CIMPA School: Perspectives in non-commutative algebras

Model structures on categories

Lecture 1: Fundamental properties

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Definition of a model structure on a category

By f is a retract of g, one means that $\exists$ a commutative diagram such that $\psi_1\varphi_1 = \mathrm{Id}_X$, $\psi_2\varphi_2 = \mathrm{Id}_Y$:

$$\begin{tikzcd}
X \arrow[r, "\varphi_1"] \arrow[d, "f"'] & X' \arrow[r, "\psi_1"] \arrow[d, "g"'] & X \arrow[d, "f"] \\
Y \arrow[r, "\varphi_2"] & Y' \arrow[r, "\psi_2"] & Y
\end{tikzcd}$$

Definition (Daniel Quillen, 1967) A model structure on a category is a triple (Cofib, Fib, Weq) of classes of morphisms, in which the morphisms are called **cofibrations, fibrations, and weak equivalence**, respectively, satisfying:

(**2 out of 3**) Let $X \xrightarrow{f} Y \xrightarrow{g} Z$. If two of f, g, gf are weak equivances, then so is the third.

(**Closed under retracts**) Let f be a retract of g. If g is a cofibration (a fibration, a weak equivalence, respectively), then so is f.

(**Lifiting**) For any commutative square with $i \in \mathrm{Cofib}$, $p \in \mathrm{Fib}$, if $i \in \mathrm{Weq}$ or $p \in \mathrm{Weq}$, then $\exists\ s : B \to X$ such that the two triangles commute:

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, hook, "i"'] & X \arrow[d, two heads, "p"] \\
B \arrow[r, "b"'] \arrow[ur, dotted, "s"] & Y
\end{tikzcd}$$

(**Factorizations**) $\forall$ morphism f, one has $f = pi = qj$, where $i \in \mathrm{Cofib} \cap \mathrm{Weq}$, $p \in \mathrm{Fib}$, $j \in \mathrm{Cofib}$, $q \in \mathrm{Fib} \cap \mathrm{Weq}$.

Any two of Fib, Cofib and Weq uniquely determine the third

A striking property of a model structure is: Any two of Fib, Cofib and Weq uniquely determine the third.

Definition Morphism i has the left lifting property (LLP) with respect to morphism p, or, p has the righ t lifting property (RLP) with respect to i, if for any commutative diagram

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "b"] \arrow[ur, dotted, "x"] & Y \end{tikzcd}$$

there is a morphism $x : B \longrightarrow X$ such that the two triangles commute.

Theorem (Quillen, 1969) Let (Fib, Cofib, Weq) be a model structure on category $\mathcal{M}$. Then

(1) Cofib is exactly the class of morphisms which has LLP with respect to any trivial fibrations.

(2) Fib is exactly the class of morphisms which has RLP with respect to any trivial cofibrations.

(3) Cofib $\cap$ Weq is exactly the class of morphisms which has LLP with respect to any fibrations.

(4) Fib $\cap$ Weq is exactly the class of morphisms which has RLP with respect to any cofibrations.

(5) $\text{Weq} = (\text{Fib} \cap \text{Weq}) \circ (\text{Cofib} \cap \text{Weq})$.

Proof

(1) By the axiom, Cofib has LLP with respect to any morphism in $\mathrm{Fib} \cap \mathrm{Weq}$.

Conversely, for an arbitrary morphism f which has LLP with respect to any morphism in $\mathrm{Fib} \cap \mathrm{Weq}$, then one has a factorization $f = p' i'$, where $i' \in \mathrm{Cofib}$ and $p' \in \mathrm{Fib} \cap \mathrm{Weq}$. So one gets a commutative square

$$\begin{tikzcd} \bullet \arrow[r, "i'"] \arrow[d, "f=p' i'"'] & \bullet \arrow[d, "p'"] \\ \bullet \arrow[r, equal] \arrow[ur, dotted, "x"] & \bullet \end{tikzcd}$$

