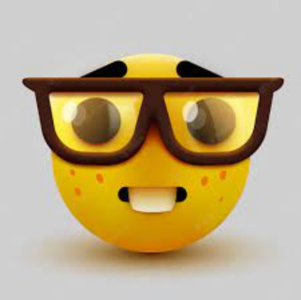
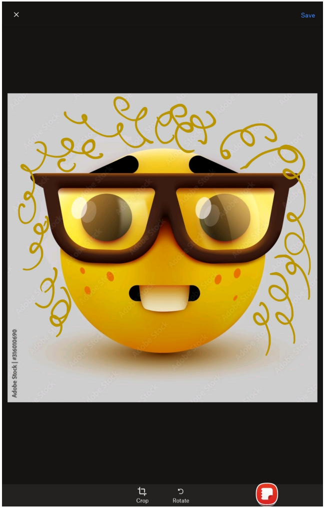
Ample groupoid algebras
Lecture 3: Steinberg algebras and Leavitt path algebras

Lisa Orloff Clark

Te Herenga Waka
Victoria University of Wellington, New Zealand

September 17, 2025





Save
Adobe Stock | #316010690
Crop
Rotate

Algebras	C^*-**algebras**
Leavitt algebras, 1957	

Algebras	C^***-algebras**
Leavitt algebras, 1957	Cuntz C^*-algebras, 1977

Algebras	C^***-algebras**
Leavitt algebras, 1957	Cuntz C^*-algebras, 1977
	Cuntz-Krieger C^*-algebras, 1980

Algebras	C^*-**algebras**
Leavitt algebras, 1957	Cuntz C^*-algebras, 1977
	Cuntz-Krieger C^*-algebras, 1980 graph C^*-algebras, 1997

Algebras	C^*-**algebras**
Leavitt algebras, 1957	Cuntz C^*-algebras, 1977
Leavitt path algebras, 2005	Cuntz-Krieger C^*-algebras, 1980 graph C^*-algebras, 1997

Algebras	C^*-**algebras**
Leavitt algebras, 1957	Cuntz C^*-algebras, 1977
Leavitt path algebras, 2005	Cuntz-Krieger C^*-algebras, 1980 graph C^*-algebras, 1997
	Groupoid C^*-algebras, 1980

Algebras	C^*-**algebras**
Leavitt algebras, 1957	Cuntz C^*-algebras, 1977
Leavitt path algebras, 2005	Cuntz-Krieger C^*-algebras, 1980 graph C^*-algebras, 1997
Steinberg Algebras, 2010	Groupoid C^*-algebras, 1980

Recap

- A groupoid is like a group except the binary operation is only partially defined.

Recap

- A groupoid is like a group except the binary operation is only partially defined.
- As a result, a groupoid can have more than one 'unit'.

Recap

- A groupoid is like a group except the binary operation is only partially defined.
- As a result, a groupoid can have more than one 'unit'.
- Given a groupoid G we have maps r and $s : G \to G^{(0)}$ such that $s(\gamma) = \gamma^{-1}\gamma$ and $r(\gamma) = \gamma\gamma^{-1}$.

Recap

- A groupoid is like a group except the binary operation is only partially defined.
- As a result, a groupoid can have more than one 'unit'.
- Given a groupoid G we have maps r and $s : G \to G^{(0)}$ such that $s(\gamma) = \gamma^{-1}\gamma$ and $r(\gamma) = \gamma\gamma^{-1}$.
- An ample groupoid is a topological groupoid that has a basis of 'compact open bisections'.
  - An open set $B \subseteq G$ is called a bisection if the source map restricts to a homeomorphism on B.

Recap

- A groupoid is like a group except the binary operation is only partially defined.
- As a result, a groupoid can have more than one 'unit'.
- Given a groupoid G we have maps r and $s : G \to G^{(0)}$ such that $s(\gamma) = \gamma^{-1}\gamma$ and $r(\gamma) = \gamma\gamma^{-1}$.
- An ample groupoid is a topological groupoid that has a basis of 'compact open bisections'.
  - An open set $B \subseteq G$ is called a bisection if the source map restricts to a homeomorphism on B.
- Given a directed graph E, the 'boundary path groupoid' G_E is an ample Hausdorff groupoid.

Motivation: The group algebra

- Throughout R is a commutative unital ring.
- Let H be a group.
- For $h \in H$, let $1_h : H \to R$ denote the characteristic function, that is
$$1_h(x) = \begin{cases} 1 & \text{if x=h} \\ 0 & \text{otherwise.} \end{cases}$$
- $RH := \operatorname{span}\{1_h : h \in H\}$
  - Addition and scalar multiplication of functions are defined pointwise.
  - Multiplication of generators is given by $1_{h_1} 1_{h_2} = 1_{h_1 h_2}$.

