REMARKS ON A TRIANGULATED VERSION OF AUSLANDER-KLEINER'S GREEN CORRESPONDENCE

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Dedicated to Sri Wahyuni's 60^{th} birthday

Abstract. For a finite group G and an algebraically closed field k of characteristic $p > 0$ for any indecomposable finite dimensional kG-module M with vertex D and a subgroup H of G containing $N_G(D)$ there is a unique indecomposable kH-module N of vertex D being a direct summand of the restriction of M to H. This correspondence, called Green correspondence, was generalised by Auslander-Kleiner to the situation of pairs of adjoint functors between additive categories. In the original situation of group rings Carlson-Peng-Wheeler proved that this correspondence is actually restriction of triangle functors between triangulated quotient categories of the corresponding module categories. We review this theory and show how we got a common generalisation of the approaches of Auslander-Kleiner and Carlson-Peng-Wheeler, using Verdier localisations.

1. Introduction

Green correspondence is one of the most important tools in modular representation theory. For a field k of finite characteristic p and a finite group G we define for any indecomposable kG-module M its vertex and its source. Actually, the vertex of M is the smallest subgroup D of G such that M is a direct factor of $L \uparrow_D^G$ for some indecomposable kD-module L. This module L is called the source of M. Both D and L are essentially unique up to conjugacy, and it can be shown that D is always a p-subgroup of G. If H contains the normaliser of D in G, then Green shows [7] that for any indecomposable kG-module M with vertex D there is a unique indecomposable direct factor $f(M)$ of $M \downarrow_H^G$ with vertex D, its Green correspondent. Carlson, Peng and Wheeler showed [6] much later that this is actually the restriction of a functor of triangulated categories defined by some well-known quotient construction imitating the construction of the standard stable category. In a different direction Auslander and Kleiner showed [1] that Green correspondence actually works in a much more general setting of three additive categories and pairs of adjoint functors between them. However, they did not mention the triangulated category structure which is present underneath. In [17] we gave a construction enlarging Auslander-Kleiner's approach to a triangulated category situation. Specialising to the classical situation we get back the Carlson-Peng-Wheeler theorem, generalising hence their result to a much vaster world. In the present note we summarize and explain these approaches and discuss their links. We focus in particular on the concept of T-relative projectivity, resp. T-relative injectivity for a functor T between triangulated categories. The classical situation appears when we have a finite group G, a subgroup H and T is the restriction functor from the (abelian) category of finitely generated kG-modules to finitely generated kH-modules.

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close insight into the highly sophisticated Javanese culture by her warm and generous personality. I am most grateful to her for this experience which truly marks one of the peek points in my mathematical career. When Indah Wijayanti asked me some time later to speak on the occasion of her 60th birthday I accepted right away without a second of hesitation. The present paper deals with results around the talk I presented on the occasion of her birthday party in Yogyakarta at 30 July 2019, and I humbly wish to dedicate this paper to Sri Wahjuni.

2. A Glimpse of Modular Representation Theory of Finite Groups; The Classical Case

Let G be a finite group and let k be a field of characteristic $p > 0$. Denote by kG the group ring of G over the field k (cf e.g. [16, Chapter 1]). Let $kG - mod$ be the category of finitely generated kG-modules. Note that $kG - mod$, the category of finite-dimensional kG-modules is a Krull-Schmidt category, that is, for every finite dimensional kG-module M there are indecomposable kG-modules $M_1, \dots, M_s$ such that $M = M_1 \oplus \dots \oplus M_s$, and if there is another such set $N_1, \dots, N_t$ of indecomposable kG-modules with

$$N_1 \oplus \dots \oplus N_t \simeq M \simeq M_1 \oplus \dots \oplus M_s,$$

then $s = t$ and there is an element σ in the symmetric group $\mathfrak{S}_t$ on $\{1, \dots, t\}$ such that $M_i \simeq N_{\sigma(i)}$ for all $i \in \{1, \dots, t\}$.

Note moreover that for any algebra A a d-dimensional A-module is the same as a k-algebra homomorphism

$$A \xrightarrow{\mu} End_k(k^d).$$

Hence, for any k-algebra homomorphism $B \xrightarrow{\alpha} A$ and any d-dimensional A-module M we obtain a d-dimensional B-module $res^A_\alpha(M)$, or res^A_B for short if α is evident from the context, given by the algebra homomorphism $\mu \circ \alpha$.

For a subgroup H of G there is an algebra homomorphism $\iota : kH \longrightarrow kG$ given by the inclusion map of H in G. By the above any kG-module M induces a kH-module $res^{kG}_\iota(M)$, denoted also by $M \downarrow^G_H$. We call this the restriction of M to H. It is clear that res^G_H is a functor $kG - mod \longrightarrow kH - mod$ since if a map $M_1 \longrightarrow M_2$ is kG-linear, then it is trivially kH-linear. Moreover, res^G_H admits a left adjoint $ind^G_H : kH - mod \longrightarrow kG - mod$ in the sense that for any kG-module M and any kH-module N we have

$$Hom_{kH}(N, res^G_H(M)) \simeq Hom_{kG}(ind^G_H(N), M)$$

and this isomorphism is functorial in M and N. Actually kG is a $kH - kH$-bimodule, and then

$$res^G_H(M) = Hom_{kH}(kG, M).$$

Therefore $ind^G_H(N) = kG \otimes_{kH} N$ and the adjointness is nothing else than an incident of the usual Hom-tensor adjunction. A more detailed analysis (cf [16, Chapter 1]) shows that ind^G_H is also right adjoint to res^G_H.

If M is an indecomposable kG-module, then $M \downarrow^G_H$ does not need to be an indecomposable kH-module. An easy example is $H = \{1\}$, then $M \downarrow^G_H$ is indecomposable if and only if M is one-dimensional.

2.1. Relative Projectivity and Vertices; Classical Case.

A key notion in the above context is relative projectivity. This is classical, and the definition we use, follows Hochschild's work [11]. We shall study his concept in more detail in Section 3.1 below.

