SYMMETRY OF THE DEFINITION OF DEGENERATION IN TRIANGULATED CATEGORIES

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ABSTRACT. Module structures of an algebra on a fixed finite dimensional vector space form an algebraic variety. Isomorphism classes correspond to orbits of the action of an algebraic group on this variety and a module is a degeneration of another if it belongs to the Zariski closure of the orbit. Riedtmann and Zwara gave an algebraic characterisation of this concept in terms of the existence of short exact sequences. Jensen, Su and Zimmermann, as well as independently Yoshino, studied the natural generalisation of the Riedtmann-Zwara degeneration to triangulated categories. The definition has an intrinsic non-symmetry. Suppose that we have a triangulated category in which idempotents split and either for which the endomorphism rings of all objects are artinian, or which is the category of compact objects in an algebraic compactly generated triangulated K-category. Then we show that the non-symmetry in the algebraic definition of the degeneration is inessential in the sense that the two possible choices which can be made in the definition lead to the same concept.

INTRODUCTION

For a finite dimensional K-algebra over an algebraically closed field K the set of d-dimensional A-modules is just the space of K-algebra homomorphisms from A to the algebra of d by d matrices over K. It carries therefore the structure of an algebraic variety $mod(A, d)$, and allows a $GL_d(K)$-action given by conjugation of matrices. $GL_d(K)$-orbits correspond to isomorphism classes of modules, and we say that a d-dimensional module M corresponding to the point $m \in mod(A, d)$ degenerates to the module N with corresponding point $n \in mod(A, d)$ if n belongs to the Zariski-closure of the orbit $GL_d(K) \cdot m$. We write in this case $M \leq_{\deg} N$. It is clear that $\leq_{\deg}$ is a partial order on the set of isomorphism classes of finite dimensional A-modules. Zwara and Riedtmann defined another relation between A-modules, namely $M \leq_{\mathrm{Zwara}} N$ if and only if there is a finite dimensional A-module Z and a short exact sequence $0 \to N \to M \oplus Z \to Z \to 0$. Moreover, they showed in [10, 22]

$$M \leq_{\deg} N \Leftrightarrow M \leq_{\mathrm{Zwara}} N.$$

Zwara showed in [21] by a purely algebraic argument that in the category of finite dimensional modules over an algebra there is Z and a short exact sequence $0 \to N \to M \oplus Z \to Z \to 0$ if and only if there is Z' and a short exact sequence $0 \to Z' \to M \oplus Z' \to N \to 0$.

In joint work with Jensen and Su [5], and independently by Yoshino in [17] for the (triangulated) stable category of maximal Cohen Macaulay modules over local Gorenstein k-algebras, the concept $\leq_{\mathrm{Zwara}}$ was generalised in the obvious way to general triangulated categories. More precisely, for a triangulated category $\mathcal{T}$ we define for two objects M and N that $M \leq_{\Delta} N$ if and only if there is an object Z and a distinguished triangle $Z \xrightarrow{\binom{u}{v}} M \oplus Z \longrightarrow N \to Z[1]$. Yoshino insisted in the point that one should ask that the induced endomorphism v of Z is nilpotent. Using Fitting's lemma and possibly replacing Z

by a suitable direct summand, this is automatic if one assumes Krull-Schmidt properties and artinian endomorphism rings for all objects. We denote $M \leq_{\Delta+\text{nil}} N$ if $M \leq_\Delta N$ and the induced endomorphism v on Z is nilpotent. The concept $\leq_\Delta$ was used in an essential way in work of Keller and Scherotzke on Nakajima quiver varieties. [5] concentrated on partial order properties of $\leq_\Delta$. Further conditions guaranteeing partial order properties of $\leq_\Delta$ can be found for various situations in [19], [18] and [13]. In this latter reference a geometric setting was developed replacing the module variety $mod(A, d)$ for general triangulated categories, mimicking for this purpose Yoshino's scheme theoretic approach [17]. Various results were given that ensure that $\leq_\Delta$ or $\leq_{\Delta+\text{nil}}$ define partial orders on the isomorphism classes of objects of $\mathcal{T}$.

Some authors define $\leq_\Delta$ (resp. $\leq_{\Delta+\text{nil}}$) by the existence of a distinguished triangle

$$N \to M \oplus Z \xrightarrow{(u_r \; v_r)} Z \to N[1],$$

and some define it as the existence of a distinguished triangle

$$Z \xrightarrow{\binom{u_\ell}{v_\ell}} M \oplus Z \to N \to Z[1].$$

Passing to the opposite category the two definitions relate to each other. Note that the opposite category of a triangulated category is triangulated as well. However, we show in this paper that actually the situation is even better. The two possible definitions lead to the same relation on the isomorphism classes of objects in two important cases. Our main result is the following. In its statement and in the rest of the paper 'artinian ring' means 'left and right artinian'.

Theorem 1. *Let K be a commutative ring and let $\mathcal{T}$ be a K-linear triangulated category satisfying one of the following two hypotheses:*
   (a) *Idempotents split in $\mathcal{T}$ and all endomorphism algebras of objects are artinian,*
   (b) *$\mathcal{T}$ is the category of compact objects in an algebraic compactly generated triangulated K-category.*
Then for any objects M, N of $\mathcal{T}$, the following assertions are equivalent:
   (1) *There is an object Z_ℓ of $\mathcal{T}$ and a distinguished triangle in $\mathcal{T}$*

$$Z_\ell \xrightarrow{\binom{v_\ell}{u_\ell}} Z_\ell \oplus M \longrightarrow N \longrightarrow Z_\ell[1],$$

   where v_ℓ is a nilpotent endomorphism of Z_ℓ.
   (2) *There is an object Z_r of $\mathcal{T}$ and a distinguished triangle in $\mathcal{T}$*

$$N \longrightarrow M \oplus Z_r \xrightarrow{(u_r \quad v_r)} Z_r \longrightarrow N[1],$$

   where v_r is a nilpotent endomorphism of Z_r.

It should be noted that since any (left or right) artinian ring is semiperfect (see [14, Examples VIII.4]), under the first situation of the theorem, the category $\mathcal{T}$ is Krull-Schmidt (see [2, Theorem A.1]). Moreover, under this hypothesis the assumption that v_ℓ (resp. v_r) is nilpotent is inessential. Indeed, a Fitting lemma type argument can then by applied and this shows that we can split off a trivial distinguished triangle as direct factor such that the remaining direct factor distinguished triangle satisfies the nilpotency hypothesis (cf the proof of Theorem 12 below for more details). We cannot avoid the artinian hypothesis in the first and the nilpotency hypothesis in both cases. The proof in the first case follows Zwara's arguments in [21] in the classical case, but there are quite a few subtleties arising by the non-uniqueness in the TR3-axiom of triangulated categories. Zwara frequently uses pushouts and pullbacks and in particular universal properties which come along with these concepts. We replace these constructions by homotopy cartesian squares, and have to cope with the lack of uniqueness of the related construction. The proof in the second case is

much more involved and heavily uses the concepts developed in [12]. The main idea in this approach is to use a dualisation functor like the K-duals for ordinary K-algebras A. However the situation is more involved here. The hypothesis that the triangulated category is the category of compact objects in an algebraic and compactly generated triangulated category gives that it is actually equivalent to the category of compact objects in the derived category of some small dg-category. Then, the new approach is to see this derived category as the derived category $\mathcal{D}(A)$ of some dg algebra without unit A, but with sufficiently many idempotents in a certain sense. Then, it can be shown that one may dualise with respect to A, using the derived functor of the suitable contravariant Hom functor to A. Further we use in particular the main result of [13] in full generality. The theory of dg algebras with enough idempotents parallels in a certain sense the development of dg categories as given by Keller but the situation is new. The approach is presented in [12], and we believe that such a theory is highly useful and should provide many further applications.

The paper is organised as follows. In Section 1 we give a summary of the contents of reference [12], in order to provide the vocabulary needed to understand the proof of the main result in the main body of the paper, without being obliged to go into the full details of that reference. In Section 2 we give the relevant background, facts and definitions of degenerations of objects in module categories, as well as in triangulated categories as it was shown in our earlier papers [5, 13]. Section 3 then proves the main result Theorem 1 under the hypothesis (a), i.e. in case idempotents split and all objects in the triangulated category have artinian endomorphism ring. The final Section 4 then gives the proof of Theorem 1 under hypothesis (b), i.e. in the case of a triangulated category which is the category of compact objects in an algebraic compactly generated triangulated K-category.

1. Review on triangulated categories, dg-categories and dg-algebras with enough idempotents

For the proof of Theorem 1 under hypothesis (b), which will cover Section 4, we shall need some concepts and statements from the theory of dg-algebras, dg-categories and triangulated categories in general which are not standard. In particular in case of categories which do not satisfy Krull-Schmidt theorem, we proceed by considering dg algebras without units, but having enough idempotents. The complete theory can be found in [12]. In order to facilitate the reading we summarize the results of this latter reference and introduce this way also the notations used in Section 4. All throughout the rest of the paper, let K be a commutative ring with unit and all categories which appear are assumed to be K-categories. The unadorned symbol $\otimes$ will stand for the tensor product over K.

1.1. **Dg categories and dg functors.** Recall that a *differential graded (dg) K-module* is a $\mathbb{Z}$-graded K-module V with a graded endomorphism $d : V \longrightarrow V$ of degree 1 and square 0, called the *differential* (here and all throughout the paper, when the term 'differential' is used to denote a graded map d, it will be assumed, without further remark, that $d \circ d = 0$ and that d is graded and of degree $+1$). We denote by Dg-K the category of dg K-modules. The morphism space $\mathrm{HOM}_K(V, W)$ in this category is again a dg K-module, where the homogeneous component of degree n, denoted $\mathrm{HOM}_K^n(V, W)$, consists of the homogeneous morphisms of degree n. The differential is given by $d_{Hom}(\alpha) = d_W \circ \alpha - (-1)^{|\alpha|}\alpha \circ d_V$, for any homogeneous morphism $\alpha \in \mathrm{HOM}_K(V, W)$, where $| \cdot |$ denotes the degree.

A *dg category* $\mathcal{A}$ (see [6] or [7]) is a category such that the morphism spaces $\mathrm{Hom}_{\mathcal{A}}(A, B) = \mathcal{A}(A, B)$ are dg K-modules, for all $A, B \in \mathrm{Ob}(\mathcal{A})$, and the composition map $\mathrm{Hom}_{\mathcal{A}}(B, C) \otimes \mathrm{Hom}_{\mathcal{A}}(A, B) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(A, C)$ satisfies Leibniz rule $d(g \circ f) = d(g) \circ f + (-1)^{|g|}g \circ d(f)$, for all homogeneous morphisms $f \in \mathrm{Hom}_{\mathcal{A}}(A, B)$ and $g \in \mathrm{Hom}_{\mathcal{A}}(B, C)$, where, abusing notation, we have denoted by d the differential on any of the appearing Hom spaces. The category Dg-K (denoted by $\mathcal{C}_{dg}K$ in [7]) is the prototype of a dg category. With any such category, one canonically associates its 0-*cycle category* $Z^0\mathcal{A}$ and its 0-*homology category* $\mathcal{H}^0\mathcal{A}$. Both of them have the same objects as $\mathcal{A}$, and as morphisms one puts

$\operatorname{Hom}_{Z^0\mathcal{A}}(A, A') = Z^0(\mathcal{A}(A, A'))$ and $\operatorname{Hom}_{H^0\mathcal{A}}(A, A') = H^0(\mathcal{A}(A, A'))$, for all $A, A' \in \operatorname{Ob}(\mathcal{A})$, the composition of morphisms in both cases being induced by the composition in $\mathcal{A}$. A *dg functor* $F : \mathcal{A} \longrightarrow \mathcal{B}$ between dg categories is just a functor which preserves the grading and the differential of Hom spaces. Any dg functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ induces corresponding functors $F = Z^0F : Z^0\mathcal{A} \longrightarrow Z^0\mathcal{B}$ and $F = H^0F : H^0\mathcal{A} \longrightarrow H^0\mathcal{B}$.

Associated to $\mathcal{A}$, there is also the *opposite dg category* $\mathcal{A}^{op}$ and, given another dg category $\mathcal{B}$, there is a definition of *tensor product of dg categories* $\mathcal{A} \otimes \mathcal{B}$, where the objects are pairs $(A, B) \in \operatorname{Ob}(\mathcal{A}) \times \operatorname{Ob}(\mathcal{B})$, the morphism spaces are $\operatorname{Hom}_{\mathcal{A} \otimes \mathcal{B}}((A, B), (A', B')) = \operatorname{Hom}_{\mathcal{A}}(A, A') \otimes \operatorname{Hom}_{\mathcal{B}}(B, B')$, with its differential as tensor product of dg K-modules, and the composition of homogeneous morphisms is given by $(\alpha_1 \otimes \beta_1) \circ (\alpha_2 \otimes \beta_2) = (-1)^{|\alpha_2||\beta_1|}(\alpha_1 \circ \alpha_2) \otimes (\beta_1 \circ \beta_2)$. Note that this tensor product construction applies to all categories considered in this paper since they are K-categories (=dg categories concentrated in degree zero). A *homological natural transformation* of dg functors $\tau : F \longrightarrow G$ is a natural transformation such that $\tau_A \in Z^0(\operatorname{Hom}_{\mathcal{B}}(F(A), G(A)))$, for any object $A \in \mathcal{A}$. If we have dg functors $F : \mathcal{A} \longrightarrow \mathcal{B}$ and $G : \mathcal{B} \longrightarrow \mathcal{A}$, then we have induced dg functors $\mathcal{A}^{op} \otimes \mathcal{B} \longrightarrow \operatorname{Dg-}K$, given by $\operatorname{Hom}_{\mathcal{B}}(F(?), ?)$ and $\operatorname{Hom}_{\mathcal{A}}(?, G(?))$. A *dg adjunction* is just an adjunction (F, G) of dg functors such that the natural isomorphism $\operatorname{Hom}_{\mathcal{B}}(F(?), ?) \xrightarrow{\cong} \operatorname{Hom}_{\mathcal{A}}(?, G(?))$ is a homological natural transformation. See [6] and [12, Section 1] for the details concerning dg categories and dg functors.

1.2. Dg categories and dg algebras with enough idempotents.
Any small K-category can be viewed as an *algebra with enough idempotents*. The latter is a K-algebra A with a distinguished family $(e_i)_{i \in I}$ of orthogonal idempotents such that $\bigoplus_{i \in I} e_i A = A = \bigoplus A e_i$. When such an algebra comes with a grading (as an algebra) such that the e_i are homogeneous of zero degree, and with a differential $d : A \longrightarrow A$ such that $d(e_i) = 0$, for all $i \in I$, and d satisfies Leibniz rule, then A or the pair (A, d) is called a *differential graded (dg) algebra with enough idempotents*. It is also shown in [12] that such an algebra may be viewed as a small dg category with I as set of objects. To any such algebra A one canonically associates a (non-small) dg category $\operatorname{Dg-}A$, whose objects are *right dg A-modules*. A right dg A module is just a graded right A-module M together with a differential $d_M : M \longrightarrow M$ such that $d_M(xa) = d_M(x)a + (-1)^{|x|}x d_A(a)$, for all homogeneous elements $x \in M$ and $a \in A$. Here and in the rest of the paper, unless otherwise specified, all modules are assumed to be unitary. That is, we assume that $M = MA$ in our case. We denote by $\operatorname{Gr-}A$ the category with objects the graded right A-modules and morphisms the graded A-homomorphisms of degree zero. This category comes with a canonical equivalence $?[1] : \operatorname{Gr-}A \longrightarrow \operatorname{Gr-}A$, and we put by $?[n] := (?[1])^n$ for each $n \in \mathbb{Z}$. Then, for each pair (M, N) of right dg A-modules, the corresponding space of morphisms in $\operatorname{Dg-}A$ is given by $\operatorname{HOM}_A(M, N) = \bigoplus_{n \in \mathbb{Z}} \operatorname{HOM}_A^n(M, N)$, where $\operatorname{HOM}_A^n(M, N) := \operatorname{Hom}_{\operatorname{Gr-}A}(M, N[n])$. The differential of $\operatorname{HOM}_A(M, N)$ is the restriction of the differential of $\operatorname{HOM}_K(M, N)$ (see the first paragraph of Section 1.1.) One similarly defines the opposite dg algebra with enough idempotents A^{op} and the tensor product $A \otimes B$ of dg algebras with enough idempotents. One then defines the dg category $A\operatorname{-Dg}$ of left dg modules and that of dg $A - B-$bimodules, which are equivalent to $\operatorname{Dg-}A^{op}$ and $\operatorname{Dg-}(B \otimes A^{op})$, respectively. This allows to treat the theories of left dg modules or dg bimodules over dg algebras with enough idempotents just as right dg modules.

