# Gorenstein algebras and Hochschild theory

18th Sept.

## I Something left over from last time:

Recall: $HH_u(S/k, S) = 0 = HH_v(S/k, S)$

for integers $u, v \geq 0$ of different parity, then $S/k$ smooth.

Here is a cohomological version:

Thm: If there exist integers $u, v$ of different parity such that $HH^{u+i}(S/k, S) = 0 = HH^{v+i}(S/k, S)$ $\forall$ $i = 0, \ldots, \dim S$, then $S/k$ is smooth.

- Proof: Use local duality to convert hypotheses to vanishing of $HH_u(S/k, -)$ & $HH_v(S/k, -)$ and then apply previous thm.
  - But need a version with coefficients.

Obvious question: Can we get away with less vanishing?

One more aspect:

$S/k$ smooth $\Rightarrow$ $\Lambda^* \Omega_{S/k} \xrightarrow{\cong} HH_*(S/k)$

In particular, the $S$-algebra $HH_*(S/k)$ is finitely generated (as a graded-comm alg.)

$\boxed{\text{Converse holds: If the } S\text{-algebra } HH_*(S/k) \text{ is finitely generated, then } S/k \text{ smooth.}}$

– Quite complicated to prove.

What about $HH^*(S/k)$?

$S/k$ smooth $\Rightarrow$ $HH^*(S/k) \cong \Lambda^* \mathrm{Der}_k(S,S)$.

In particular, $HH^*(S/k)$ is also f.g.?

We know $HH^*(S/k)$ f.g. also in other cases.

– $\boxed{\text{When exactly does this property hold?}}$

Will return to this in tomorrow's talk.

Let's move on.

II $\sigma: k \longrightarrow S$ flat + e.f.t.

We say $\sigma$ (or $S/k$) is Gorenstein if it fibers are Gorenstein (as rings), i.e. $\forall\ p \in \operatorname{Spec} k$

$$\operatorname{injdim}(S \otimes_k k(p)) < \infty.$$

– Then $\operatorname{injdim}(S \otimes_k k(p)) = \operatorname{Krull\ dim}(S \otimes_k k(p))$.

Comments: $k$ field + $S$ e.f.t. $/k$.

Then $\operatorname{injdim} S < \infty \Rightarrow \operatorname{injdim}(S \otimes_k l) < \infty$ (check)

$\forall\ k \hookrightarrow l$ extensions of fields.

Thus $\sigma$ Gorenstein = $\sigma$ "geometrically" Gorenstein.

Examples: ① $S/k$ smooth $\Rightarrow$ $S/k$ Gorenstein.

($\because$ Regular rings are Gorenstein, in CA)

② $k \hookrightarrow k[x]/(x^n)$ Gorenstein, but not smooth (unless $n = 1$)

③ $G$ finite abelian group

Then $K \hookrightarrow KG \leftarrow$ group algebra

This is a Gorenstein mcp for any $K$.

[With the right defn. this would be a Gor. mcp for any finite $G$]

(II) Quasi-Gorenstein mcps:

$S$ comm. noeth. ring

An $S$-module $L$ is invertible if

- $L$ is a f.g. projective of rank 1

Equivalently, $L$ is a f.g. projective and there exist a f.g. module $L'$ s.t. $L \otimes_S L' \cong S$

- We say a graded module $L = \{L^i\}$ is invertible if $\bigoplus_i L^i$ is invertible.

An $S$-cx. $L$ is invertible if

- $L \in D^b(\text{mod } S)$ and $\exists$ $L' \in D^b(\text{mod } S)$ s.t. $L \otimes_S^L L' \simeq S$

$\Leftrightarrow$ $H(L)$ is a graded invertible module.

Fact: If $\operatorname{Spec} S$ is connected, then

$L$ invertible $\Leftrightarrow$ $L \simeq \Sigma^g P$ for some invertible module $P$

—x—

$\pi: R \twoheadrightarrow S$ surjective map of comm. noeth. rings.

Then $\pi$ is quasi-Gorenstein if

$\mathrm{RHom}_R(S, R) \in D(S)$ is invertible.

Example: ① $\pi: R \twoheadrightarrow S$  $R$ & $S$ Gorenstein rings.

Then $\pi$ quasi-Gor.

Conversely, $\pi$ quasi-Gor. Then $R$ is Gor.

$\Leftrightarrow$ $S$ is Gorenstein (assuming $R$ local).

② $(R, m, k) \twoheadrightarrow k$ is Gor. $\Leftrightarrow$ $R$ Gor.

$\boxed{\text{Thm: } S/k \text{ Gor.} \iff \mu: S^e \to S \text{ is quasi-Gor.}}$

Sketch: Say $k = \mathbb{k}$ a field.

$S/\mathbb{k}$ Gorenstein $\Rightarrow$ $S^e/S$ Gorenstein

check $\Rightarrow$ $S^e/\mathbb{k}$ Gorenstein

$\Rightarrow$ $S^e$ Gorenstein

[Just saying $S$ and $T$ Gorenstein rings e.f.t/$\mathbb{k}$

then $S \otimes_{\mathbb{k}} T$ is Gorenstein as well]

Then $S^e \xrightarrow{\mu} S$ map of Gorenstein rings, so

it is Gorenstein.