Definition 1. Let G be a finite group, let H be a subgroup of G, and let M be a finite dimensional kG-module. Then M is relatively H-projective if for all kG-modules N and all epimorphisms $\alpha : N \longrightarrow M$ such that the induced morphism $res(\alpha) : res^G_H(N) \longrightarrow res^G_H(M)$ is split, then also α is split.

The most interesting result is now the following statement, known as Higman's lemma.

Lemma 2. *Let G be a finite group, let k be a field, and let H be a subgroup of G. Let M be a finite-dimensional indecomposable kG-module. Then the following statements are equivalent.*

- *M is relatively H-projective*
- *M is a direct summand of $ind_H^G(res_H^G(M))$*
- *There is a finite-dimensional indecomposable kH-module L such that M is a direct summand of $ind_H^G(L)$.*

A first observation is that M is a direct summand of $ind_1^G(res_1^G(M))$ if and only if M is projective. Actually, any module of the form $ind_1^G(L)$ is free of rank $\dim_k L$, and the inverse implication follows by Frobenius reciprocity and the fact that vector spaces are always free. Now, the question arises, for a given indecomposable and not necessarily projective kG-module M, what are minimal subgroups H such that M is a direct factor of $ind_H^G(res_H^G(M))$.

Using Lemma 2 and Mackey's decomposition, i.e. the decomposition of kG into indecomposable $kH-kH$-bimodules, the following consequence is immediate.

Theorem 3. *Let G be a finite group, let k be a field of characteristic $p>0$, and let H be a subgroup of G. Then for any indecomposable kG-module M the set of minimal elements in the partial ordered set of subgroups H such that M is relatively H-projective forms a G-conjugacy class of p-subgroups of G.*

Let now k be a field of characteristic $p > 0$. For an indecomposable kG-module M we know by Theorem 3 that if M is relatively D-projective and if D is minimal with this property, then D is a p-subgroup of G, and two of these subgroups are conjugate in G. We call such a subgroup D a *vertex of M*. Moreover Higman's Lemma 2 shows that for an indecomposable kG-module M with vertex D there is an indecomposable kD-module L such that M is a direct summand of M. We call L a *source of M*. Arguments using essentially the Krull-Schmidt theorem shows that also a source is basically unique.

2.2. **Green Correspondence; The Classical Case.** One of the most important and basic tools in representation theory of finite groups, namely Green correspondence, is attached to these concepts. As usual for a group G, a subgroup H of G and an element $g\in G$ we denote ${}^gH := \{ghg^{-1} \mid h\in H\}$. The following Theorem 4 is called Green correspondence.

Theorem 4. *(Green [7]) Let G be a finite group, let k be a field of characteristic $p>0$, and let D be a p-subgroup of G. Let H be a subgroup of G containing $N_G(D)$. Put*

$$\mathfrak{X} := \{S \le D\cap {}^gD \mid g\in G\smallsetminus H\}$$
$$\mathfrak{Y} := \{S \le H\cap {}^gD \mid g\in G\smallsetminus H\}$$

Then

- *for any indecomposable kG-module M with vertex D there is a unique indecomposable direct summand $f(M)$ of $res_H^G(M)$ with vertex D. All other indecomposable direct summands of $res_H^G(M)$ have vertex in $\mathfrak{Y}$.*
- *for any indecomposable kH-module N with vertex D there is a unique indecomposable direct summand $g(N)$ of $ind_H^G(N)$ with vertex D. All other indecomposable direct summands of $ind_H^G(N)$ have vertex in $\mathfrak{X}$.*
- *$fg = id$ and $gf = id$.*

Example 5. Consider the special case of D being cyclic of order p. Then D has only one proper subgroup, namely the trivial group. Then, for any indecomposable kG-module with vertex D, by Theorem 4 $res_H^G(M)$ has a unique non projective direct summand $f(M)$, and all other indecomposable direct factors are projective. A similar special case is given for G being a group with trivial intersection property of Sylow subgroups D, that is $\{{}^gD\cap D \mid g\in G\}\subseteq\{1,D\}$. There, the same statement holds.

This observation motivates the following construction, namely the well-established tool in representation theory, the stable category. For a finite dimensional algebra A let $A-\underline{mod}$ be the category whose objects are A-modules. For any two A-modules let $PHom_A(M,N)$ be the vector space of A-module homomorphisms $M \longrightarrow N$ which factor through a projective A-module. Then, the morphisms from M to N in the stable category are elements in $\underline{Hom}_A(M,N) := Hom_A(M,N)/PHom_A(M,N)$. Composition of morphisms in the stable category is given by composition of morphisms of A-modules, and it is easy to see that this gives a well-defined category. Projective A-modules are isomorphic to 0, since the endomorphism algebra in the stable category of a projective module is 0. Moreover, most interestingly, if A is self-injective, then $A-\underline{mod}$ is a triangulated category (cf e.g. [16, Chapter 3]).

Example 6. We come back to Example 5. Restriction and induction of projective modules are projective. Hence restriction and induction induce functors on the level of the stable categories.

$$ind_H^G : kH-\underline{mod} \longrightarrow kG-\underline{mod}$$
$$res_H^G : kG-\underline{mod} \longrightarrow kH-\underline{mod}$$

Consider Green correspondence for D a cyclic group of order p, or if G is a finite group with trivial intersection Sylow p-subgroups and D is a Sylow subgroup. Put H a subgroup of G containing $N_G(D)$. Then Theorem 4 shows that in these cases res_H^G and ind_H^G induce equivalences of categories of kG-modules with vertex D and kH-modules with vertex D. Observe that neither ind_H^G nor res_H^G are equivalences of categories, but they are equivalences of the additive subcategories generated by indecomposable modules of vertex D. Note that these subcategories are not triangulated subcategories of the stable category in general.

2.3. **Green Correspondence is the Trace of Triangle Functors.** In [6] Carlson-Peng-Wheeler showed that Green correspondence is actually the restriction of a triangle functor between certain triangulated subcategories of the corresponding module categories over the relevant groups.

