Ample groupoid algebras
Lecture 5: How can can wewe tell if a given algebra is a Steinberg algebra? ... and twisted Steinberg algebras

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Outline

- How can we tell if a given algebra is a Steinberg algebra?
  - Universal property

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  - Uniqueness theorems

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- Twisted Steinberg algebras

How can we tell if a given algebra is a Leavitt path algebra?

LPA's as universal algebras

Let E be an arbitrary directed graph and R be a commutative ring with identity. Then the Leavitt path algebra $L_R(E)$ is the universal R-algebra generated by a Leavitt E-family, that is, a set $\{p_v, s_e, s_{e^*} : v \in E^0 \text{ and } e \in E^1\}$ satisfying the relations:

1. $p_v p_w = \delta_{v,w} p_v$ for all $v, w \in E^0$;
2. $p_{r(e)} s_e = s_e = s_e p_{s(e)}$ for all $e \in E^1$;
3. $p_{s(e)} s_{e^*} = s_{e^*} = s_{e^*} p_{r(e)}$ for all $e \in E^1$;
4. $s_{e^*} s_{e'} = \delta_{e,e'} p_{s(e)}$ for all $e, e' \in E^1$; and
5. $p_v = \sum_{\{e \in E^1 | r(e) = v\}} s_e s_{e^*}$ for every regular vertex v.

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3. $p_{s(e)} s_{e^*} = s_{e^*} = s_{e^*} p_{r(e)}$ for all $e \in E^1$;
4. $s_{e^*} s_{e'} = \delta_{e,e'} p_{s(e)}$ for all $e, e' \in E^1$; and
5. $p_v = \sum_{\{e \in E^1 | r(e) = v\}} s_e s_{e^*}$ for every regular vertex v.

If A is an R-algebra containing a set of elements $\{q_v, t_e, t_{e^*}\}$ satisfying (1)–(5), then there exists an algebra homomorphism $\pi : L_R(E) \to A$ such that $\pi(p_v) = q_v$, $\pi(s_e) = t_e$ and $\pi(s_{e^*}) = t_{e^*}$.

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Then we used a uniqueness theorem to show injectivity.

Uniqueness Theorems for LPAs

Theorem (The graded uniqueness theorem, Raeburn, Tomforde)

Let E be an arbitrary directed graph and A be a $\mathbb{Z}$-graded ring. Then a graded ring homomorphism $\pi : L_R(E) \to A$ is injective if and only if it is injective on the diagonal.

Theorem (Cuntz-Krieger Uniqueness theorem, Raeburn, Tomforde)

Let E be an arbitrary directed graph that satisfies condition (L) and A be a ring. Then a ring homomorphism $\pi : L_R(E) \to A$ is injective if and only if it is injective on the diagonal.

Theorem (Generalised Uniqueness theorem, Gil Canto - Nasr-Isfahani)

Let E be an arbitrary directed graph and A be a ring. Then a ring homomorphism $\pi : L_R(E) \to A$ is injective if and only if it is injective on the commutative core.

Recall: Steinberg Algebra

- Let G be an ample groupoid and R be a commutative unital ring.
- For $B \subseteq G$, define $1_B : G \to R$ such that $1_B(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{otherwise.} \end{cases}$
- The Steinberg Algebra:

  $A_R(G) := \text{span}\{1_B : B \text{ is a compact open bisection}\}$

  - Addition and scalar multiplication of functions are defined pointwise
  - Multiplication is given by convolution: $f * g(\gamma) = \sum_{\alpha\beta=\gamma} f(\alpha)g(\beta)$
  - Multiplication on generators reduces to $1_B 1_D = 1_{BD}$

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$= \{f : G \to R \ : f \text{ is locally constant and has compact support}\}.$

Universal property

- Let $\mathcal{B}$ the collection of compact open bisections in G.
- Let A be an R-algebra.
- A representation of $\mathcal{B}$ in A is a family $\{t_B : B \in \mathcal{B}\} \subseteq A$ satisfying
  - (R1) $t_\emptyset = 0$;
  - (R2) $t_B t_D = t_{BD}$ for all $B, D \in \mathcal{B}$; and
  - (R3) $t_B + t_D = t_{B \cup D}$ whenever B and D are disjoint elements of $\mathcal{B}$ such that $B \cup D$ is a compact open bisection.

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- $\{1_B : B \in \mathcal{B}\}$ is a representation of $\mathcal{B}$ that spans $A_R(G)$.

Theorem (C-Edie Michell)

The Steinberg algebra $A_R(G)$ is universal for representations of $\mathcal{B}$ in the following way: if A is an R-algebra containing a representation $\{t_B : B \in \mathcal{B}\}$ of $\mathcal{B}$, then there is a unique algebra homomorphism $\pi : A_R(G) \to A$ such that $\pi(1_B) = t_B$ for all $B \in \mathcal{B}$.

