CIMPA School: Perspectives in non-commutative algebras

Model structures on categories

Lecture 5: Homological dimensions and model structures

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Gorenstein-projective modules

Definition Let $\mathcal{A}$ be an abelian category, with enough projective objects. An object M is **Gorenstein-projective**, if it is a syzygy of exact sequence of projective objects, i.e.,

$$\begin{tikzcd} P^\bullet : & \cdots \arrow[r] & P^{-1} \arrow[r, "d^{-1}"] & P^0 \arrow[rr, "d^0"] \arrow[dr, two heads] & & P^1 \arrow[r] & \cdots \\ & & & & M \arrow[ur, hook] & & \end{tikzcd}$$

s.t. $\mathrm{Hom}_{\mathcal{A}}(P^\bullet, Q)$ for any projective object Q.

If $\mathcal{A} = R$-Mod, where R is a ring, or, $\mathcal{A} = R$-mod, where R is a two-sided noetherian ring, then Gorenstein-projective objects are called **Gorenstein-projective modules**.

If $\mathcal{A} = R$-Mod, this def. is due to E. Enochs - O. Jenda, 1995.

If $\mathcal{A} = R$-mod for a two-sided noetherian ring R, M. Auslander introduced in 1967 conditions (G1), (G2), (G3), for fin. gen. modules, under the name of **modules of G-dim. zero**. This is an equiv. def..

Reflexive modules

Let R be a ring, M a left R-module M. Put $M^* = \mathrm{Hom}_R(M, R)$, the R-dual of M. This is a right R-module.

A canonical R-homomorphism $\phi_M : M \longrightarrow M^{**}$:

$$\phi_M(m)(f) = f(m), \ \forall\ m \in M,\ f \in M^*.$$

A module M is **torsionless**, if ϕ_M is injective.

A module M is **reflexive**, if ϕ_M is bijective.

Semi-Gorenstein-projective modules

Let R be a ring. An R-module M is **semi-Gorenstein-projective**, if

$$M \in {}^{\perp}R, \quad \text{i.e.,} \quad \mathrm{Ext}^i_R(M, R) = 0, \ \forall\ i \geq 1.$$

Note that

$$\{\text{projective module}\} \subseteq \{\text{Gorenstein-projective module}\} \subseteq {}^{\perp}R.$$

If R is Gorenstein, i.e., R is a two-sided notherian ring with

$$\mathrm{inj.dim}_R R < \infty, \text{ and } \mathrm{inj.dim} R_R < \infty$$

then

$$\mathcal{GP} = {}^{\perp}R.$$

Gorenstein-projective modules (continued)

Theorem (M. Auslander, E. E. Enochs - O. M. G. Jenda, L. W. Christensen, L. L. Avramov, R.-O. Buchweitz, A. Martsinkovsky, I. Reiten) Let A be an artin algebra, and $M \in A$-mod. Then the following are equivalent.

(1) M satisfies the conditions (G1), (G2) and (G3) :

  (G1) M is semi-Gor.-proj., i.e., $\mathrm{Ext}^i_A(M, A) = 0, \ \forall \ i \geq 1$.

  (G2) $M^* = \mathrm{Hom}(M, \ _AA)$ is semi-Gor.-proj..

  (G3) M is reflexive.

(2) M is Gorenstein-projective.

(3) M and $\mathrm{Tr}M$ are semi-Gorenstein-projective.

Remark **Theorem** holds also for a two-sided noetherian ring.

If M is not fin. gen., then the Theorem above does not hold.

A property of Gorenstein-projective objects

Let $\mathcal{A}$ be an abelian category with enough projective objects. Then the full subcategory $\mathcal{GP}$ of Gorenstein-projective objects is

- closed under direct summands; the kernel of epimorphisms; extensions. (M. Auslander - M. Bridger, 1969)

- a Forbenius category, and the projective-injective objects of $\mathcal{GP}$ is precisely the projective objects. Thus the stable category $\mathcal{GP}/\mathcal{P}$ is triangulated.