1.3. Stable and derived category of a dg algebra with enough idempotents.
The 0-cycle (resp. 0-homology) category of $\operatorname{Dg-}A$ is denoted by $\mathcal{C}(A)$ (resp. $\mathcal{H}(A)$). The category $\mathcal{C}(A)$ is a bicomplete abelian category, with exact sequences as in $\operatorname{Gr-}A$, and, apart from this abelian structure, it also has a Quillen exact structure, called the *semi-split exact structure*, where the conflations (=admissible short exact sequences) are those exact sequences which split in $\operatorname{Gr-}A$ (see [1] for the terminology and main properties of exact categories). With this latter structure $\mathcal{C}(A)$ is Frobenius, that is, $\mathcal{C}(A)$ has enough projectives and injectives and

the injective objects coincide with the projective ones. These projective (=injective) objects are precisely the *contractible dg A-modules*, which are the direct summands in $\mathcal{C}(A)$ of cones of identity maps (see [12, Remark 6.2 and Proposition 6.3]). The stable category of $\mathcal{C}(A)$, which is then triangulated (see [3]), is precisely $\mathcal{H}(A)$. This latter (triangulated) category is called the *homotopy category* of A. The class of *quasi-isomorphisms* in $\mathcal{H}(A)$ (i.e. those morphisms which induce isomorphisms on homology) is a multiplicative system compatible with the triangulation in the terminology of Verdier (see [15]). The localization of $\mathcal{H}(A)$ with respect to the class of quasi-isomorphism, denoted $\mathcal{D}(A)$, is the *derived category* of A. It then has a unique structure of triangulated category such that the canonical functor $q : \mathcal{H}(A) \longrightarrow \mathcal{D}(A)$ is triangulated. In [12] (see, Theorem 3.1 in that reference) it is proved that the theory of dg modules over dg algebras with enough idempotents and their homotopy and derived categories is equivalent to the corresponding theory over small dg categories (see [6] and [7] for the details of this latter theory). As a consequence of Keller's famous theorem (see [6, Theorem 4.3]), one gets that any algebraic compactly generated triangulated category is equivalent to $\mathcal{D}(A)$, for some dg algebra with enough idempotents A (see [12, Corollary 6.10]). Recall that a triangulated category $\mathcal{T}$ is *algebraic* when it is equivalent to the stable category of some Frobenius exact category, and that it is called *compactly generated* when $\mathcal{T}$ has coproducts and there is a set of compact objects $\mathcal{C}$ in $\mathcal{T}$ such that $\bigcap_{C \in \mathcal{C}, n \in \mathbb{Z}} \mathrm{Ker}(\mathrm{Hom}_{\mathcal{T}}(C[n], ?)) = 0$. Recall that an object C is *compact* when the functor $\mathrm{Hom}_{\mathcal{T}}(C, ?) : \mathcal{T} \longrightarrow Ab$ preserves arbitrary coproducts.

1.4. **Derived functors.** Recall that a dg A-module is *acyclic* when it has zero homology or, equivalently (see e.g. [20, Lemma 3.5.32] or [15]), when it is the cone of a quasi-isomorphism. A dg A-module P (resp. I) is *homotopically projective* (resp. *homotopically injective*) when the functor $\mathrm{HOM}_A(P, ?) : \mathrm{Dg}\text{-}A \longrightarrow \mathrm{Dg}\text{-}K$ (resp. $\mathrm{HOM}_A(?, I) : (\mathrm{Dg}\text{-}A)^{op} \longrightarrow \mathrm{Dg}\text{-}K$) preserves acyclic dg modules, something which is equivalent to saying that the induced functor $\mathrm{Hom}_{\mathcal{H}(A)}(P, ?) : \mathcal{H}(A) \longrightarrow \mathrm{Mod} - K$ (resp. $\mathrm{Hom}_{\mathcal{H}(A)}(?, I) : \mathcal{H}(A)^{op} \longrightarrow \mathrm{Mod} - K$) vanishes on acyclic dg A-modules. As in the case of small dg categories, the canonical functor $q_A : \mathcal{H}(A) \longrightarrow \mathcal{D}(A)$ has a left adjoint functor $\Pi_A : \mathcal{D}(A) \longrightarrow \mathcal{H}(A)$, called the *homotopically projective resolution functor*, and a right adjoint $\Upsilon_A : \mathcal{D}(A) \longrightarrow \mathcal{H}(A)$, called the *homotopically injective resolution functor*, both of which are fully faithful and triangulated. They are so named because $\mathrm{Im}(\Pi_A)$ (resp. $\mathrm{Im}(\Upsilon_A)$) consists of homotopically projective (resp. homotopically injective) dg A-modules. The counit $\pi : \Pi_A \circ q_A \longrightarrow 1_{\mathcal{H}(A)}$ (resp. unit $\iota : 1_{\mathcal{H}(A)} \longrightarrow q_A \circ \Upsilon_A$) of the adjunction (Π_A, q_A) (resp. (q_A, Υ_A)) has the property that π_M (resp. ι_M) is a quasi-isomorphism, for each dg module M, and it is even an isomorphism when M is homotopically projective (resp. homotopically injective). Given a dg functor $F : \mathrm{Dg}\text{-}A \longrightarrow \mathrm{Dg}\text{-}B$ which preserves contractible dg modules, one defines its *left derived functor* (resp. *right derived functor*) $\mathbb{L}F : \mathcal{D}(A) \longrightarrow \mathcal{D}(B)$ (resp. $\mathbb{R}F : \mathcal{D}(A) \longrightarrow \mathcal{D}(B)$), as the composition

$$\mathcal{D}(A) \xrightarrow{\Pi_A} \mathcal{H}(A) \xrightarrow{H^0 F} \mathcal{H}(B) \xrightarrow{q_B} \mathcal{D}(B)$$

(resp. $\mathcal{D}(A) \xrightarrow{\Upsilon_A} \mathcal{H}(A) \xrightarrow{H^0 F} \mathcal{H}(B) \xrightarrow{q_B} \mathcal{D}(B)$). When the dg functor is contravariant, meaning that $F : (\mathrm{Dg}\text{-}A)^{op} \longrightarrow \mathrm{Dg}\text{-}B$ is a dg functor which preserves contractibility, then we define its *right derived functor* $\mathbb{R}F$ as the composition

$$\mathcal{D}(A)^{op} \xrightarrow{\Pi_A^o} \mathcal{H}(A)^{op} \xrightarrow{H^0 F} \mathcal{H}(B) \xrightarrow{q_B} \mathcal{D}(B).$$

All these derived functors are triangulated since they are composition of triangulated functors. If moreover $G : \mathrm{Dg}\text{-}A \longrightarrow \mathrm{Dg}\text{-}B$ is another dg functor as above and $\tau : F \longrightarrow G$ is a homological natural transformation of dg functors, then one obtains corresponding natural transformations of triangulated functors, still denoted the same, $\tau : \mathbb{L}F \longrightarrow \mathbb{L}G$ and $\tau : \mathbb{R}F \longrightarrow \mathbb{R}G$ in the covariant case, and just $\tau : \mathbb{R}F \longrightarrow \mathbb{R}G$ in the contravariant case.

Not only that, but any dg adjunction (F, G) of dg functors gives rise to a corresponding triangulated adjunction $(\mathbb{L}F, \mathbb{R}G)$ in the covariant case, and $((\mathbb{R}F)^o, \mathbb{R}G)$ in the contravariant case (see [12, Proposition 7.13]).

This somehow classical picture was exposed in [4, Section 1.6] for the development of dg bifunctors, and is extended in [12] to the case of dg algebras with enough idempotents. Concretely, if A, B and C are dg algebras with enough idempotents and $F : (\text{Dg-}A) \otimes (\text{Dg-}C) \longrightarrow \text{Dg-}B$ is a dg functor which preserves contractibility on both variables, then one defines

$$\mathbb{L}F : \mathcal{D}(A) \otimes \mathcal{D}(C) \overset{\Pi_A \otimes \Pi_C}{\longrightarrow} \mathcal{H}(A) \otimes \mathcal{H}(C) \overset{H^0 F}{\longrightarrow} \mathcal{H}(B) \overset{q_B}{\longrightarrow} \mathcal{D}(B),$$

and

$$\mathbb{R}F : \mathcal{D}(A) \otimes \mathcal{D}(C) \overset{\Upsilon_A \otimes \Upsilon_C}{\longrightarrow} \mathcal{H}(A) \otimes \mathcal{H}(C) \overset{H^0 F}{\longrightarrow} \mathcal{H}(B) \overset{q_B}{\longrightarrow} \mathcal{D}(B).$$

When F is contravariant on the first variable, i.e. when $F : (\text{Dg-}A)^{op} \otimes (\text{Dg-}C) \longrightarrow \text{Dg-}B$ is a dg functor, one also defines

$$\mathbb{R}F : \mathcal{D}(A)^{op} \otimes \mathcal{D}(C) \overset{\Pi_A^o \otimes \Upsilon_C}{\longrightarrow} \mathcal{H}(A) \otimes \mathcal{H}(C) \overset{H^0 F}{\longrightarrow} \mathcal{H}(B) \overset{q_B}{\longrightarrow} \mathcal{D}(B).$$

The point is that, under suitable conditions (see [12, Proposition 7.17] for details), these later bifunctors are triangulated on each variable.

1.5. **Derived Hom and $\otimes$ functors.** Given dg algebras with enough idempotents A, B and C and dg bimodules $_C M_A$, $_B X_A$ and $_C U_B$, the dg K-modules $\text{HOM}_A(M, X)$ and $U \otimes_B X$ have canonical structures of dg $B - C-$bimodule and dg $C - A-$bimodule, respectively, but the first one is non-unitary. Recall that we assumed in Section 1.2 that, unless otherwise specified, all modules are unital. This forces to define the 'unitarization'

$$\overline{\text{HOM}}_A(M, X) := B\text{HOM}_A(M, X)C,$$

which is then a (now unitary!) dg $B - C-$bimodule. It is proved in [12] that the assignments $(M, X) \rightsquigarrow \overline{\text{HOM}}_A(M, X)$ and $(U, X) \rightsquigarrow U \otimes_B X$ are the definition on objects of dg functors

$$\overline{\text{HOM}}_A(?, ?) : (C\text{-Dg-}A)^{op} \otimes (B\text{-Dg-}A) \longrightarrow B\text{-Dg-}C$$
$$? \otimes_B ? : (C\text{-Dg-}B) \otimes (B\text{-Dg-}A) \longrightarrow C\text{-Dg-}A.$$

One then puts

$$? \otimes_B^{\mathbb{L}} X := \mathbb{L}(? \otimes_B X) : \mathcal{D}(B \otimes C^{op}) \longrightarrow \mathcal{D}(A \otimes C^{op}),$$
$$U \otimes_B^{\mathbb{L}} ? := \mathbb{L}(U \otimes_B ?) : \mathcal{D}(A \otimes B^{op}) \longrightarrow \mathcal{D}(A \otimes C^{op}),$$
$$\mathbb{R}\text{Hom}_A(M, ?) := \mathbb{R}(\overline{\text{HOM}}_A(M, ?)) : \mathcal{D}(A \otimes B^{op}) \longrightarrow \mathcal{D}(C \otimes B^{op})$$
$$\mathbb{R}\text{Hom}_A(?, X) := \mathbb{R}(\overline{\text{HOM}}_A(?, X)) : \mathcal{D}(A \otimes C^{op})^{op} \longrightarrow \mathcal{D}(C \otimes B^{op}).$$

By [12, Theorems 9.1 and 9.5], the pairs

$$(? \otimes_B X : C\text{-Dg-}A \longrightarrow C\text{-Dg-}A, \overline{\text{HOM}}_A(X, ?) : C\text{-Dg-}A \longrightarrow C\text{-Dg-}B)$$

and

$$(\overline{\text{HOM}}_{B^{op}}(?, X)^o : B\text{-Dg-}C \to (C\text{-Dg-}A)^{op}, \overline{\text{HOM}}_A(?, X) : (C\text{-Dg-}A)^{op} \to B\text{-Dg-}C)$$

are dg adjunctions and, hence, we get adjunctions of triangulated functors

$$(? \otimes_B^{\mathbb{L}} X : \mathcal{D}(B \otimes C^{op}) \longrightarrow \mathcal{D}(A \otimes C^{op}), \mathbb{R}\text{Hom}_A(X, ?) : \mathcal{D}(A \otimes C^{op}) \longrightarrow \mathcal{D}(B \otimes C^{op}))$$

and

$$(\mathbb{R}\text{Hom}_{B^{op}}(?, X)^o : \mathcal{D}(C \otimes B^{op}) \longrightarrow \mathcal{D}(A \otimes C^{op}))^{op}, \mathbb{R}\text{Hom}_A(?, X) : \mathcal{D}(A \otimes C^{op}))^{op} \longrightarrow \mathcal{D}(C \otimes B^{op})).$$

By the previous paragraph, one also defines

$$\mathbb{R}\text{HOM}_A(?, ?) := \mathbb{R}(\overline{\text{HOM}}_A(?, ?)) : \mathcal{D}(A \otimes C^{op})^{op} \otimes \mathcal{D}(A \otimes B^{op}) \longrightarrow \mathcal{D}(C \otimes B^{op}),$$

which is then a functor which is triangulated in each variable. Moreover, precise conditions are given in [12, Corollary 9.7] to have a natural isomorphisms triangulated functors

$\mathbb{R}\mathrm{HOM}_A(M, ?) \cong \mathbb{R}\mathrm{Hom}_A(M, ?)$ and $\mathbb{R}\mathrm{HOM}_A(?, X) \cong \mathbb{R}\mathrm{Hom}_A(?, X)$. In particular, by taking $C = K$ in [12, Corollary 9.7] and its proof, one gets the following consequence, which will frequently be used in Section 4:

Proposition 2. *Let A and B be dg algebras with enough idempotents and let*

$$\mathbb{R}HOM_A(?, ?) := \mathbb{R}(\overline{HOM}_A(?, ?)) : \mathcal{D}(A)^{op} \otimes \mathcal{D}(A \otimes B^{op}) \longrightarrow \mathcal{D}(B^{op})$$

be the associated bi-triangulated functor. There are natural isomorphisms of triangulated functors, for all dg $B - A-$bimodules X and all right dg A-modules M:

  (1) $\mathbb{R}HOM_A(?, X) \cong \mathbb{R}Hom_A(?, X) : \mathcal{D}(A)^{op} \longrightarrow \mathcal{D}(B^{op})$.
  (2) $\mathbb{R}HOM_A(M, ?) \cong \mathbb{R}Hom_A(\Pi_A(M), ?) : \mathcal{D}(A \otimes B^{op}) \longrightarrow \mathcal{D}(B^{op})$.

On the other hand, when $X = {}_A A_A$ is the regular dg bimodule associated to the dg algebra with enough idempotents A, one has that the adjunction $(\mathbb{R}\mathrm{Hom}_{A^{op}}(?, A)^o : \mathcal{D}(A^{op}) \longrightarrow \mathcal{D}(A)^{op}, \mathbb{R}\mathrm{Hom}_A(?, A) : \mathcal{D}(A)^{op} \longrightarrow \mathcal{D}(A^{op}))$ gives by restriction quasi-inverse dualities $\mathrm{per}(A^{op}) \xrightarrow{\cong^o} \mathrm{per}(A)$, where $\mathrm{per}(A) = \mathcal{D}^c(A)$ (resp. $\mathrm{per}(A^{op}) = \mathcal{D}^c(A^{op})$) is the *right (resp. left) perfect derived category* of A, i.e. the full subcategory of $\mathcal{D}(A)$ (resp. $\mathcal{D}(A^{op})$) consisting of the compact objects. It is this duality what will allow us to pass from the left version of degeneration to the right version, and vice versa, in the proof of Theorem 1 under hypothesis (b).

2. Review of degeneration in triangulated categories

We start to recall from [5, 13] a few facts on the concept of degeneration of objects in triangulated categories.

2.1. **The module case.** We first recall a classical result due to Zwara and Riedtmann [10, 22]. Let A be a finite dimensional algebra over an algebraically closed field k and let M and N be two finite dimensional A-modules. Then N belongs to the closure of the orbit of M if and only if there is a finite dimensional A-module Z_r and a short exact sequence

$$0 \longrightarrow N \longrightarrow M \oplus Z_r \xrightarrow{\begin{pmatrix} u_r & v_r \end{pmatrix}} Z_r \longrightarrow 0,$$

where v_r is a nilpotent endomorphism of Z_r. We say in this case that M degenerates to N. Zwara shows in [21, Theorem 5] that M degenerates to N if and only if there is a finite dimensional A-module Z_ℓ of $\mathcal{T}$ and an exact sequence triangle

$$0 \longrightarrow Z_\ell \xrightarrow{\begin{pmatrix} v_\ell \\ u_\ell \end{pmatrix}} Z_\ell \oplus M \longrightarrow N \longrightarrow 0,$$

where v_ℓ is a nilpotent endomorphism of Z_ℓ.

2.2. **Generalising degeneration to triangulated categories.** In our previous work [5, 13], and independently by work of Yoshino [17] in the case of stable categories of maximal Cohen-Macaulay modules of local Gorenstein algebras, the concept of degeneration for modules was generalised to triangulated categories. Yoshino discovered in particular the importance of the hypothesis that the induced endomorphism of Z_ℓ (resp. Z_r) is nilpotent.

Definition 3. For two objects M and N of a triangulated category $\mathcal{T}$ we say that M degenerates to N in the triangle sense and write $M \leq_{\Delta+nil} N$ if and only if there is an object Z_ℓ of $\mathcal{T}$ and a distinguished triangle in $\mathcal{T}$

$$Z_\ell \xrightarrow{\begin{pmatrix} u_\ell \\ v_\ell \end{pmatrix}} M \oplus Z_\ell \longrightarrow N \longrightarrow Z_\ell[1],$$

where v_ℓ is a nilpotent endomorphism of Z_ℓ.

The main purpose of [13] was to define a geometric notion of degeneration along the lines of [17], and to prove that this notion is equivalent with the notion of degeneration in the triangle sense. More precisely we gave the following definition.

Definition 4. Let K be a commutative ring and let $\mathcal{C}_K^\circ$ be a K-linear triangulated category with split idempotents.

