DEGENERATING 0 IN TRIANGULATED CATEGORIES

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Abstract. In previous work, based on work of Zwara and Yoshino, we defined and studied degenerations of objects in triangulated categories analogous to degeneration of modules. In triangulated categories $\mathcal{T}$ it is surprising that the zero object may degenerate. We show that the triangulated subcategory of $\mathcal{T}$ generated by the objects which are degenerations of zero coincides with the triangulated subcategory of $\mathcal{T}$ consisting of the objects with vanishing image in the Grothendieck group $K_0(\mathcal{T})$ of $\mathcal{T}$.

Introduction

Degeneration of modules were intensively studied by e.g. Gabriel [5], Huisgen-Zimmermann, Riedtmann [14], Zwara [22, 23] since at least 1974, and was highly successful in various constructions. Degeneration of modules is defined by the following setting. Let k be an algebraically closed field, and let A be a finite dimensional k-algebra. Then the A-module structures on the vector space k^d form an affine algebraic variety $\mathrm{mod}(A,d)$ on which $GL_d(k)$ acts by conjugation. Isomorphism classes correspond to orbits under this action and an A-module M degenerates to N if the point corresponding to N belongs to the Zariski closure of the $GL_d(k)$-orbit of the point corresponding to M. We write $M \leq_{\mathrm{deg}} N$ in this case. Riedtmann and Zwara showed that $M \leq_{\mathrm{deg}} N$ if and only if there is an A-module Z and a short exact sequence $0 \longrightarrow Z \longrightarrow Z \oplus M \longrightarrow N \longrightarrow 0.$ In collaboration with Jensen and Su [8] the second named author started to study an analogous concept for derived categories with a geometrically inspired concept based on orbit closures, and then in [9] more generally for triangulated categories based on Zwara's characterisation replacing short exact sequences by distinguished triangles. This last relation is denoted by the symbol $\leq_\Delta$. Both concepts were highly successfully used in many places, cf e.g. [10, 11, 12, 3, 4, 6, 7, 21]. Independently Yoshino [20] gave a scheme theoretic definition for degenerations in the (triangulated) stable category of maximal Cohen-Macaulay modules, and he highlighted that in $M \leq_\Delta N$ one should assume that the induced endomorphism on Z should be nilpotent. We denote the relation by $\leq_{\Delta+\mathrm{nil}}$ in this case. Yoshino's scheme theoretic approach was a model for us to give a more general geometric definition for degeneration, which was achieved in [16] by introducing a scheme theoretic degeneration $\leq_{\mathrm{cdeg}}$.

We then showed that, in case $\mathcal{T}$ has split idempotents, $M \leq_{\mathrm{cdeg}} N$ always implies $M \leq_{\Delta+\mathrm{nil}} N$, for objects $M, N \in \mathcal{T}$, the converse being also true when $\mathcal{T}$ is the subcategory of compact objects of a compactly generated algebraic triangulated category. Obviously, $M \leq_{\Delta+\mathrm{nil}} N$ implies $M \leq_\Delta N$. We further see right from the definition that $M \leq_\Delta N$ implies that M and N have the same image in the Grothendieck group of $\mathcal{T}$.

A striking phenomenon is that, unlike in the module case, in triangulated categories $\mathcal{T}$ one may have non zero objects M with $0 \leq_{\Delta+\mathrm{nil}} M$, namely cones of nilpotent endomorphisms of objects of $\mathcal{T}$.

By the above, $0 \leq_\Delta N$ implies that N has vanishing image in the Grothendieck group of the triangulated category. We then show as our main result that the full triangulated subcategory of $\mathcal{T}$ consisting of the objects with image 0 in the Grothendieck group of $\mathcal{T}$ coincides with the full triangulated subcategory of $\mathcal{T}$ generated by objects being degenerations of the zero object of $\mathcal{T}$.

We prove this by showing that both categories actually coincide with the full triangulated subcategory generated by objects of the form $\bigoplus_{i=1}^r (X_i \oplus X_i[t_i])$ for pairwise different odd integers t_i, and objects X_i in some fixed set of generators of $\mathcal{T}$.

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Furthermore, we use our result to give examples showing that $M \leq_\Delta N$ does not imply $M \leq_{\Delta+\mathrm{nil}} N$ and that an object M with image 0 in the Grothendieck group is not necessarily a degeneration of 0, not even in the transitive hull of the relation $\leq_{\Delta+\mathrm{nil}}$.

The paper is organised as follows. In Section 1 we recall the necessary concepts on the various types of degeneration and recall the implications which we proved essentially in our earlier work [16, 17]. In Section 2 we study the image of triangle degenerations of 0 in the Grothendieck group, prove our main result Theorem 7 and give the examples mentioned above.

1. Review on Degenerations in Triangulated Categories

We have different degeneration concepts. The first one, the triangle degeneration, is a triangular category analogue of Zwara's definition of degeneration in the case of module categories. Zwara says [22, 23] that for a k-algebra A an A-module M degenerates to an A-module N if and only if there is an A-module Z and a short exact sequence $0 \longrightarrow Z \longrightarrow Z \oplus M \longrightarrow N \longrightarrow 0$. Yoshino [20] highlighted the importance of assuming that the induced endomorphism of Z should be nilpotent. In case of a category where Fitting's lemma holds we can always assume this fact.