Steinberg Algebra

- Let G be an ample groupoid and R be a commutative unital ring.
- For $B \subseteq G$, define $1_B : G \to R$ such that
$$1_B(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{otherwise.} \end{cases}$$
- The Steinberg Algebra:
$A_R(G) := \operatorname{span}\{1_B : B \text{ is a compact open bisection}\}$

Steinberg Algebra

- Let G be an ample groupoid and R be a commutative unital ring.
- For $B \subseteq G$, define $1_B : G \to R$ such that
$$1_B(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{otherwise.} \end{cases}$$
- The Steinberg Algebra:
$A_R(G) := \text{span}\{1_B : B \text{ is a compact open bisection}\}$
  - Addition and scalar multiplication of functions are defined pointwise
  - Multiplication is given by convolution:
$f * g(\gamma) = \sum_{\alpha\beta=\gamma} f(\alpha)g(\beta)$
  - Multiplication on generators orangeuces to $1_B 1_D = 1_{BD}$

Steinberg Algebra

- Let G be an ample groupoid and R be a commutative unital ring.
- For $B \subseteq G$, define $1_B : G \to R$ such that
  $1_B(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{otherwise.} \end{cases}$
- The Steinberg Algebra:
  $A_R(G) := \text{span}\{1_B : B \text{ is a compact open bisection}\}$
  - Addition and scalar multiplication of functions are defined pointwise
  - Multiplication is given by convolution:
    $f * g(\gamma) = \sum_{\alpha\beta=\gamma} f(\alpha)g(\beta)$
  - Multiplication on generators orangeuces to $1_B 1_D = 1_{BD}$
- Introduced by Steinberg in 2010 to model discrete inverse semigroup algebras.

Steinberg Algebra

- Let G be an ample groupoid and R be a commutative unital ring.
- For $B \subseteq G$, define $1_B : G \to R$ such that
  $1_B(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{otherwise.} \end{cases}$
- The Steinberg Algebra:
  $A_R(G) := \text{span}\{1_B : B \text{ is a compact open bisection}\}$
  - Addition and scalar multiplication of functions are defined pointwise
  - Multiplication is given by convolution:
    $f * g(\gamma) = \sum_{\alpha\beta=\gamma} f(\alpha)g(\beta)$
  - Multiplication on generators orangeuces to $1_B 1_D = 1_{BD}$
- Introduced by Steinberg in 2010 to model discrete inverse semigroup algebras.
- $\overline{A_{\mathbb{C}}(G)} \cong C^*(G)$

Examples of Steinberg Algebras

- A commutative unital ring

Examples of Steinberg Algebras

- A commutative unital ring
- (Discrete) group algebras RH

Examples of Steinberg Algebras

- A commutative unital ring
- (Discrete) group algebras RH
- The algebra of $n \times n$ matrices with entries in R
  - Suppose X contains n elements
  - Let $G := X \times X$, the full equivalence relation
  - G is ample with respect to the discrete topology
  - $\{1_{(x_i, x_j)}\}$ is a set of matrix units that generate $A_R(G)$

Examples of Steinberg Algebras

- A commutative unital ring
- (Discrete) group algebras RH
- The algebra of $n \times n$ matrices with entries in R
  - Suppose X contains n elements
  - Let $G := X \times X$, the full equivalence relation
  - G is ample with respect to the discrete topology
  - $\{1_{(x_i,x_j)}\}$ is a set of matrix units that generate $A_R(G)$
- Leavitt path algebras (as we will see)

Examples of Steinberg Algebras

- A commutative unital ring
- (Discrete) group algebras RH
- The algebra of $n \times n$ matrices with entries in R
  - Suppose X contains n elements
  - Let $G := X \times X$, the full equivalence relation
  - G is ample with respect to the discrete topology
  - $\{1_{(x_i,x_j)}\}$ is a set of matrix units that generate $A_R(G)$
- Leavitt path algebras (as we will see)
- Kumjian-Pask algebras associated to higher-rank graphs

Examples of Steinberg Algebras

- A commutative unital ring
- (Discrete) group algebras RH
- The algebra of $n \times n$ matrices with entries in R
  - Suppose X contains n elements
  - Let $G := X \times X$, the full equivalence relation
  - G is ample with respect to the discrete topology
  - $\{1_{(x_i, x_j)}\}$ is a set of matrix units that generate $A_R(G)$
- Leavitt path algebras (as we will see)
- Kumjian-Pask algebras associated to higher-rank graphs
- Exel-Pardo algebras

Examples of Steinberg Algebras

- A commutative unital ring
- (Discrete) group algebras RH
- The algebra of $n \times n$ matrices with entries in R
  - Suppose X contains n elements
  - Let $G := X \times X$, the full equivalence relation
  - G is ample with respect to the discrete topology
  - $\{1_{(x_i, x_j)}\}$ is a set of matrix units that generate $A_R(G)$
- Leavitt path algebras (as we will see)
- Kumjian-Pask algebras associated to higher-rank graphs
- Exel-Pardo algebras
- Discrete inverse semigroup algebras

The Steinberg algebra $A_R(G_E)$

- Let E be a directed graph and G_E be the boundary path groupoid:
- $G_E := \{(y, k, z) : y, z \in X \text{ and } y \sim_k z\}$

The Steinberg algebra $A_R(G_E)$

- Let E be a directed graph and G_E be the boundary path groupoid:
- $G_E := \{(y, k, z) : y, z \in X \text{ and } y \sim_k z\}$
  $= \{(\mu x, |\mu| - |\nu|, \nu x) : \mu, \nu \in F(E), x \in X, s(\mu) = s(\nu) = r(x)\}$

The Steinberg algebra $A_R(G_E)$

- Let E be a directed graph and G_E be the boundary path groupoid:
- $G_E := \{(y, k, z) : y, z \in X \text{ and } y \sim_k z\}$
  $= \{(\mu x, |\mu| - |\nu|, \nu x) : \mu, \nu \in F(E), x \in X, s(\mu) = s(\nu) = r(x)\}$
- G_E has a basis of compact open bisections of the form $Z(\mu, \nu)$
- $A_R(G_E) = \operatorname{span}\{1_{Z(\mu,\nu)}\} \neq \{0\}$

