KÜLSHAMMER IDEALS OF ALGEBRAS OF QUATERNION TYPE

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Abstract. For a symmetric algebra A over a field K of characteristic $p > 0$ Külshammer constructed a descending sequence of ideals of the centre of A. If K is perfect this sequence was shown to be an invariant under derived equivalence and for algebraically closed K the dimensions of their image in the stable centre was shown to be invariant under stable equivalence of Morita type. Erdmann classified algebras of tame representation type which may be blocks of group algebras, and Holm classified Erdmann's list up to derived equivalence. In both classifications certain parameters occur in the classification, and it was unclear if different parameters lead to different algebras. Erdmann's algebras fall into three classes, namely of dihedral, semidihedral and of quaternion type. In previous joint work with Holm we used Külshammer ideals to distinguish classes with respect to these parameters in case of algebras of dihedral and semidihedral type. In the present paper we determine the Külshammer ideals for algebras of quaternion type and distinguish again algebras with respect to certain parameters.

Introduction

Erdmann gave in [4] a list of basic symmetric algebras of tame representation type which include all the algebras which may be Morita equivalent to blocks of finite groups of tame representation type. She obtained these algebras by means of properties of the Auslander-Reiten quiver which are known to hold for blocks of group rings with tame representation type. These algebras are subdivided into three classes, those of dihedral type, of semidihedral type and of quaternion type, corresponding to the possible defect group in case of group algebras, and actually defined by the behaviour of their Auslander-Reiten quiver. Holm refined in [7] Erdmann's classification of those algebras which may occur as blocks of group algebras to a classification up to derived equivalence. However, in [4, 7] the algebras are defined by quivers with relations, and the relations involve certain parameters, corresponding mostly to deformations of the socle of the algebras. It was unclear in some cases if different parameters lead to different derived equivalence classes of algebras. The question of non trivial socle deformations appears to be a very subtle one in this special case, but also in general, and little progress was made on this question until very recently.

In [18] we showed that a certain sequence of ideals of the centre of a symmetric algebra defined previously by Külshammer [12] is actually invariant under derived equivalences if the base field is perfect. We call this sequence of ideals the Külshammer ideals. In joint work [14] with Liu and Zhou we showed that if the base field is algebraically closed, then the dimension of the image of this invariant in the stable centre is also an invariant under stable equivalences of Morita type. In joint work [9] with Holm we observed that the Külshammer ideals behave in a very subtle manner with respect to the deformation parameters. Using this observation we showed that some of the parameters are invariants under derived equivalence for certain families of algebras of dihedral and of semidihedral type. The present paper is a continuation and completion of [9].

In order to apply the theory of Külshammer ideals we need to use the symmetrising form explicitly, and in the present work we progress in avoiding the ad-hoc arguments used in our previous work to determine the symmetrising form. In this note we compute the Külshammer ideals for algebras of quaternion type and distinguish this way the derived equivalence classes of the algebras with two simple modules. Over algebraically closed fields of characteristic different from 2 we can classify completely the derived equivalence classes of the algebras of quaternion type occurring in Holm's

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list, except for a case of very small parameters. If the field is algebraically closed of characteristic 2 then we have an almost complete classification in case of two simple modules. The result in this case is displayed in Corollary 10 and Theorem 7. We also deal with the case of algebras of quaternion type with three simple modules, where Külshammer ideals distinguish the isomorphism classes of algebras in characteristic 2 with parameter d according to whether or not d is a square in K. The invariance of Külshammer ideals under derived or Morita equivalence is proved only in case K is perfect, which implies that all elements of K are squares when K is of characteristic 2. Hence, we cannot say more about this case, and the derived equivalence classification remains open for this class of 20-dimensional algebras. Derived equivalent local algebras are actually Morita equivalent (cf [22, Proposition 6.7.4]), so that the derived equivalence classification of the class of algebras of quaternion type with one simple module coincides with its classification up to isomorphism. Isomorphic algebras have isomorphic Külshammer ideal structure.

For the reader's convenience we give the somewhat technical result for the class of algebras with two simples here. Blocks of quaternion type with two simple modules are derived equivalent to an algebra $A^{k,s}(a,c)$ for parameters $a \in K^\times$ and $c \in K$ and integers $s \geq 3$ and $k \geq 1$.

- In particular, if K is an algebraically closed field of characteristic different from 2, then there is $a' \in K^\times$ such that $A^{k,s}(a,c) \simeq A^{k,s}(a',0)$, and if $(k,s) \neq (1,3)$, then $A^{k,s}(a,c) \simeq A^{k,s}(1,0)$. Moreover, if $A^{k,s}(1,0)$ and $A^{k',s'}(1,0)$ are derived equivalent, then $(k,s) = (k',s')$ or $(k,s) = (s',k')$.
- If K is a perfect field of characteristic 2, we have the following situation. The algebra $A^{k,s}(a,c)$ is not derived equivalent to $A^{k,s}(a',0)$ for any $a,a',c \in K^\times$. If K is algebraically closed of characteristic 2, and if $c \neq 0$, then $A^{k,s}(a,c)$ is isomorphic to $A^{k,s}(a'',1)$ for some $a'' \in K^\times$, and if $(k,s) \neq (1,3)$, then $A^{k,s}(a,0) \simeq A^{k,s}(1,0)$. Further, again for algebraically closed K, if $A^{k,s}(a',c')$ is derived equivalent to $A^{k',s'}(a'',c'')$ then $(k,s) = (k',s')$ or $(k,s) = (k',s')$.

  We do not know for which parameters $a,a' \in K^\times$ we get $A^{k,s}(a,1)$ is derived equivalent to $A^{k,s}(a',1)$, and we do not know when $A^{(1,3)}(a,0)$ is derived equivalent to $A^{(1,3)}(a',0)$ for $a,a' \in K^\times$.

The Külshammer ideal structure depends in a quite subtle way on the parameters, and we want to stress the fact that we need to compute the ideals as ideals, and as in [9] it is not sufficient to consider the dimensions only.

The paper is organised as follows. In Section 1 we recall basic facts about Külshammer ideals and improve the general methods needed to compute the Külshammer ideal structure for symmetric algebras. In Section 2 we apply the general theory to algebras of quaternion type, and we prove our main result Theorem 7 there.

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1. Review on Külshammer ideals and how to compute them

The aim of this section is to briefly give the necessary background on Külshammer ideals, as introduced by B. Külshammer [12]. Morita invariance of Külshammer ideals (then named generalised Reynolds' ideals) was shown in [3, 6] for perfect fields K. Külshammer ideals were proved to be a derived invariant in [18], were used in [8, 11, 2] to classify weakly symmetric algebras of polynomial growth or domestic type up to derived equivalences, in [9] for a derived equivalence classification of algebras of dihedral or semidihedral type, in [10] for deformed preprojective algebras of type L, and in [17] for the derived equivalence classification of certain special biserial algebras. The concept was generalised to general finite-dimensional algebras in [1], to an invariant of Hochschild (co)homology for symmetric algebras [19] and in [20] for general algebras. The image of the Külshammer ideals in the stable centre were shown to be an invariant under stable equivalences of Morita type [14, 13]. An overview is given in [21, 22].

Let K be a field of characteristic $p > 0$. Any finite-dimensional symmetric K-algebra A has an associative, symmetric, non-degenerate K-bilinear form $\langle -,-\rangle : A \times A \to K$. For any K-linear subspace M of A we denote the orthogonal space by $M^\perp$ with respect to this form. Moreover, let $[A, A]$ be the K-subspace of A generated by all commutators $[a, b] := ab - ba$, where $a, b \in A$. For any $n \geq 0$ set

$$T_n(A) = \left\{ x \in A \mid x^{p^n} \in [A, A] \right\}.$$

Then, by [12], for any $n \geq 0$, the orthogonal space $T_n(A)^\perp$ is an ideal of the center $Z(A)$ of A, called *n-th Külshammer ideal*. These ideals form a descending sequence

$$Z(A) = [A, A]^\perp = T_0(A)^\perp \supseteq T_1(A)^\perp \supseteq T_2(A)^\perp \supseteq \ldots \supseteq T_n(A)^\perp \supseteq \ldots$$

with intersection of all ideals $T_n(A)^\perp$ for $n \in \mathbb{N}$ being the Reynolds' ideal $R(A) = Z(A) \cap \mathrm{soc}(A)$. In [6] it has been shown that if K is perfect, then the sequence of Külshammer ideals is invariant under Morita equivalences. Later, it was shown that the sequence of Külshammer ideals is invariant under derived equivalences, and the image of the sequence of Külshammer ideals in the stable centre is invariant under stable equivalences of Morita type. The following theorem recalls part of what is known.

Theorem 1.

- [18, Theorem 1] *Let A and B be finite-dimensional symmetric algebras over a perfect field K of positive characteristic p. If A and B are derived equivalent, then there is an isomorphism $\varphi : Z(A) \to Z(B)$ between the centers of A and B such that $\varphi(T_n(A)^\perp) = T_n(B)^\perp$ for all positive integers n.*
- *(cf e.g.* [22, Proposition 6.8.9]*) Let A and B be derived equivalent finite dimensional K-algebras over a field K, which is a splitting field for A and for B. Then the elementary divisors of the Cartan matrices of A and of B coincide. In particular, the determinant of the Cartan matrices coincides.*
- [14, Corollary 6.5] *If A and B are stably equivalent of Morita type, and if K is an algebraically closed field, then* $\dim_K(T_n(A)/[A, A]) = \dim_K(T_n(B)/[B, B])$.

We note that in the proof of [18, Theorem 1] the hypothesis that K is algebraically closed is never used. The assumption on the field K to be perfect is sufficient.

The aim of the present note is to show how these derived invariants can be applied to some subtle questions in the derived equivalence classifications of algebras of quaternion type.

In order to compute the Külshammer ideals we need a symmetrising form. However, the Külshammer ideals do not depend on the choice of the symmetrising form if K is perfect (cf [18, Proof of Claim 3]). We showed in [10] (see also [22]) that every Frobenius form arises as in the following proposition.

Proposition 2. [9, 10] *Let A be a basic Frobenius algebra such that K is a splitting field for A, and let $\{e_1, \ldots, e_n\}$ be primitive idempotents with $\sum_{i=1}^n e_i = 1$. Let $\mathcal{B}_{i,j}$ be bases of e_iAe_j such that $\mathcal{B} = \bigcup_{i,j=1}^n \mathcal{B}_{i,j}$ is a basis of A containing a basis of $soc(A)$.*

(1) *Then the K-linear mapping ψ defined on the basis elements by*

$$\psi(b) = \begin{cases} 1 & \text{if } b \in soc(A) \\ 0 & \text{otherwise} \end{cases}$$

for $b \in \mathcal{B}$ gives an associative non-degenerate K-bilinear form $\langle -,-\rangle$ for A by $\langle x, y\rangle := \psi(xy)$ for all $x, y \in A$.