More precisely, for an additive category $\mathcal{A}$ and an additive subcategory $\mathcal{S}$ denote by $\mathcal{A}/\mathcal{S}$ the category with the same objects as $\mathcal{A}$. A morphism f in $\mathcal{A}(X,Y)$ is said to be in $\mathcal{A}^{\mathcal{S}}(X,Y)$ if there an object Z of $\mathcal{S}$ and morphisms $g \in \mathcal{A}(X,Z)$ and $h \in \mathcal{A}(Z,Y)$ such that $f = h \circ g$. Then put

$$(\mathcal{A}/\mathcal{S})(X,Y) := \mathcal{A}(X,Y)/\mathcal{A}^{\mathcal{S}}(X,Y)$$

and composition is given by composition of representatives of classes. It is clear that this is well-defined, such as in the remarks following Example 5.

Carlson-Peng-Wheeler define for a fixed finite dimensional kG-module W a finite dimensional module M to be W-projective if M is a direct factor of $W \otimes_k W^* \otimes M$, where as usual $-^*$ denotes the k-linear dual. Then, they show

Proposition 7. [6, Section 6] *Let k be a field and let G be a finite group. Let W be a finitely generated kG-module and let $kG-mod^W$ be the full subcategory of W-projective kG-modules. Then $kG-mod/kG-mod^W$ carries the structure of a triangulated category.*

As a consequence they show that Green correspondence is the restriction of triangle functors of quotient categories of $kG-mod$ respectively $kH-mod$ for appropriate choices of W. More precisely

Theorem 8. [6] *Let k be a field of characteristic $p > 0$, let G be a finite group, let D be a p-subgroup of G, and let H be a subgroup of G containing $N_G(D)$. For any group Γ denote by k the trivial $k\Gamma$-module. Put*

$$\mathfrak{X} := \{S \le D \cap {}^gD \mid g \in G \setminus H\}$$
$$\mathfrak{Y} := \{S \le H \cap {}^gD \mid g \in G \setminus H\}$$

and $W_G := \bigoplus_{X\in\mathcal{X}} k\uparrow_X^G$ *and* $W_H := \bigoplus_{Y\in\mathcal{Y}} k\uparrow_Y^H$. *Then restriction* $\downarrow_H^G$ *induces a triangle functor*

$$kG-mod/kG-mod^{W_G} \longrightarrow kH-mod/kH-mod^{W_H}$$

and induction uar_H^G *induces a triangle functor*

$$kH-mod/kH-mod^{W_H} \longrightarrow kG-mod/kG-mod^{W_G}.$$

These functors restrict to equivalences between the full additive subcategories generated by indecomposable modules of vertex D.

Note that this version of Green correspondence is not an equivalence of triangulated categories, but that the Green correspondence is the restriction of triangle functors between triangulated categories.

3. Notions of Relative Projectivity

We observe that in order to generalize Green correspondence to a categorical setting we first need to generalise and formulate the notion of relative projectivity. The classical case provides two such settings.

3.1. **Recall Hochschild's concept.** We recall Hochschild's approach [11, Section 1] to relative projectivity (respectively injectivity). He considers an algebra R and a subalgebra S and says that an exact sequence

$$\cdots \longrightarrow M_{i+2} \xrightarrow{t_{i+2}} M_{i+1} \xrightarrow{t_{i+1}} M_i \xrightarrow{t_i} M_{i-1} \longrightarrow \dots$$

of R-module homomorphisms is said to be (R,S)-exact if the kernel of t_i is a direct summand of M_i for all i. Equivalently the sequence is (R,S)-exact if $t_i \circ t_{i+1} = 0$ for all $i \in \mathbb{Z}$ and in addition there are S-module homomorphisms $h_i : M_i \to M_{i+1}$ such that $t_{i+1}\circ h_i + h_{i-1}\circ t_i = id_{M_i}$ for all $i \in \mathbb{Z}$. Hochschild continues that an R-module A is called (R,S)-injective if for every (R,S)-exact sequence

$$0 \longrightarrow U \xrightarrow{p} V \xrightarrow{q} W \longrightarrow 0$$

and every R-module homomorphism $h : U \to A$ there is an R-module homomorphism $h' : V \to A$ with $h = h' \circ p$. Dually, an R-module A is (R,S)-projective, if for each R-module homomorphism $g : A \to W$ there is an R-module homomorphism $g' : A \to V$ such that $g = q \circ g'$.

What precisely are short (R,S)-exact sequences?

Lemma 9. *An exact sequence*

$$0 \longrightarrow U \xrightarrow{p} V \xrightarrow{q} W \longrightarrow 0$$

of R*-modules is* (R,S)*-exact if, and only if, it splits when considered as sequence of* S*-modules.*

Proof. If the sequence is R-split, then there is an S-homotopy $h_1 : W \to V$ such that $h_1 \circ q = id_W$, whence the sequence splits. Conversely if the restriction of the sequence splits, then by definition the kernel of q is an S-direct summand of V, namely $p(U)$. □

Lemma 10. A *is* (R,S)*-injective if, and only if, each* (R,S)*-exact sequence*

$$0 \longrightarrow A \xrightarrow{p} V \xrightarrow{q} W \longrightarrow 0$$

splits as sequence of R*-modules. Similarly,* A *is* (R,S)*-projective if, and only if, each* (R,S)*-exact sequence*

$$0 \longrightarrow U \xrightarrow{p} V \xrightarrow{q} A \longrightarrow 0$$

splits as sequence of R*-modules.*

Proof. This is analogous to the usual argument in homological algebra. Indeed, if A is (R,S)-projective, and

$$0 \longrightarrow U \xrightarrow{p} V \xrightarrow{q} A \longrightarrow 0$$

is an (R,S)-exact sequence, then by definition, for id_A there is an R-module homomorphism $s : A \to V$ with $id_A = q \circ s$. This is tantamount to say that the sequence splits. Conversely, suppose that each (R,S)-exact sequence

$$0 \longrightarrow U' \xrightarrow{p'} V' \xrightarrow{q'} A \longrightarrow 0$$

splits. Let

$$0 \longrightarrow U \xrightarrow{p} V \xrightarrow{q} W \longrightarrow 0$$

be an (R,S)-exact sequence and consider an R-module homomorphism $g : A \to W$. Then form the pullback of the sequence

$$0 \longrightarrow U \xrightarrow{p} V \xrightarrow{q} W \longrightarrow 0$$

along g to get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U & \xrightarrow{p} & V & \xrightarrow{q} & W & \longrightarrow & 0 \\ & & \| & & \uparrow x & & \uparrow g & & \\ 0 & \longrightarrow & U & \xrightarrow{p'} & V' & \xrightarrow{q'} & A & \longrightarrow & 0 \end{array}$$

and the bottom sequence is again (R,S)-exact, and hence splits, by hypothesis. Therefore there is an R-module homomorphism $s : A \to V'$ with $q' \circ s = id_A$. Hence $q \circ x \circ s = g \circ q' \circ s = g$ as required.

