Smooth algebras & HH – II

17th Sept.

(I) As usual S/k flat + e.f.t.
k comm. noeth. ring

Last time: S/k smooth $\Leftrightarrow$ $\mu : S^e \twoheadrightarrow S$ l.c.i.

$$\Rightarrow h^M : (\Lambda^* \Omega_{S/k}) \otimes_S M \xrightarrow{\cong} HH_*(S/k; M)$$

$\forall$ M symmetric

Also: S/k smooth $\Rightarrow$ the S-module $\Omega_{S/k}$ is projective.

Always finitely generated.

Today: $h^S : \Lambda^*(\Omega_{S/k}) \xrightarrow{\cong} HH_*(S/k) \Rightarrow S/k$ smooth.

Hypothesis $\Rightarrow$ $HH_i(S/k) = 0$ $\forall i \gg 0$

[Ex: W any f.g. S-module $\Rightarrow \Lambda^i W = 0$ $\forall i \gg 0$]

Theorem: S/k as before. If for some integers $i, j \geq 0$

$$HH_{2i}(S/k, S) = 0 = HH_{2j+1}(S/k, S)$$

then S/k is smooth.

Corollary: S/k smooth $\Leftrightarrow$ $\operatorname{projdim}_{S^e} S < \infty$.

- $R \twoheadrightarrow S$ c.i. $\Rightarrow K(\underline{r}; R) \xrightarrow{\simeq} S$ for a suitable choice of $\underline{r}$.

- And $\operatorname{projdim}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} < \infty \quad \forall\ \mathfrak{p} \in \operatorname{Spec} R$

$\Rightarrow \quad \operatorname{projdim}_R M < \infty$

[Here M is any f.g. R-module]

— x —

Corollary says S/k smooth $\Leftrightarrow$ the diagonal is compact.

The theorem is optimal:

Example: $\mathbb{Z} \hookrightarrow \dfrac{\mathbb{Z}[x]}{(x^2-2)} =: S$. Then

$$HH_n(S/\mathbb{Z}; S) = \begin{cases} S & n = 0 \\ 0 & n \geq 2 \text{ and even} \\ S/(2x) & n \geq 1 \text{ and odd.} \end{cases}$$

But:

Question: Say k is a field. Then if for some $n \geq 0$ $HH_n(S/k, S) = 0$, then S/k smooth.

- OK when char $k = 0$ + $S = k \oplus S_1 \oplus \cdots$
graded, connected.

(Proof uses cyclic homology)

- Also when S/k is l.c.i.

Open in general.

The theorem itself is a special case of:

Theorem: $S \longrightarrow R \xrightarrow{\pi} S$ algebra retract

If for some integers $i, j \geq 0$ one has
$\operatorname{Tor}^R_{2i}(S,S) = 0 = \operatorname{Tor}^R_{2j+1}(S,S)$, then π is l.c.i.

- The hypothesis localizes: $\forall\ \mathfrak{q} \in \operatorname{Spec} S \subseteq \operatorname{Spec} R$

$$\operatorname{Tor}^R_n(S,S)_{\mathfrak{q}} \cong \operatorname{Tor}^{R_{\mathfrak{q}}}_n(S_{\mathfrak{q}}, S_{\mathfrak{q}})$$

and the conclusion is local. So we may

assume $R \twoheadrightarrow S$ is a map of local rings.

Key impact: Nakayama Lemma applies

- So $\operatorname{rank}_l(M/\mathfrak{m}_S M) =$ min. # of generators of M as an S-mod.

$l = S/\mathfrak{m}_S$.

The proof goes through by explicitly constructing homology classes in $\operatorname{Tor}^R(S,S)$ assuming S/R not c.i. This uses dg algebra resolutions.

II A a dg R-algebra

- $A = \{A_i\}_{i \geq 0}$
- A strictly graded-comm.
- Say $H_0(A) = S$

[For example, $A = K(\underline{x}; R)$ where $S = R/\underline{x}R$]

Pick $z \in A_n$ cycle (so $\partial z = 0$)

Will construct a dg algebra $A\langle x \rangle$

- again strictly graded-comm. R-algebra

with $A \hookrightarrow A\langle x\rangle$ inclusion of dg algebras s.t.

$$H_i(A) \to H_i(A\langle x\rangle)$$

- bijective $\forall\ i < n$
- $\dfrac{H_n(A)}{[z]\, H_0(A)} \xrightarrow{\cong} H_n(A\langle x\rangle)$.

