An Introduction to Noncommutative Gröbner-Shirshov(=GS) Bases with Applications to Homological Algebra Lecture III: Application to (Two-sided) Anick Resolutions

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Plan

- Anick chains and (two-sided) Anick resolutions
- Algebraic Morse theory
- (Two-sided) Anick resolutions revisited

- Let X be a nonempty set. Denote by X^* the free monoid generated by X The monoid algebraof X^* is the vector space generated by X^* endowed with the multiplication induced by concatenation of monomials, denoted by $\mathbb{K}\langle X\rangle$. $\mathcal{B} = X^*$ is a multiplicative $\mathbb{K}$-basis of $\mathbb{K}\langle X\rangle$. Write $\mathcal{B}_+ = \mathcal{B} \setminus \{1\}$.
- Fix a monomial order $\prec$ on $\mathcal{B}$.
- For a polynomial $f \in \mathbb{K}\langle X\rangle$, its leading monomial $\mathrm{LM}(f)$ is by definition the maximal monomial appearing with nonzero coefficients in f.
- For a nonempty subset S of $\mathbb{K}\langle X\rangle$,
$$\mathrm{LM}(S) = \{\mathrm{LM}(f) \mid f \in S, f \neq 0\}.$$
- Let I be a two-sided ideal in $\mathbb{K}\langle X\rangle$ contained in $\mathbb{K}\mathcal{B}_+$.
- Write $A = \mathbb{K}\langle X\rangle/I$ and $A_+ = \mathbb{K}\mathcal{B}_+/I$. Then $\mathbb{K} = A/A_+$ is the trivial module. Write $\mathrm{NonLM}(I)$ for the complement set of $\mathrm{LM}(I)$ in $\mathcal{B}$. So $\mathrm{NonLM}(I)$ is a basis of the quotient algebra $A = \mathbb{K}\langle X\rangle/I$.
- A GS basis of I with respect to the monomial order $\prec$ is a subset $\mathcal{G} \subseteq I$ such that $\mathrm{LM}(\mathcal{G})$ generates the ideal $\langle \mathrm{LM}(I)\rangle$.
- A GS basis $\mathcal{G} \subseteq I$ is reduced if it consists of sharp elements in I and an element of $\mathrm{LM}(\mathcal{G})$ can not divide one another.

Fix a reduced GS basis $\mathcal{G} \subseteq I$.
Denote $W^{(-1)} = \{1\}$, $W^{(0)} = X$ and $W = W^{(1)} := \mathrm{LM}(\mathcal{G})$.
The Ufnarovskiĭ graph $\mathcal{U}$ with vertex set V and arrow set E as follows:

$$V = \{1\} \cup X \cup \{u \in \mathcal{B} \mid u \text{ is a proper right factor of some } v \in W\},$$

and E is the union of $\{1 \to x \mid x \in X\}$ with

$$\{u \to v \mid uv \in \mathcal{B}, uv \in \langle \mathrm{LM}(I)\rangle, w \notin \langle \mathrm{LM}(I)\rangle \text{ for all proper left factors } w \text{ of } uv$$

Definition

The set of i-chains $W^{(i)}, i \geq 0$ (also called Anick i-chains) consists of all sequences $(v_1, \cdots, v_i, v_{i+1})$ in $\mathcal{B}_+^{i+1}$ such that

$$1 \to v_1 \to \cdots \to v_i \to v_{i+1}$$

is a path in $\mathcal{U}$.

V. A. Ufnarovski, *Combinatorial and assymptotic methods in algebra*. Algebra VI Encycl. Math. Sci. 57, 1-196, Springer-Verlag, Heidelberg, 1995.

Part I.1: Polynomial algebras as examples

Let

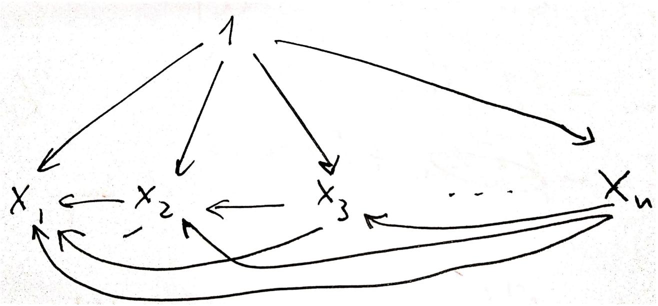
$$A = \mathbb{K}[x_1, x_2, \cdots, x_N] \simeq \mathbb{K}\langle x_1, x_2, \cdots, x_N\rangle / (x_j x_i - x_i x_j, 1 \leq i < j \leq N)$$

be the polynomial algebra in N variables.