The case of (R,S)-injective is dual. □

Corollary 11. *An R-module A is (R,S)-projective if and only if each short exact sequence of R-modules*

$$0 \longrightarrow U \xrightarrow{p} V \xrightarrow{q} A \longrightarrow 0$$

which is known to split as sequence of S-modules, is also split as sequence of R-modules.

An R-module A is (R,S)-projective if and only if each short exact sequence of R-modules

$$0 \longrightarrow A \xrightarrow{p} V \xrightarrow{q} W \longrightarrow 0$$

which is known to split as sequence of S-modules, is also split as sequence of R-modules.

In more modern terms, denoting by $res^R_S : R-Mod \longrightarrow S-Mod$ the restriction functor, then A is (R,S)-projective if and only if the induced functor

$$Ext^1_R(A,-) \longrightarrow Ext^1_S(res^R_S(A), res^R_S(-))$$

is a monomorphism in the functor category $R-Mod \longrightarrow \mathbb{Z}-Mod$. An R-module A is (R,S)-injective if and only if the induced functor

$$Ext^1_R(-,A) \longrightarrow Ext^1_S(res^R_S(-), res^R_S(A))$$

is a monomorphism in the functor category $R-Mod \longrightarrow \mathbb{Z}-Mod$.

3.2. **The new concept for triangulated categories.** In this subsection we shall give an alternative proof for the results in [17, Section 2]. Let $\mathcal{S}$ and $\mathcal{T}$ be exact categories, and let $S : \mathcal{T} \to \mathcal{S}$ be an exact functor. Denote by $Ext^1_{\mathcal{T}}(X,Y)$ be the set of equivalence classes of short exact sequences

$$0 \longrightarrow Y \xrightarrow{\iota} E \xrightarrow{\pi} X \longrightarrow 0,$$

where as usual two such sequences

$$0 \longrightarrow Y \xrightarrow{\iota_1} E_1 \xrightarrow{\pi_1} X \longrightarrow 0,$$

and

$$0 \longrightarrow Y \xrightarrow{\iota_2} E_2 \xrightarrow{\pi_2} X \longrightarrow 0,$$

are equivalent if and only if there is a homomorphism $E_1 \longrightarrow E_2$ making the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y & \xrightarrow{\iota_1} & E_1 & \xrightarrow{\pi_1} & X & \longrightarrow & 0 \\ & & \| & & \downarrow & & \| & & \\ 0 & \longrightarrow & Y & \xrightarrow{\iota_2} & E_2 & \xrightarrow{\pi_2} & X & \longrightarrow & 0 \end{array}$$

commutative. Note that by [5, Corollary 3.2] any such homomorphism $E_1 \longrightarrow E_2$ is an isomorphism. This way S induces a morphism

$$S : Ext^1_{\mathcal{T}}(X, Y) \longrightarrow Ext^1_{\mathcal{S}}(SX, SY)$$

Now, if $\mathcal{S} = A - mod$ for some algebra A, then denoting by $D^b(A)$ the derived category of bounded complex of finitely generated A-modules (cf e.g. [16, Chapter 3]),

$$Ext^1_{\mathcal{S}}(X, Y) = Ext^1_A(X, Y) = Hom_{D^b(A)}(X, Y[1]).$$

Consider Definition 1. A module M is relative H-projective if for all modules X the map

$$res^G_H : Ext^1_{kG}(M, X) \longrightarrow Ext^1_{kH}(res^G_H(M), res^G_H(X))$$

is injective. In other words, a module M is relative H-projective if the natural transformation

$$res^G_H : Ext^1_{kG}(M, -) \longrightarrow Ext^1_{kH}(res^G_H(M), res^G_H(-))$$

is a monomorphism in the category of functors $kG - mod \longrightarrow k - mod$.

If we want to enlarge the notion of relative projectivity to the derived category we first observe that res^G_H is a triangle functor $S : \mathcal{T} \longrightarrow \mathcal{S}$ from the triangulated category $\mathcal{T} := D^b(kG)$ to the triangulated category $\mathcal{S} = D^b(kH)$, and the notion should be formulated for triangle functors. Then we see that an object M in $D^b(A)$ is S-relative projective if and only if the natural transformation

$$S : Hom_{\mathcal{T}}(M, -[1]) \longrightarrow Hom_{\mathcal{S}}(SM, S - [1])$$

is a monomorphism of functors $\mathcal{T} \longrightarrow k - mod$. Since this functor can be evaluated on all objects of $\mathcal{T}$, and since $[1]$ is an auto-equivalence of $\mathcal{T}$, we can just omit $[1]$ in the above formula.

Definition 12. [17] Let $\mathcal{T}$ and $\mathcal{S}$ be triangulated categories, and let $S : \mathcal{T} \longrightarrow \mathcal{S}$ be a triangle functor. Then

- an object M of $\mathcal{T}$ is S-relatively projective if the natural transformation
$$S : Hom_{\mathcal{T}}(M, -) \longrightarrow Hom_{\mathcal{S}}(SM, S-)$$
is a monomorphism in the functor category $\mathcal{T} \longrightarrow k - mod$.
- an object N of $\mathcal{T}$ is S-relatively injective if the natural transformation
$$S : Hom_{\mathcal{T}}(-, N) \longrightarrow Hom_{\mathcal{S}}(S-, SN)$$
is a monomorphism in the functor category $\mathcal{T}^{op} \longrightarrow k - mod$.

Of course, this is quite a large generalisation of the classical notion of relative projectivity. The classical case is found for self-injective algebras, such as group algebras, for the stable category rather than the module category. Indeed, we omitted by purpose the shift of degree by 1. But then Ext^0 is part of our study, and this should be the stable homomorphisms, and not just the ordinary homomorphisms.