So this process precisely kills the class $[z]$.

Case when n is even:

Then $A\langle x\rangle = A \underset{R}{\otimes} R\langle x\rangle$

Exterior alg. on x
$|x| = n+1$

$= A \oplus Ax$ as dg A-modules

- $\partial(x) = z$
- Extend to all of $A\langle x\rangle$ by:

$$\partial(a+bx) = \partial^A(a) + \partial^A(b)x + (-1)^{|b|} b z$$

Then $0 \to A \hookrightarrow A\langle x\rangle \to Ax \to 0$

Exact seq. of dg A-modules

$(Ax)_i = A_{i-(n+1)} = 0$ for $i \leq n$.

One gets:

$$\to H_{i+1}(Ax) \to H_i(A) \to H_i(A\langle x\rangle) \to H_i(Ax) \to$$

For $i \leq n-1$ one gets $H_{i+1}(Ax) = 0 = H_i(Ax)$

so $H_i(A) \xrightarrow{\cong} H_i(A\langle x\rangle)$

$i = n$ one gets

$$\begin{tikzcd} H_{n+1}(Ax) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & H_n(A) \arrow[r] & H_n(A\langle x\rangle) \arrow[r] & 0 \\ H_0(A) \arrow[ur] & & & \end{tikzcd}$$

$$1 \longmapsto \pm[z]$$

$$\therefore \quad H_n(A\langle x\rangle) \cong H_n(A) \big/ [z]\, H_0(A)$$

Case when n is odd:

Then $A\langle x\rangle = A \otimes_R R\langle x\rangle \qquad |x| = n+1$

free divided powers alg. on x (for $R\langle x\rangle$)

$$R\langle x\rangle = \bigoplus_{i \geq 0} R x^{(i)}$$

$$x^{(i)} \cdot x^{(j)} = x^{(i+j)} \frac{(i+j)!}{i!\,j!}$$

Check $R\langle x\rangle$ strictly graded-comm. R-alg.

So the same is true of $A\langle x\rangle$

As an A-module $A\langle x\rangle = \bigoplus_{i \geq 0} A x^{(i)}$

- Introduce diff. by extending that on A

and setting $\partial(x^{(i)}) = z x^{(i-1)} \quad \forall i \geq 0$

$(x^{(0)} = 1$ so $d(x^{(0)}) = 0)$

[If $\mathbb{Q} \subseteq R$, then $R\langle x \rangle \cong R[x]$

$$x^{(i)} = \frac{x^i}{i!} \quad]$$

Again

$$0 \to A \to A\langle x \rangle \xrightarrow{\theta} \Sigma^{n+1} A\langle x \rangle \to 0$$

$$\theta\left(\sum a_i x^{(i)}\right) = \sum a_i x^{(i-1)}$$

exact seq. of dg A-modules.

Exercise: $H_i(A) \to H_i(A\langle x \rangle)$ iso $\forall\, i \leq n-1$

$$\frac{H_n(A)}{[z] H_0(A)} \xrightarrow{\cong} H_n(A\langle x \rangle).$$

— x —

Back to $R \twoheadrightarrow S = R/(r_1, \ldots, r_c)$

Pick $r_1, \ldots, r_c$ min. gen. set for $\operatorname{Ker}(R \twoheadrightarrow S)$

- Start with $A = R$ and kill $r_1, \ldots, r_c$

$R\langle x_1 \mid \partial x_1 = r_1 \rangle$ $\quad$ $H_0(A\langle x_1\rangle) = R/r_1R$

$= K(r_1; R)$

Then $R\langle x_1, x_2 \mid \partial x_1 = r_1, \ \partial x_2 = r_2 \rangle$

$= K(r_1, r_2; R)$

Finally we get $R\langle x_1, x_2, \dots, x_c \mid \partial x_i = r_i \rangle$

$=$ Koszul cx. $K(\underline{r}; R)$

Set $X_1 = \{x_1, \dots x_c\}$

Suppose φ is not c.i.

$\equiv \quad H_1(R\langle X_1\rangle) \neq 0$

Pick cycles $z_1, \dots, z_s$ in $R\langle X_1\rangle_1$

s.t. the class $[z_1], \dots, [z_s]$ minimally generate the $H_0(R\langle X_1\rangle) = S$ module

$H_1(R\langle X_1\rangle)$.