Take $x_1 < x_2 < \cdots < x_N$ and extend it to the left length lexicographic order.
A GS basis of $I = (x_j x_i - x_i x_j, 1 \leq i < j \leq N)$ is given by

$$\mathcal{G} = \{x_j x_i - x_i x_j, 1 \leq i < j \leq N\},$$

and $W = \mathrm{LM}(\mathcal{G}) = \{x_j x_i, j > i\}$.
A basis of A is given be $\{x_{i_1} x_{i_2} \cdots x_{i_p} \mid 1 \leq i_1 \leq i_2 \leq \cdots \leq i_p \leq N, p \geq 0\}$.



For $n \geq -1$, Anick n-chains are

- $W^{(-1)} = \{1\}$;
- for $0 \leq k \leq N-1$, $W^{(k)} = \{(x_{i_1}, x_{i_2}, \cdots, x_{i_{k+1}}), N \geq i_1 > i_2 > \cdots > i_{k+1} \geq 1\}$;
- for $k \geq N$, $W^{(k)} = \emptyset$.

Part I.1: Another example

Let

$$A = \mathbb{K}\langle x, y\rangle/(x^2 - yx).$$

Take $y > x$ and extend it to the left length lexicographic order.
A GS basis of I is given by

$$\mathcal{G} = \{yx - x^2\},$$

and $W = \mathrm{LM}(\mathcal{G}) = \{yx\}$.
The Ufnarovskiĭ graph $\mathcal{U}$ is given by

$$\begin{tikzcd} & 1 \arrow[dl] \arrow[dr] & \\ x & & y \arrow[ll] \end{tikzcd}$$

So $W^{(-1)} = \{1\}, W^{(0)} = \{x, y\}, W^{(1)} = \{(y, x)\}, W^{(p)} = \emptyset, \forall p \geq 2.$

Let

$$A = \mathbb{K}\langle x, y\rangle/(x^2 - yx).$$

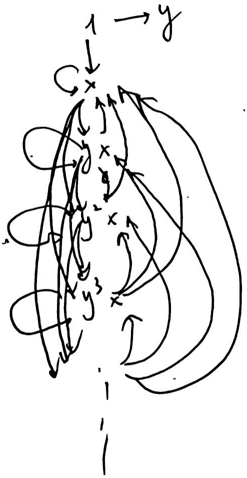
Take $x > y$ and extend it to the left length lexicographic order.
A GS basis of I is given by

$$\mathcal{G} = \{xy^n x - y^{n+1}x, n \geq 0\}$$

and $W = \mathrm{LM}(\mathcal{G}) = \{xy^n x, n \geq 0\}$.

Part I.1: Another example

The Ufnarovskiĭ graph $\mathcal{U}$ is given by



$\forall i,j \geq 0$

$\exists$ arrow : $y^i x \to y^j x$

So $W^{(-1)} = \{1\}$, $W^{(0)} = \{x, y\}$, $W^{(1)} = \{(x, y^m x), m \geq 0\}$, $W^{(p)} = \{(x, y^{m_1}x, y^{m_2}x, \ldots, y^{m_p}x), m_1, \ldots, m_p \geq 0\}, \forall p \geq 2.$

Part I.2: (Two-sided) Anick Resolutions

Theorem (Anick 86, Anick-Green 87, Sköldberg 2005, Chen-Liu-Z. 2024)

Let $A = \mathbb{K}\langle X\rangle/I$ and let $\mathcal{G}$ be a reduced GS basis of I. Then there exists a free resolution of the trivial module $\mathbb{K} = A/A_+$ of the form:

$$\cdots \to P_n = A \otimes \mathbb{K}W^{(n-1)} \to \cdots \to P_1 = A \otimes \mathbb{K}X \to P_0 = A \to \mathbb{K} \to 0$$

and a free resolution of A as bimodule:

$$\cdots \to Q_n = A\otimes\mathbb{K}W^{(n-1)}\otimes A \to \cdots \to Q_1 = A\otimes\mathbb{K}X\otimes A \to Q_0 = A\otimes A \to A \to 0$$

D. J. Anick, *On the Homology of associative algebras*. Trans. Amer. Math. Soc. **296** (1986) 641-659.

D. J. Anick and E. L. Green, *On the homology of quotients of path algebras*. Comm. Algebra **15** (1987), no. 1-2, 309-341.

E. Sköldberg, *Morse Theory from an Algebraic Viewpoint*. Trans. Amer. Math. Soc. **358** (2005), 115-129.

Jun Chen (陈骏), Yuming Liu (刘玉明), Guodong Zhou (周国栋), *Algebraic Morse theory via homological perturbation lemma*, arXiv:2404.10165, J. Pure Appl. Algebra accepted.