(2) *Conversely, any Frobenius form arises this way for some choice of idempotents and some choice of bases.*

Note that different forms may necessitate different choices of primitive idempotents. Note that the hypothesis in [9, 10] is slightly different, however equivalent to the one given here.

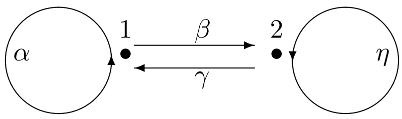
If A is a basic symmetric algebra over an algebraically closed field K, then $A = KQ/I$ and we want to determine those bases $\mathcal{B}_s$ of $\mathrm{soc}(A)$ which yield a symmetric form. This problem is addressed in previous papers dealing with Külshammer ideals (cf [10, Remark 2.9], [9, Remark 3.2]). The following remark indicates a necessary condition for the problem.

Remark 3. If A is an indecomposable, basic symmetric algebra over an algebraically closed field K and let $\{e_1, \ldots, e_n\}$ be a choice of orthogonal primitive idempotents with $\sum_{i=1}^n e_i = 1$. Suppose that $\mathcal{B}_s$ is a K-basis of $\mathrm{soc}(A)$ and suppose that for each $b \in \mathcal{B}_s$ there is a unique e_i such that $e_i b e_i = b$. Using [22, Proposition 2.7.4] it is not hard to see that we can always find a basis $\mathcal{B}_s$ of $\mathrm{soc}(A)$ such that the difference of two elements of $\mathcal{B}_s$ is in the commutator subspace. Moreover, since the elements of $\mathcal{B}_s$ are uniquely determined up to scalars by this property, Proposition 2 then shows that we can complete the basis $\mathcal{B}_s$ to a basis $\mathcal{B}$ as in the proposition. If ψ is a K-linear map as in Proposition 2, then $\psi([A, A]) = 0$. In particular, if $b, b' \in \mathrm{soc}(A)$ with $b - b' \in [A, A]$, then $\psi(b) = \psi(b')$.

2. Algebras of quaternion type

2.1. **Two simple modules.** Erdmann gave a classification of algebras which could appear as blocks of tame representation type. These algebras fall in three classes, the algebras of dihedral, the algebras of semidihedral and the algebras of quaternion type. Erdmann's classification was up to Morita equivalence. Holm [7, Appendix B] gave a classification up to derived equivalence and obtained for non-local algebras of quaternion type two families, one containing algebras with two simple modules, one containing algebras with three simple modules. The algebras in each family share a common quiver, and the relations depend on a number of parameters.

The quiver for the algebras with two simples is the following.



Let $k \geq 1, s \geq 3$ integers and $a \in K^\times, c \in K$. Then we get an algebra $Q(2\mathfrak{B})_1^{k,s}(a, c)$ by the above quiver with relations

$$\beta\eta = (\alpha\beta\gamma)^{k-1}\alpha\beta, \quad \eta\gamma = (\gamma\alpha\beta)^{k-1}\gamma\alpha, \quad \alpha^2 = a \cdot (\beta\gamma\alpha)^{k-1}\beta\gamma + c \cdot (\beta\gamma\alpha)^k,$$
$$\gamma\beta = \eta^{s-1}, \quad \alpha^2\beta = 0, \quad \gamma\alpha^2 = 0.$$

Remark 4. Using [7] we see that the centre of this algebra is of dimension $k + s + 2$ and the Cartan matrix is $\begin{pmatrix} 4k & 2k \\ 2k & k+s \end{pmatrix}$ with determinant $4ks$. Hence, using Theorem 1, if $D^b(Q(2\mathfrak{B})_1^{k,s}(a, c)) \simeq D^b(Q(2\mathfrak{B})_1^{k',s'}(a', c'))$, then $4ks = 4k's'$ and $k + s + 2 = k' + s' + 2$. Therefore $(k + s)^2 = (k' + s')^2$ and $(k - s)^2 = (k' - s')^2$, which implies $k = k'$ and $s = s'$, or $k = s'$ and $k' = s$.

Lemma 5. *Let K be a field, and let $A^{k,s}(a, c) := Q(2\mathfrak{B})_1^{k,s}(a, c)$. Then, $Z(A^{k,s}(a, c))$ has a K-basis formed by the disjoint union*

$$\{\eta - (\alpha\beta\gamma)^{k-1}\alpha\} \,\dot\cup\, \{\eta^t \mid 2 \leq t \leq s\} \,\dot\cup\, \{(\alpha\beta\gamma)^u + (\beta\gamma\alpha)^u + (\gamma\alpha\beta)^u \mid 1 \leq u \leq k-1\} \,\dot\cup\, \{1, (\alpha\beta\gamma)^k, \alpha^2\}$$

and is isomorphic, as commutative K-algebra, with

$$K[U, Y, S, T]/(Y^{s+1}, U^k - Y^s - 2T, S^2, T^2, SY, SU, ST, UY, UT, YT)$$

where

$$\begin{aligned} U &:= (\alpha\beta\gamma) + (\beta\gamma\alpha) + (\gamma\alpha\beta) \\ Y &:= \eta - (\alpha\beta\gamma)^{k-1}\alpha \\ S &:= \alpha^2 \\ T &:= (\alpha\beta\gamma)^k \end{aligned}$$

Proof. First, $\eta-(\alpha\beta\gamma)^{k-1}\alpha$ commutes trivially with η, since $\eta\alpha=0=\alpha\eta$. Now

$$\begin{aligned}\alpha(\eta-(\alpha\beta\gamma)^{k-1}\alpha)-(\eta-(\alpha\beta\gamma)^{k-1}\alpha)\alpha &= \alpha^2(\beta\gamma\alpha)^{k-1}-(\alpha\beta\gamma)^{k-1}\alpha^2\\ &= (a(\beta\gamma\alpha)^{k-1}\beta\gamma+c(\beta\gamma\alpha)^k)(\beta\gamma\alpha)^{k-1}\\ &\quad -(\beta\gamma\alpha)^{k-1}(a(\beta\gamma\alpha)^{k-1}\beta\gamma+c(\beta\gamma\alpha)^k)\\ &= a\left((\beta\gamma\alpha)^{k-1}\beta\gamma(\beta\gamma\alpha)^{k-1}-(\beta\gamma\alpha)^{2k-2}\beta\gamma\right)\\ &= a(\beta\gamma\alpha)^{k-1}\left(\beta\gamma(\beta\gamma\alpha)^{k-1}-(\beta\gamma\alpha)^{k-1}\beta\gamma\right).\end{aligned}$$

This is trivially 0 if $k=1$, and if $k>1$, then

$$(\beta\gamma\alpha)^{2k-2}\beta\gamma=\beta(\gamma\alpha\beta)^{2k-2}\gamma=\beta(\gamma\alpha\beta)^k(\gamma\alpha\beta)^{k-2}\gamma=\beta\eta^s(\gamma\alpha\beta)^{k-2}\gamma=0.$$

Hence

$$\begin{aligned}a(\beta\gamma\alpha)^{k-1}\left(\beta\gamma(\beta\gamma\alpha)^{k-1}-(\beta\gamma\alpha)^{k-1}\beta\gamma\right) &= a(\beta\gamma\alpha)^{k-1}\beta\gamma(\beta\gamma\alpha)(\beta\gamma\alpha)^{k-2}\\ &= a(\beta\gamma\alpha)^{k-1}\beta\eta^{s-1}(\gamma\alpha)(\beta\gamma\alpha)^{k-2}\\ &= a(\beta\gamma\alpha)^{k-1}\beta\eta^{s-2}(\gamma\alpha\beta)^{k-1}\gamma\alpha^2(\beta\gamma\alpha)^{k-2}\\ &= 0\end{aligned}$$

since $\gamma\alpha^2=0$. The relations $\beta\eta=(\alpha\beta\gamma)^{k-1}\alpha\beta$ and $\eta\gamma=(\gamma\alpha\beta)^{k-1}\gamma\alpha=\gamma(\alpha\beta\gamma)^{k-1}\alpha$ show that $\eta-(\alpha\beta\gamma)^{k-1}\alpha$ commutes with β and with γ. Now, if $k>1$, then $\left(\eta-(\alpha\beta\gamma)^{k-1}\alpha\right)^2=\eta^2$, and if $k=1$, then $\left(\eta-(\alpha\beta\gamma)^{k-1}\alpha\right)^2=\eta^2+\alpha^2$. Since $\alpha^2\beta=\gamma\alpha^2=0$, it is clear that α^2 is central. Hence η^t is central for each $t\geq 2$. Now,

$$\alpha\beta\gamma\beta=\alpha\beta\eta^{s-1}=\alpha(\alpha\beta\gamma)^{k-1}\alpha\beta\eta^{s-2}=0$$

since $\alpha^2\beta=0$. Likewise $\gamma\beta\gamma\alpha=0$. Hence

$$\beta U=\beta\left((\alpha\beta\gamma)+(\beta\gamma\alpha)+(\gamma\alpha\beta)\right)=\beta\gamma\alpha\beta=\left((\alpha\beta\gamma)+(\beta\gamma\alpha)+(\gamma\alpha\beta)\right)\beta=U\beta$$

and

$$\gamma U=\gamma\left((\alpha\beta\gamma)+(\beta\gamma\alpha)+(\gamma\alpha\beta)\right)=\gamma\alpha\beta\gamma=\gamma\left((\alpha\beta\gamma)+(\beta\gamma\alpha)+(\gamma\alpha\beta)\right)=U\gamma.$$

Now,

$$\begin{aligned}\eta U &= \eta\left((\alpha\beta\gamma)+(\beta\gamma\alpha)+(\gamma\alpha\beta)\right)\\ &= \eta(\gamma\alpha\beta)=(\gamma\alpha\beta)^{k-1}\gamma\alpha^2\beta\\ &= 0\\ &= \gamma\alpha(\alpha\beta\gamma)^{k-1}\alpha\beta\\ &= \gamma\alpha\beta\eta\\ &= \left((\alpha\beta\gamma)+(\beta\gamma\alpha)+(\gamma\alpha\beta)\right)\eta\\ &= U\eta\end{aligned}$$

and

$$\alpha U=\alpha\left((\alpha\beta\gamma)+(\beta\gamma\alpha)+(\gamma\alpha\beta)\right)=\alpha\beta\gamma\alpha=\left((\alpha\beta\gamma)+(\beta\gamma\alpha)+(\gamma\alpha\beta)\right)\alpha=U\alpha.$$

Hence U is central, and then we only need to compute U^u to get the result. Finally, socle elements in basic symmetric algebras over splitting fields are always central, and 1 is central of course. We know by [7] that the centre is $(2+k+s)$-dimensional, and obtain therefore the result. ∎

Remark 6. Erdmann and Skowroński show in [15, Lemma 5.7] that if K is an algebraically closed field, then $Q(2\mathfrak{B})_1^{k,s}(a,c)\simeq Q(2\mathfrak{B})_1^{k,s}(1,c')$ for some $c'\in K$ and if K is of characteristic different from 2, then $Q(2\mathfrak{B})_1^{k,s}(a,c)\simeq Q(2\mathfrak{B})_1^{k,s}(a,0)$. We can examine their computations again to get slightly better results. We assume here $k+s>4$.

