

# GENERALIZED REYNOLDS IDEALS FOR NON-SYMMETRIC ALGEBRAS

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ABSTRACT. We show how to extend the theory of generalized Reynolds ideals, as introduced by B. Külshammer, from symmetric algebras to arbitrary finite-dimensional algebras (in positive characteristic). This provides new invariants of the derived module categories of finite-dimensional algebras, extending recent results in [6] for symmetric algebras.

## 1. INTRODUCTION

Let  $\Lambda$  be a finite-dimensional algebra over a field  $k$  of characteristic  $p > 0$ . The commutator space  $K(\Lambda)$  is the  $k$ -vector space generated by all  $[a, b] := ab - ba$  where  $a, b \in \Lambda$ . For any  $n \geq 0$  set  $T_n(\Lambda) := \{x \in \Lambda \mid x^{p^n} \in K(\Lambda)\}$ . In [3], B. Külshammer defined for any symmetric  $k$ -algebra  $\Lambda$  the generalized Reynolds ideals  $T_n\Lambda^\perp$  as the orthogonal spaces with respect to the symmetrizing form on the symmetric algebra  $\Lambda$ . For an arbitrary finite-dimensional algebra  $\Lambda$ , its trivial extension  $\mathbf{T}(\Lambda)$  is a symmetric algebra. For further details, definitions and background on generalized Reynolds ideals and on trivial extensions we refer to Sections 2 and 3 below.

The following are the main results of this note. They extend the main result of [6] from symmetric algebras to arbitrary finite-dimensional algebras.

**Theorem 1.1.** *Let  $\Lambda$  be a finite-dimensional algebra over a field of characteristic  $p > 0$ . Let  $\mathbf{T}(\Lambda) = \Lambda \ltimes \Lambda^*$  be the trivial extension. Then the series of generalized Reynolds ideals  $T_i(\mathbf{T}(\Lambda))^\perp$  of the trivial extension takes the form*

$$Z(\Lambda) \ltimes \text{Ann}_{\Lambda^*}(K(\Lambda)) \supseteq 0 \ltimes \text{Ann}_{\Lambda^*}(T_1\Lambda) \supseteq 0 \ltimes \text{Ann}_{\Lambda^*}(T_2\Lambda) \supseteq \dots$$

By a result of Rickard [5, Theorem 3.1], derived equivalent algebras have derived equivalent trivial extension algebras. As a consequence of this fact and our theorem above we obtain some new numerical invariants of the derived module categories of finite-dimensional algebras in positive characteristic.

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**Corollary 1.2.** *Let  $\Lambda$  be a finite-dimensional algebra over a perfect field of characteristic  $p > 0$ . Then, for any  $n \geq 0$ , the codimension  $\dim \Lambda - \dim T_n(\Lambda)$  is invariant under derived equivalences.*

Note that the case  $i = 0$  gives the dimension of  $\Lambda/K(\Lambda)$  which is isomorphic to the degree zero Hochschild homology; the latter is well-known to be invariant under derived equivalences. But for  $i > 0$  the above codimensions seem to be new derived invariants, and do not seem to have an interpretation in terms of other well-known invariants.

We can even prove the following more general and precise statement, involving the structure as modules over the center.

**Corollary 1.3.** *Let  $\Lambda$  and  $\Gamma$  be finite dimensional  $k$ -algebras over a perfect field  $k$  of characteristic  $p > 0$ . If the bounded derived categories of  $\Lambda$  and of  $\Gamma$  are equivalent as triangulated categories then the isomorphism induced between  $Z(\Lambda)$  and  $Z(\Gamma)$  by a twosided tilting complex induces an isomorphism of the sequence of  $Z(\Lambda)$ -modules*

$$\mathrm{Ann}_{\Lambda^*}(T_1(\Lambda)) \supseteq \mathrm{Ann}_{\Lambda^*}(T_2(\Lambda)) \supseteq \mathrm{Ann}_{\Lambda^*}(T_3(\Lambda)) \supseteq \cdots$$

and the sequence of  $Z(\Gamma)$ -modules

$$\mathrm{Ann}_{\Gamma^*}(T_1(\Gamma)) \supseteq \mathrm{Ann}_{\Gamma^*}(T_2(\Gamma)) \supseteq \mathrm{Ann}_{\Gamma^*}(T_3(\Gamma)) \supseteq \cdots$$

The paper is organized as follows. In Section 2 we recall the definition and some known facts on generalized Reynolds ideals for symmetric algebras. In Section 3 we collect several properties of trivial extensions which are crucial for our purposes. Section 4 is the core part of this note, containing the proofs of our main results.