Part I.2: (Two-sided) Anick Resolutions

Example

Let

$$A = \mathbb{K}\langle x_1, x_2, \cdots, x_N\rangle/(x_j x_i - x_i x_j, 1 \le i < j \le N) \cong \mathbb{K}[x_1, x_2, \cdots, x_N]$$

be the polynomial algebra in N variables. Take $x_1 < x_2 < \cdots < x_N$ and extend it to the left length lexicographic order. Let $V = \oplus_{i=1}^{N} kx_i$. Then for $1 \le k \le N$,

$$W^{(k-1)} = \{(x_{i_1}, x_{i_2}, \cdots, x_{i_k}), N \ge i_1 > i_2 > \cdots > i_k \ge 1\},$$

so $\mathbb{K}W^{(k-1)} \cong \Lambda^k(V)$.
The two-sided Anick resolution has the form

$$0 \to A \otimes \Lambda^N(V) \otimes A \to \cdots \to A \otimes \Lambda^1(V) \otimes A \to A \otimes V \otimes A \to A \to 0,$$

which is very much like (and in fact isomorphic to) the Koszul resolution.

Example

Let

$$A = \mathbb{K}\langle x, y\rangle/(x^2 - yx).$$

Take $y > x$ and extend it to the left length lexicographic order. The two-sided Anick resolution has the form

$$0 \to A \otimes \mathbb{K}(y, x) \otimes A \to A \otimes \mathbb{K}x \otimes A \oplus A \otimes \mathbb{K}y \otimes A \to A \to 0$$

or

$$0 \to A^e \to A^e \oplus A^e \to A^e \to A \to 0.$$

Example

Let

$$A = \mathbb{K}\langle x, y\rangle/(x^2 - yx).$$

Take $x > y$ and extend it to the left length lexicographic order. The two-sided Anick resolution has the form

$$\cdots \to \oplus_{m=0}^{\infty} A \otimes \mathbb{K}(x, y^m x) \otimes A \to A \otimes \mathbb{K}x \otimes A \oplus A \otimes \mathbb{K}y \otimes A \to A \to 0.$$

Part I.2: (Two-sided) Anick Resolutions

Problem

How to describe the differential on the (two-sided) Anick resolution?

Answer

There are no explicit formulae, but four recursive methods:

- *orginal method of Anick and Anick-Green*
- *rewriting system due to Chouhy-Solotar*
- *polygraphic resolutions due to Guiraud-Hoffbeck-Malbos*

D. J. Anick, *On the Homology of associative algebras*. Trans. Amer. Math. Soc. **296** (1986) 641-659.

D. J. Anick and E. L. Green, *On the homology of quotients of path algebras.* Comm. Algebra **15** (1987), no. 1-2, 309-341.

Sergio Chouhy, Andrea Solotar, *Projective resolutions of associative algebras and ambiguities*. J. Algebra **432** (2015), 22-61.

Yves Guiraud, Eric Hoffbeck, Philippe Malbos, *Convergent presentations and polygraphic resolutions of associative algebras*, Math. Z. **293** (2019), no. 1-2, 113-179.

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How to describe the differential on the (two-sided) Anick resolution?

Answer

There are no explicit formulae, but four recursive methods:

- *algebraic Morse theory due to Sköldberg*

E. Sköldberg, *Morse Theory from an Algebric Viewpoint*. Trans. Amer. Math. Soc. **358** (2005), 115-129.

Jun Chen (陈骏), Yuming Liu (刘玉明), Guodong Zhou (周国栋), *Algebraic Morse theory via homological perturbation lemma*, arXiv:2404.10165, J. Pure Appl. Algebra accepted.

Part I.3: Monomial algebras as examples

Let $A = T(V)/(\mathcal{G})$ where $\mathcal{G}$ is a "minimal" set of monomials. Then $\mathcal{G}$ is a reduced GS basis of A.

Theorem (Bardzell 1997)

Let $w_{1,n} = (w_1, \ldots, w_n)$ be an $(n-1)$-chain, then

(i) *for n odd,*

$$d_n(w_{1,n}) = 1 \otimes w_{1,n-1} \otimes w_n - u \otimes w'_{1,n-1} \otimes 1$$

with $w_1 \cdots w_n = uw'_1 \cdots w'_{n-1}$ and $w'_{1,n-1} \in W^{(n-2)}$;

(ii) *for n even,*

$$d_n(w_{1,n}) = \sum_{w'_{1,n-1} \in W^{(n-2)} \text{ with } w_1 \cdots w_n = uw'_1 \cdots w'_{n-1} v} u \otimes w'_{1,n-1} \otimes v.$$

Michael J. Bardzell, *The alternating syzygy behavior of monomial algebras*, J. Algebra **188** (1997), no. 1, 69-89.

Part I.3: Monomial algebras as examples

Fact

The two-sided Anick resolution is isomorphic to the Bardzell resolution.

E. Sköldberg, *Morse Theory from an Algebraic Viewpoint*. Trans. Amer. Math. Soc. **358** (2005), 115-129.

Severin Barmeier, Jun Chen (陈骏), Zhengfang Wang (汪正方), and Guodong Zhou (周国栋), *Comparison morphisms for monomial algebras via algebraic Morse theory*, preprint 2021.