Suppose that K admits any k-th root, i.e. for all $x\in K$ there is $y\in K$ with $y^k=x$. We want to simplify the parameters a,c. Replace α by $x_\alpha\alpha$, β by $x_\beta\beta$, γ by $x_\gamma\gamma$ and η by $x_\eta\eta$ for non zero scalars $x_\alpha, x_\beta, x_\gamma, x_\eta$. Then the relations above are equivalent to

$$\begin{aligned}
x_\eta\beta\eta &= x_\alpha^k(x_\beta x_\gamma)^{k-1}(\alpha\beta\gamma)^{k-1}\alpha\beta,\\
x_\eta\eta\gamma &= x_\alpha^k(x_\beta x_\gamma)^{k-1}(\gamma\alpha\beta)^{k-1}\gamma\alpha,\\
x_\alpha^2\alpha^2 &= a\cdot x_\alpha^{k-1}(x_\beta x_\gamma)^k(\beta\gamma\alpha)^{k-1}\beta\gamma + c\cdot x_\alpha^k(x_\beta x_\gamma)^k(\beta\gamma\alpha)^k,\\
x_\gamma x_\beta\gamma\beta &= x_\eta^{s-1}\eta^{s-1},\\
\alpha^2\beta &= 0,\\
\gamma\alpha^2 &= 0.
\end{aligned}$$

We first choose x_β such that $x_\beta x_\gamma = x_\eta^{s-1}$ to get the system

$$\begin{aligned}
\beta\eta &= x_\alpha^k x_\eta^{(k-1)(s-1)-1}(\alpha\beta\gamma)^{k-1}\alpha\beta,\\
\eta\gamma &= x_\alpha^k x_\eta^{(k-1)(s-1)-1}(\gamma\alpha\beta)^{k-1}\gamma\alpha,\\
\alpha^2 &= a\cdot x_\alpha^{k-3}x_\eta^{k(s-1)}(\beta\gamma\alpha)^{k-1}\beta\gamma + c\cdot x_\alpha^{k-2}x_\eta^{k(s-1)}(\beta\gamma\alpha)^k,\\
\gamma\beta &= \eta^{s-1},\\
\alpha^2\beta &= 0,\\
\gamma\alpha^2 &= 0.
\end{aligned}$$

Then we put $x_\alpha = x_\eta^{-\frac{(k-1)(s-1)-1}{k}}$ and obtain the system

$$\begin{aligned}
\beta\eta &= (\alpha\beta\gamma)^{k-1}\alpha\beta,\\
\eta\gamma &= (\gamma\alpha\beta)^{k-1}\gamma\alpha,\\
\alpha^2 &= a\cdot x_\eta^{-\frac{(k-1)(s-1)-1}{k}\cdot(k-3)+k(s-1)}(\beta\gamma\alpha)^{k-1}\beta\gamma + c\cdot x_\eta^{-\frac{(k-1)(s-1)-1}{k}(k-2)+k(s-1)}(\beta\gamma\alpha)^k,\\
\gamma\beta &= \eta^{s-1},\\
\alpha^2\beta &= 0,\\
\gamma\alpha^2 &= 0.
\end{aligned}$$

Now, $-\frac{(k-1)(s-1)-1}{k}\cdot(k-3)+k(s-1)=0$ implies $k=3$ and $s=1$, or $k=1$ and $s=3$, which are excluded parameters, where the first case is already excluded since the algebra is defined only for $s\geq 3$, and where both cases are excluded by our hypothesis. This number can be simplified and we therefore define $u(k,s) := (k-3)+(4k-3)(s-1) > 0$ for our admissible parameters. Hence if there is an element $y_a\in K$ such that $y_a^{u(k,s)} = a^{-k}$, then we may choose such a parameter x_η, such that we may assume $a=1$. This holds in particular if K is algebraically closed. We obtain this way $A^{k,s}(a,0)\simeq A^{k,s}(1,0)$ if K is sufficiently big, i.e. there is an element y_a satisfying $y_a^{u(k,s)} = a^{-k}$. Moreover, since $s\geq 3$ and $k\geq 1$ we get

$$-\frac{(k-1)(s-1)-1}{k}(k-2)+k(s-1) = \frac{(k-2)+(3k-2)(s-1)}{k} \geq \frac{(k-2)+2(3k-2)}{k} = \frac{7k-6}{k} > 0.$$

Let $v(k,s) := (k-2)+(3k-2)(s-1)$. If there is $y_c\in K$ such that $y_c^{v(k,s)} = c^{-k}$, then we can therefore choose x_η such that we can assume $c=1$ if $c\neq 0$. Again, this is trivially true if K is algebraically closed.

As a consequence, combining our computation and the result [15, Lemma 5.7], if K is algebraically closed of characteristic different from 2, then $Q(2\mathfrak{B})_1^{k,s}(a,c)\simeq Q(2\mathfrak{B})_1^{k,s}(1,0)$.

Theorem 7. *Let $A^{k,s}(a,c)$ be the algebra $Q(2\mathfrak{B})_1^{k,s}(a,c)$ over a field K of characteristic p. Let $a,c\in K\setminus\{0\}$. We get the following cases.*

(1) *Suppose $p=2$.*
   (a) *If $k=1$, and*
      (i) *if s is even or if a is not a square in K, then*
      $$\dim_K(T_1^\perp(A^{k,s}(a,c))) = \dim_K(T_1^\perp(A^{k,s}(a,0))),$$
      (ii) *if s is odd and a is a square in K, then*
      $$\dim_K(T_1^\perp(A^{k,s}(a,c))) = \dim_K(T_1^\perp(A^{k,s}(a,0))) - 1.$$

(b) *If $k > 1$ is odd, and*
   (i) *if s is even and if c is a square in K, then*
$$\dim_K(T_1^\perp(A^{k,s}(a,c))) = \dim_K(T_1^\perp(A^{k,s}(a,0))),$$
   (ii) *if s is odd or if c is not a square in K, then*
$$\dim_K(T_1^\perp(A^{k,s}(a,c))) = \dim_K(T_1^\perp(A^{k,s}(a,0))) - 1.$$
(c) *If k is even, and*
   (i) *if c is a square in K, then*
$$\dim_K(T_1^\perp(A^{k,s}(a,c))) = \dim_K(T_1^\perp(A^{k,s}(a,0))),$$
   (ii) *if c is not a square in K, then*
$$\dim_K(T_1^\perp(A^{k,s}(a,c))) = \dim_K(T_1^\perp(A^{k,s}(a,0))) - 1.$$

(2) *Suppose K is a perfect field of characteristic $p = 2$.*
(a) *Then $A^{k,s}(a,0) \simeq A^{k,s}(1,0)$ and*
   (i) *if k and s are even, then*
$$Z(A^{k,s}(a,0))/T_1^\perp(A^{k,s}(a,0)) \simeq K[U,Y,S]/(U^{k/2} - Y^{s/2}, S^2, UY, US, YS),$$
   (ii) *if $k > 1$ or s is odd, then*
$$Z(A^{k,s}(a,0))/T_1^\perp(A^{k,s}(a,0)) \simeq K[U,Y,S]/(U^{\lceil k/2\rceil}, Y^{\lceil s/2\rceil}, S^2, UY, US, YS).$$
(b) *If $c \neq 0$, then*
   (i) *if k and s are even, then*
$$Z(A^{k,s}(a,c))/T_1^\perp(A^{k,s}(a,c)) \simeq K[U,Y]/(U^{k/2} - Y^{s/2}, UY),$$
   (ii) *if $k > 1$ or s is odd, then*
$$Z(A^{k,s}(a,c))/T_1^\perp(A^{k,s}(a,c)) \simeq K[U,Y]/(U^{\lceil (k+1)/2\rceil}, Y^{\lceil (s+1)/2\rceil}, UY).$$
(c) *If $k = 1$, then*
   (i) *if (s is odd and $c = 0$) or (s is even and $c \neq 0$),*
$$Z(A^{1,s}(a,c))/T_1^\perp(A^{1,s}(a,c)) \simeq K[Y,S]/(Y^{\lceil (s+1)/2\rceil}, S^2, YS),$$
   (ii) *if (s is odd and $c \neq 0$) or (s is even and $c = 0$),*
$$Z(A^{1,s}(a,c))/T_1^\perp(A^{1,s}(a,c)) \simeq K[Y]/(Y^{\lceil s/2\rceil}).$$

(3) *If $p \neq 2$, and if K is algebraically closed, then $A^{k,s}(a,c) \simeq A^{k,s}(a',0)$ for some $a' \in K^\times$, and if moreover $(k,s) \neq (1,3)$, then $A^{k,s}(a,c) \simeq A^{k,s}(1,0)$.*

(4) *If $p > 3$ or $n \geq 2$, then the dimension of the Külshammer ideal $T_n(A^{k,s}(a,c))^\perp$ does not depend on the parameters a, c.*

Suppose that K is algebraically closed. Then $A^{k,s}(a,0) \simeq A^{k,s}(1,0)$ if $(k,s) \neq (1,3)$ and if $c \neq 0$ then $A^{k,s}(a,c) \simeq A^{k,s}(a',1) \simeq A^{k,s}(1,c')$ for some $a', c' \in K^\times$.

Remark 8. Consider the case $s = 3$ and $k = 1$. The computations in Remark 6 show that if K is algebraically closed, and $c \neq 0$, then for each a' there is a such that $A^{1,3}(a',c) \simeq A^{1,3}(a,1)$.

This case is quite particular which allows an alternative argument for distinguishing derived equivalence classes. If $c = 0$, then the relations of $A^{1,3}(a,0)$ are homogeneous. Therefore the algebra $A^{1,3}(a,0)$ is graded by path lengths with semisimple degree 0 component. A theorem of Rouquier [16, Theorem 6.1] shows that if $A^{1,3}(a,0)$ is derived equivalent to another algebra B, then the induced stable equivalence of Morita type induces a grading on B. Moreover, by [16, Lemma 5.21] the degree 0 component of $A^{1,3}(a,0)$ is of finite global dimension if and only if the degree 0 component of B is of finite global dimension.

