CIMPA School: Perspectives in non-commutative algebras

Model structures on categories

Lecture 4: Weakly projective model structures and $\mathcal{W}$-model structures

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Weakly proj. model structures on exact categories

Definition (Beligiannis - Reiten, 2007) A model structure (Cofib, Fib, Weq) on exact category $\mathcal{A}$ is **weakly projective**, if

(i) $\mathrm{CoFib} = \{\text{inflation } f \mid \ \mathrm{Coker}\, f \in \mathcal{C}\}$,

(ii) $\mathrm{Fib} \cap \mathrm{Weq} \subseteq \{\text{deflation}\}$,

(iii) $\mathcal{F} = \mathcal{A}$.

Compare with yesterday: **An exact model structure**:

(i) $\mathrm{Cofib} = \{\text{inflation } f \mid \mathrm{Coker}\, f \in \mathcal{C}\}$;

(ii) $\mathrm{Fib} = \{\text{deflation } f \mid \mathrm{Ker}\, f \in \mathcal{F}\}$.

In this case, one has also

$\mathrm{Cofib} \cap \mathrm{Weq} = \{\text{inflation } f \mid \ \mathrm{Coker}\, f \in \mathcal{C} \cap \mathcal{W}\}$;

$\mathrm{Fib} \cap \mathrm{Weq} = \{\text{deflation } f \mid \ \mathrm{Ker}\, f \in \mathcal{F} \cap \mathcal{W}\}$.

Contravariantly finite subcategories

Definition (M. Auslander) $\mathcal{B}$: a full subcat. of cat. $\mathcal{A}$, $X \in \mathcal{A}$

• A morphism $B \xrightarrow{f} X$ with $B \in \mathcal{B}$ is **a right $\mathcal{B}$-approaximation of** X, if for any morphism $B' \xrightarrow{g} X$ with $B' \in \mathcal{B}$, there is a morphism $B' \xrightarrow{h} B$ s.t. $g = f \circ h$.

• Dually, a morphism $X \xrightarrow{f} B$ with $B \in \mathcal{B}$ is **a left $\mathcal{B}$-approaxim. of** X, if for any morphism $X \xrightarrow{g} B'$ with $B' \in \mathcal{B}$, there is a morphism $B \xrightarrow{h} B'$ s.t. $g = h \circ f$.

• $\mathcal{B}$ is **contravariantly finite** in $\mathcal{A}$, if each object $X \in \mathcal{A}$ has a right $\mathcal{B}$-approaximation.

• Dually, $\mathcal{B}$ is **covariantly finite** in $\mathcal{A}$, if each object $X \in \mathcal{A}$ has a left $\mathcal{B}$-approaximation.

• $\mathcal{B}$ is **functorially finite** in $\mathcal{A}$, if $\mathcal{B}$ is both contravariantly finite in $\mathcal{A}$ and covariantly finite in $\mathcal{A}$.

Examples: Contravariantly finite subcategories

- A: finite-dimensional algebra, $M \in A$-mod. Then add$M$ is functorially finite in A-mod.

  In particular, A-proj and A-inj are functorially finite in A-mod.

- If (CoFib, Fib, Weq) is a model structure on category $\mathcal{M}$ with zero object. Then $\mathcal{C}$ and $\mathcal{C} \cap \mathcal{W}$ are contravariantly finite in $\mathcal{M}$.

- If (CoFib, Fib, Weq) is a model structure on category $\mathcal{M}$ with zero object. Then $\mathcal{F}$ and $\mathcal{F} \cap \mathcal{W}$ are covariantly finite in $\mathcal{M}$.

Beligiannis-Reiten correspondence

Theorem (Beligiannis - Reiten, Mem. AMS, 2007) $\mathcal{A}$: weakly idempotent complete exact category

S_C: the class **hereditary and complete** ctps $(\mathcal{X}, \mathcal{Y})$ in $\mathcal{A}$, with $\omega = \mathcal{X} \cap \mathcal{Y}$ contravariantly finite in $\mathcal{A}$.

S_M: the class of weakly projective model structures on $\mathcal{A}$. Then

$(\mathcal{X}, \mathcal{Y}) \mapsto (\mathrm{CoFib}_\omega,\ \mathrm{Fib}_\omega,\ \mathrm{Weq}_\omega)$ is a 1-1 corresp. betw. S_C and S_M, where

$\mathrm{CoFib}_\omega = \{\text{inflation } f \mid \ \mathrm{Coker}\, f \in \mathcal{X}\}$,

$\mathrm{Fib}_\omega = \{\text{morphism } f \mid f \text{ is } \mathrm{Hom}_{\mathcal{A}}(\omega, f)\text{-epic}\}$.