## 2. GENERALIZED REYNOLDS IDEALS

In this section we briefly recall the definition of the sequence of generalized Reynolds ideals as introduced by B. Külshammer [3], [4]. For interesting recent developments on this invariant we also refer to [1], [2], [6], [7].

Let  $k$  a perfect field of characteristic  $p > 0$ . Let  $A$  be a finite-dimensional symmetric  $k$ -algebra with associative, symmetric, nondegenerate  $k$ -bilinear form  $\langle -, - \rangle : A \times A \rightarrow k$ . For any subspace  $M$  of  $A$  we denote by  $M^\perp$  the orthogonal space of  $M$  in  $A$  with respect to the form  $\langle -, - \rangle$ . Moreover, let  $K(A)$  be the  $k$ -subspace of  $A$  generated by all commutators  $[a, b] := ab - ba$ ,  $a, b \in A$ . For any  $n \geq 0$  set

$$T_n(A) = \{x \in A \mid x^{p^n} \in K(A)\}.$$

Then, by [3], for any  $n \geq 0$ , the orthogonal space  $T_n(A)^\perp$  is an ideal of the center  $Z(A)$  of  $A$ . These ideals are called *generalized Reynolds ideals*. They form a descending sequence

$$Z(A) = K(A)^\perp = T_0(A)^\perp \supseteq T_1(A)^\perp \supseteq T_2(A)^\perp \supseteq T_3(A)^\perp \supseteq \cdots$$

In fact, B. Külshammer showed in [3, 4] that there is a mapping  $\xi_n : Z(A) \rightarrow Z(A)$  so that the equation

$$\langle \xi_n(z), x \rangle^{p^n} = \langle z, x^{p^n} \rangle$$

holds for any  $z \in Z(A)$  and  $x \in A/K(A)$ . Moreover, he proved that  $\xi_n(Z(A)) = T_n(A)^\perp$ . Dually, for every  $n \geq 1$ , B. Külshammer proved in [3, 4] that the equation

$$\langle z, \kappa_n(x) \rangle^{p^n} = \langle z^{p^n}, x \rangle$$

for any  $z \in Z(A)$  and  $x \in A/K(A)$  defines a mapping  $\kappa_n : A/K(A) \rightarrow A/K(A)$ . Considering the kernel and the image of these maps leads to the following invariants,

$$\ker \kappa_n = \{ x \in A/K(A) \mid \langle z^{p^n}, x \rangle = 0 \ \forall z \in Z(A) \} =: P_n(Z(A))^\perp / K(A)$$

and

$$\begin{aligned} \kappa_n(A/K(A)) &= T_n(Z(A))^\perp / K(A) \\ &= \{ x \in A/K(A) \mid \forall z \in Z(A) : z^{p^n} = 0 \Rightarrow \langle z, x \rangle = 0 \}. \end{aligned}$$

In [2] it has been shown that the sequence of generalized Reynolds ideals is invariant under Morita equivalences. More generally, the following theorem has been proven recently by the third author.

