TUTORIAL SESSION FOR SRI'S TALKS

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1. Problems

(1) Compute the fibers of the map $\mathbb{Z} \to \mathbb{Z}[x]/(x^3 - d)$, where d is an integer. Is this map flat? When is this map smooth? Gorenstein?

(2) Let k be a field of characteristic 0. Compute the fibers of the map
$$k[t] \longrightarrow \frac{k[t,x,y]}{(y^3 - x(x-t)(x-2t))}\,.$$
Again, is this map flat? When is this map smooth? Gorenstein?

(3) Let k be a field and set $S = k[x]/(x^n)$, for some integer $n \geq 1$. Compute $\mathrm{HH}_*(S/k)$ and $\mathrm{HH}^*(S/k)$ both as S-modules and as S-algebras.

(4) Set $S = \mathbb{Z}[x]/(x^2 - d)$, where d is a positive integer. Computing $\mathrm{HH}_*(S/\mathbb{Z}, S)$ and $\mathrm{HH}^*(S/\mathbb{Z}, S)$, as S-modules and as algebras.

(5) Let k be a field of positive characteristic $p > 0$, such that there exists an element a in k that is not in $k^{1/p}$, that is to say, the equation $x^p - a$ has not root in k. What is an example of such a field?

Set $\ell = k[x]/(x^p - a)$. Verify that ℓ is a field so $\operatorname{gldim} l = 0$. Verify also that ℓ is not smooth as k-algebra, by computing $\ell \otimes_k \ell$.

(6) Let k be field and A a k-algebra; not necessarily commutative. Given an A-module, or an A-complex M, one has a natural map of graded k-algebras
$$\chi_M \colon \mathrm{HH}^*(A/k, A) \longrightarrow \mathrm{Ext}^*_A(M,M)\,.$$
Work out this map in any example you like. Here is a suggestion: Take $A = k[x]$ and $M = k[x]/(x^n)$ for $n \geq 0$.

(7) Let k be field, $R = k[x,y]$ and $S = R/(x,y) \cap (x-1)$. Verify that the surjection $R \to S$ is locally complete intersection but not a complete intersection.

(8) Compute the first few steps in the acyclic closure of the maps
$$k[x,y] \twoheadrightarrow \frac{k[x,y]}{(x^2,y^2)} \quad \text{and} \quad k[x,y] \twoheadrightarrow \frac{k[x,y]}{(x^2,xy,y^2)}\,.$$

(9) Verify that the composition of smooth (respectively, Gorenstein) maps is smooth (respectively, Gorenstein).

(10) Let k be a field, S a commutative K-algebra, flat essentially of finite type, and M a symmetric S-bimodule. Verify the isomorphisms
$$\mathrm{HH}_1(S/k, M) \cong \Omega_{S/k} \otimes_S M \quad \text{and} \quad \mathrm{HH}^1(S/k, M) \cong \mathrm{Der}_k(S, M)$$
where $\Omega_{S/k}$ is the module of Kähler differentials of S over k, and $\mathrm{Der}_k(S,M)$ is the module of k-linear derivations of S with values in M.

(11) Let S be a commutative ring and W an S-module that can be generated by n elements. Verify that $\wedge^i_S W = 0$ for $i \geq n+1$.

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(12) Let $R \to S$ be a map of commutative ring, and $A \xrightarrow{\simeq} S$ a quasi-isomorphism of dg R-algebras with A free (or just projective) as a complex of R-modules. Here one views S as a dg algebra concentrated in degree 0 and with differential 0. Prove that the product on $\mathrm{Tor}^R_*(S,S)$ viewed as $\mathrm{H}_*(A \otimes_R S)$ coincides with the $\pitchfork$-product, by using that the multiplication map

$$A \otimes_R A \longrightarrow A$$

is a lift of the multiplication map $S \otimes_R S \to S$.

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