Tuesday: Smooth algebras I

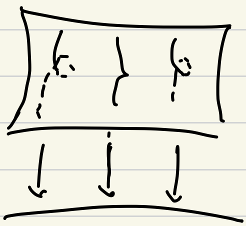
16th Sept.

(I) K comm. noeth. ring

S K-algebra, flat + e.f.t

$\sigma : K \longrightarrow S$ structure map

Visually:



$$\begin{array}{c} \operatorname{Spec} S \\ \downarrow \sigma^a \\ \operatorname{Spec} K := \{ \mathfrak{p} \subseteq K \mid \mathfrak{p} \text{ prime ideal} \} \end{array}$$

Can visualize it as a family of algebraic varieties parameterized by $\operatorname{Spec} K$.

- The flatness of S/K makes this a reliable picture.

Fix $\mathfrak{p} \in \operatorname{Spec} K$. The residue field at $\mathfrak{p}$ is $k(\mathfrak{p}) := K_{\mathfrak{p}} / \mathfrak{p} K_{\mathfrak{p}}$.

The fibre over $\mathfrak{p}$ is the $k(\mathfrak{p})$-algebra

$$S(\mathfrak{p}) := S \otimes_K k(\mathfrak{p})$$

- This is essentially of finite type $/ k(\mathfrak{p})$:

$$\begin{tikzcd} K \arrow[r] \arrow[d] & S \arrow[d] \\ k(\mathfrak{p}) \arrow[r] & S \otimes_K k(\mathfrak{p}) = S(\mathfrak{p}) \end{tikzcd}$$

It's relevance is that

$$\operatorname{Spec}(S(\mathfrak{p})) = \{ \mathfrak{q} \in \operatorname{Spec} S \mid \sigma^a(\mathfrak{q}) = \mathfrak{p} \}$$
$$= (\sigma^a)^{-1}(\mathfrak{p})$$

– i.e. the set-theoretic fibe / $\mathfrak{p}$.

Pictorially:

$$\begin{tikzcd} & \operatorname{Spec} S(\mathfrak{p}) \arrow[r, hook] \arrow[d] & \operatorname{Spec} S \arrow[d] \\ \operatorname{Spec} k(\mathfrak{p}) = & \{\mathfrak{p}\} & \subseteq \operatorname{Spec} K \end{tikzcd}$$

Example: What are the fibers of the maps?

① $\mathbb{Z} \hookrightarrow \mathbb{Z}[x]/(x^3 - d)$ $\quad d \in \mathbb{Z}$.

② $\mathbb{R}[t] \hookrightarrow \dfrac{\mathbb{R}[t, x, y]}{(y^2 - x(x-t)(x-2t))}$

The geometric fibers of σ are

$$S(\mathfrak{p}) \underset{k(\mathfrak{p})}{\otimes} l \qquad \text{where } k(\mathfrak{p}) \hookrightarrow l \text{ any extension of fields.}$$

$$= S \underset{K}{\otimes} l \qquad \forall \text{ maps of rings } K \to l \text{ where } l \text{ is a field.}$$

- Tantamount to solving equations after allowing field extensions

Example: $K := \mathbb{R} \overset{\sigma}{\hookrightarrow} \dfrac{\mathbb{R}[x]}{(x^2+1)} =: S$ (i.e. $\mathbb{C}$)

Then $\operatorname{Spec} S = \{(0)\}$ a single pt.

- The fiber of σ is $\operatorname{Spec} S$
- The geometric fiber also includes

$S \underset{\mathbb{R}}{\otimes} \mathbb{C}$ and $\operatorname{Spec}(S \underset{\mathbb{R}}{\otimes} \mathbb{C}) = \{(x+i), (x-i)\}$

Definition: $\sigma : K \to S$ as always. Then σ (or S/K) is smooth if all the geometric fibers have finite global dimension:

$$\operatorname{gldim}(S \otimes_K \ell) < \infty \quad \forall\, K \to \underset{\text{field.}}{\ell}$$

$\equiv$ $S \otimes_K \ell$ is regular (in the sense of comm. algebra)

Special Case $K = k$ a field.

Then S/k smooth $\Rightarrow$ $\operatorname{gldim} S < \infty$

Converse holds if, for example $\operatorname{char} k = 0$ or k is algebraically closed, but not always.