A degeneration data for $\mathcal{C}_K^\circ$ is given by

- a triangulated category $\mathcal{C}_K$ with split idempotents and a fully faithful embedding $\mathcal{C}_K^\circ \longrightarrow \mathcal{C}_K$,
- a triangulated category $\mathcal{C}_V$ with split idempotents and a full triangulated subcategory $\mathcal{C}_V^\circ$,
- triangulated functors $\uparrow_K^V \colon \mathcal{C}_K \longrightarrow \mathcal{C}_V$, which we write after the arguments, and $\Phi \colon \mathcal{C}_V^\circ \to \mathcal{C}_K$, so that $(\mathcal{C}_K^\circ) \uparrow_K^V \subseteq \mathcal{C}_V^\circ$, when we view $\mathcal{C}_K^\circ$ as a full subcategory of $\mathcal{C}_K$,
- a natural transformation $\mathrm{id}_{\mathcal{C}_V} \xrightarrow{\ t\ } \mathrm{id}_{\mathcal{C}_V}$ of triangulated functors

These triangulated categories and functors should satisfy the following axioms:

(1) For each object M of $\mathcal{C}_K^\circ$ the morphism $\Phi(M \uparrow_K^V) \xrightarrow{\ \Phi(t_{M\uparrow_K^V})\ } \Phi(M \uparrow_K^V)$ is a split monomorphism in $\mathcal{C}_K$.

(2) For all objects M of $\mathcal{C}_K^\circ$ we get $\Phi(\mathrm{cone}(t_{M\uparrow_K^V})) \simeq M$.

Degeneration is then given by the following concept.

Definition 5. Given two objects M and N of $\mathcal{C}_K^\circ$ we say that M degenerates to N in the categorical sense if there is a degeneration data for $\mathcal{C}_K^\circ$ and an object Q of $\mathcal{C}_V^\circ$ such that

$$p(Q) \simeq p(M \uparrow_K^V) \text{ in } \mathcal{C}_V^\circ[t^{-1}] \text{ and } \Phi(\mathrm{cone}(t_Q)) \simeq N,$$

where $p \colon \mathcal{C}_V^\circ \longrightarrow \mathcal{C}_V^\circ[t^{-1}]$ is the canonical functor. In this case we write $M \leq_{cdeg} N$.

Example 6. An analogous definition of degeneration was given by Yoshino in [16] in the case of degeneration in the category of modules over a ring. There Yoshino asked the object corresponding to Q to be flat over the power series ring. We do not ask such a property here. The reason is the following observation.

Yoshino observed that in a triangulated category $\mathcal{T}$ for all objects X we get $0 \leq_{\Delta+nil} X \oplus X[1]$. Indeed, $X \to 0 \to X[1] \xrightarrow{id} X[1]$ and $0 \to X \xrightarrow{id} X \to 0$ are distinguished triangles for each object X. Hence their direct sum $X \xrightarrow{0} X \oplus 0 \to X \oplus X[1] \to X[1]$ is a distinguished triangle as well. Taking $Z = X$ we get the result (in the left version of $\leq_{\Delta+nil}$).

Now, what about the degeneration data interpretation, which is equivalent to the triangle version in important cases? Then there is an object Q in some triangulated category $\mathcal{C}_V^\circ$ and an element t in its centre such that $\Phi(\mathrm{cone}(t_Q)) \simeq X \oplus X[1]$ and $p(Q) \simeq 0$ in $\mathcal{C}_V^\circ[t^{-1}]$, where p is the localisation functor. The latter isomorphism is equivalent to the fact that t_Q is nilpotent on Q. Hence we cannot assume, and actually do not assume, that Q is t-flat, as Yoshino does in the case of modules [16].

2.3. **When triangle degeneration is the same as categorical degeneration.** The main result of [13] is the following.

Theorem 7. *Let K be a commutative ring and let $\mathcal{C}_K^\circ$ be a triangulated K-category with split idempotents. If M and N are objects of $\mathcal{C}_K^\circ$, then $M \leq_{cdeg} N \Rightarrow M \leq_{\Delta+nil} N$. When $\mathcal{C}_K^\circ$ is equivalent to the category of compact objects of a compactly generated algebraic triangulated K-category, the converse is also true.*

In order to prove that $\leq_{\Delta+nil}$ implies $\leq_{cdeg}$ for the category of compact objects of a compactly generated algebraic triangulated K-category, we need to construct a degeneration data. By a result of Keller [6], we know that $\mathcal{T}$ is equivalent to the category $\mathcal{D}^c(\mathcal{A})$ of compact objects of the derived category $\mathcal{D}(\mathcal{A})$ of some small dg-category $\mathcal{A}$. We construct then the

degeneration data for $\mathcal{C}_K^\circ = \mathcal{D}^c(\mathcal{A})$ very explicitly, constructing a dg category $\mathcal{A}[[T]]$ from $\mathcal{A}$ and our proof of Theorem 1 under hypothesis (b), which is given in Section 4, uses this construction. More precisely, recall from the proof of [13, Proposition 9] that if $\mathcal{A}$ is a small dg category, then, considering a variable T, one can form a new dg category $\mathcal{A}[[T]]$ with the same set of objects as $\mathcal{A}$ and where one defines

$$\mathrm{Hom}^n_{\mathcal{A}[[T]]}(A, A') = \{\sum_{k\in\mathbb{N}} \alpha_k T^k \colon \alpha_k \in \mathrm{Hom}^n_{\mathcal{A}}(A, A'), \text{ for all } k \in \mathbb{N}\},$$

for all $n \in \mathbb{Z}$. Moreover, one gets a canonical functor

$$? \hat{\otimes} V : \mathcal{C}(\mathcal{A}) \longrightarrow \mathcal{C}(\mathcal{A}[[T]])$$

which takes a right dg $\mathcal{A}$-module M to the right dg $\mathcal{A}[[T]]$-module $M[[T]] : \mathcal{A}^{op} \longrightarrow \mathrm{Dg}\text{-}K$ acting on objects as $M[[T]]^n(A) = M^n(A)[[T]]$, for all $n \in \mathbb{Z}$ and all $A \in \mathcal{A}$. The degeneration data for $\mathcal{C}_K^\circ$ is then given by taking $\mathcal{C}_K = \mathcal{D}(\mathcal{A})$, with the corresponding inclusion functor as $\mathcal{C}_K^\circ \longrightarrow \mathcal{C}_K$, $\mathcal{C}_V = \mathcal{D}(\mathcal{A}[[T]])$, $\mathcal{C}_V^\circ = \mathcal{D}^c(\mathcal{A}[[T]])$, the triangulated version of $? \hat{\otimes} V$ as functor $\uparrow_K^V : \mathcal{C}_K = \mathcal{D}(\mathcal{A}) \longrightarrow \mathcal{D}(\mathcal{A}[[T]]) = \mathcal{C}_V$, the restriction of scalars functor as $\phi : \mathcal{C}_V^\circ = \mathcal{D}^c(\mathcal{A}[[T]]) \longrightarrow \mathcal{D}(\mathcal{A})$ and the natural transformation $t : id_{\mathcal{C}_V} \longrightarrow id_{\mathcal{C}_V}$ is defined by the maps $t_Q : Q \longrightarrow Q$, for each dg $\mathcal{A}[[T]]$-module Q, given by multiplication by T.

Note that in [13] the category $\mathcal{A}[[T]]$ is graded and the module $M[[T]]$ is graded. Therefore, as it is customary done and assumed implicitly in the proofs there, when writing elements one uses homogeneous elements and morphisms.

3. The case of triangulated categories whose objects have artinian endomorphism algebras

In this section we shall prove the first part of the theorem. We will mimic Zwara's proof to give the analogous statement for triangle degeneration.

3.1. Generalities on homotopy cartesian squares in triangulated categories.
As in [21] we first need some preparation. Throughout this section let $\mathcal{T}$ be a triangulated K-category for a commutative base ring K. The crucial concept is that of a *homotopy cartesian square*. Recall from [9, 1.4] that a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{a} & C \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle c} \\ B & \xrightarrow{d} & D \end{array}$$

is homotopy cartesian if there is a map $e : D \to A[1]$ such that

$$A \xrightarrow{\binom{b}{a}} B \oplus C \xrightarrow{(-c,d)} D \xrightarrow{e} A[1]$$

is a distinguished triangle. In the rest of this section, denote by C_f the cone of any morphism f in $\mathcal{T}$. By [9, Lemma 1.4.3], if a commutative square

$$\begin{array}{ccc} A & \xrightarrow{a} & C \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle c} \\ B & \xrightarrow{d} & D \end{array}$$

is homotopy cartesian, then there is an isomorphism $\sigma : C_a \xrightarrow{\cong} C_d$ such that

$$\begin{array}{ccccccc} A & \xrightarrow{a} & C & \longrightarrow & C_a & \longrightarrow & A[1] \\ {\scriptstyle b}\downarrow & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle b[1]} \\ B & \xrightarrow{d} & D & \longrightarrow & C_d & \longrightarrow & B[1] \end{array}$$

is commutative with rows being distinguished triangles. [9, Lemma 1.4.4] gives a partial converse. If

$$
\begin{array}{ccccccc}
A & \xrightarrow{a} & C & \longrightarrow & C_a & \longrightarrow & A[1] \\
{\scriptstyle b}\downarrow & & {\scriptstyle c}\downarrow & & {\scriptstyle \sigma}\downarrow & & \downarrow{\scriptstyle b[1]} \\
B & \xrightarrow{d} & D & \longrightarrow & C_d & \longrightarrow & B[1]
\end{array}
$$

is commutative, where the rows are distinguished triangles and σ is an isomorphism, then there is a possible different $b' : A \to B$ such that still

$$
\begin{array}{ccccccc}
A & \xrightarrow{a} & C & \longrightarrow & C_a & \longrightarrow & A[1] \\
{\scriptstyle b'}\downarrow & & {\scriptstyle c}\downarrow & & {\scriptstyle \sigma}\downarrow & & \downarrow{\scriptstyle b'[1]} \\
B & \xrightarrow{d} & D & \longrightarrow & C_d & \longrightarrow & B[1]
\end{array}
$$

is commutative and furthermore

$$
\begin{array}{ccc}
A & \xrightarrow{a} & C \\
{\scriptstyle b'}\downarrow & & \downarrow{\scriptstyle c} \\
B & \xrightarrow{d} & D
\end{array}
$$

is homotopy cartesian. We may alternatively modify c instead of b.

This problem has the annoying consequence that if we have two homotopy cartesian squares

$$
\begin{array}{ccc}
A & \xrightarrow{a} & C \\
{\scriptstyle b}\downarrow & & \downarrow{\scriptstyle c} \\
B & \xrightarrow{d} & D
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
C & \xrightarrow{e} & E \\
{\scriptstyle c}\downarrow & & \downarrow{\scriptstyle f} \\
D & \xrightarrow{g} & F
\end{array}
$$

then there are b' and f' fitting in certain morphisms of triangles, such that

$$
\begin{array}{ccc}
A & \xrightarrow{ea} & E \\
{\scriptstyle b'}\downarrow & & \downarrow{\scriptstyle f'} \\
B & \xrightarrow{gd} & F
\end{array}
$$

is a homotopy cartesian square. We would like to be able to assume that $b' = b$ and $f' = f$. However, we do not know if this is true. Nevertheless, we prove a weaker statement which satisfies our needs.

Lemma 8. *Let*

$$
\begin{array}{ccc}
X_1 & \xrightarrow{u_1} & X_2 \\
\downarrow & & \downarrow{\scriptstyle f_2} \\
0 & \longrightarrow & Y_2
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
X_2 & \xrightarrow{\nu_1} & X_3 \\
{\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle f_3} \\
Y_2 & \xrightarrow{\nu_2} & Y_3
\end{array}
$$

be homotopy cartesian squares. Then

$$
\begin{array}{ccc}
X_1 & \xrightarrow{\nu_1 u_1} & X_3 \\
\downarrow & & \downarrow{\scriptstyle f_3} \\
0 & \longrightarrow & Y_3
\end{array}
$$

is a homotopy cartesian square.

Proof. We first apply the octahedron axiom to the composition $\nu_1 u_1$. We hence obtain a morphism $v : X_3 \to C_{\nu_1 u_1}$, a morphism $\omega : Y_2 \to C_{\nu_1 u_1}$, a morphism $C_{\nu_1 u_1} \to C_{\nu_1}$, and $C_{\nu_1} \to Y_2[1]$ as indicated below,

$$
\begin{array}{ccccccc}
X_1 & \xrightarrow{u_1} & X_2 & \xrightarrow{\quad f_2 \quad} & Y_2 & \xrightarrow{\ +\ } & X_1[1] \\
 & & & \searrow{\nu_1} & \vdots & & \\
 & \searrow{\nu_1 u_1} & & & \vdots\,\omega & & \\
 & & X_3 & & \downarrow & & \\
 & & & \searrow{v} & C_{\nu_1 u_1} & \xrightarrow{\ +\ } & \\
 & & & & \vdots & & X_1[1] \\
 & & & & C_{\nu_1} & \searrow{+} & \\
 & & & & \vdots\,+ & \searrow & \\
 & & & & Y_2[1] & & X_2[1]
\end{array}
$$

such that all straight sequences represent distinguished triangles, and such that the diagram is commutative. In particular, $v\nu_1 = \omega f_2$. Neeman's interpretation of the octahedral axiom [9, Proposition 1.4.6] implies that we may choose v and ω such that

$$
\begin{array}{ccc}
X_2 & \xrightarrow{\nu_1} & X_3 \\
\downarrow{f_2} & & \downarrow{v} \\
Y_2 & \xrightarrow{\omega} & C_{\nu_1 u_1}
\end{array}
$$

is a homotopy cartesian square. Since

$$
\begin{array}{ccc}
X_2 & \xrightarrow{\nu_1} & X_3 \\
\downarrow{f_2} & & \downarrow{f_3} \\
Y_2 & \xrightarrow{\nu_2} & Y_3
\end{array}
$$

is homotopy cartesian by hypothesis, there is an isomorphism $\varphi : C_{\nu_1 u_1} \xrightarrow{\ \cong\ } Y_3$ such that

$$
\begin{array}{ccc}
X_2 & \xrightarrow{\nu_1} & X_3 \\
\downarrow{f_2} & \quad v\downarrow & \searrow{f_3} \\
Y_2 & \xrightarrow{\omega} & C_{\nu_1 u_1} \\
 & \searrow{\nu_2} & \downarrow{\varphi} \\
 & & Y_3
\end{array}
$$

is commutative. This then shows that

$$
X_1 \xrightarrow{\nu_1 u_1} X_3 \xrightarrow{f_3} Y_3 \longrightarrow X_1[1]
$$

is a distinguished triangle, and hence

$$
\begin{array}{ccc}
X_1 & \xrightarrow{\nu_1 u_1} & X_3 \\
\downarrow & & \downarrow{f_3} \\
0 & \longrightarrow & Y_3
\end{array}
$$

is a homotopy cartesian square as claimed. $\qquad\qquad\square$

A partial converse is true in general however.

Lemma 9. *Let*

$$
\begin{array}{ccc}
X_1 & \xrightarrow{u_1} & X_2 \\
{\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} \\
Y_1 & \xrightarrow{u_2} & Y_2
\end{array}
\qquad and \qquad
\begin{array}{ccc}
X_2 & \xrightarrow{\nu_1} & X_3 \\
{\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle f_3} \\
Y_2 & \xrightarrow{\nu_2} & Y_3
\end{array}
$$

be commutative diagrams. If the first square and

$$
\begin{array}{ccc}
X_1 & \xrightarrow{\nu_1 u_1} & X_3 \\
{\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_3} \\
Y_1 & \xrightarrow{\nu_2 u_2} & Y_3
\end{array}
$$

are homotopy cartesian squares, then there is a morphism $\hat{\nu}_2$ with $\nu_2 u_2 = \hat{\nu}_2 u_2$ such that

$$
\begin{array}{ccc}
X_2 & \xrightarrow{\nu_1} & X_3 \\
{\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle f_3} \\
Y_2 & \xrightarrow{\hat{\nu}_2} & Y_3
\end{array}
$$

is a homotopy cartesian square.

Proof. We supposed that

$$
\begin{array}{ccc}
X_1 & \xrightarrow{u_1} & X_2 \\
{\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} \\
Y_1 & \xrightarrow{u_2} & Y_2
\end{array}
\qquad and \qquad
\begin{array}{ccc}
X_1 & \xrightarrow{\nu_1 u_1} & X_3 \\
{\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_3} \\
Y_1 & \xrightarrow{\nu_2 u_2} & Y_3
\end{array}
$$

are homotopy cartesian squares. Hence we get distinguished triangles

$$
Y_2[-1] \xrightarrow{\sigma_1[-1]} X_1 \xrightarrow{\ \ \ }
\begin{array}{c}
\qquad X_2 \\
{\scriptstyle u_1}\nearrow \qquad \searrow {\scriptstyle -f_2} \\
\qquad\qquad\quad \oplus \qquad\qquad Y_2 \xrightarrow{\sigma_1} X_1[1] \\
{\scriptstyle f_1}\searrow \qquad \nearrow {\scriptstyle u_2} \\
\qquad Y_1
\end{array}
$$

and

$$
Y_3[-1] \xrightarrow{\sigma_2[-1]} X_1 \xrightarrow{\ \ \ }
\begin{array}{c}
\qquad X_3 \\
{\scriptstyle \nu_1 u_1}\nearrow \qquad \searrow {\scriptstyle -f_3} \\
\qquad\qquad\quad \oplus \qquad\qquad Y_3 \xrightarrow{\sigma_2} X_1[1] \\
{\scriptstyle f_1}\searrow \qquad \nearrow {\scriptstyle \nu_2 u_2} \\
\qquad Y_1
\end{array}
$$

We further get a morphism of distinguished triangles by the fact that the second comes from a square which factors through the first one.
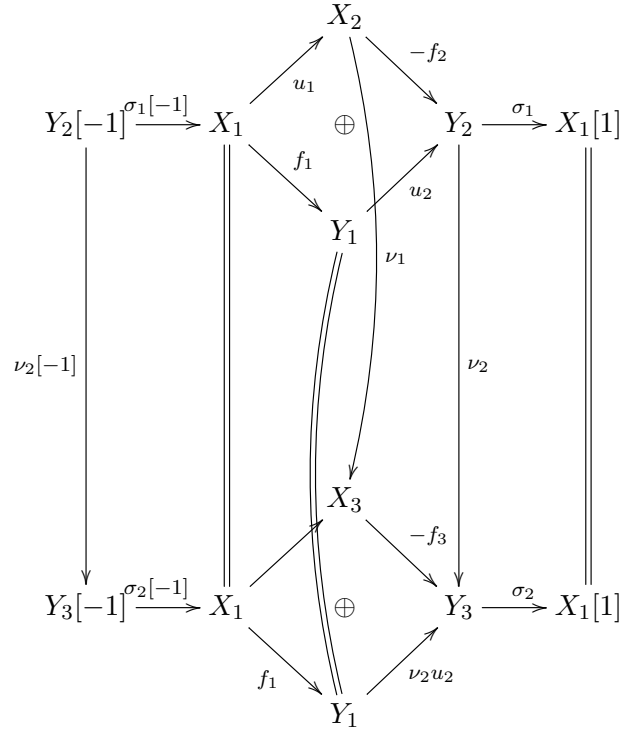
By Neeman's axiom TR4' we maybe need to modify ν_2 to another map $\hat{\nu}_2$, forming still a map of distinguished triangles, so that the cone of this commutative diagram is a distinguished triangle.