The Steinberg algebra $A_R(G_E)$

- Let E be a directed graph and G_E be the boundary path groupoid:
- $G_E := \{(y,k,z) : y,z \in X \text{ and } y \sim_k z\}$
  $= \{(\mu x, |\mu| - |\nu|, \nu x) : \mu, \nu \in F(E), x \in X, s(\mu) = s(\nu) = r(x)\}$
- G_E has a basis of compact open bisections of the form $Z(\mu,\nu)$
- $A_R(G_E) = \text{span}\{1_{Z(\mu,\nu)}\} \neq \{0\}$
- Multiplication looks like:
  1. $1_{Z(v)}1_{Z(w)} = 1_{Z(v)Z(w)} = 1_{Z(v)\cap Z(w)} = \delta_{v,w}1_{Z(v)}$

The Steinberg algebra $A_R(G_E)$

- Let E be a directed graph and G_E be the boundary path groupoid:
- $G_E := \{(y,k,z) : y,z \in X \text{ and } y \sim_k z\}$
  $= \{(\mu x, |\mu| - |\nu|, \nu x) : \mu,\nu \in F(E), x \in X, s(\mu) = s(\nu) = r(x)\}$
- G_E has a basis of compact open bisections of the form $Z(\mu,\nu)$
- $A_R(G_E) = \text{span}\{1_{Z(\mu,\nu)}\} \neq \{0\}$
- Multiplication looks like:
  1. $1_{Z(v)}1_{Z(w)} = 1_{Z(v)Z(w)} = 1_{Z(v)\cap Z(w)} = \delta_{v,w}1_{Z(v)}$
  2. $1_{Z(r(e))}1_{Z(e,s(e))} = 1_{Z(r(e),r(e))Z(e,s(e))} = 1_{Z(e,e)Z(e,s(e))} = 1_{Z(e,s(e))}$

The Steinberg algebra $A_R(G_E)$

- Let E be a directed graph and G_E be the boundary path groupoid:
- $G_E := \{(y,k,z) : y,z \in X \text{ and } y \sim_k z\}$
  $= \{(\mu x, |\mu| - |\nu|, \nu x) : \mu, \nu \in F(E), x \in X, s(\mu) = s(\nu) = r(x)\}$
- G_E has a basis of compact open bisections of the form $Z(\mu,\nu)$
- $A_R(G_E) = \text{span}\{1_{Z(\mu,\nu)}\} \neq \{0\}$
- Multiplication looks like:
  1. $1_{Z(v)}1_{Z(w)} = 1_{Z(v)Z(w)} = 1_{Z(v)\cap Z(w)} = \delta_{v,w}1_{Z(v)}$
  2. $1_{Z(r(e))}1_{Z(e,s(e))} = 1_{Z(r(e),r(e))Z(e,s(e))} = 1_{Z(e,e)Z(e,s(e))} = 1_{Z(e,s(e))}$
  3. $1_{Z(s(e),e)}1_{Z(r(e))} = 1_{Z(s(e),e)Z(r(e),r(e))} = 1_{Z(s(e),e)Z(e,e)} = 1_{Z(s(e),e)}$

The Steinberg algebra $A_R(G_E)$

- Let E be a directed graph and G_E be the boundary path groupoid:
- $G_E := \{(y, k, z) : y, z \in X \text{ and } y \sim_k z\}$
  $= \{(\mu x, |\mu| - |\nu|, \nu x) : \mu, \nu \in F(E), x \in X, s(\mu) = s(\nu) = r(x)\}$
- G_E has a basis of compact open bisections of the form $Z(\mu, \nu)$
- $A_R(G_E) = \text{span}\{1_{Z(\mu,\nu)}\} \neq \{0\}$
- Multiplication looks like:
  1. $1_{Z(v)} 1_{Z(w)} = 1_{Z(v)Z(w)} = 1_{Z(v) \cap Z(w)} = \delta_{v,w} 1_{Z(v)}$
  2. $1_{Z(r(e))} 1_{Z(e,s(e))} = 1_{Z(r(e),r(e))Z(e,s(e))} = 1_{Z(e,e)Z(e,s(e))} = 1_{Z(e,s(e))}$
  3. $1_{Z(s(e),e)} 1_{Z(r(e))} = 1_{Z(s(e),e)Z(r(e),r(e))} = 1_{Z(s(e),e)Z(e,e)} = 1_{Z(s(e),e)}$
  4. $1_{Z(s(e),e)} 1_{Z(e,s(e))} = 1_{Z(s(e),e)Z(e,s(e))} = 1_{Z(s(e),s(e))}$

