Hochschild (co)homology of commutative algebras

15th Sept.

Monday: Overview.

- HH^* already discussed by Soltan & Zimmermann
- I will focus on comm. algebras, but also on morphisms of such algebras.

I General setup and notation.

K comm. noeth. ring

S a K-algebra that is:

- essentially of finite type (e.f.t.) i.e.

$\sigma : K \longrightarrow S$ structure map

$S \cong$ localization of a f.g. K-algebra

so: $S \cong U^{-1}(K[x_1, \dots, x_n]/I)$

where $K[\underline{x}]$ polynomial ring in $\underline{x}$, $I \subseteq K[\underline{x}]$ ideal

$U \subseteq K[\underline{x}]/I$ mult. closed

In particular, S is also commutative and noetherian.

- S/K is flat i.e. S is flat as a K-module

The phenomena & results I discuss are also present and nontrivial when $K = k$, a field. But even if one cares only about this case, one is often forced to consider general K.

- This will become apparent soon.

Set $S^e := S \otimes_K S$, again commutative

- The enveloping algebra

(N.B. $S = S^{op}$)

Then
$$\begin{tikzcd} K \arrow[r, hook, "\sigma"] \arrow[d] & S \arrow[d] \\ S \arrow[r, hook, "S \otimes_K \sigma"'] & S \otimes_K S \end{tikzcd}$$

σ e.f.t $\Rightarrow$ $(S \otimes_K \sigma)$ e.f.t

$\Rightarrow$ $(S \otimes_K S)$ also noetherian.

Set $\mu : S^e \longrightarrow S$ multiplication map.

$$s_1 \otimes s_2 \longmapsto s_1 s_2$$

S comm. $\Rightarrow$ μ is a map of rings.

M an S-bimodule ($=$ (left) S^e-module).

$$HH_*(S/K; M) := \mathrm{Tor}_*^{S^e}(S, M)$$

- Hochschild homology with coefficients in M

$$HH^*(S/K; M) := \mathrm{Ext}_{S^e}^*(S, M)$$

- Hochschild cohomology

(strictly speaking need S/K projective..)

Mainly focus on symmetric M:

$$sm = ms \quad \forall \; m \in M, \; s \in S$$

S^e acts on M via μ

but important exception $M = S^e$.

(II) Why care?

① $K = k$ a field

$$S = k[x_1, .., x_n]/I$$

- coordinate ring of $V = V(I) \subseteq k^n$

Then S^e = coordinate ring of $V \times V \subseteq k^n$

And $\mu: S^e \longrightarrow S$ corresponds to the diagonal embedding $\Delta: V \hookrightarrow V \times V$

Important object in Alg. Geom.

e.g. Reduction to diagonal.

Say $W_1, W_2 \subseteq V$ subvarieties

Then $W_1 \cap W_2 \rightsquigarrow (W_1 \times W_2) \cap \Delta(V)$

Algebraic avatar:

$$\mathrm{Tor}^S_*(M, N) \cong \mathrm{Tor}^{S^e}(S, M \otimes_k N)$$
$$\| $$
$$HH_*(S/k, M \otimes_k N)$$

(2) $HH^*(S/k)$ graded-comm. ring

That acts on $D(S)$ via maps of rings:

$$HH^*(S/k) \longrightarrow \mathrm{Ext}^*_S(M, M) \quad M \in D(S)$$

- Will discuss this in the tutorial.

Basis for developing variety theory.

Raises obvious questions:

- When is $HH^*(S/k)$ f.g. (= noetherian)?
- What can be said about its action on $\mathrm{Ext}_S^*(M,M)$?

③ Deformation theory

- In the CA context the more relevant theory is André-Quillen, but these are closely connected, and often $HH^*(S/k,-)$ is easier to compute.

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What about $HH_*(S/k,-)$

① Already explained connection to reduction to the diagonal.

② Closely connected with other theories like K-theory & cyclic homology

Also has topological connection:

$$HH_*(C^*(X)/\mathbb{Q}) \cong H^*(\mathcal{L}X; \mathbb{Q})$$

X top. space — Space of free loops on X.

This connection has important: methods & insights (but I won't focus on this).