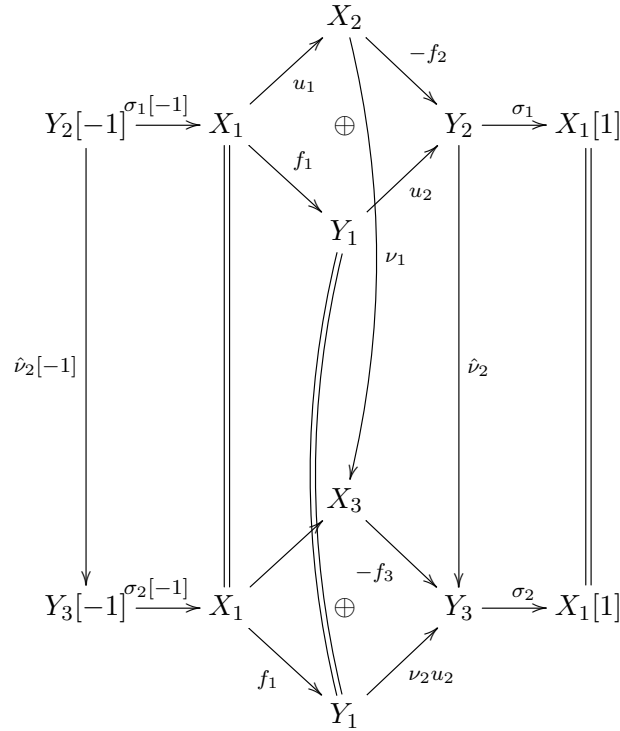This implies in particular that $\nu_2 u_2 = \hat{\nu}_2 u_2$. The cone of this has a direct factor isomorphic to

$$
\begin{array}{ccc}
& X_3 & \\
\nu_1 \nearrow & & \searrow -f_3 \\
Y_2[-1] \xrightarrow{\sigma_3[-1]} X_2 & \oplus & Y_3 \xrightarrow{\sigma_3} X_2[1] \\
f_2 \searrow & & \nearrow \hat{\nu}_2 \\
& Y_2 &
\end{array}
$$

which is therefore a distinguished triangle. This proves the statement. $\qquad\square$

3.2. **Triangle degeneration is symmetric with respect to left or right existence of the object** Z**.** We now give the main technical argument of this section.

Proposition 10. *Let* $\mathcal{T}$ *be a triangulated category with split idempotents, and let* M *and* N *be objects of* $\mathcal{T}$*. If there is a distinguished triangle*

$$
Z_\ell \xrightarrow{\binom{u_\ell}{v_\ell}} M \oplus Z_\ell \longrightarrow N \longrightarrow Z_\ell[1]
$$

with v_ℓ *a nilpotent endomorphism, then there is a distinguished triangle*

$$
N \longrightarrow M \oplus Z_r \xrightarrow{(u_r,v_r)} Z_r \longrightarrow N[1] \ .
$$

Proof. Set $N_1 := N$. Let Z_ℓ be an object and let

$$
Z_\ell \xrightarrow{\binom{u_\ell}{v_\ell}} M \oplus Z_\ell \xrightarrow{(\nu_1,h_1)} N_1 \xrightarrow{\rho_0} Z_\ell[1]
$$

be a distinguished triangle with nilpotent endomorphism v_ℓ. Then we form the homotopy pushout (cf [9, Lemma 1.4.3 and 1.4.4])

$$
\begin{array}{ccccccc}
Z_\ell & \xrightarrow{\binom{u_\ell}{v_\ell}} & M \oplus Z_\ell & \xrightarrow{(\nu_1,h_1)} & N_1 & \xrightarrow{\rho_0} & Z_\ell[1] \\
\downarrow{h_1} & & \downarrow{(\nu_2,h_2)} & & \downarrow{\mathrm{id}} & & \downarrow{h_1[1]} \\
N_1 & \xrightarrow{s_1} & N_2 & \xrightarrow{t_1} & N_1 & \xrightarrow{\rho_1} & N_1[1]
\end{array}
$$

We get $h_1 = t_1 h_2$ from the commutativity of the middle square and $h_1[1]\rho_0 = \rho_1$ from the right most square.

We proceed now by induction on the degree n. Denote for short $\hat{v} := \binom{v_\ell}{u_\ell}$. For $n = 2$, putting $N_0 := 0$ the above construction of h_1, h_2, ν_1, ν_2, s_1 and t_1 implies the existence of a commutative diagram whose square faces are homotopy cartesian squares.

Suppose now for some n we have the above commutative diagram whose square faces are homotopy cartesian squares. We shall construct such a diagram for $n + 1$.

We form the homotopy pushout (on the back face of the diagram defining N_{n+1}, h_{n+1}, s_n and ν_{n+1})

$$
\begin{array}{ccc}
Z_\ell & \xrightarrow{\ \hat{v}\ } & Z_\ell \oplus M \\
\end{array}
$$

Since $(h_n, \nu_n) \circ \hat{v} = s_{n-1} \circ t_{n-1} \circ h_n$, there is a morphism $t_n : N_{n+1} \to N_n$ making the diagram

commutative. By Lemma 9 we may modify t_n without changing commutativity of what is already shown (we denote the modified t_n again by t_n), such that the bottom face of the diagram is a homotopy pushout and such the above diagram is still commutative.

We now continue as in the proof of [21, Theorem 2.3]. We define $\omega_1 := \nu_1$, $M^1 := M$ and

$$\omega_{j+1} := (s_j \omega_j, \nu_{j+1}) \in Hom_{\mathcal{T}}(M^j \oplus M, N_{j+1}) = Hom_{\mathcal{T}}(M^{j+1}, N_{j+1})$$

for all $j \geq 1$.

In the diagram

$$
\begin{array}{ccccccc}
Z_\ell & \xrightarrow{\binom{u_\ell}{v_\ell}} & M \oplus Z_\ell & \xrightarrow{(\nu_1, h_1)} & N & \xrightarrow{\rho} & Z_\ell[1] \\
\downarrow{h_j} & & \downarrow{(\nu_{j+1}, h_{j+1})} & & \downarrow{\mathrm{id}} & & \downarrow{h_2[1]} \\
N_j & \xrightarrow{s_j} & N_{j+1} & \xrightarrow{v_j} & N & \xrightarrow{\rho_j} & N_j[1]
\end{array}
$$

the left square is a homotopy cartesian square. Therefore we get a distinguished triangle

$$
Z_\ell \xrightarrow{\begin{pmatrix} u_\ell \\ v_\ell \\ h_j \end{pmatrix}} M \oplus Z_\ell \oplus N_j \xrightarrow{(\nu_{j+1}, h_{j+1}, -s_j)} N_{j+1} \xrightarrow{\sigma_j} Z_\ell[1] \ .
$$

Since

$$M^j \xrightarrow{\ \mathrm{id}\ } M^j \longrightarrow 0 \longrightarrow M^j[1]$$

is a distinguished triangle, also the direct sum of these two distinguished triangles

$$
M^j \oplus Z_\ell \xrightarrow{\begin{pmatrix} 1_{M^j} & 0 \\ 0 & u_\ell \\ 0 & v_\ell \\ 0 & h_j \end{pmatrix}} M^j \oplus M \oplus Z_\ell \oplus N_j \xrightarrow{(0, \nu_{j+1}, h_{j+1}, -s_j)} N_{j+1} \xrightarrow{(0, \sigma_j)} (M^j \oplus Z_\ell)[1]
$$

is distinguished. For the series of morphisms $\omega_j : M^j \to N_j$ satisfying $\omega_1 = \nu_1$ and $\omega_{j+1} = (s_j\omega_j, \nu_{j+1})$ for all j, this distinguished triangle is isomorphic to the triangle

$$M^j \oplus Z_\ell \xrightarrow{\begin{pmatrix} 1_{M^j} & 0 \\ 0 & u_\ell \\ 0 & v_\ell \\ \omega_j & h_j \end{pmatrix}} M^j \oplus M \oplus Z_\ell \oplus N_j \xrightarrow{(s_j\omega_j, \nu_{j+1}, h_{j+1}, -s_j)} N_{j+1} \xrightarrow{(0,\sigma_j)} (M^j \oplus Z_\ell)[1] \ .$$

Indeed, we get morphisms of triangles

$$
\begin{array}{ccccccc}
M^j & \xrightarrow{\ \mathrm{id}\ } & M^j & \longrightarrow & 0 & \longrightarrow & M^j[1] \\
\uparrow{\scriptstyle (1,0)} & & \uparrow{\scriptstyle (1,0,0,0)} & & \uparrow & & \uparrow \\
M^j \oplus Z_\ell & \xrightarrow{\begin{pmatrix} 1_{M^j} & 0 \\ 0 & u_\ell \\ 0 & v_\ell \\ \omega_j & h_j \end{pmatrix}} & M^j \oplus M \oplus Z_\ell \oplus N_j & \xrightarrow{(s_j\omega_j, \nu_{j+1}, h_{j+1}, -s_j)} & N_{j+1} & \longrightarrow & (M^j \oplus Z_\ell)[1] \\
\uparrow{\scriptstyle \begin{pmatrix} 1 \\ 0 \end{pmatrix}} & & \uparrow{\scriptstyle \begin{pmatrix} 1 \\ 0 \\ 0 \\ \omega_j \end{pmatrix}} & & \uparrow & & \uparrow \\
M^j & \xrightarrow{\ \mathrm{id}\ } & M^j & \longrightarrow & 0 & \longrightarrow & M^j[1]
\end{array}
$$

and therefore the middle triangle has a direct factor

$$M^j \xrightarrow{\ \mathrm{id}\ } M^j \longrightarrow 0 \longrightarrow M^j[1]$$

and the remaining direct factor is the original distinguished triangle.

Hence,

$$
\begin{array}{ccc}
M^j \oplus Z_\ell & \xrightarrow{\begin{pmatrix} 1_{M^j} & 0 \\ 0 & u_\ell \\ 0 & v_\ell \end{pmatrix}} & M^j \oplus M \oplus Z_\ell \\
{\scriptstyle (\omega_j, h_j)}\downarrow & & \downarrow{\scriptstyle (s_j\omega_j, \nu_{j+1}, h_{j+1}) = (\omega_{j+1}, h_{j+1})} \\
N_j & \xrightarrow{\ s_j\ } & N_{j+1}
\end{array}
$$

is a homotopy cartesian square. Moreover,

$$
\begin{array}{ccc}
Z_\ell & \xrightarrow{\begin{pmatrix} u_\ell \\ v_\ell \end{pmatrix}} & M \oplus Z_\ell \\
\downarrow & & \downarrow{\scriptstyle (\omega_1, h_1)} \\
0 & \longrightarrow & N
\end{array}
$$

is a homotopy cartesian square by hypothesis. Now, since

$$
\begin{array}{ccc}
Z_\ell & \xrightarrow{\begin{pmatrix} u_\ell \\ v_\ell \end{pmatrix}} & M \oplus Z_\ell \\
\downarrow & & \downarrow{\scriptstyle (\omega_1, h_1)} \\
0 & \longrightarrow & N
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
M^j \oplus Z_\ell & \xrightarrow{\begin{pmatrix} 1_{M^j} & 0 \\ 0 & u_\ell \\ 0 & v_\ell \end{pmatrix}} & M^j \oplus M \oplus Z_\ell \\
{\scriptstyle (\omega_j, h_j)}\downarrow & & \downarrow{\scriptstyle (s_j\omega_j, \nu_{j+1}, h_{j+1})} \\
N_j & \xrightarrow{\ s_j\ } & N_{j+1}
\end{array}
$$

are homotopy cartesian squares, applying Lemma 8, we get a homotopy cartesian square

$$
\begin{array}{ccc}
Z_\ell & \xrightarrow{\left(\begin{smallmatrix}\psi_2\\ v_\ell^2\end{smallmatrix}\right)} & M^2 \oplus Z_\ell \\
\downarrow & & \downarrow{\scriptstyle(\omega_2,h_2)} \\
0 & \longrightarrow & N_2
\end{array}
$$

and by induction on j we get that there is a morphism $\psi_j : Z_\ell \to M^j$ and a homotopy cartesian square

$$
\begin{array}{ccc}
Z_\ell & \xrightarrow{\left(\begin{smallmatrix}\psi_j\\ v_\ell^j\end{smallmatrix}\right)} & M^j \oplus Z_\ell \\
\downarrow & & \downarrow{\scriptstyle(\omega_j,h_j)} \\
0 & \longrightarrow & N_j
\end{array}
$$

for each $j \in \mathbb{N}$. Hence for all $j > 0$ there is a morphism $\psi_j : Z_\ell \to M^j$ such that

$$
Z_\ell \xrightarrow{\left(\begin{smallmatrix}\psi_j\\ v_\ell^j\end{smallmatrix}\right)} M^j \oplus Z_\ell \xrightarrow{(\omega_j,h_j)} N_j \longrightarrow Z_\ell[1]
$$

is a distinguished triangle. Now v_ℓ is nilpotent of degree k_0, say. For any $k \geq k_0$ consider the commutative diagram with distinguished triangles in the horizontal rows

$$
\begin{array}{ccccccc}
Z_\ell & \xrightarrow{\left(\begin{smallmatrix}\psi_k\\ v_\ell^k\end{smallmatrix}\right)} & M^k \oplus Z_\ell & \xrightarrow{(\omega_k,h_k)} & N_k & \longrightarrow & Z_\ell[1] \\
\downarrow & & \downarrow{\scriptstyle(0,\mathrm{id})} & & & & \downarrow \\
0 & \longrightarrow & Z_\ell & \xrightarrow{\mathrm{id}} & Z_\ell & \longrightarrow & 0
\end{array}
$$

which can be completed by a map z_k to a morphism of distinguished triangles, using TR3 (see e.g. [9, Definition 1.1.1] or [20, Definition 3.4.1]),

$$
\begin{array}{ccccccc}
Z_\ell & \xrightarrow{\left(\begin{smallmatrix}\psi_k\\ v_\ell^k\end{smallmatrix}\right)} & M^k \oplus Z_\ell & \xrightarrow{(\omega_k,h_k)} & N_k & \longrightarrow & Z_\ell[1] \\
\downarrow & & \downarrow{\scriptstyle(0,\mathrm{id})} & & \downarrow{\scriptstyle z_k} & & \downarrow \\
0 & \longrightarrow & Z_\ell & \xrightarrow{\mathrm{id}} & Z_\ell & \longrightarrow & 0
\end{array}
$$

Therefore $z_k h_k = \mathrm{id}_{Z_\ell}$, and h_k is a split monomorphism. We recall that we constructed the sequence h_j as iterated homotopy pushouts, and hence, by definition we have a homotopy cartesian square

$$
\begin{array}{ccc}
Z_\ell & \xrightarrow{\hat{v}} & Z_\ell \oplus M \\
{\scriptstyle h_k}\downarrow & & \downarrow{\scriptstyle(h_{k+1},\nu_{k+1})} \\
N_k & \xrightarrow{s_n} & N_{k+1}
\end{array}
$$

where h_k (and h_{k+1}) are split monomorphisms. This shows first (cf e.g. [20, Lemma 3.4.9]) that $N_k \simeq Z_\ell \oplus C_{h_k}$, for C_{h_k} being the mapping cone of h_k. Moreover, we get a distinguished triangle

$$
Z_\ell \xrightarrow{\left(\begin{smallmatrix}\hat{v}\\ h_k\end{smallmatrix}\right)} Z_\ell \oplus M \oplus N_k \xrightarrow{(h_{k+1},\nu_{k+1},-s_k)} N_{k+1} \longrightarrow Z_\ell[1] \ .
$$

Since h_k is split monomorphism, this distinguished triangle is isomorphic to the direct sum of the trivial distinguished triangle

$$Z_\ell \xrightarrow{\text{id}} Z_\ell \longrightarrow 0 \longrightarrow Z_\ell[1]$$

and

$$0 \longrightarrow 0 \oplus C_{h_k} \oplus Z_\ell \oplus M \longrightarrow N_{k+1} \longrightarrow 0[1].$$

Hence (cf e.g. [20, Lemma 3.4.9]),

$$(\dagger) \quad N_{k+1} \simeq C_{h_k} \oplus Z_\ell \oplus M \simeq N_k \oplus M.$$

Recall that, posing $N_0 := 0$, for all $j \geq 1$ we have by construction homotopy cartesian squares

$$
\begin{array}{ccc}
N_j & \xrightarrow{s_j} & N_{j+1} \\
{\scriptstyle t_j}\downarrow & & \downarrow{\scriptstyle t_{j+1}} \\
N_{j-1} & \xrightarrow{s_j} & N_j
\end{array}
$$

for all $j \geq 1$. Using Lemma 8 and an obvious induction as above this implies that we get a homotopy cartesian square

$$
\begin{array}{ccc}
N & \longrightarrow & N_{k+1} \\
\downarrow & & \downarrow \\
0 & \longrightarrow & N_k
\end{array}
$$

which gives a distinguished triangle

$$N \longrightarrow N_{k+1} \longrightarrow N_k \longrightarrow N[1]$$

where we have chosen k such that $v_\ell^k = 0$. By equation $(\dagger)$ we get $N_{k+1} \simeq N_k \oplus M$, which shows that there is a distinguished triangle

$$N \longrightarrow N_k \oplus M \longrightarrow N_k \longrightarrow N[1].$$

Posing $N_k =: Z_r$ this gives the statement. $\qquad\square$

Recall the following result of Xiao-Wu Chen, Yu Ye and Phu Zhang.