By the assumption f has LLP with respect to p', i.e., there exists a morphism x such that the two triangles commutes. Since $p' x = \mathrm{Id}$, it follows that f is a retract of i', as the following commutative diagram shows

$$\begin{tikzcd} \bullet \arrow[r, equal] \arrow[d, "f"'] & \bullet \arrow[r, equal] \arrow[d, "i'"'] & \bullet \arrow[d, "f"'] \\ \bullet \arrow[r, "x"] & \bullet \arrow[r, "p'"] & \bullet \end{tikzcd}$$

Since i' is a cofibration, it follows that f is a cofibration, i.e., $f \in \mathrm{Cofib}$. This proves $\mathrm{Cofib} =$

$l(\mathrm{Fib} \cap \mathrm{Weq})$.

Basic properties

Let (CoFib, Fib, Weq) be a model structure on category $\mathcal{M}$. Then

(1) The classes CoFib and Fib are closed under compositions.

(2) Isomorphisms are fibrations, cofibrations, and weak equivalences.

(3) Cofibrations are closed under pushouts: for a pushout square with $i \in$ CoFib, then $i' \in$ CoFib:

$$\begin{tikzcd} \bullet \arrow[r, "i"] \arrow[d] & \bullet \arrow[d, dotted] \\ \bullet \arrow[r, dotted, "i'"] & \bullet \end{tikzcd}$$

Also, trivial cofibrations are closed under pushouts.

(4) Fibrations are closed under pullbacks; and trivial fibrations are closed under pullbacks.

Proof

(1) Let $\bullet \xrightarrow{p} \bullet \xrightarrow{p'} \bullet$, where p and p' are fibrations. To prove $p'p$ is a fibration it suffices to prove that $p'p$ has RLP with respect to any trivial cofibration i. Assume that $bi = (p'p)a$. Then $bi = p'(pa)$. By lifting there is a morphism x s.t. $pa = xi$, $b = p'x$.

$$\begin{tikzcd}
 & & \bullet \arrow[d, "p"] \\
A \arrow[urr, "a"] \arrow[rr, "pa"] \arrow[d, "i"'] & & \bullet \arrow[d, "p'"] \\
B \arrow[rr, "b"'] \arrow[uurr, dotted, "x'"] \arrow[urr, dotted, "x"'] & & \bullet
\end{tikzcd}$$

Since $pa = xi$, again by the lifting there is an x' s.t. $a = x'i$, $x = px'$. Thus $a = x'i$, $b = p'x = (p'p)x'$. This completes the proof.

(2) Let f be an isomorphism. From the following commutative diagram

$$\begin{tikzcd}
\bullet \arrow[r, "a"] \arrow[d, "f"'] & \bullet \arrow[d, "p\in\mathrm{Fib}"] \\
\bullet \arrow[r, "b"] \arrow[ur, dotted, "af^{-1}"] & \bullet
\end{tikzcd}$$

we know that $f \in \mathrm{Cofib} \cap \mathrm{Weq}$. From the following commutative diagram

$$\begin{tikzcd}
\bullet \arrow[r, "a"] \arrow[d, "\mathrm{Cofib}\cap\mathrm{Weq}\ni i"'] & \bullet \arrow[d, "f"] \\
\bullet \arrow[r, "b"] \arrow[ur, dotted, "f^{-1}b"] & \bullet
\end{tikzcd}$$

we know that $f \in r(\mathrm{Cofib} \cap \mathrm{Weq}) = \mathrm{Fib}$. Thus $f \in \mathrm{Cofib} \cap \mathrm{Fib} \cap \mathrm{Weq}$.

