An introduction to noncommutative Gröbner-Shirshov(=GS) bases with applications to homological algebra
Talk 0: History of GS bases and plan of the course

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September 12, 2025

Plan

- History of Gröbner-shirshov (=GS) bases
- Plan of the course

Part I: Origin of Gröbner-shirshov (=GS) bases

- Francis Macaulay 1916, homogeneous polynomials
- Maurice Janet 1920, differential polynomials
- Grete Hermann 1926, polynomial ideals in computability theory

Francis S. Macaulay, *The algebraic theory of modular systems.* Revised reprint of the 1916 original. With an introduction by Paul Roberts Cambridge Math. Lib. Cambridge University Press, Cambridge, 1994. xxxii+112 pp. ISBN:0-521-45562-6

Maurice Janet, *Sur les systèmes d'équations aux dérivées partielles.* C. R. Acad. Sci. **170** (1920), 1101-1103.

Grete Hermann, *Die Frage der endlich vielen Schritte in der Theorie der Polynomideale.* Math. Ann. **95** (1926), no. 1, 736-788.

Part I: Origin of Gröbner-shirshov (=GS) bases

- Anatoly I. Shirshov 1962, Lie algebras
- Heisuke Hironaka 1964, power series
- Bruno Buchberger 1965, polynomial algebras

Anatoly I. Shirshov, *Some algorithmic problem for Lie algebras*, Sibirsk. Mat. Zh. **3** (2) (1962) 292 – 296 (in Russian). English translation: SIGSAM Bull. **33** (2) (1999) 3 – 6.

Heisuke Hironaka (广中平佑), *Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II.* Ann. of Math. (2) **79** (1964), 109 – 203; **79** (1964), 205-326.

Bruno Buchberger, *An algorithm for finding the basis elements of the residue class ring of a zero dimensional polynomial ideal.* Translated from the 1965 German original by Michael P. Abramson, J. Symbolic Comput. **41** (2006), no. 3-4, 475-511.

From the review by Martin Kreuzer on MR4211773

L. A. Bokut, Yuqun Chen (陈裕群), Kyriakos Kalorkoti, Pavel Kolesnikov,Viktor Lopatkin, *Gröbner-Shirshov bases, Normal forms, combinatorial and decision problems in algebra*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2020, xxii+285 pp. ISBN: 978-981-4619-48-6

Remark (from Martin Kreuzer)

A well-known proverb says: "Success has many fathers, while failure is an orphan." This is true for many mathematical discoveries, and the theory of Gröbner bases is no exception.
As one colleague put it provocatively: apparently the only mathematician who is not said to have invented Gröbner bases is Wolfgang Gröbner.

Part I: History of GS bases for noncommutative associative algebras

GS bases for noncommutative associative algebras

- Bokut 1976
- Bergman 1978

L.A. Bokut, *Imbeddings into simple associative algebras*, Algebra Logika **15** (1976) 117-142.

George M. Bergman, *The diamond lemma for ring theory.* Adv. in Math. **29** (1978), no. 2, 178-218.

Teo Mora, *An introduction to commutative and noncommutative Gröbner bases.* Second International Colloquium on Words, Languages and Combinatorics (Kyoto, 1992) Theoret. Comput. Sci. 134 (1994), no. 1, 131 – 173.

Edward L. Green, *An introduction to noncommutative Gröbner bases.* Computational algebra (Fairfax, VA, 1993), 167-190. Lecture Notes in Pure and Appl. Math., **151** Marcel Dekker, Inc., New York, 1994

GS bases for categories and for strict premonoidal categories

- Bokut-Chen-Li 2012, GS bases for categories
- Bokut-Chen 2024, GS bases for strict monoidal categories
- Elias 2022, GS bases for strict monoidal categories
- Li-Song-Zhou 2025, GS bases for strict premonoidal categories

L. A. Bokut, Yuqun Chen (陈裕群), Yu Li (李羽), *Gröbner-Shirshov bases for categories*. Operads and universal algebra, 1-23. Nankai Ser. Pure Appl. Math. Theoret. Phys., **9** World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2012.

