CIMPA School: Perspectives in non-commutative algebras

Model structures on categories

Lecture 2: The homotopy category of a model structure

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Recall: Definition of a model structure on a category

Definition (Daniel Quillen, 1967) A model structure on a category is a triple (Cofib, Fib, Weq) of classes of morphisms, in which the morphisms are called **cofibrations, fibrations, and weak equivalence**, respectively, satisfying:

(**2 out of 3**) Let $X \xrightarrow{f} Y \xrightarrow{g} Z$. If two of f, g, gf are weak equivances, then so is the third.

(**Closed under retracts**) Let f be a retract of g. If g is a cofibration (a fibration, a weak equivalence), then so is f.

(**Lifiting**) For any commutative square with $i \in \mathrm{Cofib}$, $p \in \mathrm{Fib}$, if $i \in \mathrm{Weq}$ or $p \in \mathrm{Weq}$, then $\exists\ s : B \to X$ such that the two triangles commute:

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, hook, "i"'] & X \arrow[d, two heads, "p"] \\ B \arrow[r, "b"] \arrow[ur, dotted, "s"] & Y \end{tikzcd}$$

(**Factorizations**) $\forall$ morphism f, one has $f = pi = qj$, where $i \in \mathrm{Cofib} \cap \mathrm{Weq}$, $p \in \mathrm{Fib}$, $j \in \mathrm{Cofib}$, $q \in \mathrm{Fib} \cap \mathrm{Weq}$.

By f is a retract of g, one means that $\exists$ a commutative diagram such that $\psi_1 \varphi_1 = \mathrm{Id}_X$, $\psi_2 \varphi_2 = \mathrm{Id}_Y$:

$$\begin{tikzcd} X \arrow[r, "\varphi_1"] \arrow[d, "f"'] & X' \arrow[r, "\psi_1"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ Y \arrow[r, "\varphi_2"] & Y' \arrow[r, "\psi_2"] & Y \end{tikzcd}$$

Localizations of categories

$\mathcal{C}$: an arbitrary category, S: any class of morphisms of $\mathcal{C}$

Definition A localization of $\mathcal{C}$ with respect to S is a category $S^{-1}\mathcal{C}$, together with a functor $L : \mathcal{C} \longrightarrow S^{-1}\mathcal{C}$, which is called the localization functor, s.t. $L(f)$ is an isomorphism in $S^{-1}\mathcal{C}$, $\forall\ f \in S$, and L has the universal property with respect to this property:

i.e., if $F : \mathcal{C} \longrightarrow \mathcal{D}$ is also a functor s.t. $F(f)$ is an isomorphism in $\mathcal{D}$, $\forall\ f \in S$, then there is a unique functor $G : S^{-1}\mathcal{C} \longrightarrow \mathcal{D}$ s.t. $F = G \circ L$.

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "L"] \arrow[dr, "F"'] & & S^{-1}\mathcal{C} \arrow[dl, dotted, "\exists ! G"'] \\ & \mathcal{D} & \end{tikzcd}$$

P. Gabriel - M. Zisman 1967: For any category $\mathcal{C}$ and any class S of morphisms of $\mathcal{C}$, the localization of $\mathcal{C}$ with respect to S always exists, and it is unique, up to an isomorphism of category!

Definition (D. Quillen, 1967) Let (Cofib, Fib, Weq) be a model structure on category $\mathcal{M}$. The localization $\mathcal{M}[\mathrm{Weq}^{-1}]$ of $\mathcal{M}$ with respect to Weq is called the homotopy category of this model structure, and denoted by $\mathrm{Ho}(\mathcal{M})$.

Localizations of categories as additive quotients

Proposition Let $\mathcal{A}$ be an additive category.

(1) Let $\mathcal{S}$ be a class of morphisms of $\mathcal{A}$ such that $\mathrm{Id}_X \in \mathcal{S}$ for every object $X \in \mathcal{A}$, and that $\mathcal{S}$ is closed under finite coproducts. Then $\mathcal{A}[\mathcal{S}^{-1}]$ is an additive category, and $L : \mathcal{A} \longrightarrow \mathcal{A}[\mathcal{S}^{-1}]$ is an additive functor.