Definition 1. [9, 16] Let K be a commutative ring and let $\mathcal{T}$ be a K-linear triangulated category. Then for two objects M and N in $\mathcal{T}$ we get $M \leq_\Delta N$ if and only if there is an object Z in $\mathcal{T}$ and a distinguished triangle

$$Z \xrightarrow{\binom{v}{u}} Z \oplus M \longrightarrow N \longrightarrow Z[1].$$

We say that $M \leq_{\Delta+\mathrm{nil}} N$ if and only if there is such a distinguished triangle with v is nilpotent.

Note that by [17, Proposition 10] $M \leq_{\Delta+\mathrm{nil}} N$ implies that there is an object Z' and a distinguished triangle

$$N \longrightarrow Z' \oplus M \xrightarrow{(v',u')} Z' \longrightarrow N[1].$$

We may write $M \leq_{\Delta,\mathrm{right}} N$ (resp. $M \leq_{\Delta+\mathrm{nil},\ \mathrm{right}} N$) if there is such a distinguished triangle (with v' nilpotent) and, for this paragraph only, write $M \leq_{\Delta,\mathrm{left}} N$ (resp. $M \leq_{\Delta+\mathrm{nil},\ \mathrm{left}} N$) in the situation of Definition 1. Note that $M \leq_{\Delta,\mathrm{left}} N$ (resp. $M \leq_{\Delta+\mathrm{nil},\mathrm{left}} N$) in $\mathcal{T}$ if and only if $N \leq_{\Delta,\mathrm{right}} M$ (resp. $N \leq_{\Delta+\mathrm{nil},\mathrm{right}} M$) in the opposite category $\mathcal{T}^{op}$. So categorical duality applies and results about $\leq_{\Delta,\mathrm{left}}$ (resp. $\leq_{\Delta+\mathrm{nil},\ \mathrm{left}}$) admit categorical dual ones, that we omit to state. Furthermore, if $\mathcal{T}$ has split idempotents and artinian endomorphism rings of objects, or if $\mathcal{T}$ is the category of compact objects in a compactly generated algebraic triangulated category, then $M \leq_{\Delta+\mathrm{nil},\ \mathrm{right}} N$ if and only if $M \leq_{\Delta+\mathrm{nil},\ \mathrm{left}} N$ (see [17, Theorem 1]).

A second concept of degeneration, motivated by Yoshino's work, is given by the following definition.

Definition 2. [16] Let K be a commutative ring and let $\mathcal{C}_K^\circ$ be a K-linear triangulated category with split idempotents.

A degeneration data for $\mathcal{C}_K^\circ$ is given by

- a triangulated category $\mathcal{C}_K$ with split idempotents and a fully faithful embedding $\mathcal{C}_K^\circ \longrightarrow \mathcal{C}_K$,
- a triangulated category $\mathcal{C}_V$ with split idempotents and a full triangulated subcategory $\mathcal{C}_V^\circ$,
- triangulated functors $\uparrow_K^V\colon \mathcal{C}_K \longrightarrow \mathcal{C}_V$, which we write after the arguments, and $\Phi : \mathcal{C}_V^\circ \to \mathcal{C}_K$, so that $(\mathcal{C}_K^\circ)\uparrow_K^V \subseteq \mathcal{C}_V^\circ$, when we view $\mathcal{C}_K^\circ$ as a full subcategory of $\mathcal{C}_K$,
- a natural transformation $\mathrm{id}_{\mathcal{C}_V} \xrightarrow{t} \mathrm{id}_{\mathcal{C}_V}$ of triangulated functors such that
- for each object M of $\mathcal{C}_K^\circ$ the morphism $\Phi(M\uparrow_K^V) \xrightarrow{\Phi(t_{M\uparrow_K^V})} \Phi(M\uparrow_K^V)$ is a split monomorphism in $\mathcal{C}_K$ with cone M.

Remark 3. Our definition of categorical degeneration, given below, is a generalisation to general triangulated categories of a definition given by Yoshino [20] for the case of stable categories of maximal Cohen-Macaulay modules over a local Gorenstein algebra. In Yoshino's work (see, e.g., [19]) he considers modules over an algebra R over a field K and defines degeneration along a suitable discrete valuation K-algebra V. Just to emphasize the similarity and facilitate the intuition of the reader, we used in [16] subindices K and V to denote our categories, but there and in Definition 2 the letters K and V play no role.

Definition 4. [16] Given two objects M and N of $\mathcal{C}_K^\circ$ we say that M degenerates to N in the categorical sense, written $M \leq_{\rm cdeg} N$, if there is a degeneration data for $\mathcal{C}_K^\circ$ and an object Q of $\mathcal{C}_V^\circ$ such that

$$p(Q) \simeq p(M \uparrow_K^V) \text{ in } \mathcal{C}_V^\circ[t^{-1}] \text{ and } \Phi(\text{cone}(t_Q)) \simeq N,$$

where $\mathcal{C}_V^\circ[t^{-1}]$ is the Gabriel-Zisman localisation at the endomorphisms t_X for all objects X of $\mathcal{C}_V^\circ$, and where $p : \mathcal{C}_V^\circ \longrightarrow \mathcal{C}_V^\circ[t^{-1}]$ is the canonical functor. In this case we write $M \leq_{\rm cdeg} N$.