The Steinberg algebra $A_R(G_E)$

- Let E be a directed graph and G_E be the boundary path groupoid:
- $G_E := \{(y,k,z) : y, z \in X \text{ and } y \sim_k z\}$
  $= \{(\mu x, |\mu| - |\nu|, \nu x) : \mu, \nu \in F(E), x \in X, s(\mu) = s(\nu) = r(x)\}$
- G_E has a basis of compact open bisections of the form $Z(\mu,\nu)$
- $A_R(G_E) = \text{span}\{1_{Z(\mu,\nu)}\} \neq \{0\}$
- Multiplication looks like:
  1. $1_{Z(v)}1_{Z(w)} = 1_{Z(v)Z(w)} = 1_{Z(v)\cap Z(w)} = \delta_{v,w}1_{Z(v)}$
  2. $1_{Z(r(e))}1_{Z(e,s(e))} = 1_{Z(r(e),r(e))Z(e,s(e))} = 1_{Z(e,e)Z(e,s(e))} = 1_{Z(e,s(e))}$
  3. $1_{Z(s(e),e)}1_{Z(r(e))} = 1_{Z(s(e),e)Z(r(e),r(e))} = 1_{Z(s(e),e)Z(e,e)} = 1_{Z(s(e),e)}$
  4. $1_{Z(s(e),e)}1_{Z(e,s(e))} = 1_{Z(s(e),e)Z(e,s(e))} = 1_{Z(s(e),s(e))}$
  5. $1_{Z(v,v)}$

The Steinberg algebra $A_R(G_E)$

- Let E be a directed graph and G_E be the boundary path groupoid:
- $G_E := \{(y,k,z) : y,z \in X \text{ and } y \sim_k z\}$
  $= \{(\mu x, |\mu| - |\nu|, \nu x) : \mu,\nu \in F(E), x \in X, s(\mu) = s(\nu) = r(x)\}$
- G_E has a basis of compact open bisections of the form $Z(\mu,\nu)$
- $A_R(G_E) = \text{span}\{1_{Z(\mu,\nu)}\} \neq \{0\}$
- Multiplication looks like:
  1. $1_{Z(v)} 1_{Z(w)} = 1_{Z(v)Z(w)} = 1_{Z(v)\cap Z(w)} = \delta_{v,w} 1_{Z(v)}$
  2. $1_{Z(r(e))} 1_{Z(e,s(e))} = 1_{Z(r(e),r(e))Z(e,s(e))} = 1_{Z(e,e)Z(e,s(e))} = 1_{Z(e,s(e))}$
  3. $1_{Z(s(e),e)} 1_{Z(r(e))} = 1_{Z(s(e),e)Z(r(e),r(e))} = 1_{Z(s(e),e)Z(e,e)} = 1_{Z(s(e),e)}$
  4. $1_{Z(s(e),e)} 1_{Z(e,s(e))} = 1_{Z(s(e),e)Z(e,s(e))} = 1_{Z(s(e),s(e))}$
  5. $1_{Z(v,v)} = \sum_{\{e \in E^1 | r(e) = v\}} 1_{Z(e,e)}$ for every regular vertex v.

Leavitt path algebras

- Let E be an arbitrary directed graph.
- The Leavitt path algebra $L_R(E)$ is an R-algebra constructed from the vertices and edges of E so that multiplication behaves like concatenation of paths.
- Introduced in 2005 by Abrams-Aranda Pino and Ara-Moreno-Pardo.

Leavitt path algebras

- Let E be an arbitrary directed graph.
- The Leavitt path algebra $L_R(E)$ is an R-algebra constructed from the vertices and edges of E so that multiplication behaves like concatenation of paths.
- Introduced in 2005 by Abrams-Aranda Pino and Ara-Moreno-Pardo.
- A lot of the theory parallels that of the C^*-algebras associated to directed graphs.

Leavitt path algebras

- Let E be an arbitrary directed graph.
- The Leavitt path algebra $L_R(E)$ is an R-algebra constructed from the vertices and edges of E so that multiplication behaves like concatenation of paths.
- Introduced in 2005 by Abrams-Aranda Pino and Ara-Moreno-Pardo.
- A lot of the theory parallels that of the C^*-algebras associated to directed graphs.
- The class of Leavitt path algebras is included in the class of 'Kumjian-Pask' algebras associated to higher-rank graphs.

Leavitt path algebras

- Let E be an arbitrary directed graph.
- The Leavitt path algebra $L_R(E)$ is an R-algebra constructed from the vertices and edges of E so that multiplication behaves like concatenation of paths.
- Introduced in 2005 by Abrams-Aranda Pino and Ara-Moreno-Pardo.
- A lot of the theory parallels that of the C^*-algebras associated to directed graphs.
- The class of Leavitt path algebras is included in the class of 'Kumjian-Pask' algebras associated to higher-rank graphs.
  - There are some very confusing differences in notational conventions (as you are about to see).

LPA's as universal algebras

Let E be an arbitrary directed graph and R be a commutative ring with identity. Then the Leavitt path algebra $L_R(E)$ is the universal R-algebra generated by a Leavitt E-family, that is, a set $\{p_v, s_e, s_{e^*} : v \in E^0 \text{ and } e \in E^1\}$ satisfying the relations:

1. $p_v p_w = \delta_{v,w} v$ for all $v, w \in E^0$;
2. $p_{r(e)} s_e = s_e = s_e p_{s(e)}$ for all $e \in E^1$;
3. $p_{s(e)} s_{e^*} = s_{e^*} = s_{e^*} p_{r(e)}$ for all $e \in E^1$;
4. $s_{e^*} s_{e'} = \delta_{e,e'} p_{s(e)}$ for all $e, e' \in E^1$; and
5. $p_v = \sum_{\{e \in E^1 | r(e) = v\}} s_e s_{e^*}$ for every regular vertex v.