Moreover, if R is a Gorenstein ring, then $\mathcal{GP}/\mathcal{P}$ is the singularity category $D^b_{\mathrm{sg}}(R) := D^b(R)/K^b(\mathcal{P})$. (R. O. Buchweitz, 1987)

Various Gorenstein modules

Let R be a ring. An R-module M is **Gorenstein-flat**, if it is a syzygy of exact sequence of flat R-modules, which is exact after tensoring with any right injective R-module.

An R-module M is **PGF** (projectively coresolved Gorenstein flat, Šaroch - Šťovíček, 2020), if it is a syzygy of exact sequence of projective R-modules, which is exact after tensoring with any right injective R-module.

$$\mathcal{PGF} \subseteq \mathcal{GF}; \quad \mathcal{PGF} \subseteq \mathcal{GP} \quad ([\mathrm{SS}]); \quad \mathcal{PGF} = \mathcal{GP} \Longleftrightarrow \mathcal{GP} \subseteq \mathcal{GF}$$

Open: $\mathcal{GP} \subseteq \mathcal{GF}$? $\mathcal{GP} \subseteq \mathcal{PGF}$?

$$\mathrm{Gpd}M \leq \mathrm{PGFd}M \leq \mathrm{pd}M; \;\; \mathrm{fd}M \leq \mathrm{pd}M; \;\; \mathrm{Gfd}M \leq \min\{\mathrm{PGFd}M, \ \mathrm{fd}M\}$$

$$\begin{tikzcd}
 & \mathcal{F}_n \arrow[dr, hook] & \\
\mathcal{P}_n \arrow[ur, hook] \arrow[dr, hook] & & \mathcal{GF}_n \\
 & \mathcal{PGF}_n \arrow[ur, hook] \arrow[r, hook] & \mathcal{GP}_n
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathcal{F}^{<\infty} \arrow[dr, hook] & \\
\mathcal{P}^{<\infty} \arrow[ur, hook] \arrow[dr, hook] & & \mathcal{GF}^{<\infty} \\
 & \mathcal{PGF}^{<\infty} \arrow[ur, hook] \arrow[r, hook] & \mathcal{GP}^{<\infty}.
\end{tikzcd}$$

Known results: Hereditary complete ctps & abelian model struc. in R-Mod

	$n=0$	n	$<\infty$
$\mathcal{P}$	$(\mathcal{P},\ R\text{-Mod})$	$(\mathcal{P}_n,\ \mathcal{P}_n^{\perp})$	$(\mathcal{P}^{<\infty},\ -)$?
$\mathcal{F}$	$(\mathcal{F},\ \mathcal{C})$ Enochs, Jenda	$(\mathcal{F}_n,\ \mathcal{F}_n^{\perp})$	$(\mathcal{F}^{<\infty},\ -)$?
$\mathcal{PGF}$	$(\mathcal{PGF},\ \mathcal{PGF}^{\perp})$ $(\mathcal{PGF},\ R\text{-Mod},\ \mathcal{PGF}^{\perp})$ Šaroch, Šťovíček, 2020	$(\mathcal{PGF}_n, -)$? $(\mathcal{PGF}_n,\ -,\ -)$?	$(\mathcal{PGF}^{<\infty},\ -)$? $(\mathcal{PGF}^{<\infty},\ -,\ -)$?
$\mathcal{GP}$	For Artin algebras $(\mathcal{GP},\ \mathcal{GP}^{\perp})$ $(\mathcal{GP},\ R\text{-Mod},\ \mathcal{GP}^{\perp})$ Beligiannis, Reiten	For Artin algebras $(\mathcal{GP}_n,\ -)$? $(\mathcal{GP}_n,\ -,\ -)$?	For Artin algebras $(\mathcal{GP}^{<\infty},\ -)$? $(\mathcal{GP}^{<\infty},\ -,\ -)$?
$\mathcal{GF}$	$(\mathcal{GF},\ \mathcal{C}\cap\mathcal{PGF}^{\perp})$ $(\mathcal{GF},\ \mathcal{C},\ \mathcal{PGF}^{\perp})$ Šaroch, Šťovíček, 2020	$(\mathcal{GF}_n,\ \mathcal{F}_n^{\perp}\cap\mathcal{PGF}^{\perp})$ $(\mathcal{GF}_n,\ \mathcal{F}_n^{\perp},\ \mathcal{PGF}^{\perp})$ Maaouy, 2023	$(\mathcal{GF}^{<\infty},\ -)$? $(\mathcal{GF}^{<\infty},\ -,\ -)$?