We end the section by recalling the connection between these various types of degeneration and with the property of having the same image in the Grothendieck group $K_0(\mathcal{T})$.

Theorem 5. *Let $\mathcal{T}$ be a skeletally small triangulated category with split idempotents and let M and N be objects of $\mathcal{T}$. Consider the following assertions:*

(1) $M \leq_{\rm cdeg} N$
(2) $M \leq_{\Delta+\rm nil} N$
(3) $M \leq_{\Delta} N$
(4) $[M] = [N]$ *in the Grothendieck group* $K_0(\mathcal{T})$.

The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) *hold true. When $\mathcal{T}$ is the subcategory of compact objects of a compactly generated triangulated category, the implication* (2) $\Rightarrow$ (1) *also holds. When the endomorphism rings of objects in $\mathcal{T}$ are all artinian, the implication* (3) $\Rightarrow$ (2) *also holds and* $\leq_\Delta = \leq_{\Delta+\rm nil}$ *is a reflexive and transitive relation in the set of isoclasses of objects of $\mathcal{T}$.*

Proof. The implication (2) $\Rightarrow$ (3) is clear. On the other hand, if assertion 3 holds and we consider the triangle of Definition 1 we then get an equality $[Z] + [M] = [Z] + [N]$ in $K_0(\mathcal{T})$, which implies assertion 4.

On the other hand, the implication (1) $\Rightarrow$ (2) and, under the extra hypothesis, the implication (2) $\Rightarrow$ (1) are [16, Propositions 8 and 9]. Finally, under artinianity of all endomorphism rings of objects, implication (3) $\Rightarrow$ (2) and the reflexive and transitive condition of $\leq_\Delta = \leq_{\Delta+\rm nil}$ are [8, Proposition 2]. □

We postpone until the next section giving counterexamples to implications (3) $\Rightarrow$ (2) and (4) $\Rightarrow$ (3) of Theorem 5.

2. Degeneration of zero and the zero objects in the Grothendieck group

Let $\mathcal{T}$ be a skeletally small triangulated category with split idempotents all through this section. By Theorem 5, we know that any object N that is a degeneration of zero in $\mathcal{T}$ has the property that $[N] = 0$ in $K_0(\mathcal{T})$. The goal of this section is to compare the subcategories $\mathcal{T}_\Delta^0$ (resp. $\mathcal{T}_{\Delta+\rm nil}^0$) and $\mathcal{T}^0$ of $\mathcal{T}$ consisting, respectively, of the objects N such that $0 \leq_\Delta N$ (resp. $0 \leq_{\Delta+\rm nil} N$) and the objects N such that $[N] = 0$ in $K_0(\mathcal{T})$.

But, before tackling the problem, let us emphasize the ubiquity of degenerations of 0.

Remark 6. Since $M \leq_\Delta N$ (resp. $M \leq_{\Delta+\rm nil} N$) if and only if there is a distinguished triangle

$$Z \xrightarrow{\binom{u}{v}} M \oplus Z \xrightarrow{(s,t)} N \longrightarrow Z[1]$$

(resp. with v nilpotent) we see that this can be written as a homotopy cartesian square. Neeman [13, Lemma 1.4.3, Lemma 1.4.4] then shows that cone$(s) \simeq$ cone(v), and so $0 \leq_\Delta$ cone(s) (resp. $0 \leq_{\Delta+\rm nil}$ cone(s)). Hence, degenerations of 0 are intrinsic in degeneration in triangulated categories.

Recall that, given full subcategories $\mathcal{U}$ and $\mathcal{V}$ of a triangulated category $\mathcal{T}$, then the subcategory $\mathcal{U} \star \mathcal{V}$ is the full subcategory of $\mathcal{T}$ consisting of the objects M that fit in a distinguished triangle $U \longrightarrow M \longrightarrow V \longrightarrow U[1]$, with $U \in \mathcal{U}$ and $V \in \mathcal{V}$. It is well-known that the operation $\star$ is associative, in the sense that $(\mathcal{U} \star \mathcal{V}) \star \mathcal{W} = \mathcal{U} \star (\mathcal{V} \star \mathcal{W})$, for all subcategories $\mathcal{U}, \mathcal{V}, \mathcal{W}$ of $\mathcal{T}$ (see [1, Lemme 1.3.10]). If one puts $\mathcal{U}^{\star n} = \underbrace{\mathcal{U} \star \cdots \star \mathcal{U}}_{n \text{ factors}}$, for each $n \geq 0$ (with the convention that $\mathcal{U}^{\star 0} = 0$), then $\mathcal{U}^{\rm ext} = \bigcup_{n \in \mathbf{N}} \mathcal{U}^{\star n}$ is the *extension closure* of $\mathcal{U}$, that is, the smallest subcategory of $\mathcal{T}$ closed under extensions that contains $\mathcal{U}$. The smallest triangulated subcategory of $\mathcal{T}$ that contains $\mathcal{U}$, denoted $\text{tria}_{\mathcal{T}}(\mathcal{U})$, is

$$\text{tria}_{\mathcal{T}}(\mathcal{U}) = \bigcup_{n\in\mathbb{N}} \bigcup_{(r_1,\ldots,r_n)\in\mathbb{Z}^n} \mathcal{U}[r_1]\star\cdots\star\mathcal{U}[r_n].$$