- Comparison morphisms (as well as the whole homotopy retract) between the two-sided Anick resolution and the (reduced) bar resolution

María Julia Redondo, Lucrecia Román, *Comparison morphisms between two projective resolutions of monomial algebras*, Rev. Un. Mat. Argentina **59** (2018), no. 1, 1-31.

Severin Barmeier, Jun Chen (陈骏), Zhengfang Wang (汪正方), and Guodong Zhou (周国栋), *Comparison morphisms for monomial algebras via algebraic Morse theory*, preprint 2021.

Emil Sköldberg, *A contracting homotopy for Bardzell's resolution*, Math. Proc. R. Ir. Acad. **108** (2008), no. 2, 111-117.

Jingfeng Zhang (张敬峰), Constructing comparison morphisms via weak self-homotopies with applications to monomial algebras, master thesis in ECNU, June 2021.

Part I.4: Side remarks on Anick resolutions

- Anick resolutions are of combinatorial nature, so is algorithmic. It is hard to give explicate formulae for this resolution.
- In general, Anick resolutions are not best possible.
    - Even if the algebra has finite global dimension, Anick resolution may be infinite.
    - Even if minimal resolutions exist, Anick resolution may be not minimal.

Part I.4: Side remarks on constructions of projective resolutions

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- M. C. R. Butler, A. D. King, *Minimal resolutions of algebras*, J. Algebra **212** (1999), no. 1, 323-362.
- E. L. Green, Ø. Solberg, D. Zacharia, *Minimal projective resolutions*, Trans. Amer. Math. Soc. **353** (2001), no. 7, 2915-2939.
- Sergio Chouhy, Andrea Solotar, *Projective resolutions of associative algebras and ambiguities*. J. Algebra **432** (2015), 22-61.
- Yves Guiraud, Eric Hoffbeck, Philippe Malbos, *Convergent presentations and polygraphic resolutions of associative algebras*, Math. Z. **293** (2019), no. 1-2, 113-179.

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- K.S. Brown, *The geometry of rewriting systems: a proof of the Anick-Groves-Squier theorem*. Algorithms and classification in combinatorial group theory (Berkeley, CA, 1989), 137 – 163, Math. Sci. Res. Inst. Publ., 23, Springer, New York, 1992.
- R. Forman, *Morse theory for cell complexes.* Adv. Math. **134** (1998), no. 1, 90-145.

- E. Sköldberg, *Morse Theory from an Algebraic Viewpoint*. Trans. Amer. Math. Soc. **358** (2005), 115-129.
- D. N. Kozlov, *Discrete Morse theory for free chain complexes.* C.R. Math. Acad. Sci. Paris **340** (2005), 867-872.
- M. Jöllenbeck and V. Welker,*Minimal resolutions via algebraic discrete Morse theory.* Mem. Amer. Math. Soc. **197** (2009), no. 923, vi+74 pp.
- Jun Chen (陈骏), Yuming Liu (刘玉明), Guodong Zhou (周国栋), *Algebraic Morse theory via homological perturbation lemma*, arXiv:2404.10165, J. Pure Appl. Algebra accepted.

Part II. 2: Two easy observations

Fact

Let $0 \to C_\bullet \xrightarrow{f_\bullet} D_\bullet \xrightarrow{g_\bullet} E_\bullet \to 0$ *be a short exact sequence of complexes.*

- *If* $C_\bullet$ *(resp.* $E_\bullet$*) is exact, then* $g_\bullet$ *(resp.* $f_\bullet$*) is a quasi-isomorphism.*
- *If* $C_\bullet$ *(resp.* $E_\bullet$*) is null homotopic and the short exact sequence splits degreewise, then* $g_\bullet$ *(resp.* $f_\bullet$*) is a homotopy equivalence.*

Corollary

We can throw out an exact or contractible subcomplex or quotient complex to get a smaller complex with the same homology.

Question

Can we throw out one part of a complex, which is exact, but neither a subcomplex nor a quotient complex?

Let C be a complex of the form

$$\begin{tikzcd}
\cdots \arrow[r, "d_{n+2}"] & C_{n+1} \arrow[r, "d'_{n+1}"] \arrow[dr, "k"'] & C'_n \arrow[r, "d'_n"] \arrow[drr, "h"' near start] & C'_{n-1} \arrow[r, "d'_{n-1}"] & C_{n-2} \arrow[r, "d_{n-2}"] & \cdots \\
& & R \arrow[ur, "f" near end] \arrow[rr, "g", "\cong"'] & & R \arrow[u, "l"'] &
\end{tikzcd}$$

where $C_n = C'_n \bigoplus R$, $C_{n-1} = C'_{n-1} \bigoplus R$.

