Ample groupoid algebras
Lecture 6:
Quasi-Cartan subalgebras and open questions

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September 19, 2025

Outline

- How can we tell if a given algebra is a twisted Steinberg algebra?

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    - Cartan subalgebras and beyond
- Open questions / research strategies

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Let R be a field.

Theorem (ACCCLMRSS)

An algebra A contains a quasi-Cartan subalgebra C if and only if there exists a discrete $R^\times$-twist (Σ, G) such that G satisfies the local bisection hypothesis and $A \cong A_R(G;\Sigma)$ and $C \cong A_R(G^{(0)})$.

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4. There exists a conditional expectation $P : A \to C$ that is *implemented by idempotents*, in the sense that for every $n \in N(C)$, there exists $e \in I(C)$ such that $P(n) = ne = en$.

We call a map $P : A \to C$ a conditional expectation if

- P is R-linear;
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Recall the obstruction to isomorphism problem

We know $A_R(\mathbb{Z}_4) \cong A_R(\mathbb{Z}_2 \times \mathbb{Z}_2)$

...and the isomorphism is diagonal preserving...EXERCISE...

but $\mathbb{Z}_4 \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

We can get a stronger solution to the isomorphism problem by broadening the class of algebras to twisted Steinberg algebras.

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$\implies$ is always true and the converse is true if the isomorphism preserves the diagonal and Cartan semi-group.

Open questions

When does the quasi-Cartan construction give rise to an untwisted groupoids, that is, a trivial twist?

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Is there a way to view each twisted Steinberg algebra as sitting densely inside a twisted groupoid C*-algebra?

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Why? Because if the group E is countable, the boundary path groupoid G_E is σ-compact and hence any twist comes from a 2-cocycle. Then, "because the cohomology of the integers is trivial, evey 2-cocycle is cohomologous to the trivial cocycle" so we don't get anything new. (2-cocycles that are cohomologous give isomorphic twisted Steinberg algebras)

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(We can do this for Kumjian-Pask algebras.)

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For example: C-Hazrat-Rigby used this approach to characterise strongly graded Leavitt path algebras.

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Take a result about LPA's and generalise to KPAs/Steinberg algebras/Twisted Steinberg algebras.

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Take a result about group algebras and generalise to Steinberg algebras/Twisted Steinberg algebras.

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- Can we make progress looking for quasi-Cartan subalgebras?
- Is there a connection to algebraic classifiability?

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