How what the alternative definition of relative projectivity using Green's definition coming from Higman's lemma. Is there some link, or a triangulated version of Higman's lemma? Most astonishing, this is true, at least in the correct setting. For a subcategory $\mathcal{S}$ of an additive category $\mathcal{C}$ denote by add$(\mathcal{S})$ the additive closure of $\mathcal{S}$ in $\mathcal{C}$.

Lemma 13. [17] *Let $\mathcal{T}$ and $\mathcal{S}$ be k-linear categories and let $T:\mathcal{S}\longrightarrow\mathcal{T}$ be a k-linear functor. Suppose that T has a left (respectively right) adjoint S, and denote by*

$$\varphi_{X,Y}:\mathcal{T}(X,TY)\xrightarrow{\simeq}\mathcal{S}(SX,Y)$$

the adjunction isomorphism. Then any object in $\mathrm{add}(\mathrm{im}(S))$ *is T-relative projective (respective injective).*

Proof. Let $\epsilon: id_{\mathcal{T}}\to TS$ be the unit of the adjunction. By [12, IV Theorem 1.(i)] for any $f\in\mathcal{S}(SX,Y)$ we have

$$\varphi^{-1}_{X,Y}(f)=T(f)\circ\epsilon_X.$$

We first suppose $Q=SQ'$ for some object Q' of $\mathcal{T}$. Consider the following diagram

$$\begin{array}{ccc}
\mathcal{S}(Q,-) & \xrightarrow{T_Q} & \mathcal{T}(TQ,T-)\\
\| & & \|\\
\mathcal{S}(SQ',-) & \xrightarrow{T_{SQ'}} & \mathcal{T}(TSQ',T-)\\
\uparrow{\scriptstyle \varphi_{Q',-}} & & \downarrow{\scriptstyle \mathcal{T}(\epsilon_{Q'},-)}\\
\mathcal{T}(Q',T-) & \overset{\lambda}{\dashrightarrow} & \mathcal{T}(Q',T-)
\end{array}$$

We define $\lambda:=\mathcal{T}(\epsilon_{Q'},-)\circ T_{SQ'}\circ\varphi_{Q'}$, as indicated in the above diagram. Hence

$$\lambda(f)=T_{SQ'}(\varphi_{Q',-}(f))\circ\epsilon_{Q'}=\varphi^{-1}_{Q',-}(\varphi_{Q',-}(f))=f.$$

Since $\varphi_{Q',-}$ is an isomorphism, T_Q is split mono. Let now $Q\in\mathrm{add}(\mathrm{im}(S))$ and Q is a direct factor of SQ'. We simply use that the above argument is still valid for direct factors of SQ'.

The case of relative injective is done analogously, using the counit instead of the unit. □

Proposition 14. [17] *Let $\mathcal{T}$ and $\mathcal{S}$ be triangulated categories and let $T:\mathcal{S}\longrightarrow\mathcal{T}$ be a triangle functor. Suppose that T has a left (respectively right) adjoint S. Then an object Q is T-relative projective (respectively injective) if and only if Q is in* $\mathrm{add}(\mathrm{im}(S))$.

Proof. By Lemma 13 we see that any object in $\mathrm{add}(\mathrm{im}(S))$ is T-relative projective.

Suppose now that Q is T-relative projective. Let $\eta: ST\longrightarrow id_{\mathcal{S}}$ be the counit of the adjunction. We shall show that η_Q is a split epimorphism. This then shows that Q is in $\mathrm{add}(\mathrm{im}(S))$.

Again by [12, IV Theorem 1] the composition

$$T\xrightarrow{\epsilon T}TST\xrightarrow{T\eta}T$$

is the identity, and so $T\eta$ is a split epimorphism. Let

$$\widetilde{Q}\longrightarrow STQ\xrightarrow{\eta_Q}Q\xrightarrow{\nu}\widetilde{Q}[1]$$

be a distinguished triangle. Then

$$T\widetilde{Q}\longrightarrow TSTQ\xrightarrow{T\eta_Q}TQ\xrightarrow{T\nu}T\widetilde{Q}[1]$$

is a distinguished triangle as well. Since $T\eta_Q$ is a split epimorphism, $T\nu=0$. Now, Q is T-relative projective, and hence by hypothesis

$$\mathcal{S}(Q,-)\xrightarrow{T_Q}\mathcal{T}(TQ,T-)$$

is injective. Evaluate this on $\widetilde{Q}[1]$ and obtain that $\nu=0$. Hence, η_Q is split epimorphism, which shows the statement.

The proof of the case of T-relative injective objects is done completely analogously using the sequence

$$T \xrightarrow{T\widetilde{\epsilon}} TST \xrightarrow{\widetilde{\eta}T} T$$

for the unit $\widetilde{\epsilon}$ and the counit $\widetilde{\eta}$ of the adjunction (T, S). □

Remark 15. Recall that $Q \mapsto \widetilde{Q}$ is *not* a functor.

Corollary 16. *Let $\mathcal{T}$ and $\mathcal{S}$ be triangulated categories and let $T : \mathcal{S} \longrightarrow \mathcal{T}$ be a triangle functor. Suppose that T has a left (respectively right) adjoint S, and let $\eta : ST \longrightarrow \mathrm{id}$ be the counit (respectively $\widetilde{\epsilon} : \mathrm{id} \longrightarrow ST$ the unit) of the adjunction. Then Q is T-relative projective (respectively injective) if and only if η_Q is a split epimorphism (respectively $\widetilde{\epsilon}_Q$ is a split monomorphism).*

Proof. Suppose that Q is T-relative projective. By Proposition 14 we see that Q is in $\mathrm{add}(\mathrm{im}S)$. By [12, IV Theorem 1] η_{SR} is a split epimorphism for any object R of $\mathcal{T}$. Hence η_Q is a split epimorphism.

If η_Q is a split epimorphism, then Q is in $\mathrm{add}(\mathrm{im}(S))$, and by Proposition 14 this implies that Q is T-relatively projective. □

We summarize the situation to an analogue of Higman's lemma for pairs of adjoint functors between triangulated categories.