We have

$$gk + hd'_{n+1} = 0,\ fk + d'_n d'_{n+1} = 0,\ lg + d'_{n-1} f = 0,\ lh + d'_{n-1} d'_n = 0.$$

Construct a new diagram:

$$\begin{tikzcd}
\cdots \arrow[r, "d_{n+2}"] & C_{n+1} \arrow[r, "d'_{n+1}"] \arrow[dr, "k"'] & C'_n \arrow[r, "d'_n"] \arrow[drr, "h"] & C'_{n-1} \arrow[r, "d'_{n-1}"] & C_{n-2} \arrow[r, "d_{n-2}"] & \cdots \\
 & & R \arrow[ur, "f"] & & R \arrow[ll, dotted, "-g^{-1}"] \arrow[u, "l"'] &
\end{tikzcd}$$

Define a new complex $C_\bullet^M$:

$$\cdots \xrightarrow{d_{n+2}} C_{n+1} \xrightarrow{d'_{n+1}} C'_n \xrightarrow{d''_n = d'_n - fg^{-1}h} C'_{n-1} \xrightarrow{d'_{n-1}} C_{n-2} \xrightarrow{d_{n-2}} \cdots$$

Then $C_\bullet^M$ is homotopy equivalent to $C_\bullet$

Part II.2: The second example

Consider the following complex

$$\begin{tikzcd}
\cdots \arrow[r, "\alpha"] & A_{n+1} \arrow[r, "\alpha"] \arrow[dr, "\gamma"] \arrow[ddr, "\gamma"] & A_n \arrow[r, "\alpha"] \arrow[dr, "\gamma"] & A_{n-1} \arrow[r, "\alpha"] & A_{n-2} \arrow[r, "\alpha"] & \cdots \\
& & R \arrow[ur, "\beta"] \arrow[r, "\delta_1'"] \arrow[dr, "\delta''"] & R \arrow[ur, "\beta"] & & \\
& & R \arrow[r, "\delta_2'"] & R \arrow[uur, "\beta"] & &
\end{tikzcd}$$

with δ_i', $i = 1, 2$ invertible.

Part II.2: The second example (continued)

Construct a new diagram:

$$
\begin{tikzcd}
\cdots \arrow[r, "\alpha"] & A_{n+1} \arrow[r, "\alpha"] \arrow[dr, "\gamma"] \arrow[ddr, "\gamma"'] & A_n \arrow[r, "\alpha"] \arrow[dr, "\gamma"] & A_{n-1} \arrow[r, "\alpha"] & A_{n-2} \arrow[r, "\alpha"] & \cdots \\
 & & R \arrow[ur, "\beta"] \arrow[dr, "\delta''"] & R \arrow[l, dotted, "h_1"'] \arrow[ur, "\beta"] & & \\
 & & R & R \arrow[l, dotted, "h_2"'] \arrow[uur, "\beta"'] & &
\end{tikzcd}
$$

where $h_i = -\delta_i'^{-1}, i = 1, 2$.

Define a new complex

$$\cdots \xrightarrow{\alpha} A_{n+1} \xrightarrow{\alpha} A_n \xrightarrow{\alpha+\beta h_1 \gamma} A_{n-1} \xrightarrow{\alpha} A_{n-2} \xrightarrow{\alpha} \cdots$$

which is homotopy equivalent to the original complex.

Part II.2: The third example

Consider the following complex

$$\begin{tikzcd}
\cdots \arrow[r, "\alpha"] & A_{n+1} \arrow[r, "\alpha"] \arrow[dr, "\gamma"] \arrow[ddr, "\gamma"'] & A_n \arrow[r, "\alpha"] \arrow[dr, "\gamma"] & A_{n-1} \arrow[r, "\alpha"] & A_{n-2} \arrow[r, "\alpha"] & \cdots \\
 & & R \arrow[ur, "\beta"] \arrow[r, "\delta'"] \arrow[dr, "\delta''"] & R \arrow[ur, "\beta"] & & \\
 & & R \arrow[ur, "\delta''"] \arrow[r, "\delta'"] & R \arrow[uur, "\beta"'] & &
\end{tikzcd}$$

with δ' invertible.