LPA's as universal algebras

Let E be an arbitrary directed graph and R be a commutative ring with identity. Then the Leavitt path algebra $L_R(E)$ is the universal R-algebra generated by a Leavitt E-family, that is, a set $\{p_v, s_e, s_{e^*} : v \in E^0 \text{ and } e \in E^1\}$ satisfying the relations:

1. $p_v p_w = \delta_{v,w} v$ for all $v, w \in E^0$;
2. $p_{r(e)} s_e = s_e = s_e p_{s(e)}$ for all $e \in E^1$;
3. $p_{s(e)} s_{e^*} = s_{e^*} = s_{e^*} p_{r(e)}$ for all $e \in E^1$;
4. $s_{e^*} s_{e'} = \delta_{e,e'} p_{s(e)}$ for all $e, e' \in E^1$; and
5. $p_v = \sum_{\{e \in E^1 | r(e) = v\}} s_e s_{e^*}$ for every regular vertex v.

If A is an R-algebra containing a set of elements $\{q_v, t_e, t_{e^*}\}$ satisfying (1)–(5), then there exists an algebra homomorphism $\pi : L_R(E) \to A$ such that $\pi(p_v) = q_v$, $\pi(s_e) = t_e$ and $\pi(s_{e^*}) = t_{e^*}$.

$A(G_E)$ and $L_R(E)$

- We can use $A_R(G_E)$ to show that $L_R(E)$ is nonzero.

$A(G_E)$ and $L_R(E)$

- We can use $A_R(G_E)$ to show that $L_R(E)$ is nonzero.
- For $v \in E^0$ and $e \in E^1$ define elements of $A(G_E)$ as follows: $q_v := 1_{Z(v)}$, $t_e := 1_{Z(e,s(e))}$ and $t_{e^*} = 1_{Z(s(e),e)}$.

$A(G_E)$ and $L_R(E)$

- We can use $A_R(G_E)$ to show that $L_R(E)$ is nonzero.
- For $v \in E^0$ and $e \in E^1$ define elements of $A(G_E)$ as follows: $q_v := 1_{Z(v)}$, $t_e := 1_{Z(e,s(e))}$ and $t_{e^*} = 1_{Z(s(e),e)}$.
- Then $\{q_v, t_e, t_{e^*}\}$ satisfy relations (1)–(5) in the definition of a LPA.

$A(G_E)$ and $L_R(E)$

- We can use $A_R(G_E)$ to show that $L_R(E)$ is nonzero.
- For $v \in E^0$ and $e \in E^1$ define elements of $A(G_E)$ as follows: $q_v := 1_{Z(v)}$, $t_e := 1_{Z(e,s(e))}$ and $t_{e^*} = 1_{Z(s(e),e)}$.
- Then $\{q_v, t_e, t_{e^*}\}$ satisfy relations (1)–(5) in the definition of a LPA.
- The universal property of $L_R(E)$ gives a homomorphism $\pi : L_R(E) \to A_R(G_E)$ satisfying $\pi(p_v) = q_v$, $\pi(s_e) = t_e$ and $\pi(s_{e^*}) = t_{e^*}$.

$A(G_E)$ and $L_R(E)$

- We can use $A_R(G_E)$ to show that $L_R(E)$ is nonzero.
- For $v \in E^0$ and $e \in E^1$ define elements of $A(G_E)$ as follows: $q_v := 1_{Z(v)}$, $t_e := 1_{Z(e,s(e))}$ and $t_{e^*} = 1_{Z(s(e),e)}$.
- Then $\{q_v, t_e, t_{e^*}\}$ satisfy relations (1)–(5) in the definition of a LPA.
- The universal property of $L_R(E)$ gives a homomorphism $\pi : L_R(E) \to A_R(G_E)$ satisfying $\pi(p_v) = q_v$, $\pi(s_e) = t_e$ and $\pi(s_{e^*}) = t_{e^*}$.
- The map is surjective.

$A(G_E)$ and $L_R(E)$

- We can use $A_R(G_E)$ to show that $L_R(E)$ is nonzero.
- For $v \in E^0$ and $e \in E^1$ define elements of $A(G_E)$ as follows: $q_v := 1_{Z(v)}$, $t_e := 1_{Z(e,s(e))}$ and $t_{e^*} = 1_{Z(s(e),e)}$.
- Then $\{q_v, t_e, t_{e^*}\}$ satisfy relations (1)–(5) in the definition of a LPA.
- The universal property of $L_R(E)$ gives a homomorphism $\pi : L_R(E) \to A_R(G_E)$ satisfying $\pi(p_v) = q_v$, $\pi(s_e) = t_e$ and $\pi(s_{e^*}) = t_{e^*}$.
- The map is surjective.
- Surjectivity and the fact that $A_R(G_E)$ is nonzero implies $L_R(E)$ is nonzero.

Graded rings

- Suppose that Γ is an additive abelian group. A ring A is Γ-graded if there are additive subgroups $\{A_g : g \in \Gamma\}$ satisfying:
  - $A = \bigoplus_{g\in\Gamma} A_g$ and
  - for $g, h \in \Gamma$, $A_g A_h \subseteq A_{g+h}$.

Graded rings

- Suppose that Γ is an additive abelian group. A ring A is Γ-graded if there are additive subgroups $\{A_g : g \in \Gamma\}$ satisfying:
  - $A = \bigoplus_{g \in \Gamma} A_g$ and
  - for $g, h \in \Gamma, A_g A_h \subseteq A_{g+h}$.