In other words, the objects of $\text{tria}_{\mathcal{T}}(\mathcal{U})$ are precisely those M admitting a sequence

$$0 = M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} M_{n-1} \xrightarrow{f_n} M_n = M,$$

where $\text{cone}(f_k)$ is isomorphic to $U_k[r_k]$, for some $U_k \in \mathcal{U}$ and some $r_k \in \mathbb{Z}$, for all $k = 1, ..., n$.

In our next main result we will denote by $\preceq_{\Delta+\text{nil}}$ the smallest transitive relation containing $\leq_{\Delta+\text{nil}}$. Recall that $\preceq_{\Delta+\text{nil}}=\leq_{\Delta+\text{nil}}=\leq_{\Delta}$ whenever all endomorphism rings of objects of $\mathcal{T}$ are artinian (see Theorem 5).

Theorem 7. *Let $\mathcal{S}$ be a set of objects in the triangulated category $\mathcal{T}$ such that $\mathcal{T} = \text{tria}_{\mathcal{T}}(\mathcal{S})$, let $[\mathcal{S}] := \{[S]\colon S \in \mathcal{S}\}$ denote the corresponding set of generators of the group $K_0(\mathcal{T})$ and let $\widehat{\mathcal{S}}$ be the subcategory of $\mathcal{T}$ consisting of the objects X which are finite direct sums of shifts of objects in $\mathcal{S}$ and are such that $[X] = 0$ in $K_0(\mathcal{T})$. Denote by*

- *$\mathcal{T}^0_\Delta$ (resp. $\mathcal{T}^0_{\Delta+\text{nil}}$) the full subcategory of $\mathcal{T}$ consisting of the objects X such that $0 \leq_\Delta X$ (resp. $0 \leq_{\Delta+\text{nil}} X$)*
- *and by $\mathcal{T}^0$ the (triangulated) subcategory of $\mathcal{T}$ consisting of the objects M such that $[M] = 0$ in the group $K_0(\mathcal{T})$.*

Then the following assertions hold:

1. *An object M is in $\mathcal{T}^0$ if, and only if, $M \leq_\Delta X$ (resp. $M \preceq_{\Delta+\text{nil}} X$), for some $X \in \widehat{\mathcal{S}}$. When $[\mathcal{S}]$ is a basis of $K_0(\mathcal{T})$ the objects of $\widehat{\mathcal{S}}$ are precisely the finite direct sums of shifts of objects in $\bar{\mathcal{S}} := \{S \oplus S[2k+1]\colon k \in \mathbb{Z}; S \in \mathcal{S}\}$.*
2. *$\mathcal{T}^0 = \text{tria}_{\mathcal{T}}(S \oplus S[t_S]\colon S \in \mathcal{S})$, for every choice of odd integers t_S.*
3. *$\mathcal{T}^0$ is the extension closure of $\mathcal{T}^0_\Delta$ (resp. $\mathcal{T}^0_{\Delta+\text{nil}}$).*

Proof. (1) By Theorem 5, the 'if' part of this implication is clear. For the 'only if' part, we first claim that, for each $M \in \mathcal{T}$, one has that $M \preceq_{\Delta+\text{nil}} \oplus_{S\in\mathcal{S}} \oplus_{k\in\mathbb{Z}} S[k]^{m_{S,k}}$, where the S are in $\mathcal{S}$ and the $m_{k,S}$ are nonnegative integers, all zero but a finite number. Recall that $\mathcal{T} = \text{tria}_{\mathcal{T}}(\mathcal{S})$, and so we have a finite sequence

$$0 = M_0 \xrightarrow{f_1} M_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} M_n = M \qquad (*)$$

such that $C_k := \text{cone}(f_k)$ is a shift of some object of $\mathcal{S}$, for each $k = 1, ..., n$. We will settle our claim by induction on $n > 0$, the case $n = 1$ being clear. Suppose now that $n > 0$ and consider the induced triangle

$$M_{n-1} \xrightarrow{f_n} M \xrightarrow{g} C_n \longrightarrow M_{n-1}[1],$$

where $C_n \cong S[k]$, for some $S \in \mathcal{S}$ and $k \in \mathbb{Z}$. Taking the homotopy pushout of f_n and the zero endomorphism $M_{n-1} \xrightarrow{0} M_{n-1}$, we readily see that we have a distinguished triangle

$$M_{n-1} \xrightarrow{\binom{0}{f_n}} M_{n-1} \oplus M \longrightarrow M_{n-1} \oplus C_n \longrightarrow M_{n-1}[1].$$

That is, we have $M \preceq_{\Delta+\text{nil}} M_{n-1} \oplus C_n \cong M_{n-1} \oplus S[k]$. The result then follows by induction since $A_i \preceq_{\Delta+\text{nil}} B_i$, for $i = 1, 2$, implies that $A_1 \oplus A_2 \preceq_{\Delta+\text{nil}} B_1 \oplus B_2$.

