CIMPA School: Perspectives in non-commut. algebras

Model structures on categories

Lecture 3: Exact model structures

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Non abelian subcatgories of an abelian category

Even an extension-closed subcatgory of an abelian category is **Not** abelian in general.

For example, $R = \left(\begin{smallmatrix} \mathbb{R} & 0 \\ \mathbb{R} & \mathbb{Q} \end{smallmatrix}\right)$

R-mod, the category of finite generated left R-modules

R-Mod, the category of left R-modules

Then R-mod is an extension-closed full subcategory of abelian category R-Mod, but R-mod is **Not** abelian:

For example, let $M = \begin{pmatrix} 0 & 0 \\ \bigoplus\limits_{n\geq 1} \mathbb{Q}r_n & 0 \end{pmatrix} \in R$-Mod

$$0 \longrightarrow M \longrightarrow R \xrightarrow{p} R/M \longrightarrow 0$$

the morphism p in R-mod has no kernel in R-mod.

Exact categories

An exact pair (i, d) in an additive category $\mathcal{A}$ is a pair of morphisms such that Coker i and Ker d exist, and $d =$ Coker i and $i =$ Ker d.

Definition (Quillen; Keller) **An exact category** is a pair $(\mathcal{A}, \mathcal{E})$, where $\mathcal{A}$ is an additive category, $\mathcal{E}$ is a class of exact pairs (i, d), where (i, d) is called **a conflation**, i is called an **inflation**, and d is called a **deflation**, satisfying the axioms:

(E0) $\mathcal{E}$ is closed under isomorphisms, and Id_0 is a deflation.

(E1) The composition of two deflations is a deflation.

(E2) For a deflation $d\colon Y \to Z$ and a morphism $f\colon Z' \to Z$, there is a pullback, s.t. d' is a deflation:

$$\begin{tikzcd} Y' \arrow[r, dotted, "d'"] \arrow[d, dotted, "f'"'] & Z' \arrow[d, "f"] \\ Y \arrow[r, "d"] & Z \end{tikzcd}$$

($\mathrm{E2}^{\mathrm{op}}$) For an inflation $i\colon X \to Y$ and a morphism $f\colon X \to X'$, $\exists$ a pushout s.t. i' is an inflation:

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & Y \arrow[d, dotted, "f'"] \\ X' \arrow[r, dotted, "i'"] & Y' \end{tikzcd}$$

Any full subcatgory of an abelian category which is closed under extensions is an exact category.

Weakly idempotent complete exact categories

Lemma Let $\mathcal{A}$ be an exact category. The following are equivalent:

(1) Any splitting epimorphism in $\mathcal{A}$ has a kernel.
(1') Any splitting monomorphism in $\mathcal{A}$ has a cokernel.
(2) Any splitting epimorphism in $\mathcal{A}$ is a deflation.
(2') Any splitting monomorphism in $\mathcal{A}$ is an inflation.
(3) If de is a deflation, then so is d.
(3') If ki is an inflation, then so is i.

An exact category satisfying the equivalent conditions above is called a weakly idempotent complete exact category.

Many exact categories are weakly idempotent complete; but there are indeed exact categories which are not weakly idempotent complete.

k: a field, k-Vect: the category of k-linear spaces. $\mathcal{A}$: the full subcategory of spaces of infinite dim. or even dim.. $\mathcal{A}$: exact category.

$$f\colon k\oplus k\oplus k\oplus\cdots \longrightarrow k\oplus k\oplus k\oplus\cdots\,, \quad (a_1,\ a_2,\ a_3,\cdots)\mapsto(a_2,\ a_3,\ a_4,\ \cdots)$$

is splitting epic. Since $\mathrm{Ker} f = k \notin \mathcal{A}$, $\mathcal{A}$ is not w. idemp. complete.

Exams. of weakly idempotent complete exact cats.

Any full subcategory of an abelian category which is closed under extensions and direct summands is a weakly idempotent complete exact category.

- For a ring R and a non-negative inter n, the full subcategory $\mathcal{P}_n$ consisting of R-modules of projective dimension $\leq n$ is a weakly idempotent complete exact category.