**Proposition 2.1** ([6], Theorem 1; [7] Proposition 2.3 and Corollary 2.4). *Let  $A$  and  $B$  be finite-dimensional symmetric algebras over a perfect field of positive characteristic  $p$ . If  $A$  and  $B$  are derived equivalent, then*

- (i) *there is an isomorphism  $\varphi : Z(A) \rightarrow Z(B)$  between the centers of  $A$  and  $B$  such that  $\varphi(T_n(A)^\perp) = T_n(B)^\perp$  for all positive integers  $n$ ;*
- (ii) *there is an isomorphism  $\psi : A/K(A) \rightarrow B/K(B)$  so that*

$$P_n(Z(A))^\perp / K(A) \text{ is mapped by } \psi \text{ to } P_n(Z(B))^\perp / K(B)$$

*and so that*

$$T_n(Z(A))^\perp / K(A) \text{ is mapped by } \psi \text{ to } T_n(Z(B))^\perp / K(B).$$

We note that in the proof of [6, Theorem 1] the fact that  $k$  is algebraically closed is never used. The assumption on the field  $k$  to be perfect is sufficient. Hence the sequence of generalized Reynolds ideals gives a new derived invariant for symmetric algebras over perfect fields of positive characteristic. The aim of this note is to show how one can extend this result from symmetric to arbitrary finite-dimensional algebras, using trivial extensions.

### 3. TRIVIAL EXTENSIONS

Let  $\Lambda$  be a finite-dimensional algebra over a field  $k$ . We denote by  $\Lambda^*$  the  $k$ -linear dual  $\text{Hom}_k(\Lambda, k)$  which becomes a  $\Lambda$ - $\Lambda$ -bimodule by setting  $(a\varphi)(b) = \varphi(ba)$  and  $(\varphi a)(b) = \varphi(ab)$  for all  $a, b \in \Lambda$  and all  $\varphi \in \Lambda^*$ .

The *trivial extension*  $\mathbf{T}(\Lambda) := \Lambda \ltimes \Lambda^*$  is the  $k$ -algebra defined by the multiplication

$$(a, \varphi) \cdot (b, \psi) := (ab, a\psi + \varphi b) \quad \text{for all } a, b \in \Lambda, \varphi, \psi \in \Lambda^*.$$

Recall that an algebra  $A$  is symmetric if there exists a  $k$ -linear map  $\pi : A \rightarrow k$  such that  $\pi(ab) = \pi(ba)$  for all  $a, b \in A$ , and such that the kernel of  $\pi$  does not contain any nonzero left or right ideals of  $A$ . The corresponding associative non-degenerate symmetric  $k$ -bilinear form  $\langle -, - \rangle$  on  $A$  is then given by

$$\langle a, b \rangle = \pi(ab) \quad \text{for } a, b \in A.$$

Then we have the following well-known fact, which is the crucial property of trivial extensions in our context.

**Proposition 3.1.** *The trivial extension  $\mathbf{T}(\Lambda)$  is a symmetric algebra, with respect to the map  $\pi : \mathbf{T}(\Lambda) \rightarrow k$ ,  $(a, \varphi) \mapsto \varphi(1)$ .*

*Proof.* Clearly, for any  $a, b \in \Lambda$  and  $\varphi, \psi \in \Lambda^*$  we have

$$\pi((a, \varphi) \cdot (b, \psi)) = \psi(a) + \varphi(b) = \pi((b, \psi) \cdot (a, \varphi)).$$

Now let  $I \triangleleft \mathbf{T}(\Lambda)$  be a left ideal contained in the kernel of  $\pi$ . Let  $(b, \psi) \in I$ . Since  $I$  is a left ideal of  $\mathbf{T}(\Lambda)$ , we get for all  $a \in \Lambda$  and all  $\varphi \in \Lambda^*$  that

$$(a, \varphi) \cdot (b, \psi) = (ab, a\psi + \varphi b) \in \ker \pi,$$

i.e.,  $\psi(a) + \varphi(b) = 0$  for all  $a \in \Lambda$ ,  $\varphi \in \Lambda^*$ . In particular, by setting  $\varphi = 0$  we conclude that  $\psi(a) = 0$  for all  $a \in \Lambda$ , and hence  $\psi = 0$ . Then, also  $\varphi(b) = 0$  for all  $\varphi \in \Lambda^*$ , and hence  $b = 0$  as well. Therefore,  $I = 0$ .

Similarly,  $\ker \pi$  does not contain a nonzero right ideal.  $\square$

We now collect some easy fundamental properties of trivial extensions which will be used later in the proof of the main result.