We also claim that $M \leq_\Delta \oplus_{S\in\mathcal{S}} \oplus_{k\in\mathbb{Z}} S[k]^{m_{S,k}}$, for S_k and $m_{S,k}$ as in the previous paragraph. Using again the sequence $(*)$ and bearing in mind that each cone $C_k := \text{cone}(f_k)$ is a shift of some object in $\mathcal{S}$, we consider the distinguished triangles

$$M_{k-1} \xrightarrow{f_k} M_k \longrightarrow C_k \longrightarrow M_{k-1}[1]$$

for all $k \in \{1, \dots, n-1\}$. Taking the direct sum of these distinguished triangles we get a distinguished triangle

$$\left(\bigoplus_{k=1}^{n-1} M_k\right) \xrightarrow{\oplus_{k=1}^{n} f_k} \left(M \oplus \bigoplus_{k=1}^{n-1} M_k\right) \longrightarrow \left(\bigoplus_{k=1}^{n} C_k\right) \longrightarrow \left(\bigoplus_{k=1}^{n-1} M_k\right)[1]$$

and hence $M \leq_\Delta \bigoplus_{k=1}^{n} C_k$, as desired.

The last two paragraphs show that we have $M \preceq_{\Delta+\mathrm{nil}} X$ and $M \leq_\Delta Y$, for objects X, Y which are direct sums of shift of objects of $\mathcal{S}$. When in addition $M \in \mathcal{T}^0$, by Theorem 5, we also have $[X] = [Y] = 0$ in $K_0(\mathcal{T})$. Therefore we have that $X, Y \in \hat{\mathcal{S}}$. This proves assertion (1), except for the final statement.

To prove that final statement, suppose that $[\mathcal{S}]$ is a basis of $K_0(\mathcal{T})$. We claim that in this case each object of $\hat{\mathcal{S}}$ is a direct sum of objects of the form $S[k] \oplus S[l] = (S \oplus S[l-k])[k]$, with $S \in \mathcal{S}$ and $l-k$ odd. This will end the proof. Let then take $X \in \hat{\mathcal{S}}$ and decompose it as $X = \oplus_{S \in \mathcal{S}} \oplus_{k \in \mathbb{Z}} S[k]^{m_{S,k}}$. Note that, due to the fact that $[\mathcal{S}]$ is a basis of $K_0(\mathcal{T})$, the summand $X_S = \oplus_{k \in \mathbb{Z}} S[k]^{m_{k,S}}$ also satisfies that $[X_S] = 0$ in $K_0(\mathcal{T})$, for each $S \in \mathcal{S}$. So it is not restrictive to assume that $X = S[k_1]^{m_1} \oplus S[k_2]^{m_2} \oplus \cdots \oplus S[k_r]^{m_r}$, for some pairwise different integers $k_1, \dots, k_r$, where, for simplicity, we have put $m_{k_i,S} = m_i > 0$ for $i = 1, \dots, r$. We can reorder the summands in this last direct sum, so that k_i is even, for $1 \leq i \leq t$, and k_i is odd, for $t < i \leq n$. Bearing in mind that $[S[k]] = (-1)^k [S]$ in $K_0(\mathcal{T})$, that $[\mathcal{S}]$ is a basis of $K_0(\mathcal{T})$ and that $[X] = 0$ in this latter abelian group, we conclude that $\sum_{i=1}^{t} m_i = \sum_{i=t+1}^{n} m_i$. We call $m(X)$ this last integer which is strictly positive when $X \neq 0$. An easy induction on $m(X)$ then settles our claim.

(2) Let $(t_S)_{S \in \mathcal{S}}$ be a collection of odd integers and put $\mathcal{D} := \mathrm{tria}_\mathcal{T}(S \oplus S[t_S] \colon S \in \mathcal{S})$. It follows that each object of $\mathcal{S}$ is a direct summand of an object of $\mathcal{D}$ and since each object T of $\mathcal{T} = \mathrm{tria}_\mathcal{T}(\mathcal{S})$ is a finite iterated extension of objects of the form $S[k]$, with $S \in \mathcal{S}$ and $k \in \mathbb{Z}$, it easily follows that each such T is a direct summand of an object of $\mathcal{D}$. This means that $\mathcal{D}$ is a dense triangulated subcategory of $\mathcal{T}$ in the terminology of [18]. Moreover, we clearly have $\mathcal{D} \subseteq \mathcal{T}^0$. But [18, Theorem 2.1] gives an order-preserving bijection between the dense triangulated subcategories of $\mathcal{T}$ and the subgroups of $K_0(\mathcal{T})$. Since $\mathcal{T}^0$ corresponds to 0 by this bijection we get that $\mathcal{D} = \mathcal{T}^0$, as desired.