Remark 9. The hypothesis that K is algebraically closed is stronger than required. A more precise, somewhat technical statement is given at the end of Remark 6.

Proof of Theorem 7. The isomorphisms $A^{k,s}(a,0) \simeq A^{k,s}(1,0)$ for $(k,s) \neq (1,3)$ and if $c \neq 0$ then $A^{k,s}(1,c') \simeq A^{k,s}(a,c) \simeq A^{k,s}(a',1)$ for some $a', c' \in K^\times$ follow from Remark 6.

Define the following subsets of $Q(2\mathfrak{B})_1^{k,s}(a,c)$.

$$\mathcal{B}_1 := \{\alpha(\beta\gamma\alpha)^n, (\beta\gamma\alpha)^n\beta\gamma, (\beta\gamma\alpha)^m, e_1(\alpha\beta\gamma)^\ell \mid 0 \leq \ell \leq k, 1 \leq m \leq k-1, 0 \leq n \leq k-1\},$$
$$\mathcal{B}_2 := \{e_2\eta^t, (\gamma\alpha\beta)^m \mid 0 \leq t \leq s, 1 \leq m \leq k-1\},$$
$$\mathcal{B}_3 := \{(\beta\gamma\alpha)^n\beta, \alpha(\beta\gamma\alpha)^n\beta \mid 0 \leq n \leq k-1\},$$
$$\mathcal{B}_4 := \{(\gamma\alpha\beta)^n\gamma, (\gamma\alpha\beta)^n\gamma\alpha \mid 0 \leq n \leq k-1\}.$$

The (disjoint) union of these sets forms a basis of $Q(2\mathfrak{B})_1^{k,s}(a,c)$, using the known Cartan matrix of $Q(2\mathfrak{B})_1^{k,s}(a,c)$.

As a next step we need to compute the commutator space. Clearly, non closed paths are commutators, since if $e_ipe_j \neq 0$ for some path p and $e_i \neq e_j$, then $p = e_ip - pe_i$. Hence $\mathcal{B}_3 \cup \mathcal{B}_4 \subseteq [A^{k,s}(a,c), A^{k,s}(a,c)]$. Moreover,

$$\alpha(\beta\gamma\alpha)^n = \alpha(\beta\gamma\alpha)^n - (\beta\gamma\alpha)^n\alpha \in [A(a,c), A(a,c)] \ \forall n \geq 1,$$
$$(\beta\gamma\alpha)^m = (\alpha\beta\gamma)^m = (\gamma\alpha\beta)^m \in A^{k,s}(a,c)/[A^{k,s}(a,c), A^{k,s}(a,c)] \ \forall m \geq 0,$$
$$(\beta\gamma\alpha)^m\beta\gamma = \gamma(\beta\gamma\alpha)^m\beta = 0 \in A^{k,s}(a,c)/[A^{k,s}(a,c), A^{k,s}(a,c)] \ \forall m \geq 1,$$

and hence

$$\mathcal{B}_{comm} := \{\alpha, (\alpha\beta\gamma)^m, \eta^t, e_1, e_2 \mid 1 \leq t \leq s-1, 1 \leq m \leq k\}$$

is a generating set of $A^{k,s}(a,c)/[A^{k,s}(a,c), A^{k,s}(a,c)]$. Since the dimension of the centre of $A^{k,s}(a,c)$ equals the dimension of $A^{k,s}(a,c)/[A^{k,s}(a,c), A^{k,s}(a,c)]$, the algebra $A^{k,s}(a,c)$ being symmetric, and both are of dimension $2+k+s$, by [7], we get that the classes represented by the elements $\mathcal{B}_{comm}$ form actually a basis of $A^{k,s}(a,c)/[A^{k,s}(a,c), A^{k,s}(a,c)]$.

We only need to work in $A^{k,s}(a,c)/[A^{k,s}(a,c), A^{k,s}(a,c)]$, and therefore we need to consider linear combinations of paths in $\mathfrak{B}_{comm}$ only. We can omit idempotents, since computing modulo the radical these idempotents remain idempotents, and are hence never nilpotent modulo commutators. Hence we only need to consider linear combinations of elements in the set

$$\{\eta^t, \alpha, (\beta\gamma\alpha)^m \mid 1 \leq t \leq s-1, 1 \leq m \leq k\}.$$

We deal with the case $p = 2$. Let hence $p = 2$. In the commutator quotient squaring is semilinear (cf e.g. [12],[22, Lemma 2.9.3]).

If $k > 1$ is odd, then

$$\begin{aligned} 0 &= \left(\sum_{t=1}^{s-1} x_t\eta^t + u\alpha + \sum_{m=1}^{k} y_m(\beta\gamma\alpha)^m\right)^2 \\ &= \sum_{t=1}^{s-1} x_t^2\eta^{2t} + u^2c(\beta\gamma\alpha)^k + \sum_{m=1}^{k} y_m^2(\beta\gamma\alpha)^{2m} \\ &= \sum_{1 \leq t \leq s/2} x_t^2\eta^{2t} + u^2c(\beta\gamma\alpha)^k + \sum_{m=1}^{(k-1)/2} y_m^2(\beta\gamma\alpha)^{2m}, \end{aligned}$$

which implies $x_t = 0$ for all $t \leq \frac{s}{2}$, $y_m = 0$ for all $m \leq \frac{k-1}{2}$. If $c = 0$ then there is no other constraint. Suppose $c \neq 0$. Then, if s is odd we get $u = 0$. If s is even, then $x_{s/2}^2 + cu^2 = 0$ which has a non trivial solution if and only if c is a square in K. Hence, computing in $A^{k,s}(a,c)/[A^{k,s}(a,c), A^{k,s}(a,c)]$ we get

$$T_1(A^{k,s}(a,c)) = \begin{cases} \langle \alpha, \eta^t, (\beta\gamma\alpha)^m \mid t > \frac{s}{2}, m > \frac{k-1}{2} \rangle_K & \text{if } c = 0 \\ \langle \eta^t, (\beta\gamma\alpha)^m \mid t > \frac{s}{2}, m > \frac{k-1}{2} \rangle_K & \text{if } s \text{ is odd and } c \neq 0 \\ \langle \eta^t, (\beta\gamma\alpha)^m \mid t > \frac{s}{2}, m > \frac{k-1}{2} \rangle_K & \text{if } s \text{ is even and } 0 \neq c \notin K^2 \\ \langle \eta^t, (\beta\gamma\alpha)^m, \eta^{s/2} + d\alpha \mid t > \frac{s}{2}, m > \frac{k-1}{2} \rangle_K & \text{if } s \text{ is even and } 0 \neq c = d^2 \end{cases}$$

If k is even, then

$$\begin{aligned} 0 &= \left(\sum_{t=1}^{s-1} x_t\eta^t + u\alpha + \sum_{m=1}^{k} y_m(\beta\gamma\alpha)^m\right)^2 \\ &= \sum_{t=1}^{s-1} x_t^2\eta^{2t} + u^2c(\beta\gamma\alpha)^k + \sum_{m=1}^{k} y_m^2(\beta\gamma\alpha)^{2m} \\ &= \sum_{1\le t\le (s-1)/2} x_t^2\eta^{2t} + u^2c(\beta\gamma\alpha)^k + \sum_{m=1}^{k/2} y_m^2(\beta\gamma\alpha)^{2m}, \end{aligned}$$

which implies $y_m = 0$ for $1 \le m < k/2$ and $x_t = 0$ for $t \le \frac{s-1}{2}$.

If $c = 0$, then $x_{s/2} + y_{k/2} = 0$ in case s is even, and $y_{k/2} = 0$ in case s is odd.

Suppose $c \neq 0$. If s is odd, then $y_{k/2}^2 + cu^2 = 0$, and if s is even, then $y_{k/2}^2 + x_{s/2}^2 + cu^2 = 0$. Again $y_{k/2}^2 + cu^2 = 0$ and $y_{k/2}^2 + x_{s/2}^2 + cu^2 = 0$ has non zero solutions if and only if c is a square.

Computing in $A^{k,s}(a,c)/[A^{k,s}(a,c), A^{k,s}(a,c)]$ we get

$$T_1(A(a,c)) = \begin{cases} \langle \alpha, \eta^t, (\beta\gamma\alpha)^m \mid t > \frac{s-1}{2}, m > \frac{k}{2}\rangle_K & \text{if } s \text{ is odd and } c = 0 \\ \langle \alpha, \eta^{s/2} + (\beta\gamma\alpha)^{k/2}, \eta^t, (\beta\gamma\alpha)^m \mid t > \frac{s}{2}, m > \frac{k}{2}\rangle_K & \text{if } s \text{ is even and } c = 0 \\ \langle \eta^t, (\beta\gamma\alpha)^m \mid t > \frac{s-1}{2}, m > \frac{k}{2}\rangle_K & \text{if } s \text{ is odd and } 0 \neq c \notin K^2 \\ \langle \eta^t, (\beta\gamma\alpha)^m, (\beta\gamma\alpha)^{k/2} + d\alpha \mid t > \frac{s-1}{2}, m > \frac{k}{2}\rangle_K & \text{if } s \text{ is odd and } 0 \neq c = d^2 \\ \langle \eta^t, (\beta\gamma\alpha)^m, (\beta\gamma\alpha)^{k/2} + \eta^{s/2} \mid t > \frac{s}{2}, m > \frac{k}{2}\rangle_K & \text{if } s \text{ is even and } 0 \neq c \notin K^2 \\ \langle \eta^t, (\beta\gamma\alpha)^m, (\beta\gamma\alpha)^{k/2} + \eta^{s/2}, \eta^{s/2} + d\alpha \mid t > \frac{s}{2}, m > \frac{k}{2}\rangle_K & \text{if } s \text{ is even and } 0 \neq c = d^2 \end{cases}$$

If $k = 1$, then, since $\beta\gamma = \gamma\beta = \eta^{s-1}$ in the commutator quotient,

$$0 = \left(\sum_{t=1}^{s-1} x_t\eta^t + u\alpha + y_1(\beta\gamma\alpha)\right)^2 = \sum_{t=1}^{s-1} x_t^2\eta^{2t} + u^2a\eta^{s-1} + u^2c(\beta\gamma\alpha)$$

which implies $x_t = 0$ for $1 \le t \le \frac{s-2}{2}$.