- The proof of the general case is more involved.

As is the proof of the converse.

- Will get back to this shortly.

-x-

We get $S/k$ Gorenstein $\Rightarrow$ $\operatorname{Ext}^i_{S^e}(S, S^e) = 0 \ \forall\, i \gg 0$.

Question: Is the converse true?

Say $\operatorname{rank}_k(S) < \infty$. Then

$$S \otimes_k S \cong \operatorname{Hom}_k(S^\vee, S) \qquad (\ )^\vee = \operatorname{Hom}_k(-, k)$$

as $S^e$-modules.

Then

$$\operatorname{Ext}^*_{S^e}(S, S \otimes_k S) \cong \operatorname{Ext}^*_{S^e}(S, \operatorname{Hom}_k(S^\vee, S)) \cong \operatorname{Ext}^*_S(S^\vee, S)$$

So the question can be posed with reference to $HH^*$:

> Question: Say $\operatorname{rank}_k(S) < \infty$. If $\operatorname{Ext}^i_S(S^\vee, S) = 0$ for $i \gg 0$, is then $S$ Gorenstein, i.e. injective as a module / itself.

- Comm. version of Tachikawa's Conj.
  - Still open.
- The $HH^*$ formulation is due to Avramov-Iyengar? Ashashiba.

# III Reduction Formulas:

- For simplicity $K = k$ a field

Classical Reduction isomorphisms: $\forall$ $S$-modules $M, N$ one has:

$$\operatorname{Hom}_{S^e}(S, \operatorname{Hom}_k(M, N)) \cong \operatorname{Hom}_S(M, N)$$

Natural bimodule structure

$$S \otimes_{S^e} (M \otimes_k N) \cong M \otimes_S N$$

These give rise to:

$$R\operatorname{Hom}_{S^e}(S, R\operatorname{Hom}_k(M, N)) \cong R\operatorname{Hom}_S(M, N)$$

[May need $M$ f.g. / $S$]

$$S \overset{L}{\otimes}_{S^e} (M \overset{L}{\otimes}_k N) \cong M \overset{L}{\otimes}_S N.$$

— x —

We have to compute $R\operatorname{Hom}_{S^e}(S, S \overset{L}{\otimes}_k S)$

What about $R\operatorname{Hom}_{S^e}(S, M \overset{L}{\otimes}_k N)$ in general?

Let $\omega_S :=$ a dualizing cx. of $S$:

[For example, if $\operatorname{rank}_k S < \infty$, then

$$\omega_S = \operatorname{Hom}_k(S, k)$$

if $S$ is Gorenstein, $\omega_S = S$ ]

- Not unique (!)

Write $(-)^{\dagger} := R\operatorname{Hom}_S(-, \omega_S)$

Dualizing complexes are characterized by

$$D^b(\operatorname{mod} S)^{op} \xrightarrow[\equiv]{(\ )^{\dagger}} D^b(\operatorname{mod} S)$$

- Auto equivalence

Meaning $M \xrightarrow{\simeq} (M^{\dagger})^{\dagger}$

Then: For any $M \in D^b(\operatorname{mod} S)$ and $N \in D(S)$

$$\boxed{\begin{aligned} R\operatorname{Hom}_{S^e}(S, M \overset{L}{\otimes}_k N) &\simeq R\operatorname{Hom}_S(M^{\dagger}, N) \\ S \overset{L}{\otimes}_{S^e} R\operatorname{Hom}_k(M, N) &\simeq M^{\dagger} \overset{L}{\otimes}_S N. \end{aligned}}$$

- Reduction formula for HH

Recall thm: $S/k$ Gorenstein $\iff$ $S/S^e$ quasi-Gor.

i.e. $\mathrm{RHom}_{S^e}(S, S^e)$ invertible.

From the reduction formula one gets:

$$\mathrm{RHom}_{S^e}(S, S \overset{L}{\otimes}_k S) \simeq \mathrm{RHom}_S(\omega_S, S)$$

Now $\mathrm{RHom}_{S^e}(S, S^e)$ invertible

$\iff$ $\mathrm{RHom}_S(\omega_S, S)$ invertible

Simple "$\iff$" $\omega_S$ invertible

$\iff$ $S$ Gorenstein

Also simple. ✓ $\square$

The reduction isomorphisms hold for arbitrary $S/K$ (as long as $\mathrm{flatdim}_k M < \infty$, $M \in D^b(\mathrm{mod}\, S)$)

- And this allows us to prove the general version of the theorem above.

- Involves dualizing cx's of map $\sigma: K \to S$

$\omega_\sigma$

Making a functorial assignment $\sigma \to \omega_\sigma$ is the fundamental issue in duality theory (in the sense of Grothendieck and his school).

HH opens a path to it: Recall

$$S \overset{\ell}{\underset{S^e}{\otimes}} R\mathrm{Hom}_K(M, N) \simeq R\mathrm{Hom}_S(M, \omega_\sigma) \overset{\ell}{\underset{S}{\otimes}} N$$

For $M = S = N$ this gives

$$S \overset{\ell}{\underset{S^e}{\otimes}} R\mathrm{End}_K(S) \simeq \omega_\sigma$$

Can take this as the definition of $\omega_\sigma$.

- Approached developed by Lipman and his collaborators.