For any subspace  $V \leq \Lambda$  we set

$$\text{Ann}_{\Lambda^*}(V) := \{\varphi \in \Lambda^* \mid \varphi(V) = 0\},$$

i.e., those linear maps which vanish on  $V$ .

**Proposition 3.2.** *Let  $\Lambda$  be a finite-dimensional algebra over a field  $k$ , with center  $Z(\Lambda)$  and commutator subspace  $K(\Lambda)$ . Then the center of the trivial extension  $\mathbf{T}(\Lambda) = \Lambda \ltimes \Lambda^*$  has the form*

$$Z(\mathbf{T}(\Lambda)) = Z(\Lambda) \ltimes \text{Ann}_{\Lambda^*}(K(\Lambda)).$$

*Proof.* Let  $(a, \varphi) \in Z(\mathbf{T}(\Lambda))$ . Clearly,  $a \in Z(\Lambda)$ . Moreover, we then have

$$a\psi + \varphi b = b\varphi + \psi a \quad \text{for all } b \in \Lambda, \psi \in \Lambda^*.$$

The latter equality holds if and only if for all  $c \in \Lambda$  we have

$$\psi(ca) + \varphi(bc) = \varphi(cb) + \psi(ac).$$

Since  $a \in Z(\Lambda)$ , this holds precisely when  $\varphi(bc) = \varphi(cb)$  for all  $b, c \in \Lambda$ , i.e., when  $\varphi(K(\Lambda)) = 0$ , as claimed.  $\square$

**Proposition 3.3.** *Let  $\Lambda$  be a finite-dimensional algebra over a field  $k$  of characteristic  $p > 0$ , and let  $\mathbf{T}(\Lambda) = \Lambda \ltimes \Lambda^*$  be the trivial extension. Then for any  $n \geq 0$  we get*

$$T_n(\mathbf{T}(\Lambda)) = T_n(\Lambda) \ltimes \Lambda^*.$$

*Proof.* Recall that  $T_n(\mathbf{T}(\Lambda)) = \{(a, \varphi) \in \mathbf{T}(\Lambda) \mid (a, \varphi)^{p^n} \in K(\mathbf{T}(\Lambda))\}$ . We have  $(a, \varphi)^{p^n} = ((a, 0) + (0, \varphi))^{p^n}$  which modulo the commutator subspace  $K(\mathbf{T}(\Lambda))$  becomes

$$(a, \varphi)^{p^n} \equiv (a, 0)^{p^n} + (0, \varphi)^{p^n} = (a^{p^n}, 0)$$

using that  $(0 \ltimes \Lambda^*)^2 = 0$ . In particular,  $(a, \varphi)^{p^n}$  is in  $K(\mathbf{T}(\Lambda))$  if and only if  $a^{p^n} \in K(\Lambda)$ , i.e.  $a \in T_n(\Lambda)$ , as claimed.  $\square$

We need a final preparation for the main result, namely an explicit description for the commutator subspace  $K(\mathbf{T}(\Lambda))$  of a trivial extension.

**Lemma 3.4.** *Let  $\Lambda$  be a finite-dimensional algebra over a field. Then the following holds.*

- (1)  $K(\mathbf{T}(\Lambda)) = K(\Lambda) \ltimes [\Lambda, \Lambda^*]$  where  $[\Lambda, \Lambda^*]$  denotes the commutator subspace of  $\Lambda^*$  of the algebra  $\Lambda$  acting on the bimodule  $\Lambda^*$ .
- (2) Assume that  $\Lambda$  is symmetric, with symmetrizing form  $\langle -, - \rangle$ .
  - (a) The space  $[\Lambda, \Lambda^*]$  is generated by all linear maps

$$\varphi_{[a,b]} := \langle -, ab - ba \rangle \text{ where } a, b \in \Lambda.$$

- (b) We have  $[\Lambda, \Lambda^*] = \text{Ann}_{\Lambda^*}(Z(\Lambda))$ .