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Question: Is there a Universial property for Steinberg algebras associated to non-Hausdorff groupoids?
Answer:

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- If X is a topological space that has a basis of compact open sets, then X is an ample groupoid.
- Compact open sets are open bisections.
- Let U, V be compact open sets. Then
- $A_R(X) = \mathrm{span}\{1_U : U \text{ is a compact open set in } X\}$

$$UV = \{uv : u \in U, v \in V, s(u) = r(v)\} = U \cap V$$

$1_U 1_V = 1_{UV} = 1_{U\cap V} \implies$ multiplication in the algebra is pointwise.

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- Convolution in $A_R(G^{(0)})$ reduces to component-wise multiplication: $f, g \in A_R(G^{(0)})$ then $f * g(\gamma) = f(\gamma)g(\gamma)$.
- $A_R(G^{(0)})$ is commutative.

Steinberg algebra uniqueness theorems

Theorem (Graded uniqueness theorem, C-Edie Michell)

Let $c : G \to \Gamma$ be a continuous cocycle. Suppose that G_0 is effective. Let $\pi : A_R(G) \to A$ be a graded ring homomorphism. Then π is injective if and only if π is injective on the diagonal.

Theorem (Cuntz-Krieger Uniqueness Theorem, C-Edie Michell)

Let G be an effective groupoid. Suppose $\pi : A_R(G) \to A$ is a ring homomorphism. Then π is injective if and only if π is injective on the diagonal.

Theorem (Generalized Uniqueness Theorem, C-Exel-Pardo)

Suppose that A is an R-algebra and that pi : $A_R(\mathcal{G}) \to A$ is a ring homomorphism. Then π is injective if and only if π is injective on $A_R(\mathrm{Iso}(G)^\circ)$.

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$\Longrightarrow$ is always true and the converse is true with some additional hypothesis.

Diagonal preserving isomorphisms

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Theorem (Steinberg)

Suppose that G_1 and G_2 are ample Hausdorff groupoids that satisfiy the local bisection hypothesis. TFAE:

1. $G_1 \cong G_2$ *as topological groupoids.*
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Let E and F be directed graphs. TFAE:

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Question: Is there an example of an isomorphism between Leavitt path algebras that is NOT diagonal preserving?

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Obstruction to isomorphism problem

We know $A_R(\mathbb{Z}_4) \cong A_R(\mathbb{Z}_2 \times \mathbb{Z}_2)$

...and the isomorphism is diagonal preserving...EXERCISE...

but $\mathbb{Z}_4 \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$.

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but $\mathbb{Z}_4 \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

We can get a stronger solution to the isomorphism problem by broadening the class of algebras to twisted Steinberg algebras.

Twisted groupoid algebras

Algebras	**C*-algebras**
Leavitt algebras, 1957	Cuntz C*-algebras, 1977
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$\uparrow$

(All classifiable C^*-algebras.)

Discrete Twists

Let G and Σ be ample Hausdorff groupoids, R a (discrete) commutative unital ring and $T \leq R^\times$. A discrete T-twist is a sequence

$$G^{(0)} \times T \xrightarrow{i} \Sigma \xrightarrow{q} G$$

of groupoids, with i and q homomorphisms restricting to homeomorphisms of the unit spaces, satisfying:

(DT1) This sequence is exact in the sense that i is injective, π is surjective and $\operatorname{Im} i = \ker \pi$, and $\forall x \in G^{(0)} :\ i(\{x\} \times T) = q^{-1}(x)$.

(DT2) $q : \Sigma \to G$ is a locally trivial bundle. That is, for every $\gamma \in G$ there is a COB $U_\gamma \ni \gamma$ and a continuous map $P_\gamma : U_\gamma \to \Sigma$ such that

- $q \circ P_\gamma = \operatorname{id} U_\gamma$, and
- $\varphi_{P_\gamma} : U_\gamma \times T \to q^{-1}(U_\gamma)$, $(\eta, t) \mapsto i(r(\eta), t)P_\gamma(\eta)$ is a homeomorphism.

(DT3) $\operatorname{Im}(i)$ is central in Σ in the sense that

$$\forall \sigma \in \Sigma\ \forall t \in T : i(r(\sigma), t)\sigma = \sigma i(s(\sigma), t).$$

Untwisted $A_R(G)$

For now, we will assume G is a Hausdorff ample groupoid.

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Twisted Steinberg algebras

Let R be a commutative unital ring and let T be a subgroup of $R^{\times}$.

Let $G^{(0)} \times T \xrightarrow{i} \Sigma \xrightarrow{q} G$ be a discrete twist

Recall that there is an action of T on Σ.

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We say a function $f : \Sigma \to R$ is equivariant if $f(t \cdot \sigma) = tf(\sigma)$ for every $t \in T$ and $\sigma \in \Sigma$.

