Groupoids

Let G be a set and $G^{(2)}$ be a subset of $G \times G$ such that there is a (composition) map $(\gamma, \alpha) \mapsto \gamma\alpha$ from $G^{(2)}$ to G. Suppose that there is an inverse map $\gamma \mapsto \gamma^{-1}$ on G such that $(\gamma^{-1})^{-1} = \gamma$. Then we say that G is a *groupoid* if the following are satisfied:

(G1) if $(\gamma, \alpha), (\alpha, \beta) \in G^{(2)}$, then $(\gamma\alpha, \beta), (\gamma, \alpha\beta) \in G^{(2)}$ and the following equation is satisfied: $(\gamma\alpha)\beta = \gamma(\alpha\beta)$;

(G2) for all $\gamma \in G$, $(\gamma^{-1}, \gamma) \in G^{(2)}$;

(G3) if $(\gamma, \alpha) \in G^{(2)}$, then $(\gamma^{-1}\gamma)\alpha = \alpha$ and $\gamma(\alpha\alpha^{-1}) = \gamma$.

2-cocycle

Let R be a commutative unital ring. A *continuous 2-cocycle* is a continuous function $\sigma : G^{(2)} \longrightarrow R^{x}$ that satisfies the 2-cocycle identity:

$$\sigma(\alpha,\beta)\sigma(\alpha\beta,\gamma) = \sigma(\alpha,\beta\gamma)\sigma(\beta,\gamma),$$

for all $(\alpha,\beta,\gamma) \in G^{(3)}$[1].

[1] We write $G^{(3)}$ for the set of composable triples in G, that is, $G^{(3)} = \{(\alpha,\beta,\gamma) : (\alpha,\beta),(\beta,\gamma) \in G^{(2)}\}$.

normalised:

$$\boxed{\begin{gathered}\sigma(r[\gamma], \gamma) = 1 = \sigma(\gamma, s[\gamma]) \\ \text{for all } \gamma \in G\end{gathered}}$$

Goal:

$$\sigma \Longrightarrow \Sigma$$

cocycle

discrete twist

G, Hausdorff ample groupoid

R, commutative unital ring

$\sigma : G^{(2)} \longrightarrow T \leq R^{\times}$, continuous 2-cocycle

$G \times_{\sigma} T$, $G \times T$ endowed with the product topology

For $(\alpha, \beta) \in G^{(2)}$, $z, w \in T$,

- multiplication

$$(\alpha, z)(\beta, w) = (\alpha\beta, \sigma(\alpha, \beta) z w)$$

- inversion

$$(\alpha, z)^{-1} = (\alpha^{-1}, \sigma(\alpha, \alpha^{-1})^{-1} z^{-1})$$

$$= (\alpha^{-1}, \sigma(\alpha^{-1}, \alpha)^{-1} z^{-1})$$

Task!

$G \times_\sigma T$

- Hausdorff
- ample
- groupoid

?

$$G^{(0)} \times T \xrightarrow{i} G \times_\sigma T \xrightarrow{q} G$$

Discrete Twist?

$G \times_\sigma T = \{(\alpha, z) \,;\, \alpha \in G,\ z \in T\}$

Composition :

$$m[(\alpha, z), (\beta, w)] = (\alpha\beta, \sigma(\alpha, \beta) z w)$$

inversion :

$$i(\alpha, z) = (\alpha^{-1}, \sigma(\alpha, \alpha^{-1})^{-1} z^{-1})$$

$$= (\alpha^{-1}, \sigma(\alpha^{-1}, \alpha)^{-1} z^{-1})$$

source:

$$s(\alpha, z) = (s(\alpha), \underline{1})$$

range:

$$r(\alpha, z) = (r(\alpha), \underline{1})$$

$$(G \times_\sigma T)^{(0)} = G^{(0)} \times \{1\}$$

(G1) If $(\gamma, \alpha), (\alpha, \beta) \in (G \times_\sigma T)^{(2)}$

then

$$(\gamma\alpha, \beta), (\gamma, \alpha\beta) \in (G \times_\sigma T)^{(2)}$$

and

$$(\gamma\alpha)\beta = \gamma(\alpha\beta)$$

(G2)