- If A and B are Γ-graded rings, a homomorphism $\pi : A \to B$ is Γ-graded if $\pi(A_g) \subseteq B_g$ for $g \in \Gamma$.

Graded rings

- Suppose that Γ is an additive abelian group. A ring A is Γ-graded if there are additive subgroups $\{A_g : g \in \Gamma\}$ satisfying:
  - $A = \bigoplus_{g \in \Gamma} A_g$ and
  - for $g, h \in \Gamma$, $A_g A_h \subseteq A_{g+h}$.

- If A and B are Γ-graded rings, a homomorphism $\pi : A \to B$ is Γ-graded if $\pi(A_g) \subseteq B_g$ for $g \in \Gamma$.

- $A_R(G_E)$ is a $\mathbb{Z}$-graded ring such that for each $n \in \mathbb{Z}$ we have $(A_R(G_E))_n = \operatorname{span}\{1_{Z(\mu,\nu)} : |\mu| - |\nu| = n\}$. (EXERCISE)

Graded rings

- Suppose that Γ is an additive abelian group. A ring A is Γ-graded if there are additive subgroups $\{A_g : g \in \Gamma\}$ satisfying:
  - $A = \bigoplus_{g \in \Gamma} A_g$ and
  - for $g, h \in \Gamma$, $A_g A_h \subseteq A_{g+h}$.
- If A and B are Γ-graded rings, a homomorphism $\pi : A \to B$ is Γ-graded if $\pi(A_g) \subseteq B_g$ for $g \in \Gamma$.
- $A_R(G_E)$ is a $\mathbb{Z}$-graded ring such that for each $n \in \mathbb{Z}$ we have $(A_R(G_E))_n = \operatorname{span}\{1_{Z(\mu,\nu)} : |\mu| - |\nu| = n\}$. (EXERCISE)
- $L_R(E)$ is a $\mathbb{Z}$-graded ring such that for each $n \in \mathbb{Z}$ we have $(L_R(E))_n = \operatorname{span}\{s_\mu s_{\nu^*} : |\mu| - |\nu| = n\}$

Graded rings

- Suppose that Γ is an additive abelian group. A ring A is Γ-graded if there are additive subgroups $\{A_g : g \in \Gamma\}$ satisfying:
  - $A = \bigoplus_{g \in \Gamma} A_g$ and
  - for $g, h \in \Gamma$, $A_g A_h \subseteq A_{g+h}$.
- If A and B are Γ-graded rings, a homomorphism $\pi : A \to B$ is Γ-graded if $\pi(A_g) \subseteq B_g$ for $g \in \Gamma$.
- $A_R(G_E)$ is a $\mathbb{Z}$-graded ring such that for each $n \in \mathbb{Z}$ we have $(A_R(G_E))_n = \operatorname{span}\{1_{Z(\mu,\nu)} : |\mu| - |\nu| = n\}$. (EXERCISE)
- $L_R(E)$ is a $\mathbb{Z}$-graded ring such that for each $n \in \mathbb{Z}$ we have $(L_R(E))_n = \operatorname{span}\{s_\mu s_{\nu^*} : |\mu| - |\nu| = n\}$
- Our homomorphism $\pi : L_R(E) \to A_R(G_E)$ is graded.

The uniqueness theorems for Leavitt path algebras

The uniqueness theorems for Leavitt path algebras

Theorem (The graded uniqueness theorem, Raeburn, Tomforde)

Let E be an arbitrary directed graph and A be a $\mathbb{Z}$ graded ring. Then a graded ring homomorphism $\pi : L_R(E) \to A$ is injective if and only if $\pi(rp_v) \neq 0$ for all $v \in E^0$ and nonzero $r \in R$.

Theorem (Cuntz-Krieger Uniqueness theorem, Raeburn, Tomforde)

Let E be an arbitrary directed graph that satisfies condition (L) and A be a ring. Then a ring homomorphism $\pi : L_R(E) \to A$ is injective if and only if $\pi(rp_v) \neq 0$ for all $v \in E^0$.

Graph C^*-algebras

- For this slide only, suppose $R = \mathbb{C}$.

Graph C^*-algebras

- For this slide only, suppose $R = \mathbb{C}$.
- A C^*-algebra is a complex Banach $*$-algebra that satisfies the C^*-identity.

Graph C^*-algebras

- For this slide only, suppose $R = \mathbb{C}$.
- A C^*-algebra is a complex Banach $*$-algebra that satisfies the C^*-identity.
- Given a directed graph, $C^*(E)$ is the universal C^*-algebra generated by a Cuntz-Krieger E-family, $\{p_v, s_e\}$.
- We have $C^*(E) = \overline{\{\text{the algebra generated by a Cuntz-Krieger } E\text{-family}\}}$

Graph C^*-algebras

- For this slide only, suppose $R = \mathbb{C}$.
- A C^*-algebra is a complex Banach $*$-algebra that satisfies the C^*-identity.
- Given a directed graph, $C^*(E)$ is the universal C^*-algebra generated by a Cuntz-Krieger E-family, $\{p_v, s_e\}$.
- We have $C^*(E) = \overline{\{\text{the algebra generated by a Cuntz-Krieger } E\text{-family}\}}$
- A Cuntz-Krieger E-family satisfies relations (1)–(5) of a Leavitt E-family.