Theorem 17. [17, Proposition 2.7] *Let $\mathcal{T}$ and $\mathcal{S}$ be triangulated categories and let $T : \mathcal{S} \longrightarrow \mathcal{T}$ be a triangle functor. Suppose that T has a left (respectively right) adjoint S. Let M be an indecomposable object of $\mathcal{T}$. Then the following are equivalent:*

(1) *M is T-relatively projective (respectively injective).*
(2) *M is in $\mathrm{add}(\mathrm{im}S)$.*
(3) *M is a direct factor of some $S(L)$ for some L in $\mathcal{S}$.*
(4) *M is a direct factor of $ST(M)$.*

Proof. (1) ⇔ (2) is Proposition 14.

(2) ⇔ (3) is the definition of $\mathrm{add}(\mathrm{im}S)$.

(3) ⇒ (4) is trivial.

(4) ⇒ (1) is Corollary 16. □

Remark 18. Note that Corollary 16 generalises [16, Proposition 2.1.6, Proposition 2.1.8] to this more general situation.

Note that condition (2) indicates that in case T has a left adjoint S_ℓ, a right adjoint S_r, and in case $S_\ell = S_r$, then T-relatively injective and T-relatively projective is just the same property. The situation also occurs under the weaker condition $\mathrm{add}(\mathrm{im}S_\ell) = \mathrm{add}(\mathrm{im}S_r)$.

Remark 19. In [17] we restricted the notion of T-relative projectivity respectively T-relative injectivity to the case of functors T having a left respectively right adjoint. This is caused by the fact that in [17] we are guided there by the approach of Beligiannis-Marmaridis [3], whereas here we rather use Hochschild's approach. By Theorem 17 and the corresponding statement in [17] both definitions give the same result if T has a left and a right adjoint.

4. Auslander Kleiner's Version of Green Correspondence

Auslander and Kleiner proposed in [1] a version of Green correspondence which worked for pairs of adjoint functors between abelian categories. They observed that the arguments of classical Green correspondence are essentially disguised arguments on pairs of adjoint functors.

We start with some notations.

- Let $\mathcal{U}$ be an additive category, and let $\mathcal{V}$ be a full subcategory of $\mathcal{U}$. Then we denote by $\mathcal{U}/\mathcal{V}$ the category whose objects are the same as the objects of $\mathcal{U}$, and for any two objects X and Y of $\mathcal{U}/\mathcal{V}$ the morphisms from X to Y in $\mathcal{U}/\mathcal{V}$ are $\mathcal{U}(X,Y)/I_{\mathcal{U}}^{\mathcal{V}}(X,Y)$, where $I_{\mathcal{U}}^{\mathcal{V}}(X,Y)$ is the subset of $\mathcal{U}(X,Y)$ given by those $f\in\mathcal{U}(X,Y)$ such that there is an object $Z\in\mathcal{V}$ and $h\in\mathcal{U}(X,Z)$, $g\in\mathcal{U}(Z,Y)$ such that $f=g\circ h$.
- Let $\mathcal{A}$ and $\mathcal{B}$ be additive categories, and let $F:\mathcal{A}\longrightarrow\mathcal{B}$ be a functor. Then for any full subcategory $\mathcal{C}$ of $\mathcal{B}$ let $F^{-1}(\mathcal{C}$ be the full subcategory of $\mathcal{A}$ generated by the objects X of $\mathcal{A}$ such that $F(X)$ is a direct summand of an object of $\mathcal{C}$.
- If $\mathcal{S}$ and $\mathcal{R}$ are subcategories of a Krull-Schmidt category $\mathcal{W}$, then $\mathcal{R}-\mathcal{S}$ denotes the full subcategory of $\mathcal{R}$ consisting of those objects X of $\mathcal{R}$ such that no direct factor of X is an object of $\mathcal{S}$.

Let $\mathcal{D}$, $\mathcal{H}$, $\mathcal{G}$ be three additive categories

$$\mathcal{D}\underset{T'}{\overset{S'}{\rightleftarrows}}\mathcal{H}\underset{T}{\overset{S}{\rightleftarrows}}\mathcal{G}$$

such that (S,T) and (S',T') are adjoint pairs. Let $\epsilon: id_{\mathcal{H}}\longrightarrow TS$ be the unit of the adjunction (S,T).

We assume throughout the rest of the section that there is an endofunctor U of $\mathcal{H}$ such that $TS=1_{\mathcal{H}}\oplus U$, denote by $p_1:TS\longrightarrow 1_{\mathcal{H}}$ the projection, and suppose that $p_1\circ\epsilon$ is an isomorphism. Note that we use both of the notations $id_{\mathcal{C}}$ and $1_{\mathcal{C}}$ for the identity functor on the category $\mathcal{C}$.

Theorem 20. [1, Theorem 1.10] *Assume that there is an endofunctor U of $\mathcal{H}$ such that $TS=1_{\mathcal{H}}\oplus U$, denote by $p_1:TS\longrightarrow 1_{\mathcal{H}}$ the projection, and suppose that $p_1\circ\epsilon$ is an isomorphism.*

Let $\mathcal{Y}$ be a subcategory of $\mathcal{H}$ and let $\mathcal{Z}:=(US')^{-1}(\mathcal{Y})$. Then the following two conditions $(\dagger)$ are equivalent.

