulated categories of singularities*. Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. II, 503531, Progress in Mathematics, 270, Birkhäuser Boston, Inc., Boston, MA, 2009.
- [23] Sheng-Yong Pan and Guodong Zhou, *Stable equivalences of Morita type and stable Hochschild cohomology rings*. Archiv der Mathematik (Basel) **94** (2010), no. 6, 511-518.
- [24] Zygmunt Pogorzały, *Invariance of Hochschild cohomology algebras under stable equivalences of Morita type*. Journal of the Mathematical Society of Japan **53** (2001), 913-918.
- [25] Jeremy Rickard, *Derived categories and stable equivalences*. Journal of pure and applied Algebra **61** (1989) 303-317.
- [26] Chang-Chang Xi, *Stable equivalences of adjoint type*. Forum Mathematicum **20** (2008) 81-97.
- [27] Alexander Zimmermann, *Fine Hochschild invariants of derived categories for symmetric algebras*. Journal of Algebra **308** (2007) 350-367.

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