Proof (continued)

(4) For a pullback with p a fibration $\begin{tikzcd} \bullet \arrow[r, "a"] \arrow[d, dotted, "f"'] & \bullet \arrow[d, "p"] \\ \bullet \arrow[r, "b"] & \bullet \end{tikzcd}$, to prove f is a fibration, it suffices to prove $f \in r(\mathrm{Cofib} \cap \mathrm{Weq})$. For a commut. $\begin{tikzcd} \bullet \arrow[r, "s"] \arrow[d, "i"'] & \bullet \arrow[d, "f"] \\ \bullet \arrow[r, "t"] \arrow[ur, dotted, "x"] & \bullet \end{tikzcd}$ with $i \in \mathrm{Cofib} \cap \mathrm{Weq}$, we want to find a morphism x s.t. $xi = s$ and $fx = t$. Since $p \in \mathrm{Fib}$, there is y s.t. $yi = as$ and $py = bt$.

$$\begin{tikzcd} \bullet \arrow[r, "s"] \arrow[d, "i"'] & \bullet \arrow[r, "a"] \arrow[d, "f"] & \bullet \arrow[d, "p"] \\ \bullet \arrow[r, "t"'] \arrow[urr, dotted, "y"'] & \bullet \arrow[r, "b"'] & \bullet \end{tikzcd}$$

Since $py = bt$, there is a unique morphism x such that $fx = t$ and $ax = y$:

$$\begin{tikzcd} \bullet \arrow[dr, dotted, "x"'] \arrow[drr, bend left, "y"] \arrow[ddr, bend right, "t"'] & & \\ & \bullet \arrow[r, "a"] \arrow[d, "f"'] & \bullet \arrow[d, "p"] \\ & \bullet \arrow[r, "b"] & \bullet \end{tikzcd} \qquad \begin{tikzcd} \bullet \arrow[dr, dotted, "s", "xi"'] \arrow[drr, bend left, "as"] \arrow[ddr, bend right, "ti"'] & & \\ & \bullet \arrow[r, "a"] \arrow[d, "f"'] & \bullet \arrow[d, "p"] \\ & \bullet \arrow[r, "b"] & \bullet \end{tikzcd}$$

By the universal property, we have $s = xi$. Hence $xi = s$ and $fx = t$. This completes the proof.

Ten classes of objects of study

Given a model structure $(\mathrm{Cofib}, \mathrm{Fib}, \mathrm{Weq})$ on category $\mathcal{M}$ with zero object, one considers:

the class of trivial cofibrations: $\mathrm{Cofib} \cap \mathrm{Weq}$
the class of trivial fibrations: $\mathrm{Fib} \cap \mathrm{Weq}$

the class of cofibrant objects: $\mathcal{C} = \{X \in \mathcal{M} \mid 0 \to X \text{ is a cofibration}\}$
the class of fibrant objects: $\mathcal{F} = \{ Y \in \mathcal{M} \mid Y \to 0 \text{ is a fibration}\}$
the class of trivial objects:

$$\mathcal{W} = \{ W \in \mathcal{M} \mid 0 \to W \text{ is a weak equivalence}\} = \{ W \in \mathcal{M} \mid W \to 0 \text{ is a weak equivalence}\}$$

the class of trivially cofibrant objects: $\mathcal{C} \cap \mathcal{W}$
the class of trivially fibrant objects: $\mathcal{F} \cap \mathcal{W}$

Basic properties (continued)

Let (CoFib, Fib, Weq) be a model structure on additive category $\mathcal{M}$. Then

(1) If $X \in \mathcal{C}$ (respectively, $X \in \mathrm{T}\mathcal{C}$), then any splitting monomorphism with cokernel X belongs to CoFib (respectively, TCoFib).

(2) If $X \in \mathcal{F}$ (respectively, $X \in \mathrm{T}\mathcal{F}$), then any splitting epimorphism with kernel X belongs to Fib (respectively, TFib).

(3) CoFib, Fib, TCoFib, TFib and Weq are closed under finite coproducts.

Hints of proof:

(1) LLP and RLP

(3) LLP and RLP

For Weq, using factorization of a weak equivalence as a composition of a trivial cofib. with a trivial fib.

Basic properties (continued)

Let (CoFib, Fib, Weq) be a model structure on category $\mathcal{M}$ with zero object, and $f: A \to B$ a morphism. Then

(1) $\mathcal{C}$ and T$\mathcal{C}$ are contravariantly finite in $\mathcal{M}$.

(2) $\mathcal{F}$ and T$\mathcal{F}$ are covariantly finite in $\mathcal{M}$.

