L.C.I. algebras and HH^*

19th Sept.

(I) k field

$$S = \frac{k[x_1, \dots, x_n]}{I} \quad \text{affine } k\text{-algebra}$$

S is complete intersection if, in some presentation of S, as above, the ideal I can be generated by a regular seq.

Example: (1) Hypersurfaces : $S = \dfrac{k[x_1, \dots, x_n]}{(f(\underline{x}))}$.

(2) $S = \dfrac{k[x_1, \dots, x_n]}{(x_1^{d_1}, \dots, x_n^{d_n})}$ $\quad d_i \geq 0$

- Truncated polynomial ring

- $\mathbb{F}_p((\mathbb{Z}/p)^n)$ is a special case.

The Hochschild (co)homology of such an S is well-understood.

Say $S = \dfrac{k[x_1, \dots, x_n]}{(f_1, \dots, f_c)}$

with $f_1, \dots, f_c$ a regular seq. in $k[\underline{x}]$.

Consider $L := \quad 0 \longrightarrow \overset{\text{deg } 2}{\bigoplus_{j=1}^{c} S v_j} \xrightarrow[\left(\frac{\partial f_j}{\partial x_i}\right)]{} \overset{\text{deg } 1}{\bigoplus_{i=1}^{n} S u_i} \longrightarrow 0$

Jacobian matrix

So
$$\partial(v_j) = \sum_{i=1}^{n} \frac{\partial f_j}{\partial x_i} u_i$$

[Shifted cotangent cx. of S/k]

Then $HH_*(S/k)$ = homology of the complex

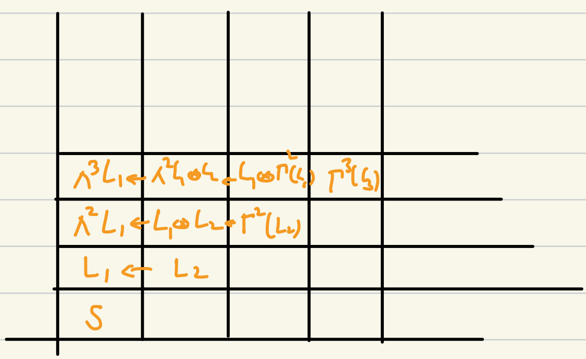
$\Gamma(L) \leftarrow$ The free divided powers algebra on L

$$= \Gamma(\oplus S v_j) \otimes_S \Lambda(\oplus S u_i)$$

$$= S\langle u_1, \dots, u_n; v_1 \dots v_c \rangle$$

$$\partial u_i = 0 \qquad \partial v_j = \sum \frac{\partial f_j}{\partial x_i} u_i$$

Has the shape:



Decomposes as $\bigoplus_{p \geq 0} \left(0 \to \Gamma^p(L_2) \to \Gamma^{p-1}(L_2) \otimes L_1 \to \cdots \to \Lambda^p L_1 \to 0 \right)$

In homology, this is the Hodge decomposition of $HH_*(S/k)$.

- we also get the multiplicative structure.

Hochschild cohomology: $|\eta_i| = 1$ $\quad |x_j| = 2$ (upper)

$$L^* := \quad 0 \longrightarrow \bigoplus_{i=1}^{n} S\eta_i \longrightarrow \bigoplus_{j=1}^{c} S x_j \longrightarrow 0$$

- Has a structure of a dg Lie algebra with

$\cdot$ $[x_j, -] = 0 \quad \forall j$ (for degree reasons)

$$\left.\begin{array}{l} \cdot \ \eta_i^2 = \displaystyle\sum_{j=1}^{c} \frac{\partial^2 f_j}{\partial x_i^2} x_j \\ \eta_i\eta_j + \eta_j\eta_i =: [\eta_i, \eta_j] = \displaystyle\sum_{h=1}^{c} \frac{\partial^2 f_h}{\partial x_i \partial x_j} x_h \end{array}\right\} (**)$$

[check the differential is compatible with $[\,,\,]$)

Then $U(L^*) =$ Universal enveloping alg.

$$= \frac{S\{\eta_i, x_j\}}{(**)\text{- relations.}}$$

Then $HH^*(S/k) = H^*(U(L^*))$ as algebras.