$\mathrm{Weq}_\omega := \{f\colon A \to B \mid \exists \ \text{ a deflation } (f, t) : A \oplus W \to B,\ W \in \omega,\ \mathrm{Ker}(f, t) \in \mathcal{Y}\}$;

$\mathrm{TCoFib}_\omega = \{\text{splitting monomorphism } f \mid \ \mathrm{Coker}\, f \in \omega\}$,

$\mathrm{TFib}_\omega = \{\text{deflations } f \mid \ \mathrm{Ker}\, f \in \mathcal{Y}\}$,

$(\mathcal{C},\ \mathcal{F},\ \mathcal{W}) = (\mathcal{X},\ \mathcal{A},\ \mathcal{Y})$, $\ \mathrm{Ho}(\mathcal{A}) = \mathcal{A}/\omega$.

This is also called a ω-model structure.

The inverse is given by $(\mathrm{CoFib}, \mathrm{Fib}, \mathrm{Weq}) \mapsto (\mathcal{C}, \mathrm{T}\mathcal{F})$.

When the ω-model structure is exact?

Prop. $\mathcal{A}$: a weakly idempotent complete exact cat. $(\mathcal{X}, \mathcal{Y})$ a hereditary complete ctp with $\omega = \mathcal{X} \cap \mathcal{Y}$ contravariantly finite.

Then the model struc. $(\mathrm{CoFib}_\omega, \mathrm{Fib}_\omega, \mathrm{Weq}_\omega)$ is exact $\Longleftrightarrow$ $\mathcal{A}$ has enough proj. objs. and $\omega = \mathcal{P}$.

Proof. $\Longleftarrow$: If $\omega = \mathcal{P}$ and $f : A \to B$ is ω-epic, taking a deflation $g : P \to B$, then $g = f \circ h$ with $h : P \to A$. Since $(\mathcal{A}, \mathcal{E})$ is w. idemp. complete, f is a deflation. So $\mathrm{Fib}_\omega = \{f \mid f \text{ is } \omega\text{-epic}\} = \{\text{deflation}\}$. Thus by the construction and the definition of an exact model structure, $(\mathrm{CoFib}_\omega, \mathrm{Fib}_\omega, \mathrm{Weq}_\omega)$ is an exact model structure.

$\Longrightarrow$: Assume that the model struc. $(\mathrm{CoFib}_\omega, \mathrm{Fib}_\omega, \mathrm{Weq}_\omega)$ is exact. Then by the construction and the Hovey correspondence one has that $(\mathcal{X}, \ \mathcal{A}, \ \mathcal{Y})$ is a Hovey triple, thus $(\omega, \mathcal{A})$ is a ctp, so $\omega = \mathcal{P}$. $\square$

Definition An exact model structure on exact category $\mathcal{A}$ is **projective**, if $\mathcal{F} = \mathcal{A}$.

Thus, on a weakly idempotent complete exact category, a model structure is both exact and weakly projective, iff it is projective.

A weakly model structure which is **not** exact

A : Artin algebra $\qquad$ T: a finitely generated tilting module

$\mathcal{P}^{<\infty}$: the full subcategory of A-mod of finite projective dimension. It is a weakly idempotent complete exact category.

$\mathcal{P}^{<\infty}$ is not abelian, in general: it is abelian iff gl.dim$\Lambda < \infty$.

Then T is a tilting object in exact category $\mathcal{P}^{<\infty}$, i.e., $\mathrm{Ext}^i(T, T) = 0, \ \forall \ i \geq 1$ and $\mathrm{Thick}(T) = \mathcal{P}^{<\infty}$.

$\widetilde{\mathrm{add}(T)} = \{M \in A\text{-mod} \mid \text{有正合列 } 0 \to M \to T_0 \to \cdots \to T_s \to 0, \ T_i \in \mathrm{add}(T)\}$

$\widehat{\mathrm{add}(T)} = \{M \in A\text{-mod} \mid \text{有正合列 } 0 \to T_s \to \cdots \to T_0 \to M \to 0, \ T_i \in \mathrm{add}(T)\}$

$(\widetilde{\mathrm{add}\, T}, \ \widehat{\mathrm{add}\, T})$ is a herditary complete cotorsion pair in $\mathcal{P}^{<\infty}$, $\omega := \widetilde{\mathrm{add}\, T} \cap \widehat{\mathrm{add}\, T} = \mathrm{add}\, T$ is contravariantly finite in $\mathcal{P}^{<\infty}$.