- *Each object of $S'T'\mathcal{Y}$ is a direct factor of an object of $\mathcal{Y}$ and of an object of $U^{-1}(\mathcal{Y})$.*
- *Each object of $TSS'T'\mathcal{Y}$ is a direct factor of an object of $\mathcal{Y}$.*

Suppose that the above conditions $(\dagger)$ hold for $\mathcal{Y}$. Then

(1) *S and T induce functors*

$$\mathcal{H}/S'T'\mathcal{Y}\overset{S}{\longrightarrow}\mathcal{G}/SS'T'\mathcal{Y}\text{ and }\mathcal{G}/SS'T'\mathcal{Y}\overset{T}{\longrightarrow}\mathcal{H}/\mathcal{Y}$$

(2) *For any object L of $\mathcal{D}$ and any object B of $U^{-1}(\mathcal{Y})$ the functor S induces an isomorphism*

$$\mathcal{H}/S'T'\mathcal{Y}(S'L,B)\longrightarrow\mathcal{G}/SS'T'\mathcal{Y}(SS'L,SB)$$

(3) *For any object L of $(US')^{-1}\mathcal{Y}$ and any object A of $\mathcal{G}$ the functor T induces an isomorphism*

$$\mathcal{G}/SS'T'\mathcal{Y}(SS'L,B)\longrightarrow\mathcal{H}/\mathcal{Y}(TSS'L,TA)$$

(4) *The restrictions of S*

$$(\mathrm{add}S'\mathcal{Z})/S'T'\mathcal{Y}\overset{S}{\longrightarrow}(\mathrm{add}SS'\mathcal{Z})/SS'T'\mathcal{Y}$$

and T

$$(\mathrm{add}SS'\mathcal{Z})/SS'T'\mathcal{Y}\overset{T}{\longrightarrow}(\mathrm{add}TSS'\mathcal{Z})/\mathcal{Y}$$

are equivalences of categories, and

$$(\mathrm{add}S'\mathcal{Z})/S'T'\mathcal{Y}\overset{TS}{\longrightarrow}(\mathrm{add}TSS'\mathcal{Z})/\mathcal{Y}$$

is isomorphic to the natural projection.

In the next corollary we shall try to make the statement of item 4 more intelligible.

Corollary 21. *Under the hypotheses of Theorem 20 we have a commutative diagram*

$$\begin{array}{ccc} (\mathrm{add} S'\mathcal{Z})/S'T'\mathcal{Y} & \xrightarrow{S} & (\mathrm{add} SS'\mathcal{Z})/SS'T'\mathcal{Y} \\ {\scriptstyle \mathrm{can}}\downarrow & \swarrow {\scriptstyle T} & \\ (\mathrm{add} TSS'\mathcal{Z})/\mathcal{Y} & & \end{array}$$

where S and T are equivalences of categories.

5. Triangulated Generalisation of Auslander Kleiner's Green Correspondence

We consider now, instead of additive categories, three triangulated categories and triangle functors

$$\mathcal{D} \underset{T'}{\overset{S'}{\rightleftarrows}} \mathcal{H} \underset{T}{\overset{S}{\rightleftarrows}} \mathcal{G}$$

such that (S, T) and (S', T') are adjoint pairs.

We then replace the additive quotient by Verdier localisation.

- For a triangulated category $\mathcal{T}$ we say that the subcategory $\mathcal{S}$ is thick in $\mathcal{T}$ if it is triangulated and stable under taking direct summands.
- If $\mathcal{S}$ is thick in $\mathcal{T}$ then there is a triangulated category $\mathcal{T}_{\mathcal{S}}$ together with a universal functor $\mathcal{T} \to \mathcal{T}_{\mathcal{S}}$ annihilating $\mathcal{S}$.
- For any thick subcategory $\mathcal{S}$ of $\mathcal{T}$ we have a canonical functor

$$L_{\mathcal{S}} : \mathcal{T}/\mathcal{S} \longrightarrow \mathcal{T}_{\mathcal{S}}$$

Theorem 22. [17] *(Green correspondence for triangulated categories) Let $\mathcal{D}$, $\mathcal{H}$, $\mathcal{G}$ be three triangulated categories and let S, S', T, T' be triangle functors*

$$\mathcal{D} \underset{T'}{\overset{S'}{\rightleftarrows}} \mathcal{H} \underset{T}{\overset{S}{\rightleftarrows}} \mathcal{G}$$

such that (S, T) and (S', T') are adjoint pairs. Let $\epsilon : id_{\mathcal{H}} \longrightarrow TS$ be the unit of the adjunction (S, T). Assume that there is an endofunctor U of $\mathcal{H}$ such that $TS = 1_{\mathcal{H}} \oplus U$, denote by $p_1 : TS \longrightarrow 1_{\mathcal{H}}$ the projection, and suppose that $p_1 \circ \epsilon$ is an isomorphism.

Let $\mathcal{Y}$ be a thick subcategory of $\mathcal{H}$, put $\mathcal{Z} := (US')^{-1}(\mathcal{Y})$, and suppose that each object of $TSS'T'\mathcal{Y}$ is a direct factor of an object of $\mathcal{Y}$.

(1) *Then S and T induce triangle functors fitting into the commutative diagram*

$$\begin{array}{ccc} (\mathrm{thick}(S'\mathcal{Z}))_{(\mathrm{thick}(S'T'\mathcal{Y}))} & \xrightarrow{S} & (\mathrm{thick}(SS'\mathcal{Z}))_{(\mathrm{thick}(SS'T'\mathcal{Y}))} \\ {\scriptstyle can}\downarrow & \swarrow {\scriptstyle T} & \\ (\mathrm{thick}(S'\mathcal{Z}))_{\mathcal{Y}} & & \end{array}$$

of Verdier localisations.

(2) *There is an additive functor S_{thick}, induced by S, and an additive functor T_{thick} induced by T, making the diagram*

$$\begin{array}{ccc} (S'\mathcal{Z})/(S'T'\mathcal{Y}) & \xrightarrow{\pi_1} & (\mathrm{thick}(S'\mathcal{Z}))/\mathrm{thick}(S'T'\mathcal{Y}) \\ \downarrow {\scriptstyle S} & & \downarrow {\scriptstyle S_{\mathrm{thick}}} \\ (SS'\mathcal{Z})/(SS'T'\mathcal{Y}) & \xrightarrow{\pi_2} & (\mathrm{thick}(SS'\mathcal{Z}))/\mathrm{thick}(SS'T'\mathcal{Y}) \\ \downarrow {\scriptstyle T} & & \downarrow {\scriptstyle T_{\mathrm{thick}}} \\ S'\mathcal{Z}/\mathcal{Y} & \xrightarrow{\pi_3} & \mathrm{thick}(S'\mathcal{Z})/\mathrm{thick}(\mathcal{Y}) \end{array}$$

commutative. Moreover, the restriction to the respective images of π_1, respectively π_2, respectively π_3 of functors S_{thick} and T_{thick} on the right is an equivalence.