Graph C^*-algebras

- For this slide only, suppose $R = \mathbb{C}$.
- A C^*-algebra is a complex Banach $*$-algebra that satisfies the C^*-identity.
- Given a directed graph, $C^*(E)$ is the universal C^*-algebra generated by a Cuntz-Krieger E-family, $\{p_v, s_e\}$.
- We have $C^*(E) = \overline{\{\text{the algebra generated by a Cuntz-Krieger } E\text{-family}\}}$
- A Cuntz-Krieger E-family satisfies relations (1)–(5) of a Leavitt E-family.
- The universal property of $L(E)$ gives a homomorphism from $L(E)$ into $C^*(E)$ and the graded uniqueness theorem says this map is injective.
- So $L_{\mathbb{C}}(E)$ is dense in $C^*(E)$.

Back to Steinberg algebras: the punchline

Our homomorphism $\pi : L_R(E) \to A_R(G_E)$ is graded and for any $v \in E^0$ and $r \in R$ we have $\pi(rp_v) = r1_{Z(v)} \neq 0$.

Back to Steinberg algebras: the punchline

Our homomorphism $\pi : L_R(E) \to A_R(G_E)$ is graded and for any $v \in E^0$ and $r \in R$ we have $\pi(rp_v) = r1_{Z(v)} \neq 0$.

Thus the graded uniqueness theorem implies π is injective so we have $\pi : L_R(E) \to A_R(G_E)$ is a graded isomorphism.

Examples

$$
\begin{tikzcd}
\bullet_v & \bullet_t \arrow[l, "e"'] & \bullet_w \arrow[l, "f"'] & E
\end{tikzcd}
$$

Examples

$$\bullet_v \xleftarrow{e} \bullet_t \xleftarrow{f} \bullet_w \qquad E$$

- $X = \{ef, f, w\}$

Examples

$$\begin{tikzcd} \bullet_v & \bullet_t \arrow[l, "e"'] & \bullet_w \arrow[l, "f"'] \end{tikzcd} \qquad E$$

- $X = \{ef, f, w\} \equiv \{(ef, 0, ef), (f, 0, f), (w, 0, w)\}$

Examples

$$
\begin{tikzcd}
\bullet_v & \bullet_t \arrow[l, "e"'] & \bullet_w \arrow[l, "f"']
\end{tikzcd}
\qquad E
$$

- $X = \{ef, f, w\} \equiv \{(ef, 0, ef), (f, 0, f), (w, 0, w)\}$
- The groupoid $G_E =$

  $\{(ef, 1, f), (ef, 2, w), (f, 1, w), (f, -1, ef), (w, -2, ef), (w, -1, f)\} \cup X$

Examples

$$\bullet_v \xleftarrow{e} \bullet_t \xleftarrow{f} \bullet_w \qquad E$$

- $X = \{ef, f, w\} \equiv \{(ef, 0, ef), (f, 0, f), (w, 0, w)\}$
- The groupoid $G_E =$

  $\{(ef, 1, f), (ef, 2, w), (f, 1, w), (f, -1, ef), (w, -2, ef), (w, -1, f)\} \cup X$
- It is the *trivial* equivalence relation on X with the discrete topology.

Examples

$$\bullet_v \xleftarrow{e} \bullet_t \xleftarrow{f} \bullet_w \qquad E$$

- $X = \{ef, f, w\} \equiv \{(ef, 0, ef), (f, 0, f), (w, 0, w)\}$
- The groupoid $G_E =$

  $\{(ef, 1, f), (ef, 2, w), (f, 1, w), (f, -1, ef), (w, -2, ef), (w, -1, f)\} \cup X$
- It is the *trivial* equivalence relation on X with the discrete topology.
- $L_R(E) \cong A_R(G_E) \cong M_3(R)$

F_1

- Let $x = eee\ldots$. Then $X_{F_1} = \{x\}$

F_1

- Let $x = eee\ldots$. Then $X_{F_1} = \{x\} \equiv \{(x, 0, x)\}$

F_1

- Let $x = eee\ldots$. Then $X_{F_1} = \{x\} \equiv \{(x, 0, x)\}$

- The groupoid $G_{F_1} = \{(x, n, x) : n \in \mathbb{Z}\}$

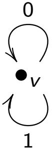
F_1

- Let $x = eee\ldots$. Then $X_{F_1} = \{x\} \equiv \{(x, 0, x)\}$
- The groupoid $G_{F_1} = \{(x, n, x) : n \in \mathbb{Z}\}$
- It is isomorphic to the discrete group $\mathbb{Z}$.

F_1

- Let $x = eee\ldots$. Then $X_{F_1} = \{x\} \equiv \{(x, 0, x)\}$
- The groupoid $G_{F_1} = \{(x, n, x) : n \in \mathbb{Z}\}$
- It is isomorphic to the discrete group $\mathbb{Z}$.
- $L_R(E) \cong A_R(G_E) \cong R\mathbb{Z} \cong R[x, x^{-1}]$

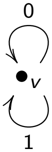
The Cuntz groupoid



F_2

- $X_{F_2} = \{$ infinite sequences of 0's and 1's $\} =$ the Cantor set

The Cuntz groupoid



F_2

- $X_{F_2} = \{$ infinite sequences of 0's and 1's $\} =$ the Cantor set
- Then $G_E = \{(\mu x, |\mu| - |\nu|, \nu x) : x \in E^{\infty} \text{and } \mu, \nu \in F(F_2)\}$.