If $c = 0$, then $x_{s/2} = 0$ in case s is even, and $x_{(s-1)/2}^2 + au^2 = 0$ in case s is odd. This last equation has non zero solutions if and only if $a \in K^2$.

Suppose $c \neq 0$. If s is even, then $x_{s/2}^2 + cu^2 = 0$. This has non zero solutions if and only if $c \in K^2$. If s is odd, then $cu^2 = 0$ and $x_{(s-1)/2}^2 + au^2 = 0$. Hence s odd implies $u = 0 = x_{(s-1)/2}$. Computing again in $A^{k,s}(a,c)/[A^{k,s}(a,c), A^{k,s}(a,c)]$,

$$T_1(A^{k,s}(a,c)) = \begin{cases} \langle \eta^{(s-1)/2} + b\alpha, \eta^t, (\beta\gamma\alpha) \mid t > \frac{s-1}{2}\rangle_K & \text{if } s \text{ is odd, } a = b^2 \text{ and } c = 0 \\ \langle \eta^t, (\beta\gamma\alpha) \mid t > \frac{s-1}{2}\rangle_K & \text{if } s \text{ is odd, } a \notin K^2 \text{ and } c = 0 \\ \langle \eta^t, (\beta\gamma\alpha) \mid t > \frac{s}{2}\rangle_K & \text{if } s \text{ is even and } c = 0 \\ \langle \eta^t, (\beta\gamma\alpha) \mid t > \frac{s-1}{2}\rangle_K & \text{if } s \text{ is odd and } c \neq 0 \\ \langle \eta^t, (\beta\gamma\alpha) \mid t > \frac{s}{2}\rangle_K & \text{if } s \text{ is even and } 0 \neq c \notin K^2 \\ \langle \eta^t, (\beta\gamma\alpha), \eta^{s/2} + d\alpha \mid t > \frac{s}{2}\rangle_K & \text{if } s \text{ is even and } 0 \neq c = d^2 \end{cases}$$

It is easy to see that computing $T_n(A^{k,s}(a,c))/[A^{k,s}(a,c), A^{k,s}(a,c)]$ for $n \ge 2$ (and any $p \ge 2$ in this case) yields expressions which are independent of a, c.

In order to compute the Külshammer ideal $T_1(A^{k,s}(a,c))^\perp$ we need to give the symmetrising form of $A^{k,s}(a,c)$. Recall that we have a basis $\mathcal{B} = \bigcup_{i=1}^4 \mathcal{B}_i$ of $A^{(k,s)}(a,c)$ given by

$$\begin{aligned} \mathcal{B}_1 &:= \{\alpha(\beta\gamma\alpha)^n, (\beta\gamma\alpha)^n\beta\gamma, (\beta\gamma\alpha)^m, e_1(\alpha\beta\gamma)^\ell \mid 0 \le \ell \le k, 1 \le m \le k-1, 0 \le n \le k-1\}, \\ \mathcal{B}_2 &:= \{e_2\eta^t, (\gamma\alpha\beta)^m \mid 0 \le t \le s, 1 \le m \le k-1\}, \\ \mathcal{B}_3 &:= \{(\beta\gamma\alpha)^n\beta, \alpha(\beta\gamma\alpha)^n\beta \mid 0 \le n \le k-1\}, \\ \mathcal{B}_4 &:= \{(\gamma\alpha\beta)^n\gamma, (\gamma\alpha\beta)^n\gamma\alpha \mid 0 \le n \le k-1\}. \end{aligned}$$

We define a trace map

$$A(a,c) \xrightarrow{\psi} K$$

by

$\psi(\eta^s) = \psi((\alpha\beta\gamma)^k) = 1,$ and $\psi(x) = 0$ if x is a path in the quiver such that $x \in \mathcal{B} \setminus \mathrm{soc}(A^{k,s}(a,c))$.

Note that $(\alpha\beta\gamma)^k = \beta\eta\gamma = (\beta\gamma\alpha)^k$. Remark 3 indicates that ψ should coincide on these socle elements for ψ to define a symmetric form. Indeed, $\eta^s = \eta\gamma\beta = (\gamma\alpha\beta)^k$ and hence

$$\eta^s - (\alpha\beta\gamma)^k = (\gamma\alpha\beta)^k - (\alpha\beta\gamma)^k = [\gamma, \alpha\beta(\gamma\alpha\beta)^{k-1}]$$

is a commutator. We need to prove that $\psi(c_1c_2) = \psi(c_2c_1)$ for all elements $c_1, c_2 \in \mathcal{B}$.

Case $(c_1, c_2) \in \mathcal{B}_1 \times \mathcal{B}_1$**:** We obtain $\alpha(\beta\gamma\alpha)^{n_1} \cdot \alpha(\beta\gamma\alpha)^{n_2} = 0$ if $n_1+n_2 > 0$, and the case $n_1 = n_2 = 0$ is clearly symmetric.

$$\alpha(\beta\gamma\alpha)^{n_1} \cdot (\beta\gamma\alpha)^{n_2}\beta\gamma = (\alpha\beta\gamma)^{n_1+n_2+1} = (\beta\gamma\alpha)^{n_2}\beta\gamma \cdot \alpha(\beta\gamma\alpha)^{n_1},$$
$$\alpha(\beta\gamma\alpha)^{n_1} \cdot (\beta\gamma\alpha)^{m_2} = (\alpha\beta\gamma)^{n_1+m_2}\alpha \in (\mathcal{B}_1 \cup \{0\}) \setminus \mathrm{soc}(A^{(k,s)}(a,c)$$

is mapped to 0 by ψ, and $m_2 > 0$ implies

$$(\beta\gamma\alpha)^{m_2} \cdot \alpha(\beta\gamma\alpha)^{n_1} = 0.$$

Now, if $\ell_2 > 0$, then

$$\alpha(\beta\gamma\alpha)^{n_1} \cdot (\alpha\beta\gamma)^{\ell_2} = 0$$

and

$$(\alpha\beta\gamma)^{\ell_2} \cdot \alpha(\beta\gamma\alpha)^{n_1} = (\alpha\beta\gamma)^{\ell_2+n_1}\alpha \in (\mathcal{B}_1 \cup \{0\}) \setminus \mathrm{soc}(A^{(k,s)}(a,c)$$

is mapped to 0 by ψ. If $\ell_2 = 0$, then the two elements commute trivially.

$$(\beta\gamma\alpha)^{n_1}\beta\gamma \cdot (\beta\gamma\alpha)^{n_2}\beta\gamma = 0 = (\beta\gamma\alpha)^{n_2}\beta\gamma \cdot (\beta\gamma\alpha)^{n_1}\beta\gamma$$

and

$$(\beta\gamma\alpha)^{n_1}\beta\gamma \cdot (\beta\gamma\alpha)^{m_2} = 0$$

whereas

$$(\beta\gamma\alpha)^{m_2} \cdot (\beta\gamma\alpha)^{n_1}\beta\gamma = (\beta\gamma\alpha)^{m_2+n_1}\beta\gamma \in (\mathcal{B}_1 \cup \{0\}) \setminus \mathrm{soc}(A^{(k,s)}(a,c)$$

is mapped to 0 by ψ. If $\ell_2 > 0$, then

$$(\beta\gamma\alpha)^{n_1}\beta\gamma \cdot (\alpha\beta\gamma)^{\ell_2} = (\beta\gamma\alpha)^{n_1+\ell_2}\beta\gamma \in (\mathcal{B}_1 \cup \{0\}) \setminus \mathrm{soc}(A^{(k,s)}(a,c)$$

is mapped to 0 by ψ, whereas

$$(\alpha\beta\gamma)^{\ell_2} \cdot (\beta\gamma\alpha)^{n_1}\beta\gamma = 0.$$

Clearly e_1 commutes with $(\beta\gamma\alpha)^{n_1}\beta\gamma$. Now, trivially

$$(\beta\gamma\alpha)^{m_1} \cdot (\beta\gamma\alpha)^{m_2} = (\beta\gamma\alpha)^{m_2} \cdot (\beta\gamma\alpha)^{m_1}$$

and

$$(\alpha\beta\gamma)^{\ell_1} \cdot (\alpha\beta\gamma)^{\ell_2} = (\alpha\beta\gamma)^{\ell_2} \cdot (\alpha\beta\gamma)^{\ell_1}.$$

Finally, if $\ell_1 > 0$ then

$$(\alpha\beta\gamma)^{\ell_1} \cdot (\beta\gamma\alpha)^{m_2} = 0 = (\beta\gamma\alpha)^{m_2} \cdot (\alpha\beta\gamma)^{\ell_1}.$$

Case $(c_1, c_2) \in \mathcal{B}_1 \times \mathcal{B}_2$**:** Since $\mathcal{B}_1 \subseteq e_1A^{k,s}(a,c)e_1$, and since $\mathcal{B}_2 \subseteq e_2A^{k,s}(a,c)e_2$ we get $\psi(c_1c_2) = \psi(c_2c_1) = 0$ for $c_1 \in \mathcal{B}_1$ and $c_2 \in \mathcal{B}_2$.

Case $(c_1, c_2) \in \mathcal{B}_1 \times \mathcal{B}_3$**:** Since $\mathcal{B}_1 \subseteq e_1A^{k,s}(a,c)e_1$, and since $\mathcal{B}_3 \subseteq e_1A^{k,s}(a,c)e_2$ we get $c_1c_2 = 0$ and $c_2c_1 \in e_1A^{k,s}(a,c)e_3$ for $c_1 \in \mathcal{B}_3$ and $c_2 \in \mathcal{B}_1$. Non closed paths are mapped to 0 by ψ.

Case $(c_1, c_2) \in \mathcal{B}_1 \times \mathcal{B}_4$**:** Since $\mathcal{B}_4 \subseteq e_2A^{k,s}(a,c)e_1$ the same arguments as in the case $(c_1, c_2) \in \mathcal{B}_1 \times \mathcal{B}_3$ apply.

Case $(c_1, c_2) \in \mathcal{B}_2 \times \mathcal{B}_2$**:** Clearly $\eta^{t_1} \cdot \eta^{t_2} = \eta^{t_2} \cdot \eta^{t_1}$ and $(\gamma\alpha\beta)^{m_1} \cdot (\gamma\alpha\beta)^{m_2} = (\gamma\alpha\beta)^{m_2} \cdot (\gamma\alpha\beta)^{m_1}$. Moreover, if $t > 0$ then

$$\eta^t \cdot (\gamma\alpha\beta)^m = 0 = (\gamma\alpha\beta)^m \cdot \eta^t.$$

If $t = 0$, then trivially $\eta^t \cdot (\gamma\alpha\beta)^m = (\gamma\alpha\beta)^m \cdot \eta^t$.