If T is not projective, then the ω-model structure on $\mathcal{P}^{<\infty}$ induced by $(\widetilde{\mathrm{add}\, T}, \ \widehat{\mathrm{add}\, T})$ is **not** exact!

$\mathcal{W}$-model structures

定义 (X.S. Lu - $\sim$, 2024) Let $\mathcal{W}$ be a class of objects of additive category $\mathcal{A}$. A model structure (CoFib, Fib, Weq) on $\mathcal{A}$ is a $\mathcal{W}$-model structure, if $(\mathcal{C},\ \mathcal{F},\ \mathcal{W}) = (\mathcal{A},\ \mathcal{A},\ \mathcal{W})$.

In this case, $\mathcal{W}$ is closed under isomorphisms, direct summands, and finite direct sums.

Theorem (X.S. Lu - $\sim$, 2024) Let (CoFib, Fib, Weq) be model structure on additive category $\mathcal{A}$, $\mathcal{W}$ a class of objects of $\mathcal{A}$. Then (CoFib, Fib, Weq) is a $\mathcal{W}$-model structure iff

- $\mathrm{CoFib} = \{\text{morphism } f \mid f \text{ is } \mathrm{Hom}_{\mathcal{A}}(-,\mathcal{W})\text{-epic}\}$,
- $\mathrm{Fib} = \{\text{morphism } f \mid f \text{ is } \mathrm{Hom}_{\mathcal{A}}(\mathcal{W},-)\text{-epic}\}$,
- $\mathrm{TCoFib} = \{\text{splitting monic } f \mid \mathrm{Coker}\, f \in \mathcal{W}\}$,
- $\mathrm{TFib} = \{\text{splitting epic } f \mid \mathrm{Ker}\, f \in \mathcal{W}\}$,
- $\mathrm{Weq} = \mathrm{TFib} \circ \mathrm{TCoFib}$.

One-one correspondence

定理 ([LZ]) $\mathcal{A}$: weakly idempotent complete additive category.

Ω_F: the class of fuctorially finite subcategories of $\mathcal{A}$, which is closed under isomorphisms, direct summands

Ω_M: the class of $\mathcal{W}$-model structure on $\mathcal{A}$

Then $\mathcal{W} \mapsto (\mathrm{CoFib}_{\mathcal{W}},\ \mathrm{Fib}_{\mathcal{W}},\ \mathrm{Weq}_{\mathcal{W}})$ gives a one-one correspondence between Ω_F and Ω_M, where

$\mathrm{CoFib}_{\mathcal{W}} := \{\text{morphism } f \mid f \text{ is } \mathrm{Hom}_{\mathcal{A}}(-, \mathcal{W})\text{-epic}\}$;

$\mathrm{Fib}_{\mathcal{W}} := \{\text{morphism } f \mid f \text{ is } \mathrm{Hom}_{\mathcal{A}}(\mathcal{W}, -)\text{-epic}\}$,

$\mathrm{Weq}_{\mathcal{W}} := \mathrm{TFib}_{\mathcal{W}} \circ \mathrm{TCoFib}_{\mathcal{W}}$, where

$\mathrm{TCoFib}_{\mathcal{W}} := \{\text{splitting monic } f \mid \mathrm{Coker}\, f \in \mathcal{W}\}$,

$\mathrm{TFib}_{\mathcal{W}} := \{\text{splitting epic } f \mid \mathrm{Ker}\, f \in \mathcal{W}\}$.

In this case $(\mathcal{C},\ \mathcal{F},\ \mathcal{W}) = (\mathcal{A},\ \mathcal{A},\ \mathcal{W})$, $\mathrm{Ho}(\mathcal{A}) = \mathcal{A}/\mathcal{W}$.

The inverse is $(\mathrm{CoFib}, \mathrm{Fib}, \mathrm{Weq}) \mapsto \mathcal{W}$, where $\mathcal{W}$ is the class of trivial objects of model structure $(\mathrm{CoFib}, \mathrm{Fib}, \mathrm{Weq})$.