The Cuntz groupoid

F_2

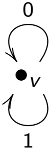
- $X_{F_2} = \{$ infinite sequences of 0's and 1's $\} =$ the Cantor set
- Then $G_E = \{(\mu x, |\mu| - |\nu|, \nu x) : x \in E^\infty \text{and } \mu, \nu \in F(F_2)\}$.
- The topology on $X = G^{(0)}$ generated by the basis $\{Z(\mu)\}$ is the usual topology on the Cantor set.

The Cuntz groupoid

F_2

- $X_{F_2} = \{$ infinite sequences of 0's and 1's $\} =$ the Cantor set
- Then $G_E = \{(\mu x, |\mu| - |\nu|, \nu x) : x \in E^{\infty}$and $\mu, \nu \in F(F_2)\}$.
- The topology on $X = G^{(0)}$ generated by the basis $\{Z(\mu)\}$ is the usual topology on the Cantor set.
- The topology generated by the basis of compact set $\{Z(\mu, \nu)\}$ makes G_{F_2} into an ample Hausdorff groupoid that is not discrete.
- $L_R(E) \cong A(G_E) \cong L_2$

The Cuntz groupoid



F_2

- $X_{F_2} = \{$ infinite sequences of 0's and 1's $\} =$ the Cantor set
- Then $G_E = \{(\mu x, |\mu| - |\nu|, \nu x) : x \in E^{\infty}$and $\mu, \nu \in F(F_2)\}$.
- The topology on $X = G^{(0)}$ generated by the basis $\{Z(\mu)\}$ is the usual topology on the Cantor set.
- The topology generated by the basis of compact set $\{Z(\mu, \nu)\}$ makes G_{F_2} into an ample Hausdorff groupoid that is not discrete.
- $L_R(E) \cong A(G_E) \cong L_2$
- Every LPA embeds into L_2 (Brownlowe-Sørensen). Does every Steinberg algebra?

F_2

- $X_{F_2} = \{$ infinite sequences of 0's and 1's $\} =$ the Cantor set
- Then $G_E = \{(\mu x, |\mu| - |\nu|, \nu x) : x \in E^\infty \text{and } \mu, \nu \in F(F_2)\}$.
- The topology on $X = G^{(0)}$ generated by the basis $\{Z(\mu)\}$ is the usual topology on the Cantor set.
- The topology generated by the basis of compact set $\{Z(\mu, \nu)\}$ makes G_{F_2} into an ample Hausdorff groupoid that is not discrete.
- $L_R(E) \cong A(G_E) \cong L_2$
- Every LPA embeds into L_2 (Brownlowe-Sørensen).
  Does every Steinberg algebra?
  Answer: ??

More general Steinberg algebras

- Question: Is the class of Steinberg algebras bigger than the class of Leavitt path algebras?
- Answer:

More general Steinberg algebras

- Question: Is the class of Steinberg algebras bigger than the class of Leavitt path algebras?
- Answer: Yes.

More general Steinberg algebras

- Question: Is the class of Steinberg algebras bigger than the class of Leavitt path algebras?
- Answer: Yes.
  - There is dichotomy within class of simple Leavitt path algebras: they are either purely infinite or ultramatricial.
  - We have examples of simple Steinberg algebras that are neither purely infinite nor ultramaricial. (C-Flynn-an Huef)
  - These examples are Kumjian-Pask algebras associated to rank-2 graphs.

More general Steinberg algebras

- Question: Is the class of Steinberg algebras bigger than the class of Leavitt path algebras?
- Answer: Yes.
  - There is dichotomy within class of simple Leavitt path algebras: they are either purely infinite or ultramatricial.
  - We have examples of simple Steinberg algebras that are neither purely infinite nor ultramaricial. (C-Flynn-an Huef)
  - These examples are Kumjian-Pask algebras associated to rank-2 graphs.
- Question: How big is the class of Steinberg algebras?
- Answer:

More general Steinberg algebras

- Question: Is the class of Steinberg algebras bigger than the class of Leavitt path algebras?
- Answer: Yes.
  - There is dichotomy within class of simple Leavitt path algebras: they are either purely infinite or ultramatricial.
  - We have examples of simple Steinberg algebras that are neither purely infinite nor ultramaricial. (C-Flynn-an Huef)
  - These examples are Kumjian-Pask algebras associated to rank-2 graphs.
- Question: How big is the class of Steinberg algebras?
- Answer: ??

Steinberg algebra research

Research in this area takes the following forms:

Steinberg algebra research

Research in this area takes the following forms:

1. Connect proprties of the groupoid to properties of the algebra (simiplicity, ideals, center, socle, grading, etc.).

Steinberg algebra research

Research in this area takes the following forms:

1. Connect proprties of the groupoid to properties of the algebra (simiplicity, ideals, center, socle, grading, etc.).

2. Show a particular class of algebras has a Steinberg algebra model and/or develop tools to do so (LPAs, KPAs, Cartan pairs, Uniqueness theorems, etc).

Steinberg algebra research

Research in this area takes the following forms:

1. Connect proprties of the groupoid to properties of the algebra (simiplicity, ideals, center, socle, grading, etc.).
2. Show a particular class of algebras has a Steinberg algebra model and/or develop tools to do so (LPAs, KPAs, Cartan pairs, Uniqueness theorems, etc).
3. Combine items (1) and (2), that is, use the groupoid model to advance the theory.

The End
Thank you for listening!