Examples k a field of $\operatorname{char} k = p > 0$ such that $\exists$ $a \in k \setminus k^{1/p}$. i.e.

$x^p - a$ has no root in k.

Then $k \hookrightarrow \dfrac{k[x]}{(x^p - a)} =: \ell$

is such that $\mathrm{gldim}(\ell) = 0$ ($\because$ ℓ field)

But $\mathrm{gldim}(\ell \otimes_k \ell) = \infty$ (Exercise).

So ℓ/k is not smooth.

-x-

Recall $\Omega_{S/k}$, the module of Kähler diff.

- This is a f.g. S-module. Concrete description:

$$S = U^{-1} \frac{k[x_1, \dots, x_n]}{(f_1, \dots, f_c)}$$

Then $\Omega_{S/k} = \mathrm{Coker}\left(S^c \xrightarrow[\left(\frac{\partial f_j}{\partial x_i}\right)]{} S^n \right)$

Jacobian matrix

Jacobian criterion: If S/k is smooth, then the S-module $\Omega_{S/k}$ is projective. The converse holds, for instance, when $\mathbb{Q} \subseteq k$.

- There are more precise statements that cover all k.

II Smooth algebras & Hochschild homology

S/k as before + M a symmetric bimodule

Recall:

$$h^M : \underbrace{\Lambda^*_S \Omega_{S/k}}_{\text{algebra of differential forms.}} \otimes_S M \longrightarrow HH_*(S/k; M)$$

Theorem: The following conditions are equivalent

1. S/k is smooth.
2. h^M is also iso. $\forall$ symmetric M
3. h^S is an iso.

1 $\Rightarrow$ 2 Hochschild - Kostant - Rosenberg (at least when $K = k$, a field)

3 $\Longleftrightarrow$ 1 André, Quillen, Grothendieck

I will present the HKR proof of 1 $\Rightarrow$ 2

And 3 $\Rightarrow$ 1 will be deduced from a more general result.

Proof of ① ⇒ ②. At least when $K = k$ a field.

So assume S/k is smooth.

– In particular $\operatorname{gldim} S < \infty$.

By base-change (check this)

$$S \otimes_k \sigma : S \longrightarrow S \otimes_k S =: S^e$$

is also smooth. Then the composition

$$k \xrightarrow{\sigma} S \xrightarrow{S \otimes_k \sigma} S^e \quad \text{is smooth}$$

$\Rightarrow$ $\operatorname{gldim} S^e < \infty$ as well.

[check this]

So we are in a situation:

$$(S^e =) \quad R \twoheadrightarrow S$$

where R and S have finite global dim.

Theorem (Chevalley): $\pi : R \twoheadrightarrow S$ surjective map of comm. noeth. rings, with $\operatorname{gldim} R$ & $\operatorname{gldim} S$ finite. Then π is locally complete intersection.

Defn: $\pi : R \twoheadrightarrow S$ is complete intersection

if $S = R/(r_1, \dots, r_c)$ where

$\underline{r}$ is a regular sequence.

This means that r_1 nzd (not a zero divisor) on R

and r_{i+1} nzd on $R/(r_1, \dots, r_i)$

$\forall \; 0 \le i \le c-1.$

Example: $S[x_1, \dots, x_c] \twoheadrightarrow S$

$x_i \longmapsto 0$

π is locally complete intersection (l.c.i.)

if $\forall \; q \in \operatorname{Spec} S \subseteq \operatorname{Spec} R$

The localization $\pi_q : R_q \twoheadrightarrow S_q$

is a complete intersection.

Example: $k[x,y] \twoheadrightarrow \dfrac{k[x,y]}{(x,y)\cap(x-1)}$

is l. c.i. but not c.i.

In summary, S/k smooth $\Rightarrow$ $\mu: S^e \twoheadrightarrow S$ locally c.i.