(2) Let $\mathcal{U}$ be a full additive subcategory of $\mathcal{A}$. Set

$$\mathcal{S} = \{f \in \mathrm{Mor}\mathcal{A} \mid \bar{f} \text{ is an isomorphism in } \mathcal{A}/\mathcal{U}\}.$$

Then the canonical functor $\mathcal{A} \longrightarrow \mathcal{A}/\mathcal{U}$ induces an isomorphism $\mathcal{A}[\mathcal{S}^{-1}] \cong \mathcal{A}/\mathcal{U}$ of additive categories.

Proof (2) By the universal property $P : \mathcal{A} \to \mathcal{A}/\mathcal{U}$ factors through $L : \mathcal{A} \to S^{-1}\mathcal{A}$. By (1), L is an additive functor. Note that $0 : U \to U$, $U \in \mathcal{U}$, is an isomorphism in $\mathcal{A}/\mathcal{U}$. Thus $L(0) = 0$ is an isomorphism, and hence $L(U) = 0$. Therefore L also facrors through P. This completes the proof.

The quotient category with respect to equivalence relation ideal

Defintion An equivalence relation ideal of category $\mathcal{C}$ is an equivalence relation $\sim$ on $\mathrm{Mor}(\mathcal{C})$, s. t.

if $f \sim g$, then $f, g \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ for some objects X and Y, and

if $f \sim g$ then $hf \sim hg$ and $fh' \sim gh'$ (if possible).

Let $\sim$ be an equivalence relation ideal. For arbitrary objects X and Y, $\sim$ induces an equivalence relation on $\mathrm{Hom}_{\mathcal{C}}(X, Y)$. Denote $\mathcal{C}(X, Y)/\sim$ the set of equivalence classes.

The quotient category $\mathcal{C}/\sim$ of $\mathcal{C}$ **with respect to equiv. relation ideal** $\sim$:

objects are the same as $\mathcal{C}$; $\qquad \mathrm{Hom}_{\mathcal{C}/\sim}(X, Y) = \mathcal{C}(X, Y)/\sim$.

An equivalence relation ideal $\sim$ of an additive category $\mathcal{C}$ is additive, if

$$f \sim g \ \text{ and } \ f' \sim g' \ \text{ imply } \ f - f' \sim g - g'.$$

In this case $\{f \in \mathrm{Hom}_{\mathcal{C}}(X, Y) \mid f \sim 0\}$ is a subgroup of $\mathrm{Hom}_{\mathcal{C}}(X, Y)$.

Lemma Let $\sim$ be an additive equivalence relation ideal of additive category $\mathcal{C}$. Then $\mathcal{C}/\sim$ is an additive category, $\mathrm{Hom}_{\mathcal{C}/\sim}(X, Y)$ is the quotient group

$$\mathrm{Hom}_{\mathcal{C}}(X, Y)/\{f \in \mathrm{Hom}_{\mathcal{C}}(X, Y) \mid f \sim 0\}$$

and the canonical functor $\pi: \ \mathcal{C} \longrightarrow \mathcal{C}/\sim$ is additive.

Left (right) homotopy

Definition Let $(\mathrm{CoFib}, \mathrm{Fib}, \mathrm{Weq})$ be a model structure on $\mathcal{M}$, $f,\ g \in \mathrm{Hom}_{\mathcal{C}}(A, B)$.

(1) f is left homotopic to g, denoted by $f \overset{l}{\sim} g$, if there is a commutative diagram

$$\begin{tikzcd} A \oplus A \arrow[r, "(f,g)"] \arrow[d, "(1,1)"'] \arrow[dr, "(\partial_0, \partial_1)"] & B \\ A & \widetilde{A} \arrow[l, "\sigma"'] \arrow[u, "h"'] \end{tikzcd}$$

such that σ is a weak equivalence, and (∂_0, ∂_1) is a cofibration. In this case, $h : \widetilde{A} \longrightarrow B$ is called a left homotopy from f to g.

