# Ample groupoid algebras
# Lecture 2: Graph Groupoids

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# Outline for lecture 2

- Groupoids associated to directed graphs
- Higher-rank graphs

# Recall from Lecture 1

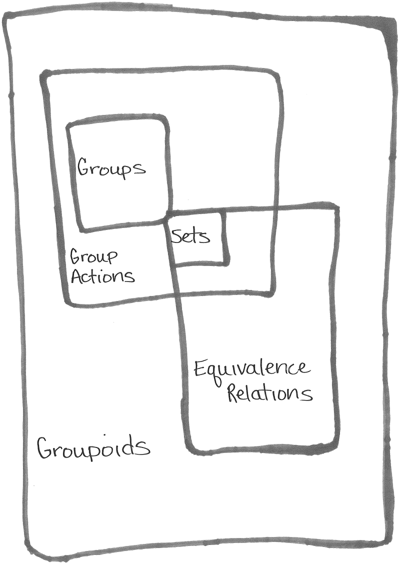
## Definition

Let $G$ be a set and $G^{(2)}$ be a subset of $G \times G$ such that there is a map $(\gamma, \alpha) \mapsto \gamma\alpha$ from $G^{(2)}$ to $G$.

Suppose there is an involution $\gamma \mapsto \gamma^{-1}$ on $G$.

Then we say $G$ is groupoid if the following are satisfied:

- If $(\gamma, \alpha)$ and $(\alpha, \beta)$ are in $G^{(2)}$, then so are $(\gamma\alpha, \beta)$ and $(\gamma, \alpha\beta)$, and the equation $(\gamma\alpha)\beta = \gamma(\alpha\beta)$ is satisfied.
- We have $(\gamma^{-1}, \gamma) \in G^{(2)}$ for every $\gamma \in G$.
- If $(\gamma, \alpha) \in G^{(2)}$, then $(\gamma^{-1}\gamma)\alpha = \alpha$ and $\gamma(\alpha\alpha^{-1}) = \gamma$.
- For $\gamma \in G$, $s(\gamma) = \gamma^{-1}\gamma$ and $r(\gamma) = \gamma\gamma^{-1}$.
- We have $(\gamma, \alpha) \in G^{(2)} \iff s(\gamma) = r(\alpha) \in G^{(0)}$.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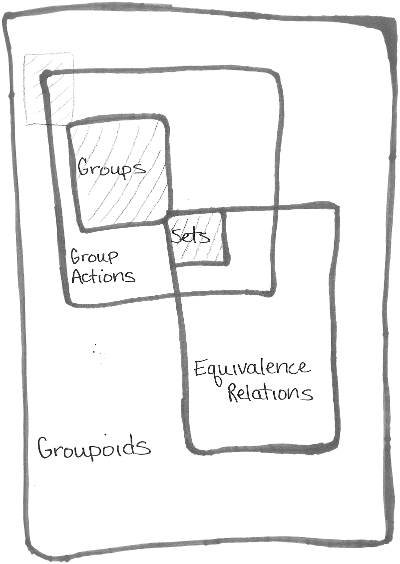
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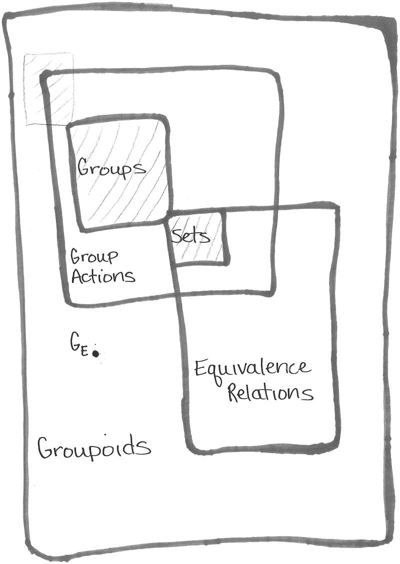
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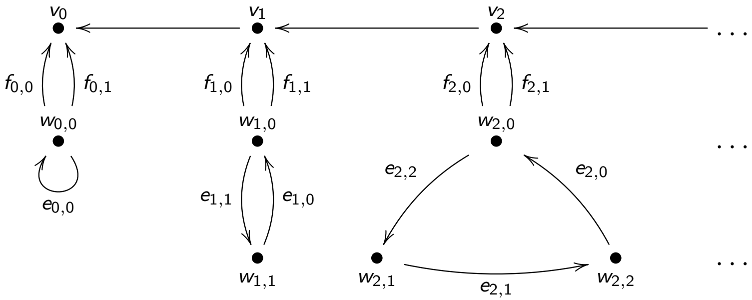
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$\dots$

# Directed graph conventions CAUTION

# What direction do your paths go?

$\longleftarrow efg$

$$\bullet \xleftarrow{e} \bullet \xleftarrow{f} \bullet \xleftarrow{g} \bullet \ldots$$

SOUTHERN CONVENTION

---

$\longrightarrow efg$

$$\bullet \xrightarrow[e]{} \bullet \xrightarrow[f]{} \bullet \xrightarrow[g]{} \bullet \ldots$$

NORTHERN CONVENTION

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- We call $v$ a regular vertex if it is neither a source nor an infinite receiver.