Part II.2: The third example (continued)

Construct a new diagram:

$$
\begin{tikzcd}
\cdots \arrow[r, "\alpha"] & A_{n+1} \arrow[r, "\alpha"] \arrow[dr, "\gamma"] \arrow[ddr, "\gamma"] & A_n \arrow[r, "\alpha"] \arrow[dr, "\gamma"] & A_{n-1} \arrow[r, "\alpha"] & A_{n-2} \arrow[r, "\alpha"] & \cdots \\
 & & R \arrow[ur, "\beta"] \arrow[dr, "\delta''"] & R \arrow[l, dotted, "h"] \arrow[ur, "\beta"] & & \\
 & & R \arrow[ur, "\delta''"] & R \arrow[l, dotted, "h"] \arrow[uur, "\beta"] & &
\end{tikzcd}
$$

where $h = -\delta'^{-1}$.
It has an oriented cycle!
Suppose that $\delta'' h \delta'' h$ is **locally nilpotent**.
Define a new complex

$$
\cdots \xrightarrow{\alpha} A_{n+1} \xrightarrow{\alpha} A_n \longrightarrow A_{n-1} \xrightarrow{\alpha} A_{n-2} \xrightarrow{\alpha} \cdots
$$

which is homotopy equivalent to the original complex.

Let R be a ring. Let $C_\bullet = (C_i, \partial_i)_{i\in\mathbb{N}}$ be a complex of free R-modules.
Choose a basis $X = \coprod_{i\geq 0} X_i$ such that $C_i \cong \bigoplus_{c\in X_i} Rc$.
Via this basis $\partial_i : C_i \to C_{i-1}$ has the form

$$\partial_i(c) = \sum_{c' \in X_{i-1}} [c : c'] \cdot c',$$

for $[c : c'] \in R$.

Definition

Define a weighted quiver $\Gamma(C_\bullet) = (V, E)$ where $V = X$ is the set of vertices of $\Gamma(C)$ and E is the set of weighted arrows:

$$(c \overset{[c:c']}{\to} c') \in E \iff c \in X_i,\ c' \in X_{i-1},\ [c : c'] \neq 0.$$

Definition

A nonempty subset $\mathcal{M} \subset E$ is a partial matching, if it satisfies the following conditions:

(1) *Each vertex $v \in V$ is incident to at most one arrow in $\mathcal{M}$.*

(2) *Each arrow $c \overset{[c:c']}{\to} c'$ in $\mathcal{M}$ has its weight $[c : c'] \in Z(R)^{\times}$.*

Define a new weighted quiver $\Gamma_{\mathcal{M}} = (V, E_{\mathcal{M}})$, where

$$E_{\mathcal{M}} = (E \setminus \mathcal{M}) \cup \{(c \overset{-\frac{1}{[c:c']}}{\dashleftarrow} c') \text{ with } (c \overset{[c:c']}{\to} c') \in \mathcal{M}\}.$$

Part II.3: Some notations

Let $\mathcal{M}$ be a partial matching in $\Gamma(C_\bullet) = (V, E)$.

- A vertex $c \in V$ is a critical vertex (with respect to $\mathcal{M}$), if c is not incident to any arrow in $\mathcal{M}$. Denote

$$X_i^{\mathcal{M}} := \{c \in X_i : c \textit{ critical}\}, X^{\mathcal{M}} = \bigcup_i X_i^{\mathcal{M}}$$

- Denote $\mathrm{Path}(c, c')$ to be the set of all paths from c to c' in $\Gamma_{\mathcal{M}}(C_\bullet)$.
- For a path $p = (c_1 \to \ldots \to c_r) \in \mathrm{Path}(c_1, c_r)$ its weight $w(p)$ is defined to be

$$w(c_1 \to \ldots \to c_r) = \prod_{i=1}^{r-1} w(c_i \to c_{i+1}),$$

$$w(c \to c') = \begin{cases} -\frac{1}{[c':c]}, & c \dashrightarrow c', \\ [c : c'], & c \to c'. \end{cases}$$

- Let $\tau(c, c') = \sum_{p \in \mathrm{Path}(c,c')} w(p)$ be the sum of the weights of all paths from c to c' in $\Gamma_{\mathcal{M}}(C_\bullet)$.

Part II.3: Morse matchings

Definition

A partial matching $\mathcal{M} \subset E$ is a Morse matching, if it satisfies the following local finiteness hypothesis:

- *Fix an arbitrary vertex c. For each vertex $c' \in X_n$, the sum of the weights of all paths from c to c' in $\Gamma_{\mathcal{M}}(C_\bullet)$*

$$\tau(c, c') = \sum_{p \in \mathrm{Path}(c, c')} w(p)$$

exists (for instance, it may be a finite sum or it is convergent in a certain norm); moreover, the number of vertices $c' \in X_n$ such that

$$\tau(c, c') \neq 0$$

is finite.