(1') f is right homotopic to g, denoted by $f \overset{r}{\sim} g$, if there is a commutative diagram

$$\begin{tikzcd} \widetilde{B} \arrow[dr, "\binom{d_0}{d_1}"] & B \arrow[l, "s"'] \arrow[d, "\binom{1}{1}"] \\ A \arrow[u, "k"] \arrow[r, "\binom{f}{g}"] & B \times B \end{tikzcd}$$

such that s is a weak equivalence, and $\binom{d_0}{d_1}$ is a fibration. In this case, $k : A \longrightarrow \widetilde{B}$ is called a right homotopy from f to g.

The Fundamental Theorem for the homotopy categories

$\overset{r}{\sim}$ and $\overset{l}{\sim}$ are not an equivalence relation. $\mathcal{M}_{cf}$: the full subcategory of cofibrant - fibrant objects. Then $\overset{l}{\sim}=\overset{r}{\sim}$ is an equivalence relation ideal of $\mathcal{M}_{cf}$, denoted by $\sim$; and one has the quotient category $\mathcal{M}_{cf}/\sim$, denoted by $\pi\mathcal{M}_{cf}$.

The embedding $\mathcal{M}_{cf} \hookrightarrow \mathcal{M}$ and the localization $\gamma : \mathcal{M} \to \mathrm{Ho}(\mathcal{M})$ give $\mathcal{M}_{cf} \to \mathrm{Ho}(\mathcal{M})$, which induces the functor $\overline{\gamma} : \pi\mathcal{M}_{cf} \to \mathrm{Ho}(\mathcal{M})$.

Theorem (Quillen 1967, A generalized version) Let (CoFib, Fib, Weq) be a model structure on category $\mathcal{M}$. Assume that $\mathcal{M}$ satisfies the conditions:

(1) $\mathcal{M}$ has the initial object, the final object, finite coproducts, and finite products;

(2) For any trivial cofibration $i : A \to B$ and morphism $u : A \to C$, $\exists$ **a weak push-out square**, such that i' is also a trivial cofibration.

(3) For any trivial fibration $p : C \to D$ and morphism $v : B \to D$, $\exists$ **a weak push-back square**, such that p' is also a trivial fibration.

Then $\overline{\gamma} : \pi\mathcal{M}_{cf} \longrightarrow \mathrm{Ho}(\mathcal{M})$ is an equivalence of categories.

Remark. If $\mathcal{M}$ is a model category (i.e., a category $\mathcal{M}$ with a model structure, and $\mathcal{M}$ has (finite) projective and inductive limits), then all the conditions above are satisfied.

Left (right) triangulated categories

Definition (A. Beligiannis, N. Marmaridis) Let $\Omega : \mathcal{C} \longrightarrow C$ be an additive functor, $\mathcal{E}$ a class of left triangles $(\Omega Z \to X \to Y \to Z)$. The triple $(\mathcal{C}, \Omega, \mathcal{E})$ is a left triangulated category, if the following are satisfied:

(LT1) $\mathcal{E}$ is closed under isomorphisms of left triangles; each morphism $w : Y \to Z$ is in some left triangle $\Omega Z \to X \to Y \stackrel{w}{\to} Z$ in $\mathcal{E}$; $0 \to X \xrightarrow{\mathrm{Id}_X} X \to 0$ is in $\mathcal{E}$ for any object $X \in \mathcal{C}$.

(LT2) If $\Omega Z \stackrel{u}{\to} X \stackrel{v}{\to} Y \stackrel{w}{\to} Z$ is in $\mathcal{E}$, then so is $\Omega Y \xrightarrow{-\Omega w} \Omega Z \stackrel{u}{\to} X \stackrel{v}{\to} Y$.

(LT3) If $(X, Y, Z, u, v, w) \in \mathcal{E}$, $(X', Y', Z', u', v', w') \in \mathcal{E}$, and $w' \circ g = h \circ w$, then there is a morphism (f, g, h) of left triangles.

