

CIMPA School: Perspectives in non-commutative algebras

Model structures on categories

Lecture 1: Fundamental properties

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Definition of a model structure on a category

By f is a retract of g , one means that \exists a commutative diagram such that $\psi_1 \varphi_1 = \text{Id}_X$, $\psi_2 \varphi_2 = \text{Id}_Y$:

$$\begin{array}{ccccc} X & \xrightarrow{\varphi_1} & X' & \xrightarrow{\psi_1} & X \\ f \Downarrow & & g \Downarrow & & \Downarrow f \\ Y & \xrightarrow{\varphi_2} & Y' & \xrightarrow{\psi_2} & Y \end{array}$$

Definition (Daniel Quillen, 1967) A model structure on a category is a triple (Cofib, Fib, Weq) of classes of morphisms, in which the morphisms are called **cofibrations**, **fibrations**, and **weak equivalence**, respectively, satisfying:

(2 out of 3) Let $X \xrightarrow{f} Y \xrightarrow{g} Z$. If two of f , g , gf are weak equivalences, then so is the third.

(Closed under retracts) Let f be a retract of g . If g is a cofibration (a fibration, a weak equivalence, respectively), then so is f .

(Lifting) For any commutative square with $i \in \text{Cofib}$, $p \in \text{Fib}$, if $i \in \text{Weq}$ or $p \in \text{Weq}$, then $\exists s : B \rightarrow X$ such that the two triangles commute:

$$\begin{array}{ccc} A & \xrightarrow{a} & X \\ i \Downarrow & \nearrow s & \Downarrow p \\ B & \xrightarrow{b} & Y \end{array}$$

(Factorizations) \forall morphism f , one has $f = pi = qj$, where $i \in \text{Cofib} \cap \text{Weq}$, $p \in \text{Fib}$, $j \in \text{Cofib}$, $q \in \text{Fib} \cap \text{Weq}$.

Any two of Fib, Cofib and Weq uniquely determine the third

A striking property of a model structure is: Any two of Fib, Cofib and Weq uniquely determine the third.

Definition Morphism i has the left lifting property (LLP) with respect to morphism p , or, p has the right lifting property (RLP) with respect to i , if for any commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{a} & X \\ i \downarrow & \nearrow x & \downarrow p \\ B & \xrightarrow{b} & Y \end{array}$$

there is a morphism $x : B \rightarrow X$ such that the two triangles commute.

Theorem (Quillen, 1969) Let $(\text{Fib}, \text{Cofib}, \text{Weq})$ be a model structure on category \mathcal{M} . Then

- (1) Cofib is exactly the class of morphisms which has LLP with respect to any trivial fibrations.
- (2) Fib is exactly the class of morphisms which has RLP with respect to any trivial cofibrations.
- (3) $\text{Cofib} \cap \text{Weq}$ is exactly the class of morphisms which has LLP with respect to any fibrations.
- (4) $\text{Fib} \cap \text{Weq}$ is exactly the class of morphisms which has RLP with respect to any cofibrations.
- (5) $\text{Weq} = (\text{Fib} \cap \text{Weq}) \circ (\text{Cofib} \cap \text{Weq})$.

Proof

(1) By the axiom, Cofib has LLP with respect to any morphism in $\text{Fib} \cap \text{Weq}$.

Conversely, for an arbitrary morphism f which has LLP with respect to any morphism in $\text{Fib} \cap \text{Weq}$, then one has a factorization $f = p' i'$, where $i' \in \text{Cofib}$ and $p' \in \text{Fib} \cap \text{Weq}$. So one gets a commutative square

$$\begin{array}{ccc}
 \bullet & \xrightarrow{i'} & \bullet \\
 \downarrow f=p'i' & \nearrow x & \downarrow p' \\
 \bullet & \xlongequal{\quad} & \bullet
 \end{array}$$

By the assumption f has LLP with respect to p' , i.e., there exists a morphism x such that the two triangles commutes. Since $p'x = \text{Id}$, it follows that f is a retract of i' , as the following commutative diagram shows

$$\begin{array}{ccccc}
 \bullet & \xlongequal{\quad} & \bullet & \xlongequal{\quad} & \bullet \\
 \downarrow f & & \downarrow i' & & \downarrow f \\
 \bullet & \xrightarrow{x} & \bullet & \xrightarrow{p'} & \bullet
 \end{array}$$

Since i' is a cofibration, it follows that f is a cofibration, i.e., $f \in \text{Cofib}$. This proves $\text{Cofib} = l(\text{Fib} \cap \text{Weq})$.

Basic properties

Let $(\text{CoFib}, \text{Fib}, \text{Weq})$ be a model structure on category \mathcal{M} . Then

(1) The classes CoFib and Fib are closed under compositions.

(2) Isomorphisms are fibrations, cofibrations, and weak equivalences.