3. $HH_*(R/k, -)$ connected with residues and duality (Lipman and his "school").
 – May comment on this later.

4. Even if one cares only about $HH^*(S/k,-)$ it is related to $HH_*(S/k,-)$ by duality. And $HH_*(S/k,-)$ is often easier to compute.

(This will come up later).

III Structures: S/k , M an S^e-module

$$HH_0(S/k;\, M) = M/JM \ ; \quad J = \mathrm{Ker}(S^e \xrightarrow[\mu]{} S).$$

$$= \frac{M}{(sm - ms \mid s \in S,\, m \in M)}$$

$$= M \quad \text{if } M \text{ is symmetric.}$$

$$HH_1(S/k;\, M) = \frac{J}{J^2} \otimes_S M \quad \text{if } M \text{ symmetric}$$

– Apply $- \otimes_{S^e} M$ to

$$0 \to J \to S^e \to S \to 0.$$

(Exercises

$J/J^2 = \Omega_{S/k}$ the module of (Kähler) differentials.

|||ly $\quad HH^0(S/k, M) = \{m \in M \mid sm = ms \ \forall\, s \in S\}$

$$= M \quad \text{if } M \text{ symmetric}$$

Then $HH^1(S/k, M) = \mathrm{Hom}_S(\Omega_{S/k}, M)$

$$= \mathrm{Der}_k(S, M)$$

– the k-linear derivations on S

with values in M.

All $HH_*(S/k, M)$ & $HH^*(S/k, M)$ are S-modules.

Also $HH^*(S/k) := HH^*(S/k; S)$ is

graded-comm., as mentioned earlier

– Not special to commutative rings.

But S commutative $\Rightarrow$

$HH_*(S/k; S)$ graded-commutative

even strictly so.

Graded-comm. means $\alpha . \beta = (-1)^{|\alpha||\beta|} \beta . \alpha$

Strictly means, in addition,

$\alpha^2 = 0$ if $|\alpha|$ is odd.

This is a special case of a general

phenomenon:

$$\begin{tikzcd} R \arrow[r] \arrow[d] & S \\ T & \end{tikzcd}$$ maps of comm. rings

Then $\operatorname{Tor}^R_*(S, T)$ is a strictly graded comm. ring.

Recall: $HH_*(S/k) = \operatorname{Tor}^{S^e}_*(S,S)$.

Good place to mention:

We are interested in the map

$$S^e \xrightarrow{\mu} S$$

One has $$S \xrightarrow{i} S^e \xrightarrow{\mu} S$$

$i = S \otimes \sigma$ (or $\sigma \otimes S$ either works)

– a map of rings with $\mu i = id^S$.

i.e. S is an algebra retract of S^e.

Much of what I have to say carries over to retracts: Maps of rings

$$S \xrightarrow{i} R \xrightarrow{\pi} S \quad \text{with } \pi \circ i = id^S$$

And with $\operatorname{Tor}^R_*(S,-)$ & $\operatorname{Ext}^*_R(S,-)$

playing the role of $HH_*(S/k,-)$ & $HH^*(S/k,-)$

Other important examples of retracts:

$$S \hookrightarrow S[x_1,\dots,x_n] \xrightarrow[\pi]{} S$$

$$\pi(x_i) = r_i$$

$$S \hookrightarrow S \ltimes M \twoheadrightarrow S$$

M an S-module
$S \ltimes M$ trivial extension.

—x—

Plan for the rest of my lectures: Discuss how properties of S/k are related to those of S/S^e. A picture that captures the general structure:

$$\begin{tikzcd} k \arrow[r, "\sigma"] \arrow[d] & S \arrow[d] & \\ S \arrow[r, "S \otimes_k \sigma"'] & S^e \arrow[r, "\mu"] & S \end{tikzcd}$$

Relate σ to $S \underset{\sigma}{\otimes} K$ and then

to μ exploiting the fact that $\mu(S \underset{\sigma}{\otimes} K) = \mathrm{id}$.

Tuesday: Smooth algebras

Wednesday: More about smooth algebras + Gorenstein algebras.

Thursday + Friday: Locally complete intersection maps.

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Extra material for Monday:

Lipman's approach to residues via HH_*.