(3) $\mathcal{C}$ and T$\mathcal{C}$ are closed under finite coproducts, direct summands, and productands. (Object X is a productand, if there is an object Y such that product $X \prod Y$ exists.)

(4) $\mathcal{F}$ and T$\mathcal{F}$ are closed under finite products, direct summands, and direct productands.

(5) If $\mathcal{M}$ is additive, then $\mathcal{W}$ is closed under finite coproducts and direct summands.

(6) If $A \in \mathcal{C}$, $f \in$ CoFib, then $B \in \mathcal{C}$.

(7) If $A \in \mathrm{T}\mathcal{C}$, $f \in$ TCoFib, then $B \in \mathrm{T}\mathcal{C}$.

(8) If $B \in \mathcal{F}$, $f \in$ Fib, then $A \in \mathcal{F}$.

(9) If $B \in \mathrm{T}\mathcal{F}$, $f \in$ TFib, then $A \in \mathrm{T}\mathcal{F}$.

The homotopy category of a model structure

Definition For a model structure $(\mathrm{Cofib}, \mathrm{Fib}, \mathrm{Weq})$ on category $\mathcal{M}$, **the homotopy category** $\mathrm{Ho}(\mathcal{M})$ is by definition the category $\mathrm{Weq}^{-1}\,\mathcal{M}$ of localization of M with respect to Weq.

Fundamental tasks:

- To obtain new model structures

- To study properties of model structures, in particular, the relations between the 10 classes of objets of study

- The structures of homotopy categories $\cdots\cdots$

Examples

- $\mathcal{A}$: additive cat., then $\mathcal{C}^b(\mathcal{A})$, $\mathcal{C}^-(\mathcal{A})$, $\mathcal{C}^+(\mathcal{A})$, and $\mathcal{C}(\mathcal{A})$ have model structures, where cofibrations are chain split monomors., fibrations are chain split epimors., weak equivs. are homotopy equivs., Ho is the corresponding homotopy category.

- J. Gillespie; Z. W. Li: $(\mathcal{F}, \mathcal{E})$ weakly idempotent complete Frobenius cat.. Put

$$\mathrm{Cofib}(\mathcal{F}) := \{\textit{inflation}\}, \quad \mathrm{Fib}(\mathcal{F}) := \{\textit{deflation}\}$$

$$\mathrm{Weq}(\mathcal{F}) := \{f \in \mathcal{F} \mid \underline{f} \text{ is isomor. in the stable cat. } \underline{\mathcal{F}}\}.$$

Then $(\mathrm{Fib}(\mathcal{F}), \mathrm{Cofib}(\mathcal{F}), \mathrm{Weq}(\mathcal{F}))$ is a model structure on $\mathcal{F}$, with $(\mathcal{F}, \mathcal{F}, \mathcal{P}), \mathrm{Ho}(\mathcal{F}) = \underline{\mathcal{F}}$.

- D. Quillen: $\mathcal{A}$: abelian cat. with enough proj. objs., then $\mathcal{C}^b(\mathcal{A})$ and $\mathcal{C}^+(\mathcal{A})$ have model structures, where cofibrations are monomors. with cokernel proj. complexes, fibrations are epimors., weak equivs. are qis., Ho is the corresponding derived category.

- $\mathcal{A}$: abelian cat. with enough proj. objs., then $\mathcal{C}^{\leq 0}(\mathcal{A})$ has the model structure, where cofibrations are monomorphisms with cokernel proj. complexes,fibrations are chain maps which are epimors., except at the 0-position, weak equivs. are qis., Ho $= \mathcal{D}^{\leq 0}(\mathcal{A})$ is a pretriangulated but not triangulated.

- M. Hovey: $\mathcal{A}$: Grothendieck cat. with proj. generators, then $\mathcal{C}(\mathcal{A})$ has the model structure, where cofibrations are monomorphisms with cokernel dg proj. complexes, fibrations are epimors., weak equivs. are qis., Ho $= D(\mathcal{A})$, the derived category.