(3) Cofibrations are closed under pushouts: for a pushout square with $i \in \text{CoFib}$, then $i' \in \text{CoFib}$:

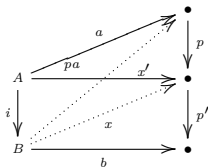
$$\begin{array}{ccc} \bullet & \xrightarrow{i} & \bullet \\ \Downarrow & \searrow i' & \Downarrow \\ \bullet & \xrightarrow{\dots\dots\dots} & \bullet \end{array}$$

Also, trivial cofibrations are closed under pushouts.

(4) Fibrations are closed under pullbacks; and trivial fibrations are closed under pullbacks.

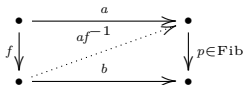
Proof

(1) Let $\bullet \xrightarrow{p} \bullet \xrightarrow{p'} \bullet$, where p and p' are fibrations. To prove $p'p$ is a fibration it suffices to prove that $p'p$ has RLP with respect to any trivial cofibration i . Assume that $bi = (p'p)a$. Then $bi = p'(pa)$. By lifting there is a morphism x s.t. $pa = xi$, $b = p'x$.

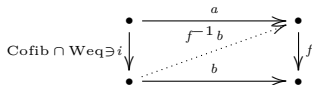


Since $pa = xi$, again by the lifting there is an x' s.t. $a = x'i$, $x = px'$. Thus $a = x'i$, $b = p'x = (p'p)x'$. This completes the proof.

(2) Let f be an isomorphism. From the following commutative diagram



we know that $f \in \text{Cofib} \cap \text{Weq}$. From the following commutative diagram



we know that $f \in \tau(\text{Cofib} \cap \text{Weq}) = \text{Fib}$. Thus $f \in \text{Cofib} \cap \text{Fib} \cap \text{Weq}$.

Proof (continued)

(4) For a pullback with p a fibration

$$\begin{array}{ccc} \bullet & \xrightarrow{a} & \bullet \\ f \downarrow & & \downarrow p \\ \bullet & \xrightarrow{b} & \bullet \end{array}$$

$f \in r(\text{Cofib} \cap \text{Weq})$. For a commut.

$$\begin{array}{ccc} \bullet & \xrightarrow{s} & \bullet \\ i \downarrow & \nearrow x & \downarrow f \\ \bullet & \xrightarrow{t} & \bullet \end{array}$$

with $i \in \text{Cofib} \cap \text{Weq}$, we want to find a morphism

x s.t. $xi = s$ and $fx = t$. Since $p \in \text{Fib}$, there is y s.t. $yi = as$ and $py = bt$.

$$\begin{array}{ccccc} \bullet & \xrightarrow{s} & \bullet & \xrightarrow{a} & \bullet \\ i \downarrow & & f \downarrow & \nearrow y & \downarrow p \\ \bullet & \xrightarrow{t} & \bullet & \xrightarrow{b} & \bullet \end{array}$$

Since $py = bt$, there is a unique morphism x such that $fx = t$ and $ax = y$:

$$\begin{array}{ccccc} & & & \nearrow y & \\ \bullet & \xrightarrow{x} & \bullet & \xrightarrow{a} & \bullet \\ & \searrow t & \downarrow f & \searrow b & \downarrow p \end{array}$$

$$\begin{array}{ccccc} & & & \nearrow as & \\ \bullet & \xrightarrow{xi} & \bullet & \xrightarrow{a} & \bullet \\ & \searrow ti & \downarrow f & \searrow b & \downarrow p \end{array}$$

By the universal property, we have $s = xi$. Hence $xi = s$ and $fx = t$. This completes the proof.

Ten classes of objects of study

Given a model structure $(\text{Cofib}, \text{Fib}, \text{Weq})$ on category \mathcal{M} with zero object, one considers:

the class of trivial cofibrations: $\text{Cofib} \cap \text{Weq}$

the class of trivial fibrations: $\text{Fib} \cap \text{Weq}$

the class of cofibrant objects: $\mathcal{C} = \{X \in \mathcal{M} \mid 0 \rightarrow X \text{ is a cofibration}\}$

the class of fibrant objects: $\mathcal{F} = \{Y \in \mathcal{M} \mid Y \rightarrow 0 \text{ is a fibration}\}$

the class of trivial objects:

$$\mathcal{W} = \{W \in \mathcal{M} \mid 0 \rightarrow W \text{ is a weak equivalence}\} = \{W \in \mathcal{M} \mid W \rightarrow 0 \text{ is a weak equivalence}\}$$

the class of trivially cofibrant objects: $\mathcal{C} \cap \mathcal{W}$

the class of trivially fibrant objects: $\mathcal{F} \cap \mathcal{W}$

Basic properties (continued)

Let $(\text{CoFib}, \text{Fib}, \text{Weq})$ be a model structure on additive category \mathcal{M} . Then

(1) If $X \in \mathcal{C}$ (respectively, $X \in \text{TC}$), then any splitting monomorphism with cokernel X belongs to CoFib (respectively, TCofib).