Proposition 11. [2, Theorem A.1] *An additive category $\mathcal{C}$ is Krull-Schmidt if and only if any idempotent splits, and the endomorphism ring of any object of $\mathcal{C}$ is semiperfect.*

Our main result of this section now is the following.

Theorem 12. *Let K be a commutative ring and let $\mathcal{T}$ be a K-linear triangulated category with split idempotents and such that the endomorphism ring of each object is artinian, and let M and N be two objects. Then there is an object Z_r and a distinguished triangle*

$$N \longrightarrow M \oplus Z_r \xrightarrow{(u_r, v_r)} Z_r \longrightarrow N[1]$$

if and only if there is an object Z_ℓ and a distinguished triangle

$$Z_\ell \xrightarrow{\binom{u_\ell}{v_\ell}} M \oplus Z_\ell \longrightarrow N \longrightarrow Z_\ell[1].$$

Proof. By [14, Examples VIII.4] any artinian ring is semiperfect. By Proposition 11 we get that $\mathcal{T}$ is Krull-Schmidt. Moreover we get Fitting's lemma for $\mathcal{T}$. This states that any endomorphism $\nu : Z \to Z$ can be decomposed into

$$\nu = \begin{pmatrix} \nu' & 0 \\ 0 & \nu'' \end{pmatrix} : Z = Z' \oplus Z'' \to Z' \oplus Z'' = Z$$

and such that ν' is an automorphism and ν'' is nilpotent. Indeed, Ringel and Pu Zhang [11, Section 5.5] give a proof for a Hom-finite Krull-Schmidt K-category. Using [20, Lemma

1.4.4] the proof there carries over to our more general situation when all objects have artinian endomorphism algebras.

For the 'if' part of the statement, splitting off the trivial triangle $Z'_\ell \xrightarrow{\nu'} Z'_\ell \to 0 \to Z'_\ell[1]$, we may assume that v_ℓ is nilpotent. By Proposition 10 we get a distinguished triangle

$$N \longrightarrow M \oplus Z_r \xrightarrow{(u_r, v_r)} Z_r \longrightarrow N[1] \ .$$

Using now Fitting decomposition

$$\nu_r = \begin{pmatrix} \nu'_r & 0 \\ 0 & \nu''_r \end{pmatrix} : Z_r = Z'_r \oplus Z''_r \to Z'_r \oplus Z''_r = Z_r$$

and splitting off the trivial triangle $0 \longrightarrow Z'_r \xrightarrow{\nu'_r} Z'r \longrightarrow 0$, we get the triangle

$$N \longrightarrow M \oplus Z''_r \xrightarrow{(u_r, v''_r)} Z''_r \longrightarrow N[1] \ ,$$

where ν''_r is a nilpotent endomorphism.

The 'only if' part is done by applying the proved 'if' part to the opposite category $\mathcal{T}^{op}$. $\square$

4. The case of a category of compact objects in an algebraic compactly generated category

Recall from Section 2.3 the construction of the category $\mathcal{A}[[T]]$. When we have a dg algebra with enough idempotents A and view it as a dg category (see [12, Section 3]), we can undertake the same construction, but there is a classical notion of power series algebra $A[[T]]$ which does not correspond to the 'power series dg category' mentioned in Section 2.3. The corresponding dg algebra with enough idempotents is the subalgebra of $A[[T]]$ given as

$$\tilde{A}[[T]] = \bigoplus_{i,j \in I} \bigoplus_{n \in \mathbb{Z}} e_i A^n e_j [[T]].$$

That is, $\tilde{A}[[T]]$ consists of the power series $\sum_{k \in \mathbb{N}} a_k T^k$ for which there are finite subsets $J \subset I$ and $F \subset \mathbb{Z}$ such that $a_k \in \bigoplus_{i,j \in J} \bigoplus_{n \in F} e_i A^n e_j$, for all $k \in \mathbb{N}$. The subalgebra $\tilde{A}[[T]]$ is made into a dg algebra by defining $\tilde{A}[[T]]^n = \bigoplus_{i,j \in I} e_i A^n e_j[[T]]$, for each $n \in \mathbb{Z}$, and by defining its differential by the rule $d(\sum_{k \in \mathbb{N}} a_k T^k) = \sum_{i \in \mathbb{N}} d(a_k) T^k$. Note that then $(e_i)_{i \in I}$ is also a distinguished family of orthogonal idempotents of $\tilde{A}[[T]]$. Moreover, we have a canonical inclusion $\iota : A \hookrightarrow \tilde{A}[[T]]$ ($a \rightsquigarrow a = aT^o$) and a canonical augmentation map $\rho : \tilde{A}[[T]] \longrightarrow A$ ($\sum_{k \in \mathbb{N}} a_k T^k \rightsquigarrow a_0$) such that $\rho \circ \iota = 1_A$. Both ι and ρ are homomorphisms of dg algebras making the codomain into a unitary bimodule over the domain. We can apply to ι and ρ the results of [12, Corollary 9.4 and Section 10] concerning homomorphisms of dg algebras with enough idempotents.

Likewise for right dg A-modules, there is a classical notion of 'module of power series' $M[[T]]$ which does not correspond the the 'dg $\mathcal{A}[[T]]$-module of power series' mentioned in Section 2.3. The corresponding right dg $\tilde{A}[[T]]$-module is

$$\tilde{M}[[T]] = \bigoplus_{i \in I} \bigoplus_{n \in \mathbb{Z}} M^n e_i [[T]],$$

which is the K-submodule of $M[[T]]$ consisting of those power series $\sum_{k \in \mathbb{N}} m_k T^k$ for which there exist finite subsets $J \subset I$ and $F \subset \mathbb{Z}$, both depending on the power series, such that $m_k \in \oplus_{i \in J, n \in F} M^n e_i$, for all $k \in \mathbb{N}$. Defining the grading by the rule $\tilde{M}[[T]]^n = \bigoplus_{i \in I} M^n e_i[[T]]$ and the differential $d : \tilde{M}[[T]] \longrightarrow \tilde{M}[[T]]$ by the rule $d(\sum_{k \in \mathbb{N}} m_k T^k) = \sum_{k \in \mathbb{N}} d(m_k) T^k$, we clearly endow $\tilde{M}[[T]]$ with a structure of right dg $\tilde{A}[[T]]$-module. We warn the reader of the possible confusion of this construction when applied to the regular right dg A-module A_A for the resulting right dg $\tilde{A}[[T]]$-module is not equal to the regular right dg module $\tilde{A}[[T]]_{\tilde{A}[[T]]}$.

4.1. **Dualising degeneration data.** Recall from Section 2.3 the definition of the functor $?\hat\otimes V : \text{Dg-}A \longrightarrow \text{Dg-}\tilde{A}[[T]]$. Here V is reminiscent of $V = K[[T]]$ the power series ring, which played the corresponding rôle in case of degeneration in the module category of a K-algebra in Yoshino's setting [16]. We now give, and actually extend, the functor corresponding to $?\hat\otimes V$ to the context of dg algebras with enough idempotents and dg modules over them.

Proposition 13. *The assignment $M \rightsquigarrow \tilde{M}[[T]]$ is the definition on objects of a dg functor $?\hat\otimes V : \text{Dg-}A \longrightarrow \text{Dg-}\tilde{A}[[T]]$ which satisfies the following properties:*

(1) *$?\hat\otimes V$ takes contractible dg modules to contractible dg modules.*
(2) *The associated functor on 0-cycle categories*
$$?\hat\otimes V : Z^0(\text{Dg-}A) = \mathcal{C}(A) \longrightarrow \mathcal{C}(\tilde{A}[[T]]) = Z^0(\text{Dg-}\tilde{A}[[T]])$$
   is exact with respect to the respective abelian structures.
(3) *$?\hat\otimes V$ preserves acyclic dg modules. In particular, it induces a triangulated functor $?\hat\otimes V : \mathcal{D}(A) \longrightarrow \mathcal{D}(\tilde{A}[[T]])$ which is both the left and right derived functor of 'itself'. We will denote this functor by $\uparrow_K^V$.*
(4) *The multiplication map $M \otimes_A \tilde{A}[[T]] \longrightarrow \tilde{M}[[T]]$ defines a homological natural transformation of dg functors $\mu : \iota^* \longrightarrow ?\hat\otimes V$*
(5) *The induced natural transformation $\mu : \mathbb{L}\iota^* \longrightarrow \uparrow_K^V$ of triangulated functors $\mathcal{D}(A) \longrightarrow \mathcal{D}(\tilde{A}[[T]])$ is a natural isomorphism when evaluated at objects of $per(A)$.*

Proof. If $f : M \longrightarrow N$ is a homogeneous morphism in Dg-A, we define $\tilde{f} := (?\hat\otimes V)(f)$ by the rule $\tilde{f}(\sum_{k\in\mathbb{N}} m_k T^k) = \sum_{k\in\mathbb{N}} f(m_k)T^k$. It is routine to verify that the assignments $M \rightsquigarrow \tilde{M}[[T]]$ and $f \rightsquigarrow \tilde{f}$ give a graded functor Gr-$A \longrightarrow$ Gr-$\tilde{A}[[T]]$. In order to check that they define a dg functor $?\hat\otimes V : \text{Dg-}A \longrightarrow \text{Dg-}\tilde{A}[[T]]$, we need to check that if M, N are right dg A-modules and $d : \text{HOM}_A(M,N) \longrightarrow \text{HOM}_A(M,N)$ and $\delta : \text{HOM}_{\tilde{A}[[T]]}(\tilde{M}[[T]], \tilde{N}[[T]]) \longrightarrow \text{HOM}_{\tilde{A}[[T]]}(\tilde{M}[[T]], \tilde{N}[[T]])$ are the respective differentials on Hom spaces, then one has $\delta(\tilde{f}) = \widetilde{d(f)}$, for any homogeneous element $f \in \text{HOM}_A(M,N)$. On one hand, we have that

$$\delta(\tilde{f}) = d_{\tilde{N}[[T]]} \circ \tilde{f} - (-1)^{|f|}\tilde{f} \circ d_{\tilde{M}[[T]]}.$$

On the other hand, if we let act $\widetilde{d(f)}$ on a homogeneous element $\sum_{k\in\mathbb{N}} m_k T^k \in \tilde{M}[[T]]$ (whence the degree $|m_k|$ is independent of k), then we get:

$$
\begin{aligned}
\widetilde{d(f)}\left(\sum_{k\in\mathbb{N}} m_k T^k\right) &= \sum_{k\in\mathbb{N}} d(f)(m_k)T^k \\
&= \sum_{k\in\mathbb{N}} \left[(d_N \circ f - (-1)^{|f|}f \circ d_M)(m_k)\right]T^k \\
&= \sum_{k\in\mathbb{N}} d_N(f(m_k))T^k - (-1)^{|f|}\sum_{k\in\mathbb{N}} f(d_M(m_k))T^k \\
&= d_{\tilde{N}[[T]]}\left(\sum_{k\in\mathbb{N}} f(m_k)T^k\right) - (-1)^{|f|}\tilde{f}\left(\sum_{k\in\mathbb{N}} d_M(m_k)T^k\right) \\
&= \left[d_{\tilde{N}[[T]]} \circ \tilde{f} - (-1)^{|f|}\tilde{f} \circ d_{\tilde{M}[[T]]}\right]\left(\sum_{k\in\mathbb{N}} m_k T^k\right).
\end{aligned}
$$

This shows that $\delta(\tilde{f}) = \widetilde{d(f)}$, as desired.

Finally, it is also routine to see that $(? \hat{\otimes} V)(\mathrm{cone}(1_M)) \cong \mathrm{cone}(1_{(? \hat{\otimes} V)(M)})$, which ends the proof of assertion (1).

(2) Let $L \xrightarrow{f} M \xrightarrow{g} N$ be an exact sequence in $\mathcal{C}(A)$, when this category is considered with its natural abelian structure, and consider the corresponding sequence $\tilde{L}[[T]] \xrightarrow{\tilde{f}} \tilde{M}[[T]] \xrightarrow{\tilde{g}} \tilde{N}[[T]]$. Since $\tilde{f}$ and $\tilde{g}$ are both morphisms in Gr-$\tilde{A}[[T]]$ we just need to check that the induced sequence

$$
\begin{array}{ccccc}
L^n e_i[[T]] & & M^n e_i[[T]] & & N^n e_i[[T]] \\
\| & & \| & & \| \\
\tilde{L}[[T]]^n e_i & \xrightarrow{\tilde{f}} & \tilde{M}[[T]]^n e_i & \xrightarrow{\tilde{g}} & \tilde{N}[[T]]^n e_i
\end{array}
$$

is exact, for all $i \in I$ and $n \in \mathbb{Z}$. But, given $\sum_{k \in \mathbb{N}} m_k T^k \in M^n e_i[[T]]$, we have that $\tilde{g}(\sum_{k \in \mathbb{N}} m_k T^k) = 0$ if and only if $g(m_k) = 0$ for all $k \in \mathbb{N}$. This in turn is equivalent to saying that, for each $k \in \mathbb{N}$, there exists a $l_k \in L^n e_i$ such that $f(l_k) = m_k$. That is, we have that $\sum_{k \in \mathbb{N}} m_k T^k \in \mathrm{Ker}(\tilde{g})$ if and only if $\sum_{k \in \mathbb{N}} m_k T^k = \tilde{f}(\sum_{k \in \mathbb{N}} l_k T^k)$, for some $\sum_{k \in \mathbb{N}} l_k T^k \in L^n e_i[[T]]$.

(3) Let M be an acyclic right dg A-module, let $\sum_{k \in \mathbb{N}} m_k T^k$ be an element of $\mathrm{Ker}(d^n : \tilde{M}[[T]]^n \longrightarrow \tilde{M}[[T]]^{n+1})$ and let $J \subset I$ be a finite subset such that $m_k \in \bigoplus_{i \in J} M^n e_i$, for all $k \in \mathbb{N}$. By the acyclicity condition of M, for each $k \in \mathbb{N}$, we have an $m'_k \in \bigoplus_{i \in J} M^{n-1} e_i$ such that $d(m'_k) = m_k$. It follows $\sum_{k \in \mathbb{N}} m'_k T^k$ is an element of $\tilde{M}[[T]]^n$ such that $d(\sum_{k \in \mathbb{N}} m'_k T^k) = \sum_{k \in \mathbb{N}} m_k T^k$. Therefore $\tilde{M}[[T]]$ is an acyclic right dg $\tilde{A}[[T]]$-module. The last comment of the assertion follows from [12, Remark 7.11].

(4) We clearly have that $\mu_M : M \otimes_A \tilde{A}[[T]] \longrightarrow \tilde{M}[[T]]$ is a morphism (of zero degree) in Gr-$\tilde{A}[[T]]$. In addition, if we denote by $d : M \otimes_A \tilde{A}[[T]] \longrightarrow M \otimes_A \tilde{A}[[T]]$ and $\tilde{d} : \tilde{M}[[T]] \longrightarrow \tilde{M}[[T]]$ the respective differentials, then, for all homogeneous elements $m \in M$ and $\sum_{k \in \mathbb{N}} a_k T^k$, we have

$$
\begin{aligned}
(\mu_M \circ d)\left[m \otimes (\sum_{k \in \mathbb{N}} a_k T^k)\right] &= \mu_M \left(d_M(m) \otimes (\sum_{k \in \mathbb{N}} a_k T^k) + (-1)^{|m|} m \otimes (\sum_{k \in \mathbb{N}} d(a_k) T^k)\right) \\
&= \sum_{k \in \mathbb{N}} d_M(m) a_k T^k + (-1)^{|m|} m d(a_k) T^k \\
&= \sum_{k \in \mathbb{N}} (d_M(m) a_k + (-1)^{|m|} m d(a_k)) T^k \\
&= \sum_{k \in \mathbb{N}} d_M(m a_k) T^k \\
&= \tilde{d}(\sum_{k \in \mathbb{N}} m a_k T^k) \\
&= (\tilde{d} \circ \mu_M) \left[m \otimes (\sum_{k \in \mathbb{N}} a_k T^k)\right].
\end{aligned}
$$

Then, once the naturality μ is proved, we will have that it is actually a homological natural transformation of dg functors (see [12, Remark 7.1]). But that naturality is clear since we

have

$$(\tilde{f} \circ \mu_M)\left[m \otimes (\sum_{k\in\mathbb{N}} a_k T^k)\right] \quad = \quad \tilde{f}(\sum_{k\in\mathbb{N}} m a_k T^k) = \sum_{k\in\mathbb{N}} f(m a_k) T^k = \sum_{k\in\mathbb{N}} f(m) a_k T^k$$

$$= \quad \mu_N\left[f(m) \otimes (\sum_{k\in\mathbb{N}} a_k T^k)\right]$$

$$= \quad (\mu_N \circ (f \otimes 1_{\tilde{A}[[T]]}))\left[m \otimes (\sum_{k\in\mathbb{N}} a_k T^k)\right],$$

for any homogeneous morphism $f : M \longrightarrow N$ in Gr-A and all homogeneous elements $m \in M$ and $\sum_{k\in\mathbb{N}} a_k T^k \in \tilde{A}[[T]]$.

