

Groupoids

Let G be a set and $G^{(2)}$ be a subset of $G \times G$ such that there is a (composition) map $(\gamma, \alpha) \mapsto \gamma\alpha$ from $G^{(2)}$ to G . Suppose that there is an inverse map $\gamma \mapsto \gamma^{-1}$ on G such that $(\gamma^{-1})^{-1} = \gamma$. Then we say that G is a *groupoid* if the following are satisfied:

- (G1) if $(\gamma, \alpha), (\alpha, \beta) \in G^{(2)}$, then $(\gamma\alpha, \beta), (\gamma, \alpha\beta) \in G^{(2)}$ and the following equation is satisfied: $(\gamma\alpha)\beta = \gamma(\alpha\beta)$;
- (G2) for all $\gamma \in G$, $(\gamma^{-1}, \gamma) \in G^{(2)}$;
- (G3) if $(\gamma, \alpha) \in G^{(2)}$, then $(\gamma^{-1}\gamma)\alpha = \alpha$ and $\gamma(\alpha\alpha^{-1}) = \gamma$.

2-cocycle

Let R be a commutative unital ring. A *continuous 2-cocycle* is a continuous function $\sigma : G^{(2)} \longrightarrow R^x$ that satisfies the 2-cocycle identity:

$$\sigma(\alpha, \beta)\sigma(\alpha\beta, \gamma) = \sigma(\alpha, \beta\gamma)\sigma(\beta, \gamma),$$

for all $(\alpha, \beta, \gamma) \in G^{(3)1}$.

¹ We write $G^{(3)}$ for the set of composable triples in G , that is, $G^{(3)} = \{(\alpha, \beta, \gamma) : (\alpha, \beta), (\beta, \gamma) \in G^{(2)}\}$.

normalised:

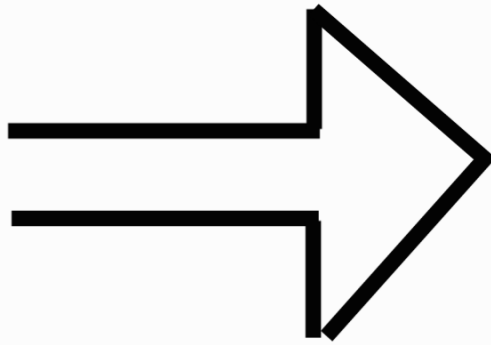
$$\sigma(r(r), r) = \underline{1} = \sigma(r, s(r))$$

for all $r \in G$

Goal:



cocycle



discrete
twist

G , Hausdorff ample groupoid
 R , commutative unital ring

$\sigma: G^{(2)} \rightarrow T \leq R^*$, continuous 2-cocycle

$G \times_{\sigma} T$, $G \times T$ endowed with the product topology

For $(\alpha, \beta) \in G^{(2)}$, $z, w \in T$,

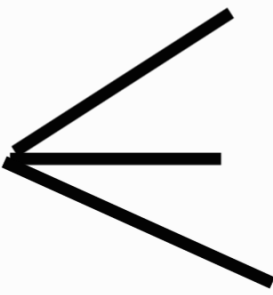
multiplication

$$(\alpha, z)(\beta, w) = (\alpha\beta, \sigma(\alpha, \beta)zw)$$

inversion

$$\begin{aligned} (\alpha, z)^{-1} &= (\alpha^{-1}, \sigma(\alpha, \alpha^{-1})^{-1}z^{-1}) \\ &= (\alpha^{-1}, \sigma(\alpha^{-1}, \alpha)^{-1}z^{-1}) \end{aligned}$$

Task !

$G \times_{\sigma} T$  Hausdorff
ample
groupoid ?

$$G^{(0)} \times T \xrightarrow{i} G \times_{\sigma} T \xrightarrow{q} G$$

Discrete Twist ?

$$G \times_\sigma T = \{(\alpha, z) : \alpha \in G, z \in T\}$$

Composition :

$$m[(\alpha, z), (\beta, w)] = (\alpha\beta, \sigma(\alpha, \beta)zw)$$

inversion :

$$\begin{aligned} i(\alpha, z) &= (\alpha^{-1}, \sigma(\alpha, \alpha^{-1})^{-1}z^{-1}) \\ &= (\alpha^{-1}, \sigma(\alpha^{-1}, \alpha)^{-1}z^{-1}) \end{aligned}$$

source :

$$s(d, z) = (s(d), \underline{1})$$

range :

$$r(d, z) = (r(d), \underline{1})$$

$$(G \times_{\sigma} T)^{(0)} = G^{(0)} \times \{1\}$$

(G1) If $(\gamma, \alpha), (\alpha, \beta) \in (G \times_{\sigma} T)^{(2)}$
then

$$(\gamma\alpha, \beta), (\gamma, \alpha\beta) \in (G \times_{\sigma} T)^{(2)}$$

and

$$(\gamma\alpha)\beta = \gamma(\alpha\beta)$$

(G^2)

$$\forall \gamma \in G, (\gamma^{-1}, \gamma) \in G^{(2)}$$

$$(G^3) \quad (r, d) \in (G \times_{\sigma} T)^{(2)}$$

then

$$(\gamma^{-1} \gamma) d = d \quad \text{and} \quad \gamma (d d^{-1}) = \gamma$$

$G \times_{\sigma} T$ is a groupoid ✓

7 Discrete Twist ?

$$G^{(0)} \times T \xrightarrow{\iota} G \times_{\sigma} T \xrightarrow{q} G$$

$$\iota(\alpha, z) = (\alpha, z)$$

$$q(\alpha, z) = \alpha$$

Discrete Twists

Let G be an ample Hausdorff groupoid, R a (discrete) commutative unital ring and $T \leq R^\times$. A **discrete T -twist** is a sequence

$$G^{(0)} \times T \xrightarrow{\iota} \Sigma \xrightarrow{\pi} G$$

of groupoids, with ι and π homomorphisms restricting to homeomorphisms of the unit spaces, satisfying:

- (DT1) This sequence is exact in the sense that ι is injective, π is surjective and $\text{Im } \iota = \ker \pi$, and $\forall x \in G^{(0)} : \iota(\{x\} \times T) = \pi^{-1}(x)$.
- (DT2) $\pi : \Sigma \rightarrow G$ is a locally trivial bundle. That is, for every $\gamma \in G$ there is a COB $U_\gamma \ni \gamma$ and a continuous map $P_\gamma : U_\gamma \rightarrow \Sigma$ such that
 - $\pi \circ P_\gamma = \text{id } U_\gamma$, and
 - $\varphi_{P_\gamma} : U_\gamma \times T \rightarrow \pi^{-1}(U_\gamma)$, $(\eta, t) \mapsto \iota(r(\eta), t)P_\gamma(\eta)$ is a homeomorphism.
- (DT3) $\text{Im } \iota$ is central in Σ in the sense that

$$\forall \sigma \in \Sigma \forall t \in T : \iota(r(\sigma), t)\sigma = \sigma\iota(s(\sigma), t).$$

i is homo :

For $\alpha, \beta \in G^{(v)}$, $z, w \in T$:

$$\begin{aligned} i((\alpha, z), (\beta, w)) &= i(\alpha\beta, \sigma(\alpha, \beta)zw) \\ &= i(\alpha\beta, \sigma(\cancel{\alpha}, \beta)zw) \\ &= i(\alpha\beta, zw) \\ &= (\alpha\beta, zw) \end{aligned}$$

$$\begin{aligned} i(\alpha, z) i(\beta, w) &= (\alpha, z)(\beta, w) \\ &= (\alpha\beta, \sigma(\alpha, \beta)zw) \\ &= (\alpha\beta, zw) \end{aligned}$$

(DT1) (exact sequence)

- q surjective

$$\forall d \in G, q^{-1}(d) = (d, 1) \in G \times_0 T$$

- i injective

since i is an inclusion map

- $\text{Im } i = \text{Ker } q$

$$\text{Im } i = G^{(0)} \times_0 T = \text{Ker } q$$

- $\forall x \in G^{(0)}, i(x \times_0 T) = q^{-1}(x)$

(DT2) (q is a locally trivial bundle)

$$\gamma \quad S_{B_\alpha} : B_\alpha \longrightarrow G \times_\sigma T$$

$$(i) \quad q \circ S_{B_\alpha} = \text{id}_{B_\alpha}$$

$$(ii) \quad \theta : B_\alpha \times T \longrightarrow q^{-1}(B_\alpha)$$

$$\theta : B_\alpha \times T \longrightarrow B_\alpha \times T$$

(DT3)

($\text{Im } i$ is central in $G \rtimes_\sigma T$)

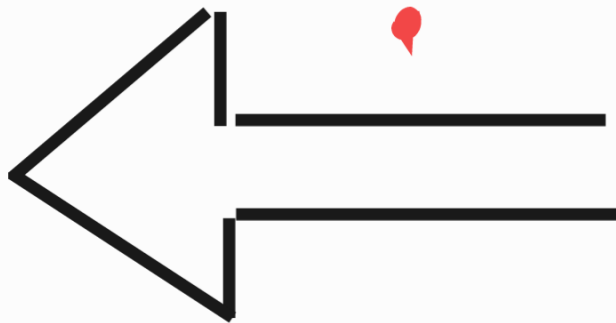
$\forall (\lambda, z) \in G \rtimes_\sigma T \quad \forall t \in T:$

$$i(r(\lambda, z), t)(\lambda, z) = (\lambda, z)i(s(\lambda, z), t)$$

7 Discrete Twist ✓

$$G^{(0)} \times T \xrightarrow{L} G \times_{\sigma} T \xrightarrow{q} G$$

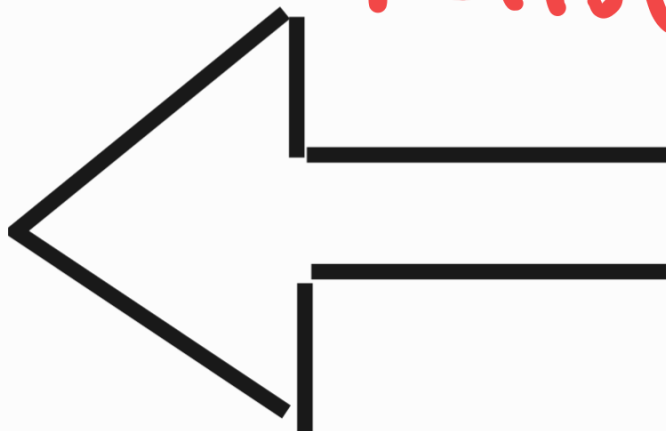
σ



?

Σ

0



False!

Σ

Thank You