(3) *S and T induce equivalences S_L and T_L of additive categories fitting into the commutative diagram*

$$\xymatrix{ (\text{thick}(S'\mathcal{Z}))_{(\text{thick}(S'T'\mathcal{Y}))} \ar[rr]^{S} \ar@/_3pc/[ddd]_{can} & & (\text{thick}(SS'\mathcal{Z}))_{(\text{thick}(SS'T'\mathcal{Y}))} \ar@/^2pc/[dddll]^{T} \\ L_{S'T'\mathcal{Y}}((S'\mathcal{Z})/(S'T'\mathcal{Y})) \ar@{^{(}->}[u] \ar[r]^{S_L} \ar[d]_{can} & L_{SS'T'\mathcal{Y}}((SS'\mathcal{Z})/(SS'T'\mathcal{Y})) \ar@{^{(}->}[ur] \ar[dl]^{T_L} & \\ L_{\mathcal{Y}}((S'\mathcal{Z})/\mathcal{Y}) \ar@{^{(}->}[d] & & \\ (\text{thick}(S'\mathcal{Z}))_{\mathcal{Y}} & & }$$

where the outer triangle consists of triangulated categories and triangle functors, and the inner triangle are full additive subcategories.

(4) *If S and T induce equivalences of additive categories*

$$\xymatrix{ (\text{thick}(S'\mathcal{Z}))/\text{thick}(S'T'\mathcal{Y}) \ar[r]^{S} \ar[d]_{can} & (\text{thick}(SS'\mathcal{Z}))/\text{thick}(SS'T'\mathcal{Y}) \ar[dl]^{T} \\ (\text{thick}(S'\mathcal{Z}))/\text{thick}\mathcal{Y} & }\ ,$$

then the restriction of S to the triangulated category $(\text{thick}(S'\mathcal{Z}))_{\text{thick}(S'T'\mathcal{Y})}$ and the restriction of T to the triangulated category $(\text{thick}(SS'\mathcal{Z}))_{\text{thick}(SS'T'\mathcal{Y})}$ are dense and full triangle functors in a commutative diagram

$$\xymatrix{ (\text{thick}(S'\mathcal{Z}))_{(\text{thick}(S'T'\mathcal{Y}))} \ar[r]^{S} \ar[d]_{can} & (\text{thick}(SS'\mathcal{Z}))_{(\text{thick}(SS'T'\mathcal{Y}))} \ar[dl]^{T} \\ (\text{thick}(S'\mathcal{Z}))_{\mathcal{Y}} & }\ .$$

- *If in addition for any morphism $t \in \text{thick}(S'\mathcal{Z})(X,Y)$ we get*

$$L_{\text{thick}SS'T'\mathcal{Y}}(St) \text{ is an isomorphism} \Rightarrow t \text{ is a plit epimorphism in } \text{thick}(S'\mathcal{Z})/(\text{thick}S'T'\mathcal{Y})$$

then

$$(\text{thick}(S'\mathcal{Z}))_{(\text{thick}(S'T'\mathcal{Y}))} \xrightarrow{S} (\text{thick}(SS'\mathcal{Z}))_{(\text{thick}(SS'T'\mathcal{Y}))}$$

is an equivalence.

- *If in addition for any morphism $t \in \text{thick}(SS'\mathcal{Z})(X,Y)$ we get*

$$L_{\text{thick}\mathcal{Y}}(Tt) \text{ is an isomorphism} \Rightarrow t \text{ is a plit epimorphism in } \text{thick}(SS'\mathcal{Z})/(\text{thick}SS'T'\mathcal{Y})$$

then

$$(\text{thick}(S'\mathcal{Z}))_{\mathcal{Y}} \xleftarrow{T} (\text{thick}(SS'\mathcal{Z}))_{(\text{thick}(SS'T'\mathcal{Y}))}$$

is an equivalence.

We shall study briefly what our concept will say for what is known as a localisation, respectively colocalisation sequence. We follow Murfet [13] for the definitions of the following concepts.

A localisation sequence is given by three categories $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and functors

$$\xymatrix{ \mathcal{A} \ar@<0.5ex>[r]^{i} & \mathcal{B} \ar@<0.5ex>[l]^{j} \ar@<0.5ex>[r]^{e} & \mathcal{C} \ar@<0.5ex>[l]^{\ell} }$$

such that in addition (j,i), (ℓ,e), are pairs of adjoint functors, and such that the counit $\ell e \longrightarrow 1$ is an isomorphism, such that the unit $1 \longrightarrow ij$ is an isomorphism, and i is the kernel of e.

A colocalisation sequence is given by three categories $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and functors

$$\mathcal{A} \underset{i}{\overset{k}{\leftrightarrows}} \mathcal{B} \underset{e}{\overset{r}{\leftrightarrows}} \mathcal{C}$$

such that in addition (k,i), (e,r), are pairs of adjoint functors, and such that the unit $1 \longrightarrow re$ is an isomorphism, such that the counit $ki \longrightarrow 1$ is an isomorphism, and such that i is the kernel of e.

A recollement diagram is given by three categories $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and functors

$$\mathcal{A} \begin{array}{c} \xleftarrow{k} \\ \xrightarrow{i} \\ \xleftarrow[j]{} \end{array} \mathcal{B} \begin{array}{c} \xleftarrow{r} \\ \xrightarrow{e} \\ \xleftarrow[\ell]{} \end{array} \mathcal{C}$$

such that in addition (j,i), (i,k), (ℓ,e), (e,ρ) are pairs of adjoint functors, and such that the unit $1 \longrightarrow re$ and the counit $\ell e \longrightarrow 1$ are isomorphisms and i is the kernel of e.

Consider now the case when

$$\mathcal{D} \underset{T'}{\overset{S'}{\rightleftarrows}} \mathcal{H} \underset{T}{\overset{S}{\rightleftarrows}} \mathcal{G}$$

is a localisation sequence. Further we need to assume that the unit $\epsilon : 1_{\mathcal{H}} \longrightarrow TS$ has the property that there is an endofunctor U with $TS = I \oplus U$ and the projection p_1 onto the first component composes to the identity with ϵ. Since $SS' = 0$ the right end of the triangle in Theorem 22 is 0. Hence, in this situation the statement of Theorem 22 is void.

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