Clearly $\quad S[x_1, \dots, x_c] \subseteq \mathcal{U}(L^*)$

$\uparrow$ central subalgebra

- $\mathcal{U}(L^*)$ f.g. $S[\underline{x}]$-module

- ∂^{L^*} is $S[\underline{x}]$-linear.

Thus $\quad S[\underline{x}] \longrightarrow H^*(\mathcal{U}(L^*)) = HH^*(S/k)$

is a finite map. In particular

$HH^*(S/k)$ f.g. S-algebra.

- The $\{x_j\}$ are the Gulliksen-Eisenbud operators.

For any $M \in D^b(\operatorname{Mod} S)$, one has:

$\theta_M : \quad HH^*(S/k) \longrightarrow \operatorname{Ext}_S^*(M, M)$

- Character map.

Then (the image of) $HH^*(S/k)$ is central in $\operatorname{Ext}_S^*(M, M)$

- and the latter is finite / it.

Can set $\quad V_S(M) = \operatorname{Proj}(\operatorname{Ker} \theta_M) \subseteq \operatorname{Proj} HH^*(S/k)$

- Cohomological variety of M
- One has a theory of cohomological support varieties.

— x —

Raise the question:

- When is $HH^*(S/k)$ f.g.? Equivalently, noetherian.

If we also assume that $\mathrm{Ext}_S^*(M,M)$ f.g. over $HH^*(S/k)$ $\forall$ $M \in D^b(\mathrm{mod}\,S)$, then

S is l.c.i.

II Question: $HH^*(S/k)$ f.g. $\Rightarrow$ S l.c.i?

- Partial answers, in ongoing work with
  - Ben Briggs & Greg Stevenson.

In the rest of my talk

$\mathrm{char}\, k = 0$ and $\mathrm{rank}_k(S) < \infty$.

Then $\mathrm{rank}_k\, HH^n(S/k) < \infty$ $\forall$ n.

If $HH^*(S/k)$ is f.g., then one gets that

$\exists\ d \in \mathbb{N}_{\geq 0}$ and $\alpha \in \mathbb{R}$ such that

$$\operatorname{rank}_k HH^n(S/k) \leq \alpha n^d$$

i.e. $(\operatorname{rank}_k HH^n(S/k))$ has polynomial growth

- Much more is true:

$$\sum_{n \geq 0} \operatorname{rank}_k HH^n(S/k)\, t^n \text{ is a rational fn.}$$

(Hilbert-Serre thm).

- Our (BIS) guess is that this cannot happen unless S is c.i.

- Indeed, say S is self-injective. Then

$$\operatorname{Hom}_k(HH^n(S/k), k) \cong HH_n(S/k)$$

So $(\operatorname{rank}_k HH^n(S/k))$ polynomial growth

$\iff$ $(\operatorname{rank}_k HH_n(S/k))$ polynomial growth.

- $HH_*(S/k)$ easier to work with.
- Better functorial prop.

Set $h_n := \operatorname{rank}_k HH_n(S/k)$.

Question: If S is not c.i., does the seq. $(h_n)_{n \geq 0}$ have exponential growth?

- Meaning is $\limsup \sqrt[n]{h_n} > 1$

($\Leftrightarrow$ a subsequence of (h_n) grows as c^n for $c > 1$).

Thm: (BIS) Yes, if S is monomial (and for some other families of rings).

- Related to results in rational homotopy theory.
- Rough idea: Start with

$$S = \frac{k[x_1, \dots, x_e]}{(x_1, \dots, x_e)^2} \qquad e \geq 2$$

- Then verify explicitly that $(h_n)_n$ grows exponentially.
- Consider the Bar complex:

$$B_n(S/k) = S \otimes_k S^{\otimes n}$$

$$x_1 [y_2 | \cdots | y_{n-1} | x_c] \quad \text{for any choice of } y_i \in \Sigma k x_j$$

$$\downarrow \partial$$

$$0$$

Since $x_c \neq x_1$, one can check that this is not a boundary.

- One has (at least) c^{n-1} options for $y_2, \ldots, y_{n-1}$

This shows that $(\operatorname{rank}_k HH_n(S/k)) \sim (c^{n-1})$.

Now track how the growth of $HH_n(S/k)$ is propagated along change of rings:

$$\varphi : R \longrightarrow S'$$

Much easier to do that with André-Quillen homology:

$$D_*(S/k, S) \hookrightarrow HH_{*+1}(S/k, S)$$

because of char $k = 0$.