$$\forall \gamma \in G, \quad (\gamma^{-1}, \gamma) \in G^{(2)}$$

(G3) $(\gamma, \alpha) \in (G \times_\sigma T)^{(2)}$

then

$(\gamma^{-1}\gamma)\alpha = \alpha$ and $\gamma(\alpha\alpha^{-1}) = \gamma$

$G \times_\sigma T$ is a groupoid ✓

7 Discrete Twist ?

$$\boxed{G^{(0)} \times T \xrightarrow{\iota} G \times_{\sigma} T \xrightarrow{q} G}$$

$$\iota(\alpha, z) = (\alpha, z)$$

$$q(\alpha, z) = \alpha$$

Discrete Twists

Let G be an ample Hausdorff groupoid, R a (discrete) commutative unital ring and $T \leq R^\times$. A discrete T-twist is a sequence

$$G^{(0)} \times T \xrightarrow{\iota} \Sigma \xrightarrow{\pi} G$$

of groupoids, with ι and π homomorphisms restricting to homeomorphisms of the unit spaces, satisfying:

(DT1) This sequence is exact in the sense that ι is injective, π is surjective and $\mathrm{Im}\ \iota = \ker \pi$, and $\forall x \in G^{(0)}: \ \iota(\{x\} \times T) = \pi^{-1}(x)$.

(DT2) $\pi : \Sigma \to G$ is a locally trivial bundle. That is, for every $\gamma \in G$ there is a COB $U_\gamma \ni \gamma$ and a continuous map $P_\gamma : U_\gamma \to \Sigma$ such that
- $\pi \circ P_\gamma = \mathrm{id}\ U_\gamma$, and
- $\varphi_{P_\gamma} : U_\gamma \times T \to \pi^{-1}(U_\gamma)$, $(\eta, t) \mapsto \iota(r(\eta), t)P_\gamma(\eta)$ is a homeomorphism.

(DT3) $\mathrm{Im}\ \iota$ is central in Σ in the sense that

$$\forall \sigma \in \Sigma\ \forall t \in T : \iota(r(\sigma), t)\sigma = \sigma\iota(s(\sigma), t).$$

i is homo :

For $\alpha, \beta \in G^{(0)}$, $z, w \in \mathbb{T}$:

$$\begin{aligned} i\big((\alpha, z), (\beta, w)\big) &= i\big(\alpha\beta, \sigma(\alpha, \beta) zw\big) \\ &= i\big(\alpha\beta, \sigma(\cancelto{1}{\alpha, \beta}) zw\big) \\ &= i(\alpha\beta, zw) \\ &= (\alpha\beta, zw) \end{aligned}$$

$$\begin{aligned} i(\alpha, z)\, i(\beta, w) &= (\alpha, z)(\beta, w) \\ &= (\alpha\beta, \sigma(\alpha, \beta) zw) \\ &= (\alpha\beta, zw) \end{aligned}$$

(DT1) (exact sequence)

- q surjective

  $\forall \alpha \in G, \quad q^{-1}(\alpha) = (\alpha, \underline{1}) \in G \times_\sigma T$

- i injective

  since i is an inclusion map

- $\mathrm{Im}\, i = \mathrm{Ker}\, q$

  $\mathrm{Im}\, i = G^{(0)} \times_\sigma T = \ker q$

- $\forall x \in G^{(0)}, \quad i(\{x\} \times T) = q^{-1}(x)$

(DT2) (q is a locally trivial bundle)

$\exists\ s_{B_\alpha} : B_\alpha \longrightarrow G \times_\sigma T$

(i) $q \circ s_{B_\alpha} = id_{B_\alpha}$

(ii) $\theta : B_\alpha \times T \longrightarrow q^{-1}(B_\alpha)$

$\theta : B_\alpha \times T \longrightarrow B_\alpha \times T$

(DT3)

(Im i is central in $G \times_\sigma T$)

$$\forall (\lambda, z) \in G \times_\sigma T \quad \forall t \in T:$$

$$i(r(\lambda, z), t)(\lambda, z) = (\lambda, z)\, i(s(\lambda, z), t)$$

> Discrete Twist

$$G^{(0)} \times T \xrightarrow{\iota} G \times_0 T \xrightarrow{q} G$$

$$\sigma \overset{?}{\Longleftarrow} \varepsilon$$

False!

$$\sigma \Longleftarrow \Sigma$$

Thank You