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Define

$$\begin{aligned} C(\Sigma, R) &:= \{f : \Sigma \to R \mid \ f \text{ is locally constant}\} \\ A_R(G; \Sigma) &:= \{f \in C(\Sigma, R) \mid \ f \text{ is equivariant and } q(\operatorname{supp}(f)) \text{ is compact.}\} \\ &= \{\sum_{B \in F} c_B \tilde{1}_B : F \text{ is a collection of compact open bisections in } \Sigma\} \end{aligned}$$

where $\tilde{1}_B(\sigma) = \begin{cases} t & \text{if } \sigma \in t \cdot B \\ 0 & \text{otherwise.} \end{cases}$

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We say a function $f : \Sigma \to R$ is equivariant if $f(t \cdot \sigma) = tf(\sigma)$ for every $t \in T$ and $\sigma \in \Sigma$.

Define

$$\begin{aligned} C(\Sigma, R) &:= \{f : \Sigma \to R \mid \ f \text{ is locally constant}\} \\ A_R(G; \Sigma) &:= \{f \in C(\Sigma, R) \mid \ f \text{ is equivariant and } q(\mathrm{supp}(f)) \text{ is compact.}\} \\ &= \{\sum_{B \in F} c_B \tilde{1}_B : F \text{ is a collection of compact open bisections in } \Sigma\} \end{aligned}$$

where $\tilde{1}_B(\sigma) = \begin{cases} t & \text{if } \sigma \in t \cdot B \\ 0 & \text{otherwise.} \end{cases}$

Twisted Steinberg algebras

$A_R(G; \Sigma)$ is the twisted Steinberg algebra.

- Addition and scalar multiplication are defined pointwise.
- Multiplication is given by a convolution formula such that $\tilde{1}_B \tilde{1}_D = \tilde{1}_{BD}$.

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A small example

Let $R = \mathbb{R}$ and $T = \{-1, 1\} \subseteq \mathbb{R}^\times$. Consider the discrete twist

$$\{[0]_2\} \times T \xrightarrow{i} \mathbb{Z}_4 \xrightarrow{q} \mathbb{Z}_2$$

such that

$$\begin{aligned} &i : ([0]_2, 1) \mapsto [0]_4, \\ &i : ([0]_2, -1) \mapsto [2]_4 \text{ and} \\ &q : [k]_4 \mapsto [k]_2. \end{aligned}$$

Then T acts on $\mathbb{Z}_4$ as follows:

$$\begin{aligned} 1 \cdot [k]_4 &= [k]_4 \\ -1 \cdot [k]_4 &= [k+2]_4. \end{aligned}$$

Thus the elements of $A_R(\mathbb{Z}_2; \mathbb{Z}_4)$ are functions $f : \mathbb{Z}_4 \to R$ satisfying

$$f(-1 \cdot [k]_4) = -f([k]_4)$$

that is, $A_R(\mathbb{Z}_2, \mathbb{Z}_4) = \{\, f \in \mathbb{R}^{\mathbb{Z}_4} \mid f([k+2]_4) = -f([k]_4) \text{ for all } k \in \mathbb{Z} \,\}$.

Lynnel's example: the trivial twist

- Let G be an ample Hausdorff groupoid and consider the trivial discrete twist

$$G^{(0)} \times T \xrightarrow{i} G \times T \xrightarrow{q} G.$$

Lynnel's example: the trivial twist

- Let G be an ample Hausdorff groupoid and consider the trivial discrete twist
$$G^{(0)} \times T \xrightarrow{i} G \times T \xrightarrow{q} G.$$

- Then $A_R(G) \cong A_R(G; G \times T)$.

Rizalyn's example: 2-cocycle twisted Steinberg algebras

- Let G be an ample Hausdorff groupoid.
- Let $\sigma : G^{(2)} \to T$ be a continuous normalised 2-cocycle.

Rizalyn's example: 2-cocycle twisted Steinberg algebras

- Let G be an ample Hausdorff groupoid.
- Let $\sigma : G^{(2)} \to T$ be a continuous normalised 2-cocycle.
- Define $A_R(G, \sigma) := A_R(G)$ as an R-module.
- The cocycle twists the convolution:

$$f *_\sigma g(\gamma) = \sum_{\alpha\beta=\gamma} \sigma(\alpha, \beta) f(\alpha) g(\beta).$$

- $A_R(G, \sigma) \cong A_R(G; G \times_\sigma T)$.

Michael's second example: the irrational rotation algebra

This twisted Steinberg algebra of the discrete irrational rotation twist is an example of a simple twisted Steinberg such that the underlying groupoid is not effective.

Michael's second example: the irrational rotation algebra

This twisted Steinberg algebra of the discrete irrational rotation twist is an example of a simple twisted Steinberg such that the underlying groupoid is not effective.

Finding necessary and sufficient conditions on the discrete twist for simplicity of the twisted Steinberg algebra is an open question.

How can we tell if a given algebra is a twisted Steinberg algebra?

Universal property for twisted Steinberg algebras:

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Uniqueness Theorems for twisted Steinberg algebras:

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Uniqueness Theorems for twisted Steinberg algebras: Rizalyn Bongcawel's PhD thesis.

Next time, we will see how twisted Steinberg algebras help to solve the isomorphism problem and give a definitive answer to the question:
How can we tell if a given algebra is a twisted Steinberg algebra?

The end
Thank you for listening!