(5) Let $\Pi_A : \mathcal{D}(A) \longrightarrow \mathcal{H}(A)$ denote the homotopically projective resolution functor. Since each $e_i A$ is homotopically projective (see [12, Example 7.6]), we have that $\Pi_A(e_i A) \cong e_i A$ in $\mathcal{H}(A)$. Moreover, we have an isomorphism $\mu_{e_i A} : e_i A \otimes_A \tilde{A}[[T]] \xrightarrow{\cong} e_i \tilde{A}[[T]] = e_i A \hat{\otimes} V$ in Dg-$\tilde{A}[[T]]$ (see the proof of Proposition 10.5 in [12]). One then gets from [12, Proposition 7.12] that $\mu_{e_i A} : \mathbb{L}\iota^*(e_i A) \longrightarrow (e_i A) \uparrow_K^V$ is an isomorphism in $\mathcal{D}(\tilde{A}[[T]])$, from which it follows that $\mu_M : \mathbb{L}\iota^*(M) \longrightarrow M \uparrow_K^V$ is an isomorphism, for all $M \in \mathrm{per}(A) = \mathrm{thick}_{\mathcal{D}(A)}(e_i A: i \in I)$. $\square$

Note that we have a canonical isomorphism of dg algebras with enough idempotents $\tilde{A}^{op}[[T]] \cong \tilde{A}[[T]]^{op}$. We will still denote by ι^* the dg functor $\tilde{A}[[T]] \otimes_A ? : A\text{-Dg} \longrightarrow \tilde{A}[[T]]\text{-Dg}$ and by $\mathbb{L}\iota^*$ its left derived functor $\mathcal{D}(A^{op}) \longrightarrow \mathcal{D}(\tilde{A}[[T]]^{op})$. We will denote by $V\hat{\otimes}? : A\text{-Dg} \longrightarrow \tilde{A}[[T]]\text{-Dg}$ and $\uparrow_K^V : \mathcal{D}(A^{op}) \longrightarrow \mathcal{D}(\tilde{A}[[T]]^{op})$ the corresponding dg functor and triangulated functor, respectively. We now get:

Corollary 14. *Consider the compositions of triangulated functors*

$$\mathcal{D}(A)^{op} \xrightarrow{\mathbb{R}Hom_A(?,A)} \mathcal{D}(A^{op}) \xrightarrow{\mathbb{L}\iota^*} \mathcal{D}(\tilde{A}[[T]]^{op})$$

and

$$\mathcal{D}(A)^{op} \xrightarrow{(\mathbb{L}\iota^*)^o} \mathcal{D}(\tilde{A}[[T]])^{op} \xrightarrow{\mathbb{R}Hom_{\tilde{A}[[T]]}(?,\tilde{A}[[T]])} \mathcal{D}(\tilde{A}[[T]]^{op}).$$

There are natural isomorphisms of triangulated functors

$$\eta : (\mathbb{L}\iota^* \circ \mathbb{R}Hom_A(?, A))_{|per(A)^{op}} \xrightarrow{\cong} [\mathbb{R}Hom_{\tilde{A}[[T]]}(?, \tilde{A}[[T]]) \circ \mathbb{L}\iota^*]_{|per(A)^{op}}$$

and

$$\eta : \uparrow_K^V \circ \mathbb{R}Hom_A(?, A))_{|per(A)^{op}} \xrightarrow{\cong} [\mathbb{R}Hom_{\tilde{A}[[T]]}(?, \tilde{A}[[T]]) \circ \uparrow_K^V]_{|per(A)^{op}}.$$

Proof. The first natural isomorphism is a direct consequence of [12, Proposition 10.5]. On the other hand, Proposition 13 and its left-right symmetric version gives natural isomorphisms $(\mathbb{L}\iota^*)_{|per(A)} \cong (\uparrow_K^V)_{|per(A)}$ and $(\mathbb{L}\iota^*)_{|per(A^{op})} \cong (\uparrow_K^V)_{|per(A^{op})}$. Using now the duality

$$\mathbb{R}Hom_A(?, A) : \mathrm{per}(A) \xrightarrow{\cong^o} \mathrm{per}(A^{op}),$$

the result follows. $\square$

We now revisit and generalize some point of [13]. Note that the variable T is not an element of $\tilde{A}[[T]]$, unless A has a unit. However, if Q is a right dg $\tilde{A}[[T]]$-module and $x \in Q$ is a homogeneous element, then the product xT makes sense. Indeed since $x = \sum_{i\in I} xe_i$, with $xe_i = 0$ for almost all $i \in I$, the element $xT := \sum_{i\in I} x(e_i T)$ is a well-defined element of Q with $deg(xT) = deg(x)$. Furthermore, if $f : Q \longrightarrow Q'$ is a morphism in Dg-$\tilde{A}[[T]]$, then we have $f(xT) = f(\sum_{i\in I} x(e_i T)) = \sum_{i\in I} f(x)e_i T = f(x)T$. We can now prove:

Lemma 15. *For each right dg $\tilde{A}[[T]]$-module Q, the map $t_Q : Q \longrightarrow Q$ ($x \rightsquigarrow xT$) is a morphism of zero degree in Dg-$\tilde{A}[[T]]$ and, when Q varies, the t_Q give a homological natural transformation of dg functors $t : 1_{\mathrm{Dg}\text{-}\tilde{A}[[T]]} \longrightarrow 1_{\mathrm{Dg}\text{-}\tilde{A}[[T]]}$.*

Proof. Let $x \in Q$ and $\sum_{k \in \mathbb{N}} a_k T^k \in \tilde{A}[[T]]$ be homogeneous elements. By definition of $\tilde{A}[[T]]$ and by the fact that $Q = \bigoplus_{i \in I} Qe_i$, we have a finite subset $F \subset I$ such that $xe_i = 0$ and $a_k e_i = 0 = e_i a_k$, for all $i \in I \setminus F$ and all $k \in \mathbb{N}$. It follows that $(x \sum_{k \in \mathbb{N}} a_k T^k)e_i = 0$, for all $i \in I \setminus F$. We then have

$$
\begin{aligned}
t_Q(x \sum_{k \in \mathbb{N}} a_k T^k) &= (x \sum_{k \in \mathbb{N}} a_k T^k)T = \sum_{i \in F}(x \sum_{k \in \mathbb{N}} a_k T^k)e_i T = \sum_{i \in F} x(\sum_{k \in \mathbb{N}} a_k e_i T^{k+1}) \\
&= x \sum_{k \in \mathbb{N}} a_k T^{k+1} = (xT) \sum_{k \in \mathbb{N}} a_k T^k = t_Q(x) \sum_{k \in \mathbb{N}} a_k T^k,
\end{aligned}
$$

which shows that t_Q is a morphism of zero degree in Dg-$\tilde{A}[[T]]$.

If $f : Q \longrightarrow Q'$ is a homogeneous morphism in Dg-$\tilde{A}[[T]]$ then we have $(t_{Q'} \circ f)(x) = f(x)T = f(xT) = (f \circ t_Q)(x)$, for each $x \in Q$. This proves that, when Q varies, the t_Q give a natural transformation of dg functors $t : 1_{\mathrm{Dg}\text{-}\tilde{A}[[T]]} \longrightarrow 1_{\mathrm{Dg}\text{-}\tilde{A}[[T]]}$. This natural transformation is homological since we have

$$
\begin{aligned}
(d_Q \circ t_Q)(x) &= d_Q(xT) = d_Q(\sum_{i \in I} x(e_i T)) = \sum_{i \in I} d_Q(x(e_i T)) \\
&= \sum_{i \in I}(d_Q(x)e_i)T = d_Q(x)T = (t_Q \circ d_Q)(x),
\end{aligned}
$$

for each homogeneous element $x \in Q$, due to the fact that $d_{\tilde{A}[[T]]}(e_i T) = d(e_i)T = 0$ (see [12, Remark 7.1]). $\qquad\square$

Note that the associated natural transformation of triangulated functors $t : 1_{\mathcal{D}(\tilde{A}[[T]])} \longrightarrow 1_{\mathcal{D}(\tilde{A}[[T]])}$ is the one given in [13], after translation to the language of dg algebras with enough idempotents. Replacing A by A^{op} in Lemma 15 and interpreting right dg modules over $\tilde{A}[[T]]^{op} \cong \tilde{A}^{op}[[T]]$ as left dg $\tilde{A}[[T]]$-modules, we get a natural transformation of dg functors $t : 1_{\tilde{A}[[T]]\text{-Dg}} \longrightarrow 1_{\tilde{A}[[T]]\text{-Dg}}$ which in turn gives a natural transformation of triangulated functors $t : 1_{\mathcal{D}(\tilde{A}[[T]]^{op})} \longrightarrow 1_{\mathcal{D}(\tilde{A}[[T]]^{op})}$. These natural transformations do not have correspondents for dg $\tilde{A}[[T]] - \tilde{A}[[T]]-$bimodules, because the action of T by multiplication on a dg $\tilde{A}[[T]] - \tilde{A}[[T]]-$bimodule need not be the same on the left and on the right. We say that a dg $\tilde{A}[[T]] - \tilde{A}[[T]]-$bimodule X is *T-symmetric* when $Tx = xT$, for each $x \in X$. Note that, with a suitable modification of the argument used in the proof of Lemma 15, one easily sees that if X is a T-symmetric dg $\tilde{A}[[T]]$-bimodule, then the assignment $x \rightsquigarrow xT = Tx$ is a morphism of $\tilde{A}[[T]] - \tilde{A}[[T]]-$bimodules, which we also denote by t_X.

Proposition 16. *Let us consider the bi-triangulated functor*

$$
\mathbb{R}HOM_{\tilde{A}[[T]]}(?, ?) : \mathcal{D}(\tilde{A}[[T]])^{op} \otimes \mathcal{D}(\tilde{A}[[T]] \otimes \tilde{A}[[T]]^{op}) \longrightarrow \mathcal{D}(\tilde{A}[[T]]^{op})
$$

(see Proposition 2) and let Q be a right dg $\tilde{A}[[T]]$-module and X be a T-symmetric dg $\tilde{A}[[T]] - \tilde{A}[[T]]-$bimodule.

Then $\mathbb{R}HOM_{\tilde{A}[[T]]}(t_Q^o, 1_X)$ and $\mathbb{R}HOM_{\tilde{A}[[T]]}(1_Q^o, t_X)$, considered as maps

$$
\mathbb{R}HOM_{\tilde{A}[[T]]}(Q, X) \longrightarrow \mathbb{R}HOM_{\tilde{A}[[T]]}(Q, X)
$$

are equal. Moreover they are equal to the evaluation of the natural transformation

$$
t : 1_{\mathcal{D}(\tilde{A}[[T]]^{op})} \longrightarrow 1_{\mathcal{D}(\tilde{A}[[T]]^{op})}
$$

at $\mathbb{R}HOM_{\tilde{A}[[T]]}(Q, X)$.

Proof. By Proposition 2, we have a natural isomorphism of triangulated functors

$$\mathbb{R}\mathrm{HOM}_{\tilde{A}[[T]]}(?, X) \cong \mathbb{R}\mathrm{Hom}_{\tilde{A}[[T]]}(?, X).$$

So in order to see that $\mathbb{R}\mathrm{HOM}_{\tilde{A}[[T]]}(t_Q^o, 1_X)$ is the evaluation of t at $\mathbb{R}\mathrm{HOM}_{\tilde{A}[[T]]}(Q, X)$ it is enough to check that $t_Q^* = \mathbb{R}\mathrm{Hom}_{\tilde{A}[[T]]}(?, X)(t_Q)$ is precisely $t_{\mathbb{R}\mathrm{Hom}_{\tilde{A}[[T]]}(Q,X)}$, where $\mathbb{R}\mathrm{Hom}_{\tilde{A}[[T]]}(Q, X) := \mathbb{R}\mathrm{Hom}_{\tilde{A}[[T]]}(?, X)(Q)$ in the rest of the proof. Note that t_Q^* is the map

$$\mathbb{R}\mathrm{Hom}_{\tilde{A}[[T]]}(Q, X) = \overline{HOM}_{\tilde{A}[[T]]}(\Pi(Q), X) \overset{\Pi(t_Q)^*}{\longrightarrow} \overline{HOM}_{\tilde{A}[[T]]}(\Pi(Q), X) = \mathbb{R}\mathrm{Hom}_{\tilde{A}[[T]]}(Q, X).$$

Here and in the rest of the proof

$$\Pi := \Pi_{\tilde{A}[[T]]} : \mathcal{D}(\tilde{A}[[T]]) \longrightarrow \mathcal{H}(\tilde{A}[[T]])$$

and

$$\Upsilon := \Upsilon_{\tilde{A}[[T]] \otimes \tilde{A}[[T]]^{op}} : \mathcal{D}(\tilde{A}[[T]] \otimes \tilde{A}[[T]]^{op}) \longrightarrow \mathcal{H}(\tilde{A}[[T]] \otimes \tilde{A}[[T]]^{op})$$

are the homotopically projective and the homotopically injective resolution functors, respectively.

It is convenient to have a careful look at a special case of the action of Π and Υ on morphisms. Let Q and X be as in the statement and let $f : Q \longrightarrow Q$ and $\alpha : X \longrightarrow X$ be morphisms in $\mathcal{H}(\tilde{A}[[T]])$ and $\mathcal{H}(\tilde{A}[[T]] \otimes \tilde{A}[[T]]^{op})$, respectively. Abusing notation, we put $q(f) = f$ and $q(\alpha) = \alpha$, where q is the functor from the homotopy to the derived category in each case. Viewing Q and X as objects of the respective derived categories, we have a counit map

$$\pi_Q : (\Pi \circ q)(Q) = \Pi(Q) \longrightarrow Q$$

and a unit map

$$\iota_X : X \longrightarrow (\Upsilon \circ q)(X) = \Upsilon(X),$$

which are quasi-isomorphism. Then $\Pi(f) := (\Pi \circ q)(f)$ is a morphism $\Pi(Q) \longrightarrow \Pi(Q)$ in $\mathcal{H}(\tilde{A}[[T]])$ such that

$$\pi_Q \circ \Pi(f) = f \circ \pi_Q \quad (*),$$

due to the naturality of the counit π. But since we have an isomorphism

$$\mathrm{Hom}_{\mathcal{H}(\tilde{A}[[T]])}(\Pi(Q), \Pi(Q)) \overset{\cong}{\longrightarrow} \mathrm{Hom}_{\mathcal{D}(\tilde{A}[[T]])}(Q, Q)$$

(which maps $\varphi \rightsquigarrow q(\pi) \circ \varphi \circ q(\pi)^{-1}$), we see that $\Pi(f)$ is the unique morphism in $\mathcal{H}(\tilde{A}[[T]])$ satisfying the equality (*). Similarly, $\Upsilon(\alpha)$ is the unique morphism $\Upsilon(X) \longrightarrow \Upsilon(X)$ in $\mathcal{H}(\tilde{A}[[T]]) \otimes \tilde{A}[[T]]^{op})$ such that

$$\Upsilon(\alpha) \circ \iota_X = \iota_X \circ \alpha.$$

By taking $f = t_Q$ in this argument, we readily see that $\Pi(t_Q) = t_{\Pi(Q)}$ since, due to the naturality of $t : 1_{\mathrm{Dg}\text{-}\tilde{A}[[T]]} \longrightarrow 1_{\mathrm{Dg}\text{-}\tilde{A}[[T]]}$, we have that $\pi \circ t_{\Pi(Q)} = t_Q \circ \pi$ in $\mathcal{H}(\tilde{A}[[T]])$. The analogous argument does not work for $\alpha = t_X$ since we do not have a correspondent of the natural transformation t for $\tilde{A}[[T]] - \tilde{A}[[T]]$-bimodules. In any case, these comments together with the previous paragraph show that $\mathbb{R}\mathrm{Hom}_{\tilde{A}[[T]]}(?, X)(t_Q) = t_{\Pi(Q)}^*$. Using the naturality of t, we see that $t_{\Pi(Q)}^*$ is a morphism $\overline{\mathrm{HOM}}_A(\Pi(Q), X) \longrightarrow \overline{\mathrm{HOM}}_A(\Pi(Q), X)$ of left dg $\tilde{A}[[T]]$-modules which maps

$$f \rightsquigarrow (-1)^{|t_{\Pi(Q)}||f|} f \circ t_{\Pi(Q)} = f \circ t_{\Pi(Q)} = t_X \circ f.$$

We then have that

$$[t_{\Pi(Q)}^*(f)](z) = (f \circ t_{\Pi(Q)})(z) = f(zT) = f(z)T = Tf(z) = (Tf)(z) = [t_{\overline{\mathrm{HOM}}_A(\Pi(Q),X)}(f)](z),$$

for all homogeneous elements $z \in \Pi(Q)$, using the definition of the left $\tilde{A}[[T]]$-module structure on $\overline{\mathrm{HOM}}_A(\Pi(Q), X)$ (see [12, Section 8]) and the T-symmetry of X. Therefore we have $\mathbb{R}\mathrm{Hom}_A(?, X)(t_Q) = t_{\mathbb{R}\mathrm{Hom}_A(Q,X)}$, as desired.