*Proof.* (1) First, for any  $a, b \in \Lambda$  we have

$$(a, 0) \cdot (b, 0) - (b, 0) \cdot (a, 0) = (ab - ba, 0),$$

i.e., this generates  $K(\Lambda) \ltimes 0$ . Secondly, for any  $a \in \Lambda$  and  $\varphi \in \Lambda^*$  we get

$$(a, 0) \cdot (0, \varphi) - (0, \varphi) \cdot (a, 0) = (0, a\varphi - \varphi a)$$

which generates  $0 \ltimes [\Lambda, \Lambda^*]$ . This shows the inclusion ' $\supseteq$ '.

Conversely, an arbitrary generator of  $K(\mathbf{T}(\Lambda))$  has the form

$$[(a, \varphi), (b, \psi)] = (ab, a\psi + \varphi b) - (ba, b\varphi + \psi a) = (ab - ba, a\psi - \psi a + \varphi b - b\varphi)$$

which clearly is in  $K(\Lambda) \ltimes [\Lambda, \Lambda^*]$ .

(2 a) The commutator space  $[\Lambda, \Lambda^*]$  is generated by all  $a\varphi - \varphi a$  where  $a \in \Lambda$ , and  $\varphi \in \Lambda^*$ . Since  $\Lambda$  is finite-dimensional, the elements of  $\Lambda^*$  are of the form  $\varphi = \varphi_b := \langle -, b \rangle$  where  $b \in \Lambda$ . It suffices to show that  $a\varphi_b - \varphi_b a = \varphi_{ab-ba}$ . This holds, since for every  $x \in \Lambda$  we have

$$\begin{aligned} (a\varphi_b - \varphi_b a)(x) &= \varphi_b(xa) - \varphi_b(ax) = \langle xa - ax, b \rangle = \langle x, ab \rangle - \langle b, ax \rangle \\ &= \langle x, ab \rangle - \langle ba, x \rangle = \langle x, ab \rangle - \langle x, ba \rangle \\ &= \varphi_{ab-ba}(x). \end{aligned}$$

(2 b) Let  $a \in \Lambda$ ,  $\varphi \in \Lambda^*$  and  $z \in Z(\Lambda)$ . Then we have

$$(a\varphi - \varphi a)(z) = \varphi(az) - \varphi(za) = \varphi(az - za) = \varphi(0) = 0,$$

i.e.,  $[\Lambda, \Lambda^*] \subseteq \text{Ann}_{\Lambda^*}(Z(\Lambda))$ .

Conversely, let  $\varphi \in \text{Ann}_{\Lambda^*}(Z(\Lambda))$ . Since  $\Lambda$  is finite-dimensional, there exists an element  $b \in \Lambda$  such that  $\varphi = \varphi_b = \langle -, b \rangle$ . By assumption on  $\varphi$  we have that

$$b \in Z(\Lambda)^\perp = (K(\Lambda)^\perp)^\perp = K(\Lambda).$$

From part (2a) it now follows that  $\varphi = \varphi_b \in [\Lambda, \Lambda^*]$ .  $\square$

#### 4. PROOF OF THE MAIN RESULTS

This section is devoted to proving the main results of this article.

The following main step shows that indeed the generalized Reynolds ideals for the trivial extension  $\mathbf{T}(\Lambda)$  are closely related to the algebra  $\Lambda$  itself.

**Theorem 4.1.** *Let  $\Lambda$  be a finite-dimensional algebra over a field of characteristic  $p > 0$ , and let  $\mathbf{T}(\Lambda) = \Lambda \ltimes \Lambda^*$  be its trivial extension.*

- (1) *We have  $T_0(\mathbf{T}(\Lambda))^\perp = Z(\mathbf{T}(\Lambda)) = Z(\Lambda) \ltimes \text{Ann}_{\Lambda^*}(K(\Lambda))$ .*
- (2) *For all  $n \geq 1$  the generalized Reynolds ideals of the trivial extension are of the form  $T_n(\mathbf{T}(\Lambda))^\perp = 0 \ltimes \text{Ann}_{\Lambda^*}(T_n\Lambda)$ .*
- (3) *For all  $n \geq 1$  we have*  

$$T_n(Z(\mathbf{T}(\Lambda)))^\perp / K(\mathbf{T}(\Lambda)) = 0 \ltimes (\text{Ann}_{\Lambda^*}(T_n(Z(\Lambda))) / [\Lambda, \Lambda^*]).$$
- (4) *For all  $n \geq 1$  we have*  

$$P_n(Z(\mathbf{T}(\Lambda)))^\perp / K(\mathbf{T}(\Lambda)) = \Lambda / K(\Lambda) \ltimes (\text{Ann}_{\Lambda^*}(P_n(Z(\Lambda))) / [\Lambda, \Lambda^*]).$$