Case $(c_1, c_2) \in \mathcal{B}_2 \times \mathcal{B}_3$**:** Since then c_1c_2 and c_2c_1 are non closed paths, the same arguments as in the case $(c_1, c_2) \in \mathcal{B}_1 \times \mathcal{B}_3$ apply.

Case $(c_1, c_2) \in \mathcal{B}_2 \times \mathcal{B}_4$**:** Again since then c_1c_2 and c_2c_1 are non closed paths, the same arguments as in the case $(c_1, c_2) \in \mathcal{B}_1 \times \mathcal{B}_3$ apply.

Case $(c_1, c_2) \in \mathcal{B}_3 \times \mathcal{B}_3$**:** Then $c_1c_2 = 0 = c_2c_1$.

Case $(c_1, c_2) \in \mathcal{B}_3 \times \mathcal{B}_4$**:**

$$(\beta\gamma\alpha)^{n_1}\beta \cdot (\gamma\alpha\beta)^{n_2}\gamma = (\beta\gamma\alpha)^{n_1+n_2}\beta\gamma \in (\mathcal{B}_1 \cup \{0\}) \setminus \mathrm{soc}(A^{(k,s)}(a, c)$$

is mapped to 0 by ψ, and

$$(\gamma\alpha\beta)^{n_2}\gamma \cdot (\beta\gamma\alpha)^{n_1}\beta = (\gamma\alpha\beta)^{n_2}(\gamma\beta)(\gamma\alpha\beta)^{n_1} = 0.$$

Now,

$$(\beta\gamma\alpha)^{n_1}\beta \cdot (\gamma\alpha\beta)^{n_2}\gamma\alpha - (\gamma\alpha\beta)^{n_2}\gamma\alpha \cdot (\beta\gamma\alpha)^{n_1}\beta = (\beta\gamma\alpha)^{n_1+n_2+1} - (\gamma\alpha\beta)^{n_1+n_2+1}.$$

and the value of ψ on each of the summands is equal.

$$\alpha(\beta\gamma\alpha)^{n_1}\beta \cdot (\gamma\alpha\beta)^{n_2}\gamma = (\alpha\beta\gamma)^{n_1+n_2+1}$$

and

$$(\gamma\alpha\beta)^{n_2}\gamma \cdot \alpha(\beta\gamma\alpha)^{n_1}\beta = (\gamma\alpha\beta)^{n_2+n_1+1}$$

both have identical values under ψ. Finally

$$\alpha(\beta\gamma\alpha)^{n_1}\beta \cdot (\gamma\alpha\beta)^{n_2}\gamma\alpha = (\alpha\beta\gamma)^{n_1+n_2+1}\alpha \in (\mathcal{B}_1 \cup \{0\}) \setminus \mathrm{soc}(A^{(k,s)}(a, c)$$

is mapped to 0 by ψ and

$$(\gamma\alpha\beta)^{n_2}\gamma\alpha \cdot \alpha(\beta\gamma\alpha)^{n_1}\beta = 0.$$

Case $(c_1, c_2) \in \mathcal{B}_4 \times \mathcal{B}_4$**:** Then $c_1c_2 = 0 = c_2c_1$.

Altogether this shows that ψ is symmetric. The fact that ψ defines a non degenerate form follows as in the proof of Proposition 2. For the reader's convenience we recall the short argument. Suppose that the form defined by ψ is degenerate. Then there is a $0 \neq x \in A^{k,s}(a, c)$ with $\psi(xy) = 0$ for all y, and since $1 = e_1 + e_2$ there is a primitive idempotent $e \in \{e_1, e_2\}$ of $A^{k,s}(a, c)$ such that we may suppose that $x \in eA^{k,s}(a, c)$. Let S be a simple submodule of $xA^{k,s}(a, c)$ and there is y such that $0 \neq s = xy \in S$. Since $S \leq eA^{k,s}(a, c)$ is one-dimensional, and included in the socle, and since $e\mathcal{B}e$ contains a basis of S we get $\psi(xy) \neq 0$. The form defined by ψ is trivially associative. Hence ψ defines a symmetrising form.

We come to the main body of the proof. Recall from Lemma 5 that $\dim_K(Z(A^{k,s}(a, c))) = k+s+2$. We proceed case by case.

$k > 1$ **odd and** $c = 0$: Recall that in this case

$$T_1(A^{k,s}(a, c))/[A^{k,s}(a, c), A^{k,s}(a, c)] = \left\langle \alpha, \eta^t, (\beta\gamma\alpha)^m \mid t > \frac{s}{2}; m > \frac{k-1}{2} \right\rangle_K.$$

Hence

$$\begin{aligned} \dim_K(T_1(A^{k,s}(a, 0))/[A^{k,s}(a, 0), A^{k,s}(a, 0)]) &= \begin{cases} 1 + \frac{s}{2} + \frac{k+1}{2} - 1 & \text{if } s \text{ is even} \\ 1 + \frac{s+1}{2} + \frac{k+1}{2} - 1 & \text{if } s \text{ is odd} \end{cases} \\ &= \begin{cases} \frac{s}{2} + \frac{k+1}{2} & \text{if } s \text{ is even} \\ \frac{s+1}{2} + \frac{k+1}{2} & \text{if } s \text{ is odd} \end{cases} \end{aligned}$$

observing that $(\beta\gamma\alpha)^k - \eta^s \in [A^{k,s}(a, c), A^{k,s}(a, c)]$. Therefore

$$\begin{aligned} \dim_K(T_1(A^{k,s}(a, 0))^\perp) &= k + s + 2 - \begin{cases} \frac{s}{2} + \frac{k+1}{2} & \text{if } s \text{ is even} \\ \frac{s+1}{2} + \frac{k+1}{2} & \text{if } s \text{ is odd} \end{cases} \\ &= \begin{cases} \frac{s}{2} + 1 + \frac{k+1}{2} & \text{if } s \text{ is even} \\ \frac{s+1}{2} + \frac{k+1}{2} & \text{if } s \text{ is odd} \end{cases} \end{aligned}$$

But, in case s is even,

$$\left\{\eta^t, (\alpha\beta\gamma)^u + (\beta\gamma\alpha)^u + (\gamma\alpha\beta)^u, (\alpha\beta\gamma)^k \mid u \geq \frac{k+1}{2}, t \geq \frac{s}{2}\right\} \subseteq T_1(A(a, c))^\perp,$$

and in case s is odd,

$$\left\{\eta^t, (\alpha\beta\gamma)^u + (\beta\gamma\alpha)^u + (\gamma\alpha\beta)^u, (\alpha\beta\gamma)^k \mid u \geq \frac{k+1}{2}, t \geq \frac{s+1}{2}\right\} \subseteq T_1(A(a, c))^\perp.$$

This is a basis of a subspace of the centre of the dimension as required, and hence the set above is a basis of $T_1(A^{k,s}(a,c))^\perp$. Hence, with these parameters, if s is even then

$$Z(A^{k,s}(a,c))/T_1^\perp(A^{k,s}(a,c)) \simeq K[U,Y,S]/(Y^{s/2}, U^{(k+1)/2}, S^2, YS, US, UY),$$

and if s is odd, then

$$Z(A^{k,s}(a,c))/T_1^\perp(A^{k,s}(a,c)) \simeq K[U,Y,S]/(Y^{(s+1)/2}, U^{(k+1)/2}, S^2, YS, US, UY).$$

$k>1$ **odd,** $c\neq 0$**, and** s **is odd**: Recall

$$T_1(A^{k,s}(a,c))/[A^{k,s}(a,c),A^{k,s}(a,c)] = \left\langle \eta^t, (\beta\gamma\alpha)^m \mid t>\frac{s}{2}; m>\frac{k-1}{2}\right\rangle_K.$$

In this case we get $\alpha^2\in T_1(A^{k,s}(a,c))^\perp$, and using the preceding discussion we get that

$$\left\{\alpha^2, (\beta\gamma\alpha)^k, \eta^t, (\alpha\beta\gamma)^u+(\beta\gamma\alpha)^u+(\gamma\alpha\beta)^u \mid u\geq \frac{k+1}{2}, t\geq\frac{s+1}{2}\right\}$$

is a K-basis of $T_1(A^{k,s}(a,c))^\perp$. Hence in this case

$$Z(A^{k,s}(a,c))/T_1^\perp(A^{k,s}(a,c)) \simeq K[U,Y]/(Y^{(s+1)/2}, U^{(k+1)/2}, UY).$$

$k>1$ **odd,** $d^2=c\neq 0$**, and** s **is even.** Recall

$$T_1(A^{k,s}(a,c))/[A^{k,s}(a,c),A^{k,s}(a,c)] = \left\langle d\alpha+\eta^{s/2}, \eta^t, (\beta\gamma\alpha)^m \mid t>\frac{s}{2}; m>\frac{k-1}{2}\right\rangle_K.$$

Then

$$\left(\eta^{s/2}+d\alpha\right)\cdot\left(\eta^{s/2}+\frac{d}{ca}\alpha^2\right) = \eta^s+\frac{d^2}{ca}\alpha^3 = \eta^s+\frac{c}{ca}\alpha^3 = \eta^s+\frac{1}{a}\cdot a(\alpha\beta\gamma)^k$$

and this is mapped to 0 by ψ. Hence,

$$\left\{\frac{d}{ca}\alpha^2+\eta^{s/2}, (\alpha\beta\gamma)^k, \eta^t, (\alpha\beta\gamma)^u+(\beta\gamma\alpha)^u+(\gamma\alpha\beta)^u \mid u\geq\frac{k-1}{2}, t\geq\frac{s}{2}+1\right\}$$

is a K-basis of $T_1(A^{k,s}(a,c))^\perp$. Therefore in this case

$$Z(A^{k,s}(a,c))/T_1^\perp(A^{k,s}(a,c)) \simeq K[U,Y]/(Y^{\frac{s}{2}+1}, U^{(k+1)/2}, UY).$$

k **even and** $c=0$ **and** s **is odd**. Recall

$$T_1(A^{k,s}(a,c))/[A^{k,s}(a,c),A^{k,s}(a,c)] = \left\langle d\alpha+(\beta\gamma\alpha)^{k/2}, \eta^t, (\beta\gamma\alpha)^m \mid t>\frac{s-1}{2}; m>\frac{k}{2}\right\rangle_K.$$