Construct, step by step,

$R\langle X_1, y_1 \dots, y_s \mid \partial y_i = z_i \rangle$

Then $R \hookrightarrow R\langle X_{\leq 2}\rangle$ satisfies

$X_2 = \{y_1, \dots, y_1\}$ $\qquad$ $H_0(R\langle X_{\leq 2}\rangle) \cong S$

$H_1(R\langle X_{\leq 2}\rangle) = 0$

Now continue by killing cycles in higher degrees

We get a chain:

$$R \subseteq R\langle X_1\rangle \subseteq R\langle X_{\leq 2}\rangle \subseteq \cdots \subseteq R\langle X_{\leq n}\rangle \subseteq \cdots$$

- has homology S in deg 0
- no homology in degrees $1 \leq i \leq n-1$

Set $R\langle X\rangle := \bigcup R\langle X_{\leq n}\rangle$

- strictly graded-comm. R-algebra with $H_0(R\langle X\rangle) = S$

Take resolution of S

- By choose minimal generators to kill at each step, we get

$$\partial x \subseteq \mathfrak{m}_R R\langle X\rangle + X^{(2)}$$

quadratic part

This is the acyclic closure of S/k.

This is rarely minimal.

Example: $k[x,y] \twoheadrightarrow \dfrac{k[x,y]}{(x^2, xy, y^2)}$

III **Thm:** If $R \overset{\pi}{\twoheadrightarrow} S$ has a section, i.e. a map $i : S \to R$ of rings s.t. $\pi i = \mathrm{id}_S$, then the acyclic closure $R\langle X\rangle \xrightarrow{\simeq} S$ is minimal, i.e. $\partial R\langle X\rangle \subseteq \mathfrak{m}_R R\langle X\rangle$.

- Key result. Proof is not easy.
- Suppose $\pi : R \twoheadrightarrow S$ has a section. But π is <u>not</u> c.i.

Let $R\langle X\rangle \xrightarrow{\simeq} S$ be an acyclic closure.

$X_1 = \{x_1, \dots, x_c\}$ $\quad (\partial x_1, \dots, \partial x_c) = \mathrm{Ker}(R \to S)$

Pick $y \in X_2$ (exists because π is not c.i.)

Then $\partial y \in \sum_{i=1}^{c} r_i x_i \qquad r_i \in R$

$$\operatorname{Tor}^R(S,S) = H_*(R\langle X\rangle \otimes_R S)$$

$$= H_*(S\langle X\rangle) \quad \text{and} \quad \partial S\langle X\rangle \subseteq m_S S\langle X\rangle$$

In $S\langle X\rangle \qquad \partial(x_i) = r_i = 0$

Set $\quad w_{2n+c} = y^{(n)} x_1 \cdots x_c \qquad \forall\ n \geq 0$

Check: $\quad \partial(w_{2n+c}) = 0 \quad$ in $S\langle X\rangle$

So w_{2n+c} is a cycle.

But $w_{2n+c} \notin m_S S\langle X\rangle$

so cannot be in $\partial(S\langle X\rangle)$

$\therefore \quad [w_{2n+c}] \neq 0 \quad$ in $\quad H_{2n+c}(S\langle X\rangle)$

$$\| $$

$$\operatorname{Tor}^R_{2n+c}(S,S)$$

So $\pi : R \twoheadrightarrow S$ not c.i.

$$\Longrightarrow \operatorname{Tor}^R_{2n+c}(S,S) \neq 0 \quad \forall\ n \geq 0$$

Also $\operatorname{Tor}^R_i(S,S) \neq 0$ for $0 \leq i \leq c$ (Check)

This is the contrapositive of the desired statement.

— x —

Back to $HH_{2i}(S/k) = 0 = HH_{2j+1}(S/k) \Rightarrow S/k$ smooth.

There is also a cohomological version:

If $\exists$ integers u & $v \geq 0$ of different parity such that

$$HH^{u+i}(S/k) = 0 = HH^{v+i}(S/k)$$

for $i = 0, \ldots, \dim S$

Then S/k smooth.

- Proof is by an application of local duality.

Again: When $K = k$ a field, is $HH^n(S/k) = 0$ for some $n \geq 0 \Rightarrow S/k$ smooth?

One more aspect: S/k smooth $\Rightarrow HH_*(S/k) \cong \Lambda^* \Omega_{S/k}$ so a f.g. S-algebra. Conversely, if the S-algebra $HH_*(S/k)$ is f.g., then S/k is smooth.