(3) Note that assertion (2) implies assertion (3). Indeed, by the comments preceding Theorem 7, assertion (2) says that $\mathcal{T}^0$ is the extension closure of $\{(S \oplus S[t_S])[n] \colon S \in \mathcal{S} \text{ and } n \in \mathbb{Z}\}$, for any choice of odd integers t_S $(S \in \mathcal{S})$. We may choose $t_S = 1$ for each S, and then we have the split triangle

$$S \xrightarrow{0} S \xrightarrow{\binom{1}{0}} S \oplus S[1] \xrightarrow{(0 \quad 1)} S[1],$$

which shows that $0 \leq_{\Delta+\mathrm{nil}} S \oplus S[1]$. Since we have inclusions

$$\{(S \oplus S[1]) \colon S \in \mathcal{S}\} \subset \mathcal{T}^0_{\Delta+\mathrm{nil}} \subset \mathcal{T}^0_\Delta \subseteq \mathcal{T}^0$$

assertion (3) immediately follows. □

Example 8. The following are examples of a triangulated category $\mathcal{T}$ and a set $\mathcal{S}$ of its objects that satisfy the hypotheses of Theorem 7 and, in addition, $[\mathcal{S}]$ is a basis of $K_0(\mathcal{T})$. Here K is a commutative ring

(1) Call a dg K-algebra A *homologically non positive* when $H^k A = 0$, for all $k > 0$, and call it *homologically finite dimensional* when $H^*(A) = \oplus_{k \in \mathbb{Z}} H^k(A)$ is a K-module of finite length. For instance, any Artin algebra is homologically non positive and homologically finite dimensional over its center, when viewed as dg algebra. Let A be a homologically non positive homologically finite dimensional dg algebra and let $\mathcal{T} = \mathcal{D}^b_{fl}(A)$ be the subcategory of the derived category $\mathcal{D}(A)$ consisting of the dg A-modules M such that $H^*(M) = \oplus_{k \in \mathbb{Z}} H^k(M)$ has finite length as a K-module. When choosing as $\mathcal{S}$ a set of representatives, up to isomorphism in $\mathcal{D}(A)$, of the dg A-modules S such that $H^*(S) = H^0(S)$ (i.e. its homology is concentrated in degree zero) and $H^0(S)$ is a simple $H^0(A)$-module, one has that $\mathcal{T}$ and $\mathcal{S}$ satisfy the hypotheses of Theorem 7 and $[\mathcal{S}]$ is a basis of $K_0(\mathcal{T})$. In particular, taking A

to be an Artin algebra, $\mathcal{T} = \mathcal{D}^b(\text{mod} - A)$ and $\mathcal{S}$ be a set of representatives, up to isomorphism, of the simple A-modules (viewed as stalk complexes in degree zero), the hypotheses of Theorem 7 hold and $[\mathcal{S}]$ is a basis of $K_0(\mathcal{T})$.

(2) Suppose that $\mathcal{A}$ is an additive category with a set $\mathcal{S}'$ of objects such that $\mathcal{A} = \text{add}(\mathcal{S}')$ and the Grothendieck group $K_0(\mathcal{A})$ is free with $\{[S]\colon S \in \mathcal{S}'\}$ as a basis. Then the bounded homotopy category $\mathcal{T} = \mathcal{K}^b(\mathcal{A})$ and the set $\mathcal{S} = \mathcal{S}'[0]$ of stalk complexes $S'[0]$, with $S' \in \mathcal{S}'$, satisfy the hypotheses of Theorem 7 (see [15, Theorems 1.1 and 1.2]). This includes the case when $\mathcal{T} = \mathcal{K}^b(A-\text{proj})$, where A is a principal ideal domain or a semiperfect ring, in particular an Artin algebra, by taking as $\mathcal{S}'$ the set of (isomorphism classes of) indecomposable projective A-modules

Our next example shows that the implication (4) $\Rightarrow$ (3) of Theorem 5 is in general not true.

Example 9. Let A be an Artin algebra and S be a simple A-module. For each $k \in \mathbb{Z}$ the complex $M = S \oplus S[2k+1]$ has the property that $[M] = 0$ in $K_0(D^b(A-\text{mod}))$. However, it is a Δ-degeneration of zero (i.e. $0 \leq_\Delta M$) if and only if $k = 0$ or $k = -1$.

Indeed, In the final paragraph of the proof of Theorem 7 it is shown that $0 \leq_{\Delta+\text{nil}} S \oplus S[1]$, which implies by shift that also $0 \leq_{\Delta+\text{nil}} S \oplus S[-1]$. We then get that $0 \leq_\Delta M$ whenever $k = 0, -1$.

Suppose now that $k \neq -1, 0$. Note that the homology module $H^i(M)$ is zero, except for $i = 0$ and $i = -2k - 1$ in which case it is equal to S. If there is a distinguished triangle

$$Z \xrightarrow{f} Z \longrightarrow M \longrightarrow Z[1]$$

in $D^b(A - \text{mod})$, the associated sequence of homologies gives an exact sequence

$$0 \longrightarrow H^0(Z) \xrightarrow{H^0(f)} H^0(Z) \longrightarrow S \longrightarrow H^1(Z) \xrightarrow{H^1(f)} H^1(Z) \longrightarrow 0,$$

which forces $H^0(f)$ and $H^1(f)$ to be isomorphisms since they are a monomorphic and an epimorphic endomorphism, respectively, of finite length modules. Therefore $S = 0$, and we get a contradiction.