L. A. Bokut, Yuqun Chen (陈裕群), *Gröbner-Shirshov bases for strict monoidal categories*. Southeast Asian Bull. Math. **48** (2024), no. 5, 635-650.

Ben Elias, *A diamond lemma for Hecke-type algebras*, Trans. Amer. Math. Soc. **375** (2022), no. 3, 1883 – 1915.

Ziling Li (李子玲), Linliang Song (宋林亮) Guodong Zhou, *Gröbner-Shirshov bases for strict premonoidal categories with applications to Lie theory*, preprint in preparation 2025.

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- L. A. Bokut, Yuqun Chen (陈裕群), Kyriakos Kalorkoti, Pavel Kolesnikov,Viktor Lopatkin, *Gröbner-Shirshov bases, Normal forms, combinatorial and decision problems in algebra*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2020, xxii+285 pp. ISBN: 978-981-4619-48-6
- Huishi Li (李会师), *Gröbner bases in ring theory.* World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2012. x+284 pp.

GS bases for operads and for operated algebras

- Hoffbeck 2010, Dotsenko-Khoroshkin 2010, GS bases for operads
- Gao-Guo 2017, GS bases for operated algebras

Eric Hoffbeck, *A Poincaré-Birkhoff-Witt criterion for Koszul operads*, Manuscripta Math. **131** (2010), 87-110.

Vladimir Dotsenko, Anton Khoroshkin, *Gröbner bases for operads.* Duke Math. J. **153** (2010), no. 2, 363 – 396.

Xing Gao (高兴), Li Guo (郭锂), *Rota's classification problem, rewriting systems and Gröbner-Shirshov bases*. J. Algebra **470** (2017), 219-253.

Xing Gao (高兴), Huhu Zhang (张虎虎), Li Guo (郭锂), *Rota's program on algebraic operators, rewriting systems and Gröbner-Shirshov bases* Adv. Math. (China) **51** (2022), no. 1, 1-31.

Part I: Rewriting systems and polygraphic resolutions

- Donald E. Knuth, Peter B. Bendix, *Simple word problems in universal algebras*. Computational Problems in Abstract Algebra (Proc. Conf., Oxford, 1967), pp. 263-297 Pergamon Press, Oxford-New York-Toronto, Ont., 1970
- Yves Guiraud, Philippe Malbos, *Higher-dimensional normalisation strategies for acyclicity*. Adv. Math., **231**(2012) no. 3-4, 2294-2351.
- Yves Guiraud, Philippe Malbos, *Polygraphs of finite derivation type.* Math. Structures Comput. Sci., **28** (2018) no. 2, 155 – 201.
- Yves Guiraud, Eric Hoffbeck, Philippe Malbos, *Convergent presentations and polygraphic resolutions of associative algebras*, Math. Z. **293** (2019), no. 1-2, 113-179.
- Philippe Malbos, Isaac Ren, *Shuffle polygraphic resolutions for operads*, J. Lond. Math. Soc. (2) **107** (2023), no. 1, 61-122.
- Zuan Liu (刘钻), Philippe Malbos, *Polygraphic resolutions for operated algebras*, arXiv:2502.16304
- Dimitri Ara, Albert Burroni, Yves Guiraud, Philippe Malbos, François Métayer, Samuel Mimram, *Polygraphs: from rewriting to higher categories*, London Math. Soc. Lecture Note Ser., **495** Cambridge University Press, Cambridge, 2025, xx+648 pp. ISBN: 978-1-009-49898-2

Lecture I: Linear GS bases

Daniel R. Farkas, Edward L. Green, *Do subspaces have distinguished bases?* Rocky Mountain J. Math. **22** (1992), no. 4, 1295-1302.

Lecture II: Noncommutative GS bases

George M. Bergman, *The diamond lemma for ring theory.* Adv. Math. **29** (1978), no. 2, 178-218.