Then the discussion of the case $k>1$ odd and $c=0$ shows that

$$\left\{\eta^t, (\alpha\beta\gamma)^k, (\alpha\beta\gamma)^u+(\beta\gamma\alpha)^u+(\gamma\alpha\beta)^u \mid u\geq\frac{k}{2}, t\geq\frac{(s+1)}{2}\right\}$$

is a K-basis of $T_1(A^{k,s}(a,c))^\perp$. Hence in this case

$$Z(A^{k,s}(a,c))/T_1^\perp(A^{k,s}(a,c)) \simeq K[U,Y,S]/(Y^{(s+1)/2}, U^{k/2}, S^2, YS, US, UY).$$

k **even and** $c=0$ **and** s **is even**. Recall

$$T_1(A^{k,s}(a,c))/[A^{k,s}(a,c),A^{k,s}(a,c)] = \left\langle \alpha, \eta^{s/2}+(\beta\gamma\alpha)^{k/2}, \eta^t, (\beta\gamma\alpha)^m \mid t>\frac{s}{2}; m>\frac{k}{2}\right\rangle_K.$$

Then

$$\left\{(\alpha\beta\gamma)^k, \eta^t, (\alpha\beta\gamma)^u+(\beta\gamma\alpha)^u+(\gamma\alpha\beta)^u, \eta^{s/2}+(\beta\gamma\alpha)^{k/2} \mid u\geq\frac{k}{2}+1, t\geq\frac{s}{2}+1\right\}$$

is a K-basis of $T_1(A^{k,s}(a,c))^\perp$. Hence in this case

$$Z(A^{k,s}(a,c))/T_1^\perp(A^{k,s}(a,c)) \simeq K[U,Y,S]/(Y^{s/2}-U^{k/2}, S^2, YS, US, UY).$$

k **even,** $0\neq c=d^2$**,** s **odd:** Recall

$$T_1(A^{k,s}(a,c))/[A^{k,s}(a,c),A^{k,s}(a,c)] = \left\langle d\alpha+(\beta\gamma\alpha)^{k/2}, \eta^t, (\beta\gamma\alpha)^m \mid t>\frac{s-1}{2}; m>\frac{k}{2}\right\rangle_K.$$

Since $\dim(Z(A^{k,s}(a,c))) = k+s+2$, and since

$$\dim(T_1(A^{k,s}(a,c)))/[A^{k,s}(a,c),A^{k,s}(a,c)] = \frac{k}{2}+\frac{s+1}{2},$$

we get

$$\dim(T_1(A^{k,s}(a,c))^{\perp}) = 2+k+s-\frac{k}{2}-\frac{s+1}{2} = \frac{k}{2}+\frac{s+1}{2}+1.$$

Moreover,

$$\left(\frac{d}{c}\alpha^2+U^{k/2}\right)\cdot\left(d\alpha+(\beta\gamma\alpha)^{k/2}\right) = \frac{d^2}{c}\alpha^3+dU^{k/2}\alpha+\frac{d}{c}(\beta\gamma\alpha)^{k/2}+(\beta\gamma\alpha)^k$$

and this maps to 0 by the map ψ. Therefore

$$\left\{(\beta\gamma\alpha)^k, \frac{d}{c}\alpha^2+(\alpha\beta\gamma)^{k/2}+(\beta\gamma\alpha)^{k/2}+(\gamma\alpha\beta)^{k/2}, \eta^t, (\alpha\beta\gamma)^u+(\beta\gamma\alpha)^u+(\gamma\alpha\beta)^u \mid u\geq \frac{k}{2}+1, t\geq\frac{s+1}{2}\right\}$$

is a K-basis of $T_1(A^{k,s}(a,c))^{\perp}$, and therefore

$$Z(A^{k,s}(a,c))/T_1(A^{k,s}(a,c))^{\perp} \simeq K[U,Y]/(U^{k/2},Y^{(s+1)/2},UY).$$

k **even,** $c=d^2\neq 0$**, and** s **even:** Recall

$$T_1(A^{k,s}(a,c))/[A^{k,s}(a,c),A^{k,s}(a,c)] = \left\langle d\alpha+\eta^{s/2}, (\beta\gamma\alpha)^{k/2}+\eta^{s/2}, \eta^t, (\beta\gamma\alpha)^m \mid t>\frac{s}{2}; m>\frac{k}{2}\right\rangle_K.$$

Then

$$\left\{\alpha^2+d\eta^{s/2}, (\beta\gamma\alpha)^k, \eta^t, (\alpha\beta\gamma)^u+(\beta\gamma\alpha)^u+(\gamma\alpha\beta)^u, \eta^{s/2}+(\beta\gamma\alpha)^{k/2} \mid u\geq\frac{k}{2}+1, t\geq\frac{s}{2}+1\right\}$$

is a K-basis of $T_1(A^{k,s}(a,c))^{\perp}$. Hence in this case

$$Z(A^{k,s}(a,c))/T_1^{\perp}(A^{k,s}(a,c)) \simeq K[U,Y]/(Y^{s/2}-U^{k/2},UY).$$

If $k=1$ **and** $c=0$ **and** s **odd:** Since $\dim(Z(A^{1,s}(a,c)) = 3+s$, and since

$$\dim(T_1(A^{1,s}(a,c))/[A^{1,s}(a,c),A^{1,s}(a,c)]) = 3+\frac{s-1}{2},$$

we obtain $\dim(T_1(A^{1,s}(a,c))^{\perp}) = \frac{s+1}{2}$. Observe that $\eta^s = (\alpha\beta\gamma)+(\beta\gamma\alpha)+(\gamma\alpha\beta) = U$. Then we get

$$\left\{\beta\gamma\alpha, \eta^t, \mid t\geq\frac{s+1}{2}\right\}$$

is a K-basis of $T_1(A^{1,s}(a,c))^{\perp}$. Therefore

$$Z(A^{1,s}(a,c))/T_1^{\perp}(A^{1,s}(a,c)) \simeq K[Y,S]/(Y^{(s+1)/2},S^2,YS).$$

If $k=1$ **and** $c=0$ **and** s **even:** Since $\dim(Z(A^{1,s}(a,c)) = 3+s$, and since

$$\dim(T_1(A^{1,s}(a,c))/[A^{1,s}(a,c),A^{1,s}(a,c)]) = 1+\frac{s}{2},$$

we obtain $\dim(T_1(A^{1,s}(a,c))^{\perp}) = 2+\frac{s}{2}$. Hence

$$\left\{\alpha^2, \beta\gamma\alpha, \eta^t, \mid t\geq\frac{s}{2}\right\}$$

is a K-basis of $T_1(A^{1,s}(a,c))^{\perp}$ and

$$Z(A^{1,s}(a,c))/T_1^{\perp}(A^{1,s}(a,c)) \simeq K[Y]/Y^{s/2}.$$

If $k=1$ **and** $c\neq 0$ **and** s **odd:** Since $\dim(Z(A^{1,s}(a,c)) = 3+s$, and since

$$\dim(T_1(A^{1,s}(a,c))/[A^{1,s}(a,c),A^{1,s}(a,c)]) = 2+\frac{s-1}{2},$$

we obtain $\dim(T_1(A^{1,s}(a,c))^{\perp}) = 1+\frac{s+1}{2}$. Hence

$$\left\{\alpha^2, \beta\gamma\alpha, \eta^t, \mid t\geq\frac{s+1}{2}\right\}$$

is a K-basis of $T_1(A^{1,s}(a,c))^\perp$ and

$$Z(A^{1,s}(a,c))/T_1^\perp(A^{1,s}(a,c)) \simeq K[Y]/Y^{(s+1)/2}.$$

If $k=1$ and $c\neq 0$ and s even: Since $\dim(Z(A^{1,s}(a,c)) = 3+s$, and since

$$\dim(T_1(A^{1,s}(a,c))/[A^{1,s}(a,c),A^{1,s}(a,c)]) = 2+\frac{s}{2},$$

we obtain $\dim(T_1(A^{1,s}(a,c))^\perp) = 1+\frac{s}{2}$. Hence

$$\left\{\beta\gamma\alpha, \eta^t, \mid t\geq 1+\frac{s}{2}\right\}$$

is a K-basis of $T_1(A^{1,s}(a,c))^\perp$ and

$$Z(A^{1,s}(a,c))/T_1^\perp(A^{1,s}(a,c)) \simeq K[Y,S]/(Y^{(s+2)/2},S^2,YS).$$

■

In order to be more concise we summarise the results from Theorem 7 and Remark 4 in case K is algebraically closed in the following corollary.

Corollary 10. *Let K be an algebraically closed field of characteristic $p\in\mathbb{N}\cup\{\infty\}$, and let a,a',c be non-zero elements in K, and let $c',c''\in K$.*

- *If $p\neq 2$, then there is $a'\in K^\times$ such that $Q(2\mathfrak{B})_1^{k,s}(a,c)\simeq Q(2\mathfrak{B})_1^{k,s}(a',0)$, and if $(k,s)\neq(1,3)$, then $Q(2\mathfrak{B})_1^{k,s}(a,c)\simeq Q(2\mathfrak{B})_1^{k,s}(1,0)$.*
- *If $p=2$, then $D^b(Q(2\mathfrak{B})_1^{k,s}(a,c))\not\simeq D^b(Q(2\mathfrak{B})_1^{k,s}(a',0))$. Moreover, there is $a''\in K^\times$ such that $Q(2\mathfrak{B})_1^{k,s}(a,c)\simeq Q(2\mathfrak{B})_1^{k,s}(a'',1)$ and if $(k,s)\neq(1,3)$, then $Q(2\mathfrak{B})_1^{k,s}(a',0)\simeq Q(2\mathfrak{B})_1^{k,s}(1,0)$.*
- *For any characteristic of K we get*

$$\left(D^b(Q(2\mathfrak{B})_1^{k,s}(a,c''))\simeq D^b(Q(2\mathfrak{B})_1^{k',s'}(a',c'))\right)\Rightarrow((k=k' \text{ and } s=s') \text{ or } (k=s' \text{ and } s=k').)$$

Proof. The first statement is an immediate consequence of Theorem 7 item (3) and [15, Lemma 5.7]. The second statement follows from Theorem 7 item (2)(a), (2)(b), (2)(c), and Theorem 1. Indeed, the isomorphism type of the centre modulo the first Külshammer ideal differs in case $c=0$ and $c\neq 0$. More precisely, the commutative algebras from case (2)(a) (i) and (2)(a) (ii) are non isomorphic since the dimensions of the socles of these algebras differ by 1. Likewise, the commutative algebras from case (2)(b) (i) and (2)(b) (ii) are non isomorphic since the dimensions of the socles of these algebras differ by 1. The dimension of the socle of the centre modulo the Külshammer ideal distinguish the algebras also in case (2)(c), i.e. $k=1$. The third statement follows from Remark 4.
■

Remark 11. Let K be an algebraically closed field of characteristic 2. We do not know for which pair of parameters $a,a'\in K^\times$ we get that $Q(2\mathfrak{B})_1^{k,s}(a',1)$ is derived equivalent to $Q(2\mathfrak{B})_1^{k,s}(a,1)$. We do not know for which parameters k,s the algebras $Q(2\mathfrak{B})_1^{k,s}(a,c)$ and $Q(2\mathfrak{B})_1^{s,k}(a,c)$ are derived equivalent.