Recall that we denote by $\preceq_{\Delta+\text{ nil}}$ the smallest transitive relation on the set of isomorphism classes of objects in $\mathcal{T}$ containing $\leq_{\Delta+\text{ nil}}$. Our final result shows that the implication (3) $\Rightarrow$ (2) of Theorem 5 is false, in a strong sense.

Proposition 10. *Let $\mathcal{A}$ be any skeletally small abelian category for which $\mathcal{D}^b(\mathcal{A})$ is well-defined, i.e. it has Hom sets as opposed to proper classes, and let us identify $\mathcal{A}$ with the subcategory of $\mathcal{D}^b(\mathcal{A})$ consisting of objects X such that $H^i(X) = 0$, for $i \neq 0$. The following assertions hold:*

1. *If $Y \leq_{\Delta+\text{nil}} X$ in $\mathcal{D}^b(\mathcal{A})$ and $X \in \mathcal{A}$, then $Y \in \mathcal{A}$.*
2. *If Z is an object of $\mathcal{A}$ and $f : Z \longrightarrow Z$ is a monomorphic endomorphism $\mathcal{A}$ which is not an isomorphism, then $X := \text{Coker}(f)$ satisfies that $0 \leq_\Delta X$ in $\mathcal{T} := \mathcal{D}^b(\mathcal{A})$ (and hence $[X] = 0$ in $K_0(\mathcal{T})$), but $0 \not\leq_{\Delta+\text{nil}} X$.*

Proof. (1) Let us consider a distinguished triangle

$$W \xrightarrow{\binom{v}{\alpha}} W \oplus Y \longrightarrow X \longrightarrow W[1]$$

in $D^b(\mathcal{A})$, where v is a nilpotent endomorphism of W and $X \in \mathcal{A}$. The long exact sequence of homologies gives that

$$H^j(W) \xrightarrow{\binom{H^j(v)}{H^j(\alpha)}} H^j(W) \oplus H^j(Y)$$

is an isomorphism, for $j \neq 0, 1$, and there is an exact sequence

$$0 \longrightarrow H^0(W) \xrightarrow{\binom{H^0(v)}{H^0(\alpha)}} H^0(W) \oplus H^0(Y) \longrightarrow X \longrightarrow H^1(W) \xrightarrow{\binom{H^1(v)}{H^1(\alpha)}} H^1(W) \oplus H^1(Y) \longrightarrow 0$$

in $\mathcal{A}$. Proving that Y has homology concentrated in zero degree reduces to prove that if $\begin{pmatrix} w \\ g \end{pmatrix} : A \longrightarrow A \oplus B$ is an epimorphism in $\mathcal{A}$, for some objects $A, B \in \mathcal{A}$, where w is a nilpotent endomorphism of A, then $A = B = 0$. This is clear when $w = 0$. But if $w \neq 0$ and m is the nilpotent index of w (i.e. $w^m = 0 \neq w^{m-1}$), then the composition

$$A \xrightarrow{\begin{pmatrix} w \\ g \end{pmatrix}} A \oplus B \xrightarrow{\begin{pmatrix} w^{m-1} & 0 \end{pmatrix}} A$$

is the zero map, which implies that $w^{m-1} = 0$, thus yielding a contradiction.

(2) We have an induced distinguished triangle

$$Z \xrightarrow{f} Z \longrightarrow X \longrightarrow Z[1]$$

in $\mathcal{D}^b(\mathcal{A})$, thus showing that $0 \leq_\Delta X = X[0]$ in the latter triangulated category. Suppose now that $0 \preceq_{\Delta+\text{nil}} X$. Then we have a sequence $0 = X_0, X_1, \ldots, X_n = X$ in $D^b(\mathcal{A})$ such that $X_{i-1} \leq_{\Delta+\text{nil}} X_i$ and $X_i \neq 0$ for $i = 1, \ldots, n$. By assertion 1, we know that all X_i are in $\mathcal{A}$. Replacing X by X_1 if necessary, we get an object $X \neq 0$ of $\mathcal{A}$ such that $0 \leq_{\Delta+\text{nil}} X$ in $\mathcal{D}^b(\mathcal{A})$. We can fix a distinguished triangle

$$Q \xrightarrow{u} Q \longrightarrow X \longrightarrow Q[1]$$

in $\mathcal{D}^b(\mathcal{A})$, where u is a nilpotent endomorphism of Q. The long exact sequence of homologies gives then an exact sequence

$$0 \longrightarrow H^0(Q) \xrightarrow{H^0(u)} H^0(Q) \longrightarrow X \longrightarrow H^1(Q) \xrightarrow{H^1(u)} H^1(Q) \longrightarrow 0$$

in $\mathcal{A}$. But it is obvious that a nilpotent endomorphism of an object $A' \in \mathcal{A}$ can be a monomorphism or an epimorphism only in case $A' = 0$. We then get $H^j(Q) = 0$ for $j = 0, 1$, which in turn implies $X = 0$ and hence a contradiction. □

Example 11. Proposition 10 applies to the case when $\mathcal{A} = R - \text{mod}$ is the category of finitely generated modules over a Noetherian integral domain R. Indeed if $U(R)$ denotes the group of units of R, then any element $x \in R \setminus U(R)$ defines by multiplication a monomorphic endomorphism $\mu = \mu_x : R \longrightarrow R$ in $R - \text{mod}$ which is not an isomorphism. Putting $X := R/Rx$, we then get that $0 \leq_\Delta X$ but $0 \not\preceq_{\Delta+\text{nil}} X$ in $\mathcal{D}^b(R - \text{mod})$.