Teo Mora, *An introduction to commutative and noncommutative Gröbner bases.* Second International Colloquium on Words, Languages and Combinatorics (Kyoto, 1992) Theoret. Comput. Sci. 134 (1994), no. 1, 131 – 173.

Edward L. Green, *An introduction to noncommutative Gröbner bases.* Computational algebra (Fairfax, VA, 1993), 167-190. Lecture Notes in Pure and Appl. Math., **151** Marcel Dekker, Inc., New York, 1994

Lecture III: Applications to (two-sided) Anick resolutions

- D. J. Anick, *On the Homology of associative algebras*. Trans. Amer. Math. Soc. **296** (1986) 641-659.
- D. J. Anick and E. L. Green, *On the homology of quotients of path algebras*. Comm. Algebra **15** (1987), no. 1-2, 309-341.
- E. Sköldberg, *Morse Theory from an Algebric Viewpoint*. Trans. Amer. Math. Soc. **358** (2005), 115-129.
- Jun Chen (陈骏), Yuming Liu (刘玉明), Guodong Zhou (周国栋), *Algebraic Morse theory via homological perturbation lemma*, arXiv:2404.10165, J. Pure Appl. Algebra accepted.

Lecture IV: Applications to Koszul algebras (and PBW deformations)

- PBW algebras are Koszul

Stewart B. Priddy, *Koszul resolutions*, Trans. Amer. Math. Soc. **152** (1970), 39-60.

Edward Green, Rosa Q. Huang, *Projective resolutions of straightening closed algebras generated by minors*, Adv. Math. **110** (1995), no. 2, 314-333.

Lecture IV: Applications to Koszul algebras (and PBW deformations)

- PBW deformations of Koszul algebras

Stewart B. Priddy, *Koszul resolutions*, Trans. Amer. Math. Soc. **152** (1970), 39-60.

A.A. Beilinson, V. Ginzburg, W. Soergel, *Koszul duality patterns in representation theory*, J. Amer. Math. Soc. **9** (1996) 473-527.
L. E. Positselski, Nonhomogeneous quadratic duality and curvature, Functional Analysis and its Applications 27 (1993), 197 – 204.

A. Polishchuk, L. Positselski, *Quadratic Algebras*, Univ. Lecture Ser., vol. 37, Amer. Math. Soc., Providence. RI. 2005.

A. Braverman, D. Gaitsgory, *Poincaré-Birkhoff-Witt theorem for quadratic algebras of Koszul type*, J. Algebra **181** (1996) 315-328.

Part II: Plan of the course and references

Lecture IV: Applications to Koszul algebras (and PBW deformations)

- PBW deformations of N-Koszul algebras

R. Berger, *Koszulity for nonquadratic algebras*, J. Algebra **239** (2001) 705-734.

Y. Ye (叶郁), P. Zhang (章璞), *Higher Koszul modules*. Sci. China Ser. A. **32** (2002) 1042-1049

E.L. Green, E.N. Marcos, R. Martínez-Villa, Pu Zhang (章璞), *D-Koszul algebras*, J. Pure Appl. Algebra **193** (1-3) (2004) 141-162.

R. Berger, V. Ginzburg, *Higher symplectic reflection algebras and nonhomogeneous N-Koszul property*, J. Algebra **304** (2006), 577-601.

G. Fløystad, J. E. Vatne, *PBW-deformations of N-Koszul algebras*. J. Algebra **302** (2006), 116-155.

E. Herscovich, A. Solotar, and M. Suárez-Álvarez, *PBW-deformations and deformation à la Gerstenhaber of N-Koszul algebras*, J. Noncommut. Geom. **8**(2) (2014), 505-539.

Lecture IV: Applications to Koszul algebras (and PBW deformations)

- Generalised PBW property (over a field)

T. Cassidy and B. Shelton, *PBW-deformation theory and regular central extensions*, J. Reine Angew. Math. **610** (2007), 1-12.

Huishi Li (李会师), *The general PBW property*, Algebra Colloq. **14** (2007), no. 4, 541-554.