III - We want to relate this property to HH_*

i.e. $\mathrm{Tor}^{S^e}(S, -)$

Again consider the general situation:

$$\pi: R \twoheadrightarrow S = R/(r_1, \dots, r_c)$$

- No hypothesis on $\underline{r}$ just yet

$K(\underline{r}; R) :=$ Koszul cx. on $\underline{r}$ (with coefficient in S).

$$K(r_1; R) = \quad 0 \to \underset{\substack{\uparrow \\ \text{degree } 1}}{R} \xrightarrow{r_1} \underset{\substack{\uparrow \\ \text{degree } 0}}{R} \to 0$$

$$K(\underline{r}; R) := \bigotimes_{i=1}^{c} K(r_i; R)$$

- This is a dg R-algebra

Can also express as
$$\Lambda^*\left(\bigoplus_{i=1}^{c} R e_i\right) \quad \text{with } \partial e_i = r_i$$
$$|e_i| = 1$$

Exterior alg.

Looks like this:

$$0 \to R \xrightarrow{\begin{bmatrix} r_1 \\ \vdots \\ r_c \end{bmatrix}} R^c \to \cdots \to R^c \xrightarrow{(r_1, \dots, r_c)} R \to 0$$

with some signs.

In particular $H_0(K(\underline{r}; R)) = \dfrac{R}{(\underline{r})R} = S$

Depth sensitivity of Koszul cx:

- If $\underline{r}$ is reg. seq, then $H_i(K(\underline{r}; R)) = 0 \ \forall i \geq 1$

so $K(\underline{r}; R) \xrightarrow{\sim} S$ is a free resoln.

- When R is local (or $\underline{r} \subseteq J(R)$)

$$H_1(K(\underline{r};R)) = 0 \implies H_i(K(\underline{r};R)) = 0 \ \forall i \geq 1$$

and $\underline{r}$ is a reg. seq.

— + —

Now $R \twoheadrightarrow S$ l. c.i. map.

M an S-module

$$h^M : \bigwedge^* \operatorname{Tor}_1^R(S,S) \otimes_S M \longrightarrow \operatorname{Tor}_*^R(S, M)$$

- The natural map.

$$\boxed{\text{Claim: } \pi \text{ l.c.i.} \implies h^M \text{ iso.}}$$

- This would settle ① ⇒ ② of Thm.

Proof of claim: Check an iso. is local on $\operatorname{Spec} S$, namely h^M iso $\iff h^M_{\mathfrak{q}}$ iso $\forall$ $\mathfrak{q} \in \operatorname{Spec} S$.

[Exercise)

So we may assume $\pi : R \twoheadrightarrow S$ is c.i.

$$S = R/(r_1, \ldots, r_c)$$

To compute $\mathrm{Tor}^R(S, M)$ we use

$$K(\underline{r}; R) \xrightarrow{\sim} S \quad (\because \ \underline{r} \ \text{reg. seq})$$

Then $\mathrm{Tor}^R_*(S, M) = H_*(K(\underline{r}; R) \otimes_R M)$

$$= H_*((K(\underline{r}, R) \otimes_R S) \otimes_S M)$$

$$= H_*(K(\underline{r}; S) \otimes_S M)$$

N.B. differential on $K(\underline{r}; S)$ is 0.

Thus $\mathrm{Tor}^R_*(S, M) = K(r; S) \otimes_S M$

$$= \mathrm{Tor}^R_*(S, S) \otimes_S M$$

$$= \Lambda_S \, \mathrm{Tor}^R_1(S, S) \otimes_S M.$$

Check this

$\square$

– We get more: $\mathrm{Tor}_i(S, S) = K(r; S)_i$

$$= S^{\binom{c}{i}}.$$

Thus Our proof showed also that

$R \twoheadrightarrow S$ l.c.i. $\Rightarrow$ $\mathrm{Tor}_i^R(S,S)$ projective S-module $\forall\, i$.

In particular, that the S-module $\mathrm{Tor}_1^R(S,S)$ is projective.

In the HH_* context, we get:

S/S^e l.c.i. $\Rightarrow$ $HH_1(S/k, S)$ projective S-module

i.e. $\Omega_{S/k}$ proj. S-module.

$\Rightarrow$ S/k smooth

Jacobian criterion (when it applies).

To summarize:

S/k smooth $\Leftrightarrow$ S/S^e l.c.i. $\Rightarrow$ h^M iso $\forall$ symmetric M

Thus to prove ③ $\Rightarrow$ ① it suffices to prove: h^S iso. $\Rightarrow$ S/S^e l.c.i.