Remark 12. Consider the situation of a triangulated category $\mathcal{C}_K^\circ$ and an object M of $\mathcal{C}_K^\circ$ with $0 \leq_{\text{cdeg}} M$. It is not hard to see that, with the notations used in Definition 2 and Definition 4, we get $M \simeq \Phi(\text{cone}(t_Q))$ with nilpotent endomorphism t_Q for some object Q in the category $\mathcal{C}_V^\circ$ corresponding to the degeneration data, and Φ the functor $\mathcal{C}_V^\circ \to \mathcal{C}_K$. This, together with Remark 6, pinpoints the difference between degeneration and flat deformations.

References

[1] Alexander A. Beilinson, Joseph Bernstein, Pierre Deligne, *Faisceaux pervers.* Astérisque 100, vol. 1. Société Mathématique de France (1982).

[2] Klaus Bongartz, *On Degenerations and Extensions of Finite Dimensional Modules*, Advances of Mathematics **121** (1996) 245-287.

[3] Florian Eisele, *The p-adic group ring of $SL_2(p^f)$*, Journal of Algebra **410** (2014) 421-459.

[4] Florian Eisele, *Blocks with a generalized quaternion defect group and three simple modules over a 2-adic ring,* Journal of Algebra **456** (2016) 294-322.

[5] Pierre Gabriel, *Finite representation type is open*, Proceedings of the International Conference on Representations of Algebras (Carleton University, Ottawa, Ontario, 1974), Paper No. 10, 23 pp. Carleton Mathematics Lecture Notes, No. 9, Carleton University, Ottawa, Ontario, 1974

[6] Naoya Hiramatsu, *Degenerations of graded Cohen-Macaulauy modules*, Journal of Commuatative algebra **7** (2015) 221-239.

[7] Naoya Hiramatsu, *On the stable hom relation and stable degenerations of Cohen-Macaulay modules*, Journal of Pure and Applied Algebra **222** (2018) 2609-2625.

[8] Bernt Tore Jensen, Xiuping Su and Alexander Zimmermann, *Degeneration for derived categories*, Journal of Pure and Applied Algebra **198** (2005) 281-295.

[9] Bernt Tore Jensen, Xiuping Su and Alexander Zimmermann, *Degeneration-like orders for triangulated categories*, Journal of Algebra and Applications **4** (2005) 587-597.
[10] Berhnard Keller, Dong Yang, and Guodong Zhou, *The Hall algebra of a spherical object*, Journal of the London Mathematical Society **80** (2009) 771-784.
[11] Bernhard Keller and Sarah Scherotzke, *Graded quiver varieties and derived categories*, Journal für die reine und angewandte Mathematik **713** (2016) 85-127.
[12] Henning Krause, *Cohomological Length Functions*, Nagoya Math. Journal **223** (2016) 136-161.
[13] Amnon Neeman, TRIANGULATED CATEGORIES, Princeton University Press, Princeton MA 2001.
[14] Christine Riedtmann, *Degenerations for representations of quivers with relations*, Annales Scientifiques de l'École Normale Supérieure **19** (1986) 275-301.
[15] David E.V. Rose *A note on the Grothendieck group of an additive category.* Vestnik Chelyabinskogo Gosudarstvennogo Universiteta. Matematika, Mekhanika, Informatika. Nauchnyĭ Zhurnal no 3 **17** (2015) 135-139. Available at https://arxiv.org/pdf/1109.2040.pdf
[16] Manuel Saorín and Alexander Zimmermann, *An Axiomatic Approach for Degenerations in Triangulated Categories*, Applied Categorical Structures **24** (2016) no 4, 385-405.
[17] Manuel Saorín and Alexander Zimmermann, *Symmetry of the definition of degeneration in triangulated categories*, Algebra Represention Theory **22** (2019) 801-836.
[18] Robert W. Thomason, *The classification of triangulated subcategories.* Compositio Mathematica **105** (1997), 1-27.
[19] Yuji Yoshino, *On degeneration of modules*, Journal of Algebra **278** (2004) 217-226.
[20] Yuji Yoshino, *Stable degeneration of Cohen-Macaulay modules*, Journal of Algebra **332** (2011) 500-521.
[21] Zhengfang Wang, *Triangle order $\leq_\Delta$ in Singular Categories*, Algebras and Representation Theory **19** (2016) 397-404.
[22] Grzegorz Zwara, *A degeneration-like order for modules*, Archiv der Mathematik **71** (1998) 437-444.
[23] Grzegorz Zwara, *Degenerations of finite dimensional modules are given by extensions*, Compositio Mathematica **121** (2000) 205-218.

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