*Proof.* (1) The first equality follows directly from the general fact that for a symmetric algebra the orthogonal space of the commutator subspace is the center (see e.g. [4]). The second equality is Proposition 3.2.

(2) Let us fix some  $n \geq 1$ . Recalling the definition of the symmetric bilinear form on  $\mathbf{T}(\Lambda)$  and using Proposition 3.3 we have

$$T_n(\mathbf{T}(\Lambda))^\perp = \{(b, \psi) \in \mathbf{T}(\Lambda) \mid \psi(a) + \varphi(b) = 0 \text{ for all } a \in T_n(\Lambda), \varphi \in \Lambda^*\}.$$

Setting  $a = 0 \in T_n(\Lambda)$  we conclude that  $b = 0$  and then we get

$$T_n(\mathbf{T}(\Lambda))^\perp = \{(0, \psi) \in \mathbf{T}(\Lambda) \mid \psi(T_n\Lambda) = 0\},$$

as claimed.

(3) First note that, since  $Z(\mathbf{T}(\Lambda))$  is commutative, the corresponding commutator space  $K(Z(\mathbf{T}(\Lambda)))$  becomes zero. In particular,

$$(a, \varphi) \in T_n(Z(\mathbf{T}(\Lambda))) \iff (0, 0) = (a, \varphi)^{p^n} = (a, 0)^{p^n} + (0, \varphi)^{p^n} = (a^{p^n}, 0).$$

Hence,  $T_n(Z(\mathbf{T}(\Lambda))) = T_n(Z(\Lambda)) \ltimes \text{Ann}_{\Lambda^*}(K(\Lambda))$ . Therefore, we can conclude for the orthogonal space that

$$(a, \varphi) \in T_n(Z(\mathbf{T}(\Lambda)))^\perp \iff \langle (a, \varphi), (z, \psi) \rangle = \varphi(z) + \psi(a) = 0$$

for all  $(z, \psi) \in T_n(Z(\Lambda)) \ltimes \text{Ann}_{\Lambda^*}(K(\Lambda))$ . In particular, setting  $z = 0$  one gets  $a \in K(\Lambda)$ . Moreover, choosing  $\psi = 0$  shows that  $\varphi \in \text{Ann}_{\Lambda^*}(T_n(Z(\Lambda)))$ . Now invoking Lemma 3.4 gives the required statement.

(4) Using the definition of the symmetrizing bilinear form on  $\mathbf{T}(\Lambda)$  and the fact that  $0 \ltimes \Lambda^*$  is a nilpotent ideal of square zero we have that

$$(a, \varphi) \in P_n(Z(\mathbf{T}(\Lambda)))^\perp / K(\mathbf{T}(\Lambda)) \iff \varphi(z^{p^n}) = 0 \text{ for all } z \in Z(\Lambda).$$

Now recall that by definition  $P_n(Z(\Lambda))$  is the space generated by all  $z^{p^n}$  where  $z \in Z(\Lambda)$ . Then we can use Lemma 3.4 to deduce the desired statement.  $\square$

Corollary 1.2, as stated in the introduction, can now be deduced from the above theorem as follows.