(LT4) (The octahedral axiom) Assume that the third row, the third column and the fourth column are in $\mathcal{E}$. Then $\exists$ morphisms s.t. the second row is also in $\mathcal{E}$, and each square commutes:

$$\begin{tikzcd}
 & \Omega C \arrow[r, "\Omega f"] \arrow[d, "u"'] & \Omega D \arrow[r, equal] \arrow[d] & \Omega D \arrow[d] \\
\Omega E \arrow[r, "u\Omega v"] \arrow[d, "\Omega v"'] & A \arrow[r, dashed] \arrow[d, equal] & F \arrow[r, dashed] \arrow[d] & E \arrow[d, "v"] \\
\Omega C \arrow[r, "u"] & A \arrow[r] & B \arrow[r, "g"] \arrow[d, "fg"'] & C \arrow[d, "f"] \\
 & & D \arrow[r, equal] & D
\end{tikzcd}$$

Dually, one has the notion of right triangulated category.

Pretriangulated categories

Definition (A. Beligiannis, I. Reiten) Let $\Sigma,\ \Omega : \mathcal{C} \longrightarrow C$ be additive functor, $\triangledown$ a class of right triangles, and $\triangle$ a class of left triangles. The datum $(\mathcal{C}, \Sigma, \Omega, \triangledown, \triangle, \varepsilon, \delta)$ is a pretriangulated category, if the following are satisfied:

(LT1) (Σ, Ω) is an adjoint pair, with counit $\varepsilon :\ \Sigma\Omega \to \mathrm{id}_{\mathcal{C}}$, and unit $\delta :\ \mathrm{id}_{\mathcal{C}} \to \Omega\Sigma$.

(LT2) $(\mathcal{C}, \Omega, \triangle)$ is a left triangulated category.

(LT3) $(\mathcal{C}, \Sigma, \triangledown)$ is a right triangulated category.

(LT4) Let the first row is in $\triangledown$, the second is in $\triangle$, and the left square commutes, then $\exists\ \ \gamma : C \to B'$ such that all the squares commute:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[r, "g"] \arrow[d, "\beta"'] & C \arrow[r, "h"] \arrow[d, dotted, "\gamma"'] & \Sigma A \arrow[d, "\varepsilon_{C'} \circ \Sigma\alpha"] \\
\Omega C' \arrow[r, "f'"] & A' \arrow[r, "g'"] & B' \arrow[r, "h'"] & C'
\end{tikzcd}$$

(LT5) Let the first row is in $\triangledown$, the second is in $\triangle$, and the right square commutes, then $\exists\ \ \gamma : B \to A'$ such that all the squares commute:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\Omega\alpha \circ \delta_A"'] & B \arrow[r, "g"] \arrow[d, dotted, "\gamma"'] & C \arrow[r, "h"] \arrow[d, "\beta"'] & \Sigma A \arrow[d, "\alpha"'] \\
\Omega C' \arrow[r, "f'"] & A' \arrow[r, "g'"] & B' \arrow[r, "h'"] & C'
\end{tikzcd}$$

The homotopy category

The homotopy category is an important object of study in algebra and topology. Although $\mathrm{Ho}(\mathcal{M})$ is not a triangulated category in general, but the most known triangulated categories arise as the homotopy categories of model structures.

Theorem (D. Quillen; M. Hovey, A. Beligiannis, I. Reiten) Let $(\mathrm{CoFib}, \mathrm{Fib}, \mathrm{Weq})$ be a model structure on additive category $\mathcal{M}$. Then $\mathrm{Ho}(\mathcal{M})$ is a pretriangulated category.

Theorem (Xue-Song Lu, $\sim$, 2024) Let $(\mathrm{CoFib}, \mathrm{Fib}, \mathrm{Weq})$ be a model structure on weakly idempotent complete additive category $\mathcal{M}$. Then

$$\mathrm{Ho}(\mathcal{M}) \cong (\mathcal{C} \cap \mathcal{F})/(\mathcal{C} \cap \mathcal{F} \cap \mathcal{W}).$$