Abelian model structures from $\mathcal{PGF}_n$ and $\mathcal{GP}_n$

Theorem ([GLZ], 2024) R: a ring, $n \geq 0$ an integer. Then

(1) $(\mathcal{PGF}_n,\ \mathcal{P}_n^{\perp} \cap \mathcal{PGF}^{\perp})$ is a hered. complete ctp in R-Mod, with core $\mathcal{P}_n \cap \mathcal{P}_n^{\perp}$.

(2) $(\mathcal{PGF}_n,\ \mathcal{P}_n^{\perp},\ \mathcal{PGF}^{\perp})$ is a hered. model structure on R-Mod, with the homotopy cat.

$$(\mathcal{PGF}_n \cap \mathcal{P}_n^{\perp})/(\mathcal{P}_n \cap \mathcal{P}_n^{\perp}) \overset{\triangle\text{ equiv.}}{\cong} \mathcal{PGF}/\mathcal{P}.$$

Theorem Let A be an Artin algebra, $n \geq 0$ an integer. Then

(1) $(\mathcal{GP}_n,\ \mathcal{P}_n^{\perp} \cap \mathcal{GP}^{\perp})$ is a hered. complete ctp in A-Mod, with core $\mathcal{P}_n \cap \mathcal{P}_n^{\perp}$.

(2) $(\mathcal{GP}_n,\ \mathcal{P}_n^{\perp},\ \mathcal{GP}^{\perp})$ is a hered. model structure on R-Mod, with the homotopy cat.

$$(\mathcal{GP}_n \cap \mathcal{P}_n^{\perp})/(\mathcal{P}_n \cap \mathcal{P}_n^{\perp}) \overset{\triangle\text{ equiv.}}{\cong} \mathcal{GP}/\mathcal{P}.$$