Part II.3: Main theorem of algebraic Morse theory

Theorem (E. Skoldberg 05, D. Kozlov 05)

Given a complex of free modules $(C_\bullet, \partial_\bullet)$, *let* $\mathcal{M}$ *be a Morse matching in* Γ. *Define*

$$C_i^{\mathcal{M}} := \bigoplus_{c \in X_i^{\mathcal{M}}} Rc,$$

$$\partial_i^{\mathcal{M}} : \begin{cases} C_i^{\mathcal{M}} \to C_{i-1}^{\mathcal{M}} \\ c \mapsto \sum_{c' \in X_{i-1}^{\mathcal{M}}} \tau(c, c')c'. \end{cases}$$

Then $(C_\bullet^{\mathcal{M}}, d_\bullet^{\mathcal{M}})$ *is a complex.*

Part II.3: Main theorem of algebraic Morse theory (continued)

Theorem (E. Skoldberg 05, D. Kozlov 05)

Define maps

$$\begin{aligned} f_p : C_p &\to C_p^{\mathcal{M}} \\ c &\mapsto f_p(c) := \sum_{c' \in X_p^{\mathcal{M}}} \tau(c, c')c' \end{aligned}$$

and

$$\begin{aligned} g_p : C_p^{\mathcal{M}} &\to C_p \\ c &\mapsto g_p(c) := \sum_{c' \in X_p} \tau(c, c')c' \end{aligned}$$

Then $f_\bullet : C_\bullet \to C_\bullet^{\mathcal{M}}$ *and* $g_\bullet : C_\bullet^{\mathcal{M}} \to C_\bullet$ *are chain maps and establish a homotopy equivalence between* $C_\bullet$ *and* $C_\bullet^{\mathcal{M}}$.

Part II.3: Main theorem of algebraic Morse theory (continued)

Theorem (E. Skoldberg 05, D. Kozlov 05)

The chain maps $f_\bullet : C_\bullet \to C_\bullet^{\mathcal{M}}$ and $g_\bullet : C_\bullet^{\mathcal{M}} \to C_\bullet$ establish homotopy equivalence between $C_\bullet$ and $C_\bullet^{\mathcal{M}}$. In fact $fg = Id$ and $gf \sim Id$ via the homotopy

$$
\begin{aligned}
\sigma_p : C_p &\to C_{p+1} \\
c &\mapsto \sigma_p(c) := \sum_{c' \in X_{p+1}} \tau(c, c')c'
\end{aligned}
$$

Remark

Algebraic Morse theory enables the use of homotopy transfer technique.

$$
\begin{tikzcd}
C_*^{\mathcal{M}} \arrow[r, bend left] & C_* \arrow[l, bend left] \arrow[loop right]
\end{tikzcd}
$$

Part III.1: The normalized two-sided bar resolution

Let $A = T(V)/(\mathcal{G})$ with a reduced GS basis $\mathcal{G}$ and the induced linear basis $\mathrm{Irr}(\mathcal{G})$. Let $\bar{A} = A/\mathbb{K}1_A$, Then $\mathrm{Irr}(\mathcal{G})_+ = \mathrm{Irr}(\mathcal{G})\backslash\{1\}$ is a linear basis of $\bar{A}$.

Recall the normalized two-sided bar resolution $B(A)$ has the form:

$$B(A)_n = A \otimes_k \bar{A}^{\otimes n} \otimes_k A \cong A^e \otimes_k \bar{A}^{\otimes n},$$

where $\bar{A} = A/k$ and differential is defined by:

$$d([a_1|\cdots|a_n]) = a_1[a_2|\cdots|a_n] + \sum_{i=1}^{n-1}(-1)^i[a_1|\cdots|a_ia_{i+1}|\cdots|a_n]$$

$$+(-1)^n[a_1|\cdots|a_{n-1}]a_n,$$

where $[a_1|\cdots|a_n] = 1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1$.

Recall that one can construct a weighted quiver $\Gamma_{B(A)}$ with vertices $[w_1|\cdots|w_n]$ for $w_1, \ldots, w_n \in \mathrm{Irr}(\mathcal{G})_+$.

Part III.1: Morse matching over the normalized two-sided bar resolution

For a vertex $[w_1|\cdots|w_n]$ in $\Gamma_{B(A)}$, let i is the largest integer such that $(w_1, \cdots, w_{i+1})$ is an i-chain.
We define a partial matching on $\Gamma_{B(A)}$ by letting arrows be the form

$$[w_1|\cdots|w_{i+2}|w''_{i+2}|\cdots|w_n] \to [w_1|\cdots|w_{i+1}|w_{i+2}|\cdots|w_n],$$

where $w'_{i+2}w''_{i+2} = w_{i+2}$ and $(w_1, \cdots, w_{i+1}, w'_{i+2})$ is an $(i+1)$-chain.

Theorem (E. Sköldberg 2006)

The partial matching $\mathcal{M}$ defined above is a Morse matching such that the set of critical vertices in n-th component is identified with the set $W^{(n-1)}$ of $n-1$-chains. The resolution $B(A)^{\mathcal{M}}$ is called the two-sided Anick resolution of A.

E. Sköldberg, *Morse Theory from an Algebric Viewpoint*. Trans. Amer. Math. Soc. **358** (2005), 115-129.

Jun Chen (陈骏), Yuming Liu (刘玉明), Guodong Zhou (周国栋), *Algebraic Morse theory via homological perturbation lemma*, arXiv:2404.10165, J. Pure Appl. Algebra accepted.