On the other hand, by Proposition 2, there is a natural isomorphism

$$\mathbb{R}\mathrm{HOM}_{\tilde{A}[[T]]}(Q,?) \cong \mathbb{R}\mathrm{Hom}_{\tilde{A}[[T]]}(\Pi(Q),?) = q \circ \overline{HOM}_{\tilde{A}[[T]]}(\Pi(Q),?) \circ \Upsilon$$

of triangulated functors

$$\mathcal{D}(\tilde{A}[[T]] \otimes \tilde{A}[[T]]^{op}) \longrightarrow \mathcal{D}(\tilde{A}[[T]]^{op}).$$

Then $\mathbb{R}\mathrm{HOM}_{\tilde{A}[[T]]}(1_Q^o, t_X)$ is the morphism

$$\overline{HOM}_{\tilde{A}[[X]]}(\Pi(Q), \Upsilon(X)) \longrightarrow \overline{HOM}_{\tilde{A}[[X]]}(\Pi(Q), \Upsilon(X))$$
$$f \rightsquigarrow (-1)^{|1_Q||q(\Upsilon(t_X)|}\Upsilon(t_X) \circ f \circ 1_Q) = \Upsilon(t_X)_*(f).$$

In other words, we have that

$$\mathbb{R}\mathrm{HOM}_{\tilde{A}[[T]]}(1_Q^o, t_X) = q(\Upsilon(t_X)_*),$$

where $q : \mathcal{H}(\tilde{A}[[T]]^{op}) \longrightarrow \mathcal{D}(\tilde{A}[[T]]^{op})$ is the canonical functor and

$$\Upsilon(t_X)_* = \overline{HOM}_A(\Pi(M),?)(\Upsilon(t_X)) : \overline{HOM}_A(\Pi(M), \Upsilon(X)) \longrightarrow \overline{HOM}_A(\Pi(M), \Upsilon(X)).$$

But the induced functor

$$\overline{HOM}_{\tilde{A}[[T]]}(\Pi(M),?) : \mathcal{H}(\tilde{A}[[T]] \otimes \tilde{A}[[T]]^{op}) \longrightarrow \mathcal{H}(\tilde{A}[[T]]^{op})$$

preserves quasi-isomorphisms since $\Pi(M)$ is homotopically projective in $\mathcal{H}(\tilde{A}[[T]])$. If now $\iota := \iota_X : X \longrightarrow \Upsilon(X)$ is as above, then

$$\iota_* := \overline{HOM}_{\tilde{A}[[T]]}(\Pi(M), \iota) : \overline{HOM}_{\tilde{A}[[X]]}(\Pi(Q), X) \longrightarrow \overline{HOM}_{\tilde{A}[[X]]}(\Pi(Q), \Upsilon(X)))$$

is a quasi-isomorphism of left dg $\tilde{A}[[T]]$-modules. Applying to the equality $\Upsilon(t_X) \circ \iota = \iota \circ t_X$ the functor

$$\overline{HOM}_A(\Pi(M),?) : \mathcal{H}(\tilde{A}[[T]] \otimes \tilde{A}[[T]]^{op}) \longrightarrow \mathcal{H}(\tilde{A}[[T]]^{op}),$$

we get the following commutative diagram in $\mathcal{H}(\tilde{A}[[T]]^{op})$, where the horizontal arrows are quasi-isomorphisms.

$$
\begin{array}{ccc}
\overline{HOM}_{\tilde{A}[[T]]}(\Pi(M), X) & \xrightarrow{\iota_*} & \overline{HOM}_{\tilde{A}[[T]]}(\Pi(M), \Upsilon(X)) \\
\downarrow{\scriptstyle (t_X)_*} & & \downarrow{\scriptstyle \Upsilon(t_X)_*} \\
\overline{HOM}_{\tilde{A}[[T]]}(\Pi(M), X) & \xrightarrow{\iota_*} & \overline{HOM}_{\tilde{A}[[T]]}(\Pi(M), \Upsilon(X))
\end{array}
$$

Moreover, the left vertical arrow takes $f \rightsquigarrow t_X \circ f$, for each homogeneous element $f \in \overline{HOM}_{\tilde{A}[[T]]}(\Pi(M), X)$. But, in turn, we have that

$$(t_X \circ f)(v) = Tf(v) = (Tf)(v) = t_{\overline{HOM}_{\tilde{A}[[T]]}(\Pi(M),X)}(f)(v),$$

for each homogeneous element $v \in \Pi(M)$. Therefore the left vertical arrow of last diagram is the evaluation of the natural transformation of dg functors $t : 1_{\tilde{A}[[T]]\text{-Dg}} \longrightarrow 1_{\tilde{A}[[T]]\text{-Dg}}$ at $\overline{HOM}_{\tilde{A}[[T]]}(\Pi(Q), X)$. The fact that t is a natural transformation of dg functors implies that we also have an equality

$$t_{\overline{HOM}_{\tilde{A}[[T]]}(\Pi(Q),\Upsilon(X))} \circ \iota_* = \iota_* \circ t_{\overline{HOM}_{\tilde{A}[[T]]}(\Pi(Q),X)}$$

in $\tilde{A}[[T]]\text{-Dg}$ and, hence, also in $\mathcal{H}(\tilde{A}[[T]]^{op})$. We then have that

$$t_{\overline{HOM}_{\tilde{A}[[T]]}(\Pi(Q),\Upsilon(X))} \circ \iota_* = \Upsilon(t_X)_* \circ \iota_*$$

in $\mathcal{H}(\tilde{A}[[T]]^{op})$. Applying the functor $q : \mathcal{H}(\tilde{A}[[T]]^{op}) \longrightarrow \mathcal{D}(\tilde{A}[[T]]^{op})$ to this last equality and bearing in mind that $q(\iota_*)$ is an isomorphism, we conclude that

$$q((\Upsilon(t_X))_*) = q(t_{\overline{HOM}_{\tilde{A}[[T]]}(\Pi(Q),\Upsilon(X))}) = t_{\mathbb{R}\mathrm{HOM}_{\tilde{A}[[T]]}(Q,X)}.$$

$\square$

For our next result we adopt the terminology of [13, Proposition 9] and, for the given dg algebra with enough idempotents A, we put $\mathcal{C}_V^o = \mathrm{per}(\tilde{A}[[T]])$, we denote by $\mathcal{C}_V^o[t^{-1}]$ the localization of $\mathcal{C}_V^o$ with respect to natural transformation t given above (see [13, Remark 2] for the definition) and we let $p : \mathcal{C}_V^o \longrightarrow \mathcal{C}_V^o[t^{-1}]$ be the canonical functor. We also put $_V\mathcal{C}^o = \mathrm{per}(\tilde{A}[[T]]^{op})$, $_V\mathcal{C}^o[t^{-1}]$ and $p : {_V\mathcal{C}^o} \longrightarrow {_V\mathcal{C}^o[t^{-1}]}$ for the corresponding concepts on the left.

Lemma 17. *Let $p : \mathcal{C}_V^o \longrightarrow \mathcal{C}_V^o[t^{-1}]$ and $p' : {_V\mathcal{C}^o} \longrightarrow {_V\mathcal{C}^o[t^{-1}]}$ be the canonical triangulated functors given by localization, and let Q_1 and Q_2 be objects of $\mathcal{C}_V^o = per(\tilde{A}[[T]])$. There is an isomorphism $p(Q_1) \cong p(Q_2)$ if, and only if, there is an isomorphism $p'(Q_1^\star) \cong p'(Q_2^\star)$, where $(?)^\star := \mathbb{R}Hom_{\tilde{A}[[T]]}(?, \tilde{A}[[T]]) : \mathcal{D}(\tilde{A}[[T]])^{op} = \mathcal{C}_V^{op} \longrightarrow {_V\mathcal{C}} = \mathcal{D}(\tilde{A}^{op}[[T]])$ is the usual triangulated functor.*

Proof. The fact that $p(Q_1)$ and $p(Q_2)$ are isomorphic in $\mathcal{C}_V^o[t^{-1}]$ means that we have morphisms $f : Q_1 \longrightarrow Q_2$ and $g : Q_2 \longrightarrow Q_1$ in $\mathcal{C}_V^o = \mathrm{per}(\tilde{A}[[T]])$ such that $g \circ f \circ t_{Q_1}^r = t_{Q_1}^s$ and $f \circ g \circ t_{Q_2}^m = t_{Q_2}^n$, for some $r, s, m, n \in \mathbb{N}$. If we now apply the duality

$$(?)^\star = \mathbb{R}\mathrm{Hom}_{\tilde{A}[[T]]}(?, \tilde{A}[[T]]) : \mathrm{per}(\tilde{A}[[T]]) = \mathcal{C}_V^o \xrightarrow{\cong^o} {_V\mathcal{C}^o} = \mathrm{per}(\tilde{A}[[T]]^{op}),$$

then we get that $(t_{Q_2}^\star)^m \circ g^\star \circ f^\star = (t_{Q_2}^\star)^n$. But Propositions 2 and 16 tell us that we have $(t_{Q_k})^\star = t_{Q_k^\star}$ for $k = 1, 2$, which implies that $p'(g^\star) \circ p'(f^\star)$ and $p'(f^\star) \circ p'(g^\star)$ are isomorphisms in $_V\mathcal{C}^o[t^{-1}]$, and hence that $p'(Q_1^\star) \cong p'(Q_2^\star)$ in $_V\mathcal{C}^o[t^{-1}]$.

The reverse implication follows by exchanging the roles of A and A^{op} and of Q_k and $Q_k^\star$, bearing in mind that Q_k is isomorphic to $Q_k^{\star\star}$, for $k = 1, 2$ (see [12, Proposition 10.4]). □

The first assertion of the following Lemma 18 seems to be folklore, but we include a short proof. A right dg $\tilde{A}[[T]]$-module Q will be called *T-torsion-free* when $t_Q : Q \longrightarrow Q$ is monomorphism in Gr-$\tilde{A}[[T]]$. Note that this is equivalent to saying that t_Q is a monomorphism for the abelian structure of $\mathcal{C}(\tilde{A}[[T]])$.

Lemma 18. *Let A be a dg algebra with enough idempotents and let $q_A : \mathcal{H}(A) \longrightarrow \mathcal{D}(A)$ be the canonical functor. The following assertions hold:*

(1) *The induced functor $q : thick_{\mathcal{H}(A)}(e_iA\colon i \in I) \longrightarrow per(A) = \mathcal{D}^c(A)$ is an equivalence of triangulated categories.*

(2) *Each $Q \in thick_{\mathcal{H}(\tilde{A}[[T]])}(e_i\tilde{A}[[T]]\colon i \in I)$ is isomorphic in $\mathcal{H}(\tilde{A}[[T]])$ to a T-torsion-free right dg $\tilde{A}[[T]]$-module.*

Proof. (1) The subcategory $\mathrm{thick}_{\mathcal{H}(A)}(e_iA\colon i \in I)$ of $\mathcal{H}(A)$ consists of homotopically projective objects and the restriction of q to the subcategory of homotopically projective objects is fully faithful. In order to prove the density, recall that $\mathrm{per}(A) = \mathrm{thick}_{\mathcal{D}(A)}(e_iA\colon i \in I)$ (see [6, Theorem 5.3]). This implies in particular that each $X \in \mathrm{per}(A)$ is a direct summand in $\mathcal{D}(A)$ of a right dg A-module P for which there is a sequence of morphisms

$$0 = P_0 \xrightarrow{f_1} P_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} P_{n-1} \xrightarrow{f_n} P_n$$

in $\mathcal{D}(A)$ such that $P_n = P$ and $\mathrm{cone}(f_k) \cong e_{i_k}A[m_k]$, for some $i_k \in I$ and some $m_k \in \mathbb{Z}$, for $k = 1, ..., n$. We will prove by induction on n that $P \cong q(Q)$, for some $Q \in \mathrm{thick}_{\mathcal{H}(A)}(e_iA\colon i \in I)$. For $n = 0$ there is nothing to prove, so we assume that $n > 1$. By the induction hypothesis, we can choose $Q_{n-1} \in \mathrm{thick}_{\mathcal{H}(A)}(e_iA\colon i \in I)$ such that $q(Q_{n-1}) \cong P_{n-1}$. We then get a distinguished triangle $Q_{n-1} \longrightarrow P \longrightarrow e_iA[m] \xrightarrow{f[1]} Q_{n-1}[1]$, for some $i \in I$, some $m \in \mathbb{Z}$ and some morphism $f : e_iA[m-1] \longrightarrow Q_{n-1}$ in $\mathcal{D}(A)$. But the functor q gives an isomorphism $\mathrm{Hom}_{\mathcal{H}(A)}(e_iA[m-1], Q_{n-1}) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{D}(A)}(e_iA[m-1], Q_{n-1})$. This means that we may view f as a morphism in $\mathcal{H}(A)$, and then the triangulated cone $Q = \mathrm{cone}_{\mathcal{H}(A)}(f)$ is in $\mathrm{thick}_{\mathcal{H}(A)}(e_iA\colon i \in I)$ and satisfies that $q(Q) \cong P$.

Let now X, P and Q be as above and let $e \in \operatorname{End}_{\mathcal{D}(A)}(P)$ be the idempotent endomorphism corresponding to the direct summand X of P. Since q gives an algebra isomorphism $\operatorname{End}_{\mathcal{H}(A)}(Q) \xrightarrow{\cong} \operatorname{End}_{\mathcal{D}(A)}(P)$, we have a unique $\epsilon = \epsilon^2 \in \operatorname{End}_{\mathcal{H}(A)}(Q)$ such that $q(\epsilon) = e$. Since $\mathcal{H}(A)$ has arbitrary (set-indexed) coproducts, we know that idempotents split in $\mathcal{H}(A)$ (see [9, Proposition 1.6.8]). We then get a direct summand Y of Q in $\mathcal{H}(A)$ corresponding to ϵ, and we clearly have that $q(Y) \cong X$.

(2) Let $Q \in \operatorname{thick}_{\mathcal{H}(\tilde{A}[[T]])}(e_i \tilde{A}[[T]]\colon i \in I)$ be any object. By the obvious adaptation of [6, Theorem 3.1] to the language of dg algebras with enough idempotents, we know that there is a chain of inflations in $\mathcal{C}(\tilde{A}[[T]])$

$$0 = P_0 \hookrightarrow P_1 \hookrightarrow ... \hookrightarrow P_n \hookrightarrow ...$$

such that $\operatorname{Coker}(P_{n-1} \hookrightarrow P_n)$ is a direct summand in $\mathcal{C}(\tilde{A}[[T]])$ of a (possibly infinite) coproduct of dg right $\tilde{A}[[T]]$-module of the form $e_i \tilde{A}[[T]][m]$, with $i \in I$ and $m \in \mathbb{Z}$, and $P = \bigcup_{n \in \mathbb{N}} P_n$ is isomorphic to Q in $\mathcal{D}(\tilde{A}[[T]])$. Since all the exact sequences

$$0 \to P_{n-1} \hookrightarrow P_n \longrightarrow P_n/P_{n-1} \to 0$$

split in Gr-$\tilde{A}[[T]]$, we readily see that P is projective in this category. In particular P is T-torsion-free. But P and Q are homotopically projective objects of $\mathcal{H}(\tilde{A}[[T]])$, which implies that the canonical functor $q : \mathcal{H}(\tilde{A}[[T]]) \longrightarrow \mathcal{D}(\tilde{A}[[T]])$ induces bijections $\operatorname{Hom}_{\mathcal{H}(\tilde{A}[[T]])}(X,Y) \xrightarrow{\cong} \operatorname{Hom}_{\mathcal{D}(\tilde{A}[[T]])}(X,Y)$, for $X, Y \in \{P, Q\}$. We deduce that any isomorphism $P \xrightarrow{\cong} Q$ in $\mathcal{D}(\tilde{A}[[T]])$ can be lifted to a corresponding isomorphism in $\mathcal{H}(\tilde{A}[[T]])$. $\square$

4.2. The main theorem under hypothesis (b).

We can now complete the proof of Theorem 1.

Theorem 19. *Let $\mathcal{C}_K^\circ$ be the category of compact objects of an algebraic compactly generated triangulated category. For any objects $M, N \in Ob(\mathcal{C}_K^\circ)$, the following assertions are equivalent:*

(1) *There is a distinguished triangle* $Z_\ell \xrightarrow{\binom{v_\ell}{u_\ell}} Z_\ell \oplus M \xrightarrow{(h_\ell \ \ j_\ell)} N \longrightarrow Z_\ell[1]$ *, where v_ℓ is a nilpotent endomorphism of Z_ℓ.*

(2) *There is a distinguished triangle* $N \xrightarrow{\binom{j_r}{h_r}} M \oplus Z_r \xrightarrow{(u_r \ \ v_r)} Z_r \longrightarrow N[1]$ *, where v_r is a nilpotent endomorphism of Z_r.*

Proof. Using the version of Keller's theorem for dg algebras with enough idempotents (see [12, Corollary 6.11]), we can and shall assume that $\mathcal{C}_K^\circ = \mathcal{D}(A)^c = \operatorname{per}(A)$, where A is a dg algebra with enough idempotents.