Remark 12. The case $p=3$ is special if K is not perfect. Let $p=3$ and use the notations used in the proof of Theorem 7. Then $\alpha^3=a(\beta\gamma\alpha)^m$. In the commutator quotient taking third power is semilinear (cf e.g. [12],[22, Lemma 2.9.3]), and therefore

$$\begin{aligned} 0 &= \left(\sum_{t=1}^{s-1}x_t\eta^t+u\alpha+\sum_{m=1}^{k}y_m(\beta\gamma\alpha)^m\right)^3 \\ &= \sum_{t=1}^{s-1}x_t^3\eta^{3t}+u^3a(\beta\gamma\alpha)^k+\sum_{m=1}^{k}y_m^3(\beta\gamma\alpha)^{3m} \\ &= \sum_{1\leq t\leq(s-1)/3}x_t^3\eta^{3t}+u^3a(\beta\gamma\alpha)^k+\sum_{1\leq m\leq k/3}y_m^3(\beta\gamma\alpha)^{3m}. \end{aligned}$$

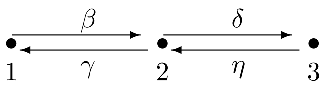
We have again various cases. If 3 does not divide k and 3 does not divide s, then $x_t=0$ for all $t\leq s/3$ and $y_m=0$ for all $m\leq k/3$ and $u=0$. If 3 does not divide k but $3|s$, then $x_t=0$ for all

$t < s/3$ and $y_m = 0$ for all $m \leq k/3$ and $x_{s/3}^3 + au^3 = 0$, which has a non zero solution if and only if a is a cube. If 3 divides k and 3 does not divide s, then $x_t = 0$ for all $t \leq s/3$ and $y_m = 0$ for all $m < k/3$ and $y_{m/3}^3 + au^3 = 0$, which has a non zero solution if and only if a is a cube. If 3 divides k and 3 divides s, then $x_t = 0$ for all $t < s/3$ and $y_m = 0$ for all $m < k/3$ and $x_{s/3}^3 + y_{m/3}^3 + au^3 = 0$, which has a non zero solution if and only if a is a cube.

As seen above, the first Külshammer ideal detects if the parameter a is a third power in case k or s is divisible by 3. This shows that the isomorphism $A^{k,s}(a, 0) \simeq A^{k,s}(1, 0)$, which we proved for algebraically closed base fields, is false if the base field is not perfect.

2.2. **Three simple modules.** Holm shows that there are two families of algebras $Q(3\mathcal{K})^{a,b,c}$ and $Q(3\mathcal{A})_1^{2,2}(d)$ with three simple modules such that any block with quaternion defect group and three simple modules is derived equivalent to an algebra in one of these families. According to [7] the derived classification of the case $Q(3\mathcal{K})^{a,b,c}$ is complete, whereas the classification for the case $Q(3\mathcal{A})_1^{2,2}(d)$ is complete up to the scalar $d \in K \setminus \{0, 1\}$.

The quiver $3\mathcal{A}$ is

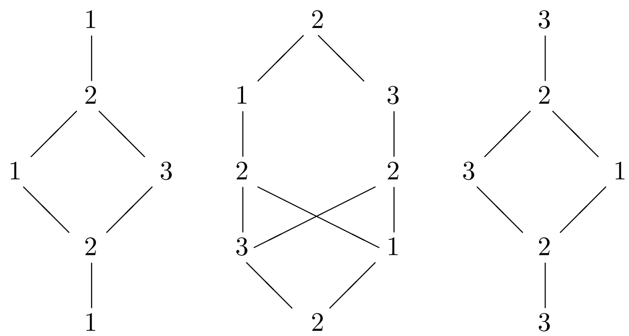


$B(d) := Q(3\mathcal{A})_1^{2,2}(d)$ is the quiver algebra of $3\mathcal{A}$ modulo the relations

$$\beta\delta\eta = \beta\gamma\beta, \quad \delta\eta\gamma = \gamma\beta\gamma, \quad \eta\gamma\beta = d \cdot \eta\delta\eta, \quad \gamma\beta\delta = d \cdot \delta\eta\delta, \quad \beta\delta\eta\delta = 0, \quad \eta\gamma\beta\gamma = 0$$

for $d \in K \setminus \{0, 1\}$.

Following [4] the Cartan matrix of $B(d)$ is $\begin{pmatrix} 4 & 2 & 2 \\ 2 & 3 & 1 \\ 2 & 1 & 3 \end{pmatrix}$ and the centre is 6-dimensional. The Loewy series of the projective indecomposable modules are given below.



We obtain a basis of the socle of $B(d)$ by

$$\{s_1 := \beta\delta\eta\gamma, s_2 := \eta\gamma\beta\delta, s_3 := \gamma\beta\delta\eta\}.$$

The closed paths of the algebra are

$$\{e_0, e_1, e_2, \beta\gamma, \gamma\beta, \delta\eta, \eta\delta, \beta\delta\eta\gamma, \eta\gamma\beta\delta, \gamma\beta\delta\eta\}.$$

The centre is formed by linear combinations of closed paths and has a basis

$$\{1, \beta\gamma + \gamma\beta + \frac{1}{d}\eta\delta, \beta\gamma + \delta\eta + \eta\delta, \beta\delta\eta\gamma, \eta\gamma\beta\delta, \gamma\beta\delta\eta\}$$

as is easily verified. Non closed paths are clearly commutators. Obviously

$$\beta\delta\eta\gamma \equiv \eta\gamma\beta\delta \equiv \gamma\beta\delta\eta \bmod [B(d), B(d)].$$

Moreover,

$$\beta\gamma - \gamma\beta \in [B(d), B(d)] \text{ and } \delta\eta - \eta\delta \in [B(d), B(d)].$$

Since the dimension of the centre of $B(d)$ coincides with the dimension of the commutator quotient, we get a basis of $B(d)/[B(d), B(d)]$ by

$$\{e_0, e_1, e_2, \beta\gamma, \delta\eta, \beta\delta\eta\gamma\}.$$

We now suppose that the characteristic p of K is $p = 2$. If x is a square in K, then denote $y = \sqrt{x}$ if $y^2 = x$. We compute

$$(\beta\gamma)^2 = \beta\gamma\beta\gamma = \beta\delta\eta\gamma \text{ and } (\delta\eta)^2 = \delta\eta\delta\eta = \frac{1}{d}\delta\eta\gamma\beta.$$

If d is a square in K, then

$$((\gamma\beta) + \sqrt{d}(\eta\delta))^2 = \gamma\eta\gamma\beta + d\eta\delta\eta\delta = \delta\eta\gamma\beta + \eta\gamma\beta\delta \in [B(d), B(d)]$$

so that $T_1(B(d))/[B(d), B(d)]$ is 1-dimensional. If d is not a square, then $T_1(B(d)) = [B(d), B(d)]$.

Let us consider the centre. Denote $\beta\gamma + \gamma\beta + \frac{1}{d}\eta\delta = x$ and $\beta\gamma + \delta\eta + \eta\delta = y$. Then we get

$$\begin{aligned} x^2 = (\beta\gamma + \gamma\beta + \frac{1}{d}\eta\delta)^2 &= \beta\delta\eta\gamma + \frac{1}{d}\delta\eta\gamma\beta + \frac{1}{d}\eta\gamma\beta\delta = s_1 + \frac{1}{d}s_2 + \frac{1}{d}s_2, \\ y^2 = (\beta\gamma + \delta\eta + \eta\delta)^2 &= \beta\delta\eta\gamma + \delta\eta\gamma\beta + \frac{1}{d^3}\eta\gamma\beta\delta = s_1 + s_2 + \frac{1}{d^3}s_3, \\ xy = (\beta\gamma + \delta\eta + \eta\delta) \cdot (\beta\gamma + \gamma\beta + \frac{1}{d}\eta\delta) &= \beta\delta\eta\gamma + \delta\eta\gamma\beta + \frac{1}{d^2}\eta\gamma\beta\delta = s_1 + s_2 + \frac{1}{d^2}s_3. \end{aligned}$$

The coefficient matrix above has determinant $\frac{(d-1)^2}{d^4}$ and since $d \neq 1$, the elements x^2, y^2, xy are linearly independent, and hence $Z(B(d)) \simeq K[x, y]/(x^3, y^3, x^2y, xy^2)$. Moreover, choose the Frobenius form given by

$$\psi(\beta\delta\eta\gamma) = \psi(\delta\eta\gamma\beta) = \psi(\eta\gamma\beta\delta) = 1 \text{ and } \psi(c) = 0 \text{ if } c \text{ is a path of length at most 3,}$$

following Remark 3. The relations are homogeneous, which shows that in order to prove symmetry of the form we only need to consider paths c_1 and c_2 such that the lengths of c_1 and c_2 sum up to 4. The verification is a trivial and short computation which can be left to the reader.

Suppose now that K is a perfect field. An elementary computation gives that $T_1^\perp(B(d))$ has a basis $\{x, s_1, s_2, s_3\}$, and therefore $Z(B(d))/T_1^\perp(B(d)) \simeq K[y]/y^2$, independently of d.

Theorem 13. *Let K be a field of characteristic 2, and let $B(d)$ be the algebra $Q(3\mathcal{A})_1^{2,2}(d)$. Then* $\dim_K(T_1^\perp(B(d))/R(B(d))) = 1$ *if d is a square in K, and* $\dim_K(T_1^\perp(B(d))/R(B(d))) = 0$ *if d is not a square in K.*

Proof: is done above. ■

Remark 14. Unlike in case of Theorem 7 and its Corollary 10, using Külshammer ideals we cannot distinguish the derived category of $Q(3\mathcal{A})_1^{2,2}(d)$ from the derived category of $Q(3\mathcal{A})_1^{2,2}(d')$ for two parameters d, d'. If K is perfect of characteristic 2, then all elements of K are squares. Theorem 1 needs that K is perfect for the invariance of Külshammer ideals under derived equivalences and K is even algebraically closed for the invariance under stable equivalences of Morita type. We can only say that the algebra $Q(3\mathcal{A})_1^{2,2}(d)$ is not isomorphic to the algebra $Q(3\mathcal{A})_1^{2,2}(d')$ if d is a square and d' is not.

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