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Notation: We write $F(E)$ for the set of finite paths in $E$.

$$F(E) = E^0 \cup \{e_1 e_2 ... e_n : \text{ for each } 1 \leq i < n, s(e_i) = r(e_{i+1})\}.$$

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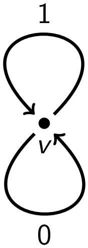
- $(y, k, z) \in G_E$ has range $(y, 0, y)$ and source $(z, 0, z)$
- $G_E^{(0)} = \{(x, 0, x) : x \in X_E\}$ which we identify with $X_E$

# The Cuntz groupoid (1980)



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- $G_E := \{(y, k, z) : y, z \in X_E \text{ and } y \sim_k z\}$
- $zG_Ez \cong 2\mathbb{Z}$.

# The algebraic structure of $G_E$ is special

$G_E := \{(y, k, z) : y, z \in X_E \text{ and } y \sim_k z\}$

Isotropy: Fix $x \in X_E$. The isotropy group at $x$ is:

$$\begin{aligned} xG_Ex &= \{\gamma \in G_E \text{ such that } s(\gamma) = r(\gamma) = x\} \\ &= \{(x, k, x) \in G_E \text{ for some } k \in \mathbb{Z}\}. \end{aligned}$$

The operation on the group $xG_Ex$ is:

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So all isotropy groups are subgroups of $\mathbb{Z}$.

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Terminology: A groupoid in which all isotropy groups are trivial is called a principal groupoid.

Every principal groupoid is algebraically isomorphic to an equivalence relation groupoid in $G^{(0)} \times G^{(0)}$.

However, the topology need not be the product topology.

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- For $\mu, \nu \in F(E)$ with $s(\mu) = s(\nu)$, define
  $Z(\mu, \nu) := \{(\mu x, |\mu| - |\nu|, \nu x) : x \in X_E, s(\mu) = r(x)\}$, and
- The collection of $Z(\mu, \nu)$ form a basis of compact open bisections for a Hausdorff topology on $G_E$.
- Since we view $G^{(0)} \subseteq G$, we identify (for example) $Z(v)$ and $Z(v, v)$.

# The topology on $G_E$ is special

- $$Z(\mu) \cap Z(\nu) = \begin{cases} Z(\nu) & \text{if } \nu = \mu\mu' \\ Z(\mu) & \text{if } \mu = \nu\nu' \\ \emptyset & \text{otherwise.} \end{cases}$$

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- Answer:

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- Question: given an ample groupoid $G$, how can you tell if $G \cong G_E$ for some directed graph $E$?

- Answer: ??

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  If $n > 0$, then $Z(\underset{n}{e...e}, v) = \{(x, n, x)\}$.

  If $n < 0$, then $Z(v, \underset{-n}{e...e}) = \{(x, n, x)\}$.

  If $n = 0$, then $Z(v, v) = \{(x, 0, x)\}$.

$$
\begin{tikzcd}
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\end{tikzcd}
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## Exercise

# Path category

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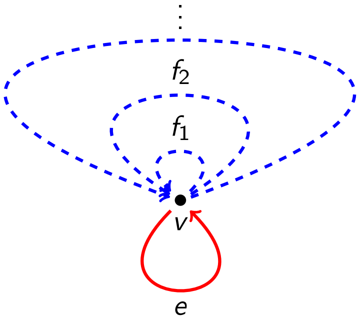
- Our path conventions help here.

- We can view length of a path is a functor $d : F(E) \to \mathbb{N}$ such that if $d(\lambda) = n + m$, then there exist unique $\nu$ and $\mu$ in $F(E)$ with $d(\nu) = n$, $d(\mu) = m$ and $\lambda = \nu\mu$.



$\vdots$
$f_2$
$f_1$
$v$
$e$

# Higher-rank graphs: the idea

- A higher rank graph or *$k$-graph* is a category $\Lambda$ with objects $\Lambda^0$ (called vertecies) such that there is a functor $d : \Lambda \to \mathbb{N}^k$ that satisfies the unique factorisation property:

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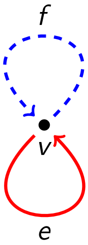
  - $G_\Lambda := \{(\mu x, d(\mu) - d(\nu), \nu x) : \mu, \nu \in \Lambda, x \in \Lambda^\infty, s(\mu) = s(\nu) = r(x)\}$

# Example of a 2-graph via its skeleton

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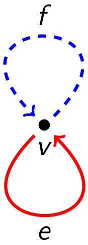


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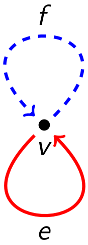


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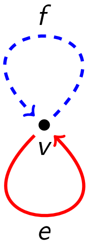


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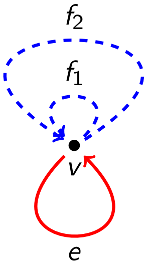
# Another 2-graph

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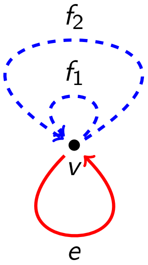


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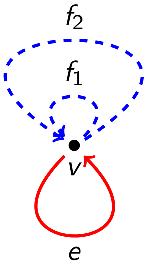


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# $k$-graph groupoids are significantly more general

They are still algebraically special: the isotropy groups are all subgroups of $\mathbb{Z}^k$.

The topology is far less special in comparison to just 1-graphs.

The End

Thank you for listening!