The global view: Hereditary complete ctps & abelian model struc. in R-Mod

	$n = 0$	n	$< \infty$
$\mathcal{P}$	$(\mathcal{P},\ R\text{-Mod})$	$(\mathcal{P}_n,\ \mathcal{P}_n^{\perp})$	If Fpd $< \infty$, then $(\mathcal{P}^{<\infty},\ (\mathcal{P}^{<\infty})^{\perp})$
$\mathcal{F}$	$(\mathcal{F},\ \mathcal{C})$ Enochs, Jenda	$(\mathcal{F}_n,\ \mathcal{F}_n^{\perp})$	If Fpd $< \infty$, then $(\mathcal{F}^{<\infty},\ (\mathcal{F}^{<\infty})^{\perp})$
$\mathcal{PGF}$	$(\mathcal{PGF},\ \mathcal{PGF}^{\perp})$ $(\mathcal{PGF},\ R\text{-Mod},\ \mathcal{PGF}^{\perp})$ Šaroch, Šťovíček, 2020	$(\mathcal{PGF}_n,\ \mathcal{P}_n^{\perp} \cap \mathcal{PGF}^{\perp})$ $(\mathcal{PGF}_n,\ \mathcal{P}_n^{\perp},\ \mathcal{PGF}^{\perp})$	If Fpd $< \infty$, then $(\mathcal{PGF}^{<\infty},\ (\mathcal{P}^{<\infty})^{\perp} \cap \mathcal{PGF}^{\perp})$ $(\mathcal{PGF}^{<\infty},\ (\mathcal{P}^{<\infty})^{\perp},\ \mathcal{PGF}^{\perp})$
$\mathcal{GP}$	For Artin algebras $(\mathcal{GP},\ \mathcal{GP}^{\perp})$ $(\mathcal{GP},\ R\text{-Mod},\ \mathcal{GP}^{\perp})$ Beligiannis, Reiten	For Artin algebras $(\mathcal{GP}_n,\ \mathcal{P}_n^{\perp} \cap \mathcal{GP}^{\perp})$ $(\mathcal{GP}_n,\ \mathcal{P}_n^{\perp},\ \mathcal{GP}^{\perp})$	For Artin algebras If Fpd $< \infty$, then $(\mathcal{GP}^{<\infty},\ (\mathcal{P}^{<\infty})^{\perp} \cap \mathcal{GP}^{\perp})$ $(\mathcal{GP}^{<\infty},\ (\mathcal{P}^{<\infty})^{\perp},\ \mathcal{GP}^{\perp})$
$\mathcal{GF}$	$(\mathcal{GF},\ \mathcal{C} \cap \mathcal{PGF}^{\perp})$ $(\mathcal{GF},\ \mathcal{C},\ \mathcal{PGF}^{\perp})$ Šaroch, Šťovíček, 2020	$(\mathcal{GF}_n,\ \mathcal{F}_n^{\perp} \cap \mathcal{PGF}^{\perp})$ $(\mathcal{GF}_n,\ \mathcal{F}_n^{\perp},\ \mathcal{PGF}^{\perp})$ Maaouy, 2023	If Fpd $< \infty$, then $(\mathcal{GF}^{<\infty},\ (\mathcal{F}^{<\infty})^{\perp} \cap \mathcal{PGF}^{\perp})$ $(\mathcal{GF}^{<\infty},\ (\mathcal{F}^{<\infty})^{\perp},\ \mathcal{PGF}^{\perp})$

Hered. complete ctps. in exact subcategories

Key Lemma Let $\mathcal{A}$ be an abelian category, and $\mathcal{B}$ a full subcategory of $\mathcal{A}$, which is closed under extensions. Suppose that $(\mathcal{X},\ \mathcal{Y})$ is a complete cotorsion pair in $\mathcal{A}$ and $\mathcal{X} \subseteq \mathcal{B}$. If $\mathcal{B}$ is closed under the kernels of epimorphisms, then

(1) $(\mathcal{X},\ \mathcal{Y} \cap \mathcal{B})$ is a complete cotorsion pair in exact category $\mathcal{B}$.

(2) If $(\mathcal{X},\ \mathcal{Y})$ is hereditary, then $(\mathcal{X},\ \mathcal{Y} \cap \mathcal{B})$ is hereditary.

This observation is one of the main methods to obtain hered. complete cotorsion pairs in exact subcategories, and to obtain exact model structures in exact subcategories.

Application: Exact model structures on $\mathcal{PGF}_n$

For $n \geq 0$, one has in R-Mod hered. complete ctps $(\mathcal{X},\ \mathcal{Y})$ satisfying $\mathcal{X} \subseteq \mathcal{PGF}_n$:

$$(\mathcal{P}_m,\ \mathcal{P}_m^{\perp}), \qquad (\mathcal{PGF}_m,\ \mathcal{P}_m^{\perp} \cap \mathcal{PGF}^{\perp}), \qquad 0 \leq m \leq n.$$

Applying **Key Lemma** $\mathcal{A}$: abelian cat., $\mathcal{B}$: a full subcat. which is closed under extensions. Suppose that $(\mathcal{X},\ \mathcal{Y})$ is a (hereditary) complete ctp in $\mathcal{A}$ and $\mathcal{X} \subseteq \mathcal{B}$. If $\mathcal{B}$ is closed under the kernels of epimorphisms, then $(\mathcal{X},\ \mathcal{Y} \cap \mathcal{B})$ is a (hereditary) complete ctp in $\mathcal{B}$.