Part III.2: Polynomial algebras as examples

Let k be a field, $N \geq 1$ be an integer, V be a vector space of dimension N with basis $\{x_1, x_2, \cdots, x_N\}$. Let

$$A = S(V^*) = k[x_1, x_2, \cdots, x_n] \simeq k\langle x_1, x_2, \cdots, x_N\rangle/(x_j x_i - x_i x_j, 1 \leq i < j \leq N)$$

be the polynomial algebra in N variables.
Take $x_1 < x_2 < \cdots < x_N$ and extend it to the left length lexicographic order.
A Gröbner-Shirshov basis of $I = (x_j x_i - x_i x_j, 1 \leq i < j \leq N)$ is given by

$$\mathcal{G} = \{x_j x_i - x_i x_j, 1 \leq i < j \leq N\},$$

and $W = \mathrm{L}\mathcal{M}(\mathcal{G}) = \{x_j x_i, j > i\}$.
A basis of A is given be $\mathrm{Irr}(\mathcal{G}) = \{x_1^{k_1} x_2^{k_2} \cdots x_N^{k_N} \mid k_1, k_2, \ldots, k_N \geq 0\}$.

Fact

The two-sided Anick resolution is isomorphic to the Koszul resolution.

For $p \geq 0$, let $\Lambda^p(V)$ be the p-th exterior product of V For $0 \leq p \leq N$, let $K_p = A \otimes \Lambda^p(V) \otimes A$, and the differential

$$d_p : K_p = A \otimes \Lambda^p(V) \otimes A \longrightarrow K_{p-1} = A \otimes \Lambda^{p-1}(V) \otimes A$$

sends $1 \otimes (x_{j_p} \wedge x_{j_{p-1}} \wedge \cdots \wedge x_{j_1}) \otimes 1$ with $N \geq j_p > \cdots > j_1 \geq 1$ to

$$\sum_{i=1}^{p} (-1)^{p-i} x_{j_i} \otimes (x_{j_p} \wedge \cdots \wedge \hat{x}_{j_i} \wedge \cdots \wedge x_{j_1}) \otimes 1$$

$$-\sum_{i=1}^{p} (-1)^{p-i} 1 \otimes (x_{j_p} \wedge \cdots \wedge \hat{x}_{j_i} \wedge \cdots \wedge x_{j_1}) \otimes x_{j_i},$$

where $\widehat{\ }$ means deleting this element.

- Comparison morphism from Koszul resolution to the normalised bar resolution

$$F_* : K_* \to B_*$$

For $p \geq 0$, define $F_p : A \otimes \bigwedge^p(V) \otimes A \to A \otimes \overline{A}^{\otimes p} \otimes A$ by

$$F_p\big(1 \otimes (x_{j_p} \wedge \cdots \wedge x_{j_1}) \otimes 1\big) = \sum_{\pi \in \mathrm{S}_p} \mathrm{sgn}(\pi) \otimes x_{j_{\pi(p)}} \otimes \cdots \otimes x_{j_{\pi(1)}} \otimes 1,$$

where $N \geq j_p > j_{p-1} > \cdots > j_1 \geq 1$.

Part III.2: Polynomial algebras as examples

- Comparison morphism from the normalised bar resolution to Koszul resolution

$$G_* : B_* \to K_*$$

A.V. Shepler and S. Witherspoon, *Quantum differentiation and chain maps of bimodule complexes*. Algebra Number Theory **5** (2011), no. 3, 339-360.

S. Witherspoon and G. Zhou, *Gerstenhaber brackets on Hochschild cohomology of quantum symmetric algebras and their group extensions*. Pacific J. Math. **283** (2016), no. 1, 223 – 255.

- Constructing the homotopy retract

$$
\begin{tikzcd}
K_* \arrow[r, shift left, "F_*"] & B_* \arrow[l, shift left, "G_*"] \arrow[loop right, "\sigma_*"]
\end{tikzcd}
$$

with $G_*F_* = Id_{K_*}, F_*G_* - Id_{B_*} = \sigma_{*-1}d_* + d_*\sigma_{*+1}$

Qiong Duan (段琼), *Constructing comparison morphisms via algebraic Morse theory*, Master thesis in ECNU, June 2018.

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- Construct the homotopy retract for cohomology complexes

M. de Wilde and P. Lecomte, *An homotopy formula for the Hochschild cohomology*, Compositio Math. **96** (1995), 99-109.

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- Construct the homotopy retract for homology complexes

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Markus J. Pflaum, Hessel Posthuma, Xiang Tang (唐翔), *On the Hochschild homology of proper Lie groupoids*, J. Noncommut. Geom. **17** (2023), no. 1, 101-162.