(1) $\Longrightarrow$ (2) : In [13, Proposition 9] we showed that if there is a distinguished triangle as in assertion 1, then the quintuple $(\mathcal{C}_K, \mathcal{C}_V, \mathcal{C}_V^\circ, \uparrow_K^V, t)$ give degeneration data for $\mathcal{C}_K^\circ$, where $\mathcal{C}_K = \mathcal{D}(A)$, $\mathcal{C}_V = \mathcal{D}(\tilde{A}[[T]])$, $\mathcal{C}_V^\circ = \mathcal{D}(\tilde{A}[[\mathcal{T}]])^c = \operatorname{per}(\tilde{A}[[T]])$ and $\uparrow_K^V \colon \mathcal{D}(A) \longrightarrow \mathcal{D}(\tilde{A}[[T]])$ and $t : 1_{\mathcal{D}(\tilde{A}[[T]])} \longrightarrow 1_{\mathcal{D}(\tilde{A}[[T]])}$ are as in Section 2.3. Moreover, in the above mentioned result [13, Proposition 9] it was also proved that there exists an object $Q \in \mathcal{C}_V^\circ = \operatorname{per}(\tilde{A}[[T]])$ so that both required conditions for categorical degeneration are satisfied, namely:

(1) If $p : \mathcal{C}_V^\circ \longrightarrow \mathcal{C}_V^\circ[t^{-1}]$ is the canonical functor, then $p(Q) \cong p(M \uparrow_K^V)$;

(2) $\phi(\operatorname{cone}(t_Q)) \cong N$, where $\phi : \mathcal{C}_V^\circ = \operatorname{per}(\tilde{A}[[T]]) \longrightarrow \mathcal{D}(A) = \mathcal{C}_K$ is the restriction of $\iota_* : \mathcal{D}(\tilde{A}[[T]]) \longrightarrow \mathcal{D}(A)$ to $\operatorname{per}(\tilde{A}[[T]])$.

With this information in mind, we give the proof of the theorem, which is divided in two steps:

Step 1: If Q_1 is a T-torsion-free right dg $\tilde{A}[[T]]$-module in $\text{thick}_{\mathcal{H}(\tilde{A}[[T]])}(e_i\tilde{A}[[T]]: i \in I)$
and if we put $Q_1^\star = \overline{HOM}_{\tilde{A}[[T]]}(Q_1, \tilde{A}[[T]])$, *then*

$$\phi(cone(t_{Q_1^\star})) \cong \mathbb{R}Hom_A(?, A)(\phi(cone(t_{Q_1})))$$

Note that Q_1 is homotopically projective, so that we also have

$$Q_1^\star = \mathbb{R}\text{Hom}_{\tilde{A}[[T]]}(?, \tilde{A}[[T]])(Q_1) \cong \mathbb{R}\text{HOM}_{\tilde{A}[[T]]}(Q_1, \tilde{A}[[T]])$$

(see Proposition 2). On the other hand, the homomorphism of dg algebras

$$\rho \otimes \rho^o : \tilde{A}[[T]] \otimes \tilde{A}[[T]]^{op} \longrightarrow A \otimes A^{op}$$

gives a restriction of scalars functor

$$(\rho \otimes \rho^o)_* : A\text{-Dg-}A \longrightarrow \tilde{A}[[T]]\text{-Dg-}\tilde{A}[[T]].$$

In particular A is a dg $\tilde{A}[[T]] - \tilde{A}[[T]]$−bimodule by defining $(\sum_{k\in\mathbb{N}} a_k T^k)a = a_0 a$ and $a(\sum_{k\in\mathbb{N}} a_k T^k)) = aa_0$, for all homogeneous elements $\sum_{k\in\mathbb{N}} a_k T^k \in \tilde{A}[[T]]$ and $a \in A$. Note that we then have an exact sequence of T-symmetric $\tilde{A}[[T]] - \tilde{A}[[T]]$−bimodules

$$0 \to \tilde{A}[[T]] \xrightarrow{t_{\tilde{A}[[T]]}} \tilde{A}[[T]] \xrightarrow{\rho} A \to 0$$

in Gr-$(\tilde{A}[[T]] \otimes \tilde{A}[[T]]^{op})$ and in (the abelian structure of) $\mathcal{C}(\tilde{A}[[T]] \otimes \tilde{A}[[T]]^{op})$.

The last sequence gives a distinguished triangle

$$\tilde{A}[[T]] \xrightarrow{t_{\tilde{A}[[T]]}} \tilde{A}[[T]] \xrightarrow{\rho} A \longrightarrow \tilde{A}[[T]][1]$$

in $\mathcal{D}(\tilde{A}[[T]] \otimes \tilde{A}[[T]]^{op})$. By Propositions 2 and 16, application of the functor $\mathbb{R}\text{Hom}_{\tilde{A}[[T]]}(Q_1, ?) :$ $\mathcal{D}(\tilde{A}[[T]] \otimes \tilde{A}[[T]]^{op}) \longrightarrow \mathcal{D}(\tilde{A}[[T]]^{op})$ to the last distinguished triangle gives a distinguished triangle

$$Q_1^\star \xrightarrow{t_{Q_1^\star}} Q_1^\star \longrightarrow \mathbb{R}\text{Hom}_{\tilde{A}[[T]]}(Q_1, A) \longrightarrow Q_1^\star[1]$$

in $\mathcal{D}(\tilde{A}[[T]]^{op})$, so that $\text{cone}(t_{Q_1^\star}) \cong \mathbb{R}\text{Hom}_{\tilde{A}[[T]]}(Q_1, A) := \mathbb{R}\text{Hom}_{\tilde{A}[[T]]}(Q_1, ?)(A)$.

It is important to notice that, by Proposition 2 again, we have isomorphisms in $\mathcal{D}(\tilde{A}[[T]]^{op})$:

$$\mathbb{R}\text{Hom}_{\tilde{A}[[T]]}(Q_1, ?)(A) \cong \mathbb{R}\text{HOM}_{\tilde{A}[[T]]}(Q_1, A) \cong \mathbb{R}\text{Hom}_{\tilde{A}[[T]]}(?, A)(Q_1).$$

When we apply the contravariant triangulated functor $\mathbb{R}\text{Hom}_{\tilde{A}[[T]]}(?, A) : \mathcal{D}(\tilde{A}[[T]]) \longrightarrow$ $\mathcal{D}(\tilde{A}[[T]]^{op})$ to the morphisms $t_{Q_1} : Q_1 \longrightarrow Q_1$ we obtain the zero map. Indeed, due to the homotopically projective condition of Q_1, we have that $\mathbb{R}\text{Hom}_{\tilde{A}[[T]]}(?, A)(Q_1) = \overline{HOM}_{\tilde{A}[[T]]}(Q_1, A)$. But since multiplication by T kills the elements of A, for each homogeneous element $f \in \overline{HOM}_{\tilde{A}[[T]]}(Q_1, A)$ we have

$$[\mathbb{R}\text{Hom}_{\tilde{A}[[T]]}(?, A)(t_{Q_1})](f) = (-1)^{|f||t_{Q_1}|}f \circ t_{Q_1} = f \circ t_{Q_1},$$

and this is a morphism of right dg $\tilde{A}[[T]]$-modules $Q_1 \longrightarrow A$ such that

$$(f \circ t_{Q_1})(x) = f(xT) = f(x)T = 0$$

for all $x \in Q_1$. On the other hand, the contravariant dg functor $\overline{HOM}_{\tilde{A}[[T]]}(?, A) : \text{Gr-}\tilde{A}[[T]] \longrightarrow$ $\tilde{A}[[T]]\text{-Gr}$ is left exact and we have an exact sequence

$$0 \to Q_1 \xrightarrow{t_{Q_1}} Q_1 \xrightarrow{p} Q_1/TQ_1 \to 0$$

in Gr-$\tilde{A}[[T]]$ (which is actually an exact sequence in $\mathcal{C}(\tilde{A}[[T]])$) due to the T-torsion-free condition of Q_1. It follows that we have an exact sequence

$$0 \to \overline{HOM}_{\tilde{A}[[T]]}(Q_1/TQ_1, A) \xrightarrow{p^*} \overline{HOM}_{\tilde{A}[[T]]}(Q_1, A) \xrightarrow{0} \overline{HOM}_{\tilde{A}[[T]]}(Q_1, A)$$

in $\tilde{A}[[T]]$-Gr. Therefore we have an isomorphism

$$p^* : \overline{HOM}_{\tilde{A}[[T]]}(Q_1/TQ_1, A) \xrightarrow{\cong} \overline{HOM}_{\tilde{A}[[T]]}(Q_1, A).$$

We will see that p^* commutes with the differentials, which will show that we have an isomorphism

$$\text{cone}(t_{Q_1^\star}) \cong \overline{HOM}_{\tilde{A}[[T]]}(Q_1/TQ_1, A)$$

in $\mathcal{D}(\tilde{A}[[T]]^{op})$. Indeed if $d : \overline{HOM}_{\mathcal{A}[[T]]}(Q_1/Q_1T, A) \longrightarrow \overline{HOM}_{\mathcal{A}[[T]]}(Q_1/Q_1T, A)$ and $\delta : \overline{HOM}_{\mathcal{A}[[T]]}(Q_1, A) \longrightarrow \overline{HOM}_{\mathcal{A}[[T]]}(Q_1, A)$ denote the respective differentials, then, for each homogeneous element $g \in \overline{HOM}_{\mathcal{A}[[T]]}(Q_1/Q_1T, A)$, we have

$$(\delta \circ p^*)(g) = \delta(g \circ p) = d_A \circ g \circ p - (-1)^{|g \circ p|} g \circ p \circ d_{Q_1} = d_A \circ g \circ p - (-1)^{|g|} g \circ p \circ d_{Q_1}.$$

But p is a morphism in $Z^0(\tilde{A}[[T]]\text{-Dg}) = \mathcal{C}(\tilde{A}[[T]]^{op})$, so that $d_{Q_1/Q_1T} \circ p - p \circ d_{Q_1} = 0$. We then get

$$(\delta \circ p^*)(g) = d_A \circ g \circ p - (-1)^{|g|} g \circ d_{Q_1/Q_1T} \circ p = d(g) \circ p = (p^* \circ d)(g),$$

and hence $\delta \circ p^* = p^* \circ d$ as desired.

If we apply the restriction of scalars functor $\phi = \iota_* : \tilde{A}[[T]]\text{-Dg} \longrightarrow A\text{-Dg}$, then $\overline{HOM}_{\tilde{A}[[T]]}(Q_1/TQ_1, A)$ is taken to $\overline{HOM}_A(\phi(Q_1/Q_1T), A)$. But, as right dg A-modules, we have an isomorphism $\phi(Q_1/Q_1T) = \rho^*(Q_1)$ where $\rho^* = ? \otimes_{\tilde{A}[[T]]} A : \text{Dg-}\tilde{A}[[T]] \longrightarrow \text{Dg-}A$ is the extension of scalars along the morphism of dg algebras with enough idempotents $\rho : \tilde{A}[[T]] \longrightarrow A$. Then the induced triangulated functor

$$\rho^* = H^0(\rho^*) : H^0(\text{Dg-}\tilde{A}[[T]]) = \mathcal{H}(\tilde{A}[[T]]) \longrightarrow \mathcal{H}(A) = H^0(\text{Dg-}A)$$

has the property that

$$\rho^*(Q_1) = \phi(Q_1/Q_1T) \in \text{thick}_{\mathcal{H}(A)}(\rho^*(e_i\tilde{A}[[T]]) : i \in I) = \text{thick}_{\mathcal{H}(A)}(e_iA : i \in I).$$

In particular, we get that $\phi(Q_1/Q_1T)$ is homotopically projective in $\mathcal{H}(A)$ and, hence, that

$$\overline{HOM}_A(\phi(Q_1/Q_1T), A) \cong \mathbb{R}\text{Hom}_A(?, A)(\phi(Q_1/Q_1T)) \cong \mathbb{R}\text{HOM}_A(\phi(Q_1/Q_1T), A).$$

(see Proposition 2). Bearing in mind that the exact sequence

$$0 \to Q_1 \xrightarrow{t_{Q_1}} Q_1 \longrightarrow Q_1/TQ_1 \to 0$$

in $\mathcal{C}(\tilde{A}[[T]])$ (with respect to the abelian exact structure of $\mathcal{C}(\tilde{A}[[T]])$) gives a distinguished triangle

$$Q_1 \xrightarrow{t_{Q_1}} Q_1 \longrightarrow Q_1/TQ_1 \longrightarrow Q_1[1]$$

in $\mathcal{D}(\tilde{A}[[T]])$, we get that $Q_1/Q_1T \cong \text{cone}(t_{Q_1})$ in $\mathcal{D}(\tilde{A}[[T]])$. We then get isomorphisms

$$\begin{aligned}
\phi(\text{cone}(t_{Q_1^\star})) &\cong \phi(\overline{HOM}_{\tilde{A}[[T]]}(Q_1/TQ_1, A)) \\
&\cong \overline{HOM}_A(\phi(Q_1/Q_1T), A) \\
&\cong \mathbb{R}\text{Hom}_A(?, A)(\phi(\text{cone}(t_{Q_1}))).
\end{aligned}$$

Step 2: End of the proof: Let now M and N be as in assertion (1) and let $Q \in \mathcal{C}_V^o = \text{per}(\tilde{A}[[T]])$ be the right dg $\tilde{A}[[T]]$-module considered in the second paragraph of this proof. Using Lemma 18, without loss of generality, we may assume that $M, N \in \text{thick}_{\mathcal{H}(A)}(e_iA : i \in I)$ and that Q is a T-torsion-free right $\tilde{A}[[T]]$-module in $\text{thick}_{\mathcal{H}(\tilde{A}[[T]])}(e_i\tilde{A}[[T]] : i \in I)$. Note that then $\tilde{M}[[T]] \in \text{thick}_{\mathcal{H}(\tilde{A}[[T]])}(e_i\tilde{A}[[T]] : i \in I)$ and $\tilde{M}[[T]]$ is T-torsion-free.

According to the Step 1, we have that

$$\phi(\text{cone}(t_{Q^\star})) \cong \mathbb{R}\text{Hom}_A(?, A)(\phi(\text{cone}(t_Q))) = \mathbb{R}\text{Hom}_A(?, A)(N) =: N^*.$$

On the other hand, by Corollary 14 and Proposition 13, we know that

$$(M \uparrow_K^V)^\star := \mathbb{R}\text{Hom}_{\tilde{A}[[T]]}(?, \tilde{A}[[T]])(M \uparrow_K^V) \cong M^* \uparrow_K^V,$$

where $M^* := \mathbb{R}\mathrm{Hom}_A(?, A)(M)$. Moreover, by Lemma 17, we get that $p'((M \uparrow_K^V)^\star)$ and $p'(Q^\star)$ are isomorphic in $_V\mathcal{C}^o[t^{-1}]$, where $p' : {}_V\mathcal{C}^o \longrightarrow {}_V\mathcal{C}^o[t^{-1}]$ is the canonical functor. It follows from this that the left dg $\tilde{A}[[T]]$-module

$$Q^\star := \mathbb{R}\mathrm{Hom}_{\tilde{A}[[T]]}(?, \tilde{A}[[T]])(Q) = \mathbb{R}\mathrm{HOM}_{\tilde{A}[[T]]}(Q, \tilde{A}[[T]])$$

defines a categorical degeneration $M^* \leq_{cdeg} N^*$, where $(?)^* = \mathbb{R}\mathrm{Hom}_A(?, A) : \mathrm{per}(A) \xrightarrow{\cong o} \mathrm{per}(A^{op})$ is the duality defined by the regular dg bimodule $X = A$ (see [12, Proposition 10.4]).

But categorical degeneration implies triangle degeneration by Theorem 7 (see [13, Proposition 8]), so that we have $M^* \leq_{\Delta+\mathrm{nil}} N^*$. That is, we have a distinguished triangle

$$U \xrightarrow{\binom{v'}{u'}} U \oplus M^* \xrightarrow{(h' \quad j')} N^* \longrightarrow U[1]$$

in $_K\mathcal{C}^o = \mathrm{per}(A^{op})$, where v' is a nilpotent endomorphism of U. Applying the duality $\mathbb{R}\mathrm{Hom}_{A^{op}}(?, A) : \mathrm{per}(A^{op}) \xrightarrow{\cong o} \mathrm{per}(A)$ to this last distinguished triangle, we obtain a distinguished triangle

$$N \xrightarrow{\binom{j'^*}{h'^*}} M \oplus U^* \xrightarrow{(u'^* \quad v'^*)} U^* \longrightarrow N[1],$$

where v'^* is clearly a nilpotent endomorphism of U^*. The proof of the implication ends by taking $Z_r := U^*$.

$(2) \Longrightarrow (1)$: By applying the duality $\mathbb{R}\mathrm{Hom}_A(?, A) : \mathcal{C}_K^\circ = \mathrm{per}(A) \xrightarrow{\cong o} \mathrm{per}(A^{op})$ to the distinguished triangle in assertion (2) we get that $M^* \leq_{\Delta+\mathrm{nil}} N^*$ in $_K\mathcal{C}^\circ := \mathrm{per}(A^{op})$. The proof of the implication $1) \Longrightarrow 2)$, when applied with A^{op} instead of A, shows that we have $M \cong M^{**} \leq_{\Delta+\mathrm{nil}} N^{**} \cong N$, so that the distinguished triangle of assertion 1) exists. $\square$

Observe that the proof actually gives the following additional statement.

Corollary 20. *Let $\mathcal{C}_K^\circ = \mathcal{D}(A)^c$ be the category of compact objects in the derived category $\mathcal{D}(A)$ of a dg algebra A with enough idempotents. For any objects $M, N \in Ob(\mathcal{C}_K^\circ)$, we get*

$$M \leq_{\Delta+nil} N \Leftrightarrow \mathbb{R}Hom_A(?, A)(M) \leq_{\Delta+nil} \mathbb{R}Hom_A(?, A)(N).$$

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