Theorem R: a ring, $m \geq 0$ and $n \geq 0$ integers, $m \leq n$. Then

(1) $(\mathcal{P}_m,\ \mathcal{P}_m^{\perp} \cap \mathcal{PGF}_n)$ is a hered. complete ctp in $\mathcal{PGF}_n$.

(2) $(\mathcal{PGF}_m,\ \mathcal{P}_m^{\perp} \cap \mathcal{P}_n)$ is a hered. complete ctp in $\mathcal{PGF}_n$.

(3) $(\mathcal{PGF}_m,\ \mathcal{P}_m^{\perp} \cap \mathcal{PGF}_n,\ \mathcal{P}_n)$ is a hered. model structure on $\mathcal{PGF}_n$, with homotopy cat. $(\mathcal{PGF}_m \cap \mathcal{P}_m^{\perp})/(\mathcal{P}_m \cap \mathcal{P}_m^{\perp}) \overset{\triangle\ \text{equiv.}}{\cong} \mathcal{PGF}/\mathcal{P}$.

The global view: Hereditary complete ctps & exact model struc. in exact categories

	$n=0$	n $0 \le m \le n$	$< \infty$ m
$\mathcal{P}$	In $\mathcal{P}$ $(\mathcal{P},\ \mathcal{P})$	In $\mathcal{P}_n$ $(\mathcal{P}_m,\ \mathcal{P}_m^{\perp} \cap \mathcal{P}_n)$	In $\mathcal{P}^{<\infty}$ $(\mathcal{P}_m,\ \mathcal{P}_m^{\perp} \cap \mathcal{P}^{<\infty})$
$\mathcal{F}$	In $\mathcal{F}$ $(\mathcal{F},\ \mathcal{C} \cap \mathcal{F})$	In $\mathcal{F}_n$ $(\mathcal{P}_m,\ \mathcal{P}_m^{\perp} \cap \mathcal{F}_n)$ $(\mathcal{F}_m,\ \mathcal{F}_m^{\perp} \cap \mathcal{F}_n)$	In $\mathcal{F}^{<\infty}$ $(\mathcal{P}_m,\ \mathcal{P}_m^{\perp} \cap \mathcal{F}^{<\infty})$ $(\mathcal{F}_m,\ \mathcal{F}_m^{\perp} \cap \mathcal{F}^{<\infty})$
$\mathcal{PGF}$	In $\mathcal{PGF}$ $(\mathcal{PGF},\ \mathcal{P})$ $(\mathcal{PGF},\ \mathcal{PGF},\ \mathcal{P})$	In $\mathcal{PGF}_n$ $(\mathcal{P}_m,\ \mathcal{P}_m^{\perp} \cap \mathcal{PGF}_n)$ $(\mathcal{PGF}_m,\ \mathcal{P}_m^{\perp} \cap \mathcal{P}_n)$ $(\mathcal{PGF}_m,\ \mathcal{P}_m^{\perp} \cap \mathcal{PGF}_n,\ \mathcal{P}_n)$	In $\mathcal{PGF}^{<\infty}$ $(\mathcal{P}_m,\ \mathcal{P}_m^{\perp} \cap \mathcal{PGF}^{<\infty})$ $(\mathcal{PGF}_m,\ \mathcal{P}_m^{\perp} \cap \mathcal{P}^{<\infty})$ $(\mathcal{PGF}_m,\ \mathcal{P}_m^{\perp} \cap \mathcal{PGF}^{<\infty},\ \mathcal{P}^{<\infty})$