*Proof of Corollary 1.2.* Suppose  $\Lambda$  and  $\Gamma$  are derived equivalent finite-dimensional algebras over a perfect field of positive characteristic. Then we consider their trivial extensions  $\mathbf{T}(\Lambda)$  and  $\mathbf{T}(\Gamma)$ , respectively. These are derived equivalent, by a result of J. Rickard [5]. Moreover, the trivial extensions are symmetric algebras. Hence, they have generalized Reynolds ideals, and by Theorem 4.1 the sequences of these take the form

$$Z(\Lambda) \ltimes \text{Ann}_{\Lambda^*}(K(\Lambda)) \supseteq 0 \ltimes \text{Ann}_{\Lambda^*}(T_1\Lambda) \supseteq 0 \ltimes \text{Ann}_{\Lambda^*}(T_2\Lambda) \supseteq \dots$$

and analogously for  $\mathbf{T}(\Gamma)$

$$Z(\Gamma) \ltimes \text{Ann}_{\Gamma^*}(K(\Gamma)) \supseteq 0 \ltimes \text{Ann}_{\Gamma^*}(T_1\Gamma) \supseteq 0 \ltimes \text{Ann}_{\Gamma^*}(T_2\Gamma) \supseteq \dots$$

According to [6], these sequences are invariants of the derived equivalent symmetric algebras  $\mathbf{T}(\Lambda)$  and  $\mathbf{T}(\Gamma)$ . In particular, the dimensions in each step have to coincide. Recall that derived equivalent algebras have isomorphic centers, so  $Z(\Lambda) \cong Z(\Gamma)$ . Now we compare the remaining dimensions in the above sequences. Note that for any  $n \geq 0$  we have  $\dim \text{Ann}_{\Lambda^*}(T_n\Lambda) = \dim \Lambda - \dim T_n\Lambda$  and similarly for  $\Gamma$ . Since these dimension have to agree, we get that the codimensions  $\dim \Lambda - \dim T_n\Lambda$  are derived invariant, as claimed.  $\square$

**Remark 4.2.** We are going to explain that our main result, Theorem 4.1 above, really extends the corresponding result from [6] for symmetric algebras. In fact, let us start with a finite-dimensional symmetric algebra  $\Lambda$ . Then the nondegenerate symmetrizing form  $\langle -, - \rangle$  on  $\Lambda$  induces an isomorphism  $\lambda : \Lambda \longrightarrow \Lambda^*$  as  $\Lambda$ - $\Lambda$ -bimodules by setting

$$\lambda(b) = (a \mapsto \langle a, b \rangle).$$

Moreover,  $K(\Lambda)$ , and hence also  $T_n(\Lambda)$ , is a  $Z(\Lambda)$ -submodule of  $\Lambda$ . We now have two series of Reynolds ideals to compare. Namely, the one for the symmetric algebra  $\Lambda$  itself

$$Z(\Lambda) \supseteq T_1(\Lambda)^\perp \supseteq T_2(\Lambda)^\perp \supseteq \dots$$

and the one for the trivial extension  $\mathbf{T}(\Lambda)$  which, by Theorem 4.1, takes the form

$$Z(\Lambda) \ltimes \text{Ann}_{\Lambda^*}(K(\Lambda)) \supseteq 0 \ltimes \text{Ann}_{\Lambda^*}(T_1(\Lambda)) \supseteq 0 \ltimes \text{Ann}_{\Lambda^*}(T_2(\Lambda)) \supseteq \dots$$

We claim that the structure of the ideals occurring in these two series are the same. In fact, for the former series, the  $T_n(\Lambda)^\perp$  are ideals of  $Z(\Lambda)$  by multiplication in the ring  $Z(\Lambda)$ . For the latter series, take elements  $(z, \psi) \in Z(\Lambda) \ltimes \text{Ann}_{\Lambda^*}(K(\Lambda))$  and  $(0, \varphi) \in 0 \ltimes \text{Ann}_{\Lambda^*}(T_n(\Lambda))$  for some  $n \geq 1$ . By the multiplication rule for the trivial extension we have  $(z, \psi) \cdot (0, \varphi) = (0, z\varphi)$  and similarly  $(0, \varphi) \cdot (z, \psi) = (0, \varphi z)$ . Hence, the ideal structure is given by the  $Z(\Lambda)$ -action on  $\text{Ann}_{\Lambda^*}(T_n(\Lambda))$ . Now our claim follows from the observation that under the above identification,  $\text{Ann}_{\Lambda^*}(T_n(\Lambda)) \cong T_n(\Lambda)^\perp$ ,  $\langle -, t \rangle \leftrightarrow t$ , the  $Z(\Lambda)$ -bimodule action on  $\text{Ann}_{\Lambda^*}(T_n(\Lambda))$  corresponds precisely to multiplication in  $Z(\Lambda)$ .