- The full subcategory $\mathcal{GP}$ ($\mathcal{GF}$, respectively) consisting of Gorensrein-projective (Gorensrein-flat, respectively) R-modules is a weakly idempotent complete exact category.

- The full subcategory $\mathcal{GP}_n$ ($\mathcal{GF}_n$, respectively) consisting of R-modules of Gorensrein-projective (Gorensrein-flat, respectively) dimension $\leq n$ is a weakly idempotent complete exact category.
- The full subcategory $\mathcal{GP}^{<\infty}$ ($\mathcal{GF}^{<\infty}$, respectively) consisting of R-modules of finite Gorensrein-projective (Gorensrein-flat, respectively) dimension is a weakly idempotent complete exact category.

Exact (Abelian) model structures

Definition (M. Hovey 2002; J. Gillespie 2011) A model structure (Cofib, Fib, Weq) on an exact category $\mathcal{A}$ is **an exact model structure**, provided that the following conditions are satisfied:

(i) $\mathrm{Cofib} = \{\text{inflation } f \mid \operatorname{Coker} f \text{ is a cofibrant object}\}$;

(ii) $\mathrm{Fib} = \{\text{deflation } f \mid \operatorname{Ker} f \text{ is a fibrant object}\}$.

If this is the case, then one has also

$\mathrm{Cofib} \cap \mathrm{Weq} = \{\text{inflation } f \mid \operatorname{Coker} f \text{ is a trivial cofibrant object}\}$;

$\mathrm{Fib} \cap \mathrm{Weq} = \{\text{deflation } f \mid \operatorname{Ker} f \text{ is a trivial fibrant object}\}$.

Cotorsion pairs in an exact category

Definition A cotorsion pair $(\mathcal{C}, \mathcal{F})$ in an exact category $\mathcal{A}$ is a pair of objects, such that

$$\mathcal{C} = {}^{\perp}\mathcal{F} = \{X \in \mathcal{A} \mid \mathrm{Ext}^1_{\mathcal{A}}(X, \mathcal{F}) = 0\}, \qquad \mathcal{F} = \mathcal{C}^{\perp}.$$

(2) A cotorsion pair $(\mathcal{C}, \mathcal{F})$ is **hereditary**, if

- $\mathcal{C}$ is closed under the kernel of deflations; and
- $\mathcal{F}$ is closed under cokernel of inflations.

(3) A cotorsion pair $(\mathcal{C}, \mathcal{F})$ is **complete**, if for any object $X \in \mathcal{A}$ there are exact sequences

$$0 \to F \to C \to X \to 0, \qquad 0 \to X \to F' \to C' \to 0$$

where $C, \ C' \in \mathcal{C}, \ F, \ F' \in \mathcal{F}$.

Lemma Let $\mathcal{A}$ be an exact category, $\mathcal{C}$ and $\mathcal{F}$ classes of objects which are closed under direct summands and isomorphisms. Then $(\mathcal{C}, \mathcal{F})$ is complete ctp in $\mathcal{A}$ iff $\mathrm{Ext}^1_{\mathcal{A}}(\mathcal{C}, \mathcal{F}) = 0$ and the two exact sequences in (3) exist. for any object $X \in \mathcal{A}$.

The Hovey triple

Definition A triple $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ in an exact category $\mathcal{A}$ is **a Hovey triple**, if

(i) $\mathcal{Z}$ is **thick**, i.e., $\mathcal{Z}$ is closed under direct summands, and if two terms in an admissible exact seq. $0 \to X \to Y \to Z \to 0$ are in $\mathcal{Z}$, then so is the third one.

(ii) $(\mathcal{X},\ \mathcal{Y} \cap \mathcal{Z})$ and $(\mathcal{X} \cap \mathcal{Z},\ \mathcal{Y})$ are complete cotorsion pairs in $\mathcal{A}$.

Definition A Hovey triple $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ in an exact category $\mathcal{A}$ is **hereditary**, if both $(\mathcal{X},\ \mathcal{Y} \cap \mathcal{Z})$ and $(\mathcal{X} \cap \mathcal{Z},\ \mathcal{Y})$ are hereditary complete cotorsion pairs in $\mathcal{A}$.