$\mathcal{GP}$	In $\mathcal{GP}$ $(\mathcal{PGF},\ \mathcal{PGF}^{\perp} \cap \mathcal{GP})$ $(\mathcal{GP},\ \mathcal{P})$ $(\mathcal{PGF},\ \mathcal{GP},\ \mathcal{PGF}^{\perp} \cap \mathcal{GP})$ $(\mathcal{GP},\ \mathcal{GP},\ \mathcal{P})$	In $\mathcal{GP}_n$ $(\mathcal{P}_m,\ \mathcal{P}_m^{\perp} \cap \mathcal{GP}_n)$ $(\mathcal{PGF}_m,\ \mathcal{P}_m^{\perp} \cap \mathcal{PGF}^{\perp} \cap \mathcal{GP}_n)$ $(\mathcal{GP}_m,\ \mathcal{P}_m^{\perp} \cap \mathcal{P}_n)$ $(\mathcal{PGF}_m,\ \mathcal{P}_m^{\perp} \cap \mathcal{GP}_n,\ \mathcal{PGF}^{\perp} \cap \mathcal{GP}_n)$ $(\mathcal{GP}_m,\ \mathcal{P}_m^{\perp} \cap \mathcal{GP}_n,\ \mathcal{P}_n)$	In $\mathcal{GP}^{<\infty}$ $(\mathcal{P}_m,\ \mathcal{P}_m^{\perp} \cap \mathcal{GP}^{<\infty})$ $(\mathcal{PGF}_m,\ \mathcal{P}_m^{\perp} \cap \mathcal{PGF}^{\perp} \cap \mathcal{GP}^{<\infty})$ $(\mathcal{GP}_m,\ \mathcal{P}_m^{\perp} \cap \mathcal{P}^{<\infty})$ $(\mathcal{PGF}_m,\ \mathcal{P}_m^{\perp} \cap \mathcal{GP}^{<\infty},\ \mathcal{PGF}^{\perp} \cap \mathcal{GP}^{<\infty})$ $(\mathcal{GP}_m,\ \mathcal{P}_m^{\perp} \cap \mathcal{GP}^{<\infty},\ \mathcal{P}^{<\infty})$
$\mathcal{GF}$	In $\mathcal{GF}$ $(\mathcal{F},\ \mathcal{C} \cap \mathcal{GF})$ $(\mathcal{GF},\ \mathcal{F} \cap \mathcal{C})$ $(\mathcal{GF},\ \mathcal{C} \cap \mathcal{GF},\ \mathcal{F})$ $(\mathcal{PGF},\ \mathcal{F})$ $(\mathcal{PGF},\ \mathcal{GF},\ \mathcal{F})$ Dalezios, Emmanouil, 2022	In $\mathcal{GF}_n$ $(\mathcal{P}_m,\ \mathcal{P}_m^{\perp} \cap \mathcal{GF}_n)$ $(\mathcal{F}_m,\ \mathcal{F}_m^{\perp} \cap \mathcal{GF}_n)$ $(\mathcal{PGF}_m,\ \mathcal{F}_n \cap \mathcal{P}_m^{\perp})$ $(\mathcal{GF}_m,\ \mathcal{F}_n \cap \mathcal{F}_m^{\perp})$ $(\mathcal{PGF}_m,\ \mathcal{P}_m^{\perp} \cap \mathcal{GF}_n,\ \mathcal{F}_n)$ $(\mathcal{GF}_m,\ \mathcal{F}_m^{\perp} \cap \mathcal{GF}_n,\ \mathcal{F}_n)$	In $\mathcal{GF}^{<\infty}$ $(\mathcal{P}_m,\ \mathcal{P}_m^{\perp} \cap \mathcal{GF}^{<\infty})$ $(\mathcal{F}_m,\ \mathcal{F}_m^{\perp} \cap \mathcal{GF}^{<\infty})$ $(\mathcal{PGF}_m,\ \mathcal{F}^{<\infty} \cap \mathcal{P}_m^{\perp})$ $(\mathcal{GF}_m,\ \mathcal{F}^{<\infty} \cap \mathcal{F}_m^{\perp})$ $(\mathcal{PGF}_m,\ \mathcal{P}_m^{\perp} \cap \mathcal{GF}^{<\infty},\ \mathcal{F}^{<\infty})$ $(\mathcal{GF}_m,\ \mathcal{F}_m^{\perp} \cap \mathcal{GF}^{<\infty},\ \mathcal{F}^{<\infty})$