(2) If $X \in \mathcal{F}$ (respectively, $X \in \text{TF}$), then any splitting epimorphism with kernel X belongs to Fib (respectively, TFib).

(3) CoFib , Fib , TCofib , TFib and Weq are closed under finite coproducts.

Hints of proof:

(1) LLP and RLP

(3) LLP and RLP

For Weq , using factorization of a weak equivalence as a composition of a trivial cofib. with a trivial fib.

Basic properties (continued)

Let $(\text{CoFib}, \text{Fib}, \text{Weq})$ be a model structure on category \mathcal{M} with zero object, and $f: A \rightarrow B$ a morphism. Then

- (1) \mathcal{C} and TC are contravariantly finite in \mathcal{M} .
- (2) \mathcal{F} and TF are covariantly finite in \mathcal{M} .
- (3) \mathcal{C} and TC are closed under finite coproducts, direct summands, and productands. (Object X is a productand, if there is an object Y such that product $X \coprod Y$ exists.)
- (4) \mathcal{F} and TF are closed under finite products, direct summands, and direct productands.
- (5) If \mathcal{M} is additive, then \mathcal{W} is closed under finite coproducts and direct summands.
- (6) If $A \in \mathcal{C}$, $f \in \text{CoFib}$, then $B \in \mathcal{C}$.
- (7) If $A \in \text{TC}$, $f \in \text{TCofib}$, then $B \in \text{TC}$.
- (8) If $B \in \mathcal{F}$, $f \in \text{Fib}$, then $A \in \mathcal{F}$.
- (9) If $B \in \text{TF}$, $f \in \text{TFib}$, then $A \in \text{TF}$.

The homotopy category of a model structure

Definition For a model structure $(\text{Cofib}, \text{Fib}, \text{Weq})$ on category \mathcal{M} , **the homotopy category** $\text{Ho}(\mathcal{M})$ is by definition the category $\text{Weq}^{-1}\mathcal{M}$ of localization of \mathcal{M} with respect to Weq .

Fundamental tasks:

- To obtain new model structures
- To study properties of model structures, in particular, the relations between the 10 classes of objects of study
- The structures of homotopy categories

Examples

- \mathcal{A} : additive cat., then $\mathcal{C}^b(\mathcal{A})$, $\mathcal{C}^-(\mathcal{A})$, $\mathcal{C}^+(\mathcal{A})$, and $\mathcal{C}(\mathcal{A})$ have model structures, where cofibrations are chain split monomors., fibrations are chain split epimors., weak eqivs. are homotopy eqivs., Ho is the corresponding homotopy category.

- J. Gillespie; Z. W. Li: $(\mathcal{F}, \mathcal{E})$ weakly idempotent complete Frobenius cat.. Put

$$\text{Cofib}(\mathcal{F}) := \{\textit{inflation}\}, \quad \text{Fib}(\mathcal{F}) := \{\textit{deflation}\}$$

$$\text{Weq}(\mathcal{F}) := \{f \in \mathcal{F} \mid \underline{f} \text{ is isomor. in the stable cat. } \underline{\mathcal{F}}\}.$$

Then $(\text{Fib}(\mathcal{F}), \text{Cofib}(\mathcal{F}), \text{Weq}(\mathcal{F}))$ is a model structure on \mathcal{F} , with $(\mathcal{F}, \mathcal{F}, \mathcal{P})$, $\text{Ho}(\mathcal{F}) = \underline{\mathcal{F}}$.

- D. Quillen: \mathcal{A} : abelian cat. with enough proj. objs., then $\mathcal{C}^b(\mathcal{A})$ and $\mathcal{C}^+(\mathcal{A})$ have model structures, where cofibrations are monomors. with cokernel proj. complexes, fibrations are epimors., weak eqivs. are qis., Ho is the corresponding derived category.

- \mathcal{A} : abelian cat. with enough proj. objs., then $\mathcal{C}^{\leq 0}(\mathcal{A})$ has the model structure, where cofibrations are monomorphisms with cokernel proj. complexes, fibrations are chain maps which are epimors., except at the 0-position, weak eqivs. are qis., $\text{Ho} = \mathcal{D}^{\leq 0}(\mathcal{A})$ is a pretriangulated but not triangulated.

- M. Hovey: \mathcal{A} : Grothendieck cat. with proj. generators, then $\mathcal{C}(\mathcal{A})$ has the model structure, where cofibrations are monomorphisms with cokernel dg proj. complexes, fibrations are epimors., weak eqivs. are qis., $\text{Ho} = D(\mathcal{A})$, the derived category.