Now let us consider the invariants occurring in Proposition 2.1, part (ii), and Theorem 4.1, parts (3),(4), respectively. We first have to compare  $T_n(Z(\Lambda))^\perp/K(\Lambda)$  and  $0 \ltimes \text{Ann}_{\Lambda^*}(T_n(Z(\Lambda)))/[\Lambda, \Lambda^*]$ . Note that the former is contained in  $\Lambda/K(\Lambda)$ , whereas the latter is contained in  $\mathbf{T}(\Lambda)/K(\mathbf{T}(\Lambda))$ . Again, we use the map  $\lambda : \Lambda \rightarrow \Lambda^*$ ,  $b \mapsto \langle -, b \rangle$ . Under this isomorphism,  $T_n(Z(\Lambda))^\perp$  corresponds to  $\text{Ann}_{\Lambda^*}(T_n(Z(\Lambda)))$ . Moreover,  $K(\Lambda)$  corresponds similarly to

$$\text{Ann}_{\Lambda^*}(K(\Lambda)^\perp) = \text{Ann}_{\Lambda^*}(Z(\Lambda)) = [\Lambda, \Lambda^*]$$

where the last equation is Lemma 3.4 (2b).

The above argument applies analogously to the spaces  $P_n(Z(\Lambda))$  and then shows that, for  $\Lambda$  symmetric,  $P_n(Z(\Lambda))^\perp/K(\Lambda)$  corresponds to the second argument in the invariant

$$P_n(Z(\mathbf{T}(\Lambda)))^\perp/K(\mathbf{T}(\Lambda)) = \Lambda/K(\Lambda) \ltimes (\text{Ann}_{\Lambda^*}(P_n(Z(\Lambda)))/[\Lambda, \Lambda^*]).$$

Taken together, the above considerations of this remark show that for a symmetric algebra  $\Lambda$  the invariants obtained from the trivial extension  $\mathbf{T}(\Lambda)$  carry the same information as the invariants obtained from  $\Lambda$  directly. In this sense, our construction genuinely extends the results from [6], [7] from symmetric to arbitrary finite-dimensional algebras.

As the final step we can now give the proof of our second corollary from the introduction.

*Proof of Corollary 1.3.* Let  $\Lambda$  and  $\Gamma$  be two derived equivalent  $k$ -algebras. By Rickard's main theorem we know that the derived equivalence can be replaced by a derived equivalence of standard type, i.e. given by tensoring with a two-sided tilting complex. Such an equivalence gives an isomorphism  $\zeta : Z(\mathbf{T}(\Lambda)) \rightarrow Z(\mathbf{T}(\Gamma))$  and by Proposition 2.1 we know that then the decreasing sequence of  $Z(\mathbf{T}(\Lambda))$ -ideals  $T_n(\mathbf{T}(\Lambda))^\perp$  is mapped to the decreasing sequence  $T_n(\mathbf{T}(\Gamma))^\perp$ . But the Remark 4.2 implies that this is just the same as saying that the isomorphism  $\zeta$  induces a  $Z(\Lambda)$ -module isomorphism  $\text{Ann}_{\Lambda^*}(K(\Lambda)) \rightarrow \text{Ann}_{\Gamma^*}(K(\Gamma))$  so that the sequence of submodules

$$\text{Ann}_{\Lambda^*}(T_1\Lambda) \supseteq \text{Ann}_{\Lambda^*}(T_2\Lambda) \supseteq \text{Ann}_{\Lambda^*}(T_3\Lambda) \supseteq \dots$$



is mapped to the sequence of submodules

$$\mathrm{Ann}_{\Gamma^*}(T_1\Gamma) \supseteq \mathrm{Ann}_{\Gamma^*}(T_2\Gamma) \supseteq \mathrm{Ann}_{\Gamma^*}(T_3\Gamma) \supseteq \dots$$

This completes the proof of the corollary.  $\square$

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