The Hovey correspondences

Theorem (M. Hovey, 2002; Gillespie, 2011; Šťovíček, 2014) Let $\mathcal{A}$ be a weakly idempotent complete exact category. Then the exact model structures on $\mathcal{A}$ are in 1-1 correspondence with the Hovey triples in $\mathcal{A}$:

$$(\mathrm{Cofib},\ \mathrm{Fib},\ \mathrm{Weq}) \mapsto (\mathcal{C},\ \mathcal{F},\ \mathcal{W})$$

with the inverse given by a Hovey triple $(\mathcal{X}, \mathcal{Y}, \mathcal{Z}) \mapsto (\mathrm{Cofib},\ \mathrm{Fib},\ \mathrm{Weq})$, where

$$\mathrm{Cofib} = \{\text{inflation with cokernel in } \mathcal{X}\}$$

$$\mathrm{Fib} = \{\text{deflation with kerenel in } \mathcal{Y}\}$$

$$\mathrm{Weq} = \{pi \mid i \text{ inflation, } \mathrm{Coker}\, i \in \mathcal{X} \cap \mathcal{Z},\ p \text{ deflation, } \mathrm{Ker}\, p \in \mathcal{Y} \cap \mathcal{Z}\}$$

$$\mathrm{Cofib} \cap \mathrm{Weq} = \{\text{inflation } f \mid \mathrm{Coker}\, f \in \mathcal{X} \cap \mathcal{Z}\}$$

$$\mathrm{Fib} \cap \mathrm{Weq} = \{\text{deflation } f \mid \mathrm{Ker}\, f \in \mathcal{Y} \cap \mathcal{Z}\}.$$

By this result, from now on, we will identify an exact model structure with a Hovey triple.

The homotopy category of an exact model structure

Theorem (H. Nakaoka, Y. Palu 2019; J. Gillespie 2025) The homotopy category of an exact model structure on exact category $\mathcal{A}$ is equivalent to the additive quotient

$$(\mathcal{C} \cap \mathcal{F})/(\mathcal{C} \cap \mathcal{F} \cap \mathcal{W})$$

, and it is a triangulated category.

Hereditary Hovey triple

Theorem (H. Becker 2014; J. Gillespie 2011) Let $(\mathcal{C},\ \mathcal{F},\ \mathcal{W})$ be a hereditary Hovey triple in exact category $\mathcal{A}$

(i.e., $\mathcal{W}$ is thick, $(\mathcal{C},\ \mathcal{F}\cap\mathcal{W})$, $(\mathcal{C}\cap\mathcal{W},\ \mathcal{F})$ are hereditary complete cotorsion pairs).

Then $\mathcal{C}\cap\mathcal{F}$ is a Frobenius category, with class of projective-injective objects $\mathcal{C}\cap\mathcal{F}\cap\mathcal{W}$; the composition $\mathcal{C}\cap\mathcal{F}\hookrightarrow\mathcal{A}\longrightarrow \mathrm{Ho}(\mathcal{A})$ induces a triangle equivalence

$$\mathrm{Ho}(\mathcal{A})\cong(\mathcal{C}\cap\mathcal{F})/(\mathcal{C}\cap\mathcal{F}\cap\mathcal{W}).$$

Examples of hereditary Hovey triples

R: ring $\quad R$-Mod: the category of left R-module

- $(\mathcal{P},\ R\text{-Mod},\ R\text{-Mod}),\ \text{Ho} = 0$
- $(R\text{-Mod},\ \mathcal{I},\ R\text{-Mod}),\ \text{Ho} = 0$
- R: Artin algebra, $\quad (\mathcal{GP}(R),\ R\text{-Mod},\ \mathcal{GP}(R)^{\perp}),\ \text{Ho} = \mathcal{GP}(R)/\mathcal{P}$
- R: Gorenstein ring, $\quad (\mathcal{GP}(R),\ R\text{-Mod},\ \mathcal{P}^{<\infty}),\ \text{Ho} = \mathcal{GP}(R)/\mathcal{P}$
- R: Artin algebra, $\quad (R\text{-Mod},\ \mathcal{GI}(R),\ {}^{\perp}\mathcal{GI}(R)),\ \text{Ho} = \mathcal{GI}(R)/\mathcal{I}$
- R: Gorenstein ring, $\quad (R\text{-Mod},\ \mathcal{GI}(R),\ \mathcal{P}^{<\infty}),\ \text{Ho} = \mathcal{GI}(R)/\mathcal{I}$
- $(\mathcal{F}(R),\ \mathcal{C},\ R\text{-Mod})$

How to obtain hereditary Hovey triples?

Definition Cotorsion pairs $(\Phi, \ \Phi^{\perp})$ and $({}^{\perp}\Psi, \ \Psi)$ in abelian category $\mathcal{A}$ are *compatible*, if

(i) $\mathrm{Ext}^1_{\mathcal{A}}(\Phi, \Psi) = 0$, i.e., $\Phi \subseteq {}^{\perp}\Psi, \ \ \Psi \subseteq \Phi^{\perp}$;

(ii) $\Phi \cap \Phi^{\perp} = {}^{\perp}\Psi \cap \Psi$.

Theorem (H. Becker 2014; J. Gillespie, 2015) $(\Phi, \ \Phi^{\perp})$, $({}^{\perp}\Psi, \ \Psi)$: hereditary, complete, compatible ctp in abelian cat. $\mathcal{A}$. Then $({}^{\perp}\Psi, \ \Phi^{\perp}, \ \mathcal{W})$ is a hereditary Hovey triple, where

$$\mathcal{W} = \{ Y \in \mathcal{A} \mid \exists \text{ exact sequence } 0 \to P \to F \to Y \to 0, \ P \in \Psi, \ F \in \Phi\};$$

$$\mathcal{W} = \{ Y \in \mathcal{A} \mid \exists \text{ exact sequence } 0 \to Y \to P' \to F' \to 0, \ P' \in \Psi, \ F' \in \Phi\}.$$

Thus $\Phi^{\perp} \cap \mathcal{W} = \Phi, \ \ \Phi^{\perp} \cap \ \mathcal{W} = \Psi$, $\mathcal{W}$ is thick.

Any hereditary Hovey triple can be obtained in this way.

If this is the case, then the corresponding model structure is

- Cofibrations are monomorphisms with cokernel in ${}^{\perp}\Psi$;
- Fibrations are epimorphisms with kernel in $\Phi^{\perp}$;
- Weakly equivalences are all the compositions pi, where i is a monomorphism with cokernel in Φ, p is an epimorphism with kernel in Ψ.
- $\mathrm{Ho}(\mathcal{A}) = ({}^{\perp}\Psi \cap \Phi^{\perp})/({}^{\perp}\Psi \cap \Psi) = ({}^{\perp}\Psi \cap \Phi^{\perp})/(\Phi \cap \Phi^{\perp})$.

Weq is uniquely determined by $\mathcal{W}$ for exact model structures

Theorem (X. S. Lu, $\sim$, 2024) Let $\mathcal{A}$ be a weakly idempotent complete exact category, and (CoFib, Fib, Weq) an exact model structure on $\mathcal{A}$. Put

$$\mathcal{S} = \{f \in \mathrm{Mor}(\mathcal{A}) \mid f \text{ is admissible}, \ \mathrm{Ker}\, f \in \mathcal{W}, \ \mathrm{Coker}\, f \in \mathcal{W}\}.$$

Then

$$\mathrm{Weq} = \mathcal{S} \circ \mathcal{S} := \{f \in \mathrm{Mor}(\mathcal{A}) \mid f = f_2 f_1, \ f_i \in \mathcal{S}, \ i = 1, 2\}.$$

In particular $\mathcal{S} \subseteq \mathrm{Weq}$.

Thus, if both $(\mathcal{C}, \ \mathcal{F}, \ \mathcal{W})$ and $(\mathcal{C}', \ \mathcal{F}', \ \mathcal{W})$ are exact model structures on $\mathcal{A}$, then the corresponding homotopy category are the same.