Ample groupoid algebras
Lecture 1: Ample Groupoids

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Outline for course

- Groupoids

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- Groupoids
- Ample groupoids and inverse semigroups

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- Groupoids
- Ample groupoids and inverse semigroups
- Groupoids, directed graphs and higher-rank graphs

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    - Special case: Leavitt path algebras

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- How can you tell if a given algebra is a (twisted) Steinberg algebra?

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  - Uniqueness theorems

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  - Uniqueness theorems
  - Cartan subalgebras and generalisations

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- Open questions

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In this series, Algebras will be associative R-algebras where R is a commutative unital ring. The algebras need not be commutative, unital or finite-dimensional (but can be).

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- Steinberg algebra development

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- Groupoids
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- Isotropy

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- Isotropy
- Connection to inverse semigroups

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- "Groupoid C*-algebras have been studied for years. A systematic developement of the fundamentals of groupoid C*-algebras was provided by J. Renault in 1980. More or less at the same time, A. Connes showed how groupoid C*-algebras have to be used in the study of geometric objects. In Connes' work, various groupoids arise... Furthermore groupoids themselves turn out to take into account , somewhat unexpectedly, various knds of geometric phenomena."
  Khoshkam and Skandalis 2002

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- *A groupoid approach to discrete inverse semigroup algebras*, Benjamin Steinberg 2010
- *A groupoid geralisation of Leavitt path algebras*, C-Farthing-Sims-Tomforde, 2014
- *Simplicity of algebras associated to étale groupoids*, Brown-Clark-Farthing-Sims, 2014

Definition

Let G be a set and $G^{(2)}$ be a subset of $G \times G$ such that there is a map $(\gamma, \alpha) \mapsto \gamma\alpha$ from $G^{(2)}$ to G.

Suppose there is an involution $\gamma \mapsto \gamma^{-1}$ on G.

Then we say G is groupoid if the following are satisfied:

- If (γ, α) and (α, β) are in $G^{(2)}$, then so are $(\gamma\alpha, \beta)$ and $(\gamma, \alpha\beta)$, and the equation $(\gamma\alpha)\beta = \gamma(\alpha\beta)$ is satisfied.
- We have $(\gamma^{-1}, \gamma) \in G^{(2)}$ for every $\gamma \in G$.
- If $(\gamma, \alpha) \in G^{(2)}$, then $(\gamma^{-1}\gamma)\alpha = \alpha$ and $\gamma(\alpha\alpha^{-1}) = \gamma$.

We call $G^{(2)}$ the set of composable pairs.

Example

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$$M := \bigcup_{n,m \in \mathbb{N} \setminus \{0\}} M_{n,m}(R).$$

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Then M is a groupoid with repect to matrix addition.

Terminology

- Let G be a groupoid.
- Define maps r and $s : G \to G$ such that if $\gamma \in G$,
  - $s(\gamma) = \gamma^{-1}\gamma$ called the source of γ
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- Let $G^{(0)}$ denote the image of r (which is equal to the image of s), called the set of units.

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- Let $G^{(0)}$ denote the image of r (which is equal to the image of s), called the set of units.
- Each $\gamma \in G$ can be viewed as a morphisms from $s(\gamma)$ to $r(\gamma)$.

$$\begin{tikzcd} \bullet r(\gamma) & \bullet s(\gamma) \arrow[l, bend right, "\gamma"'] \end{tikzcd}$$

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Side note: An alternate (equivalent) definition

- A groupoid is a small category in which every morphism has an inverse.
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- A groupoid is a small category in which every morphism has an inverse.

- What is a category?

  It is a collection of objects along with a collection of morphism between the objects such that:
  - morphisms can be composed when the source and range line up;
  - composition is associative;
  - each object has an identity morphism.

- A small catergory means the collection of objects is a set.

Examples

Let G be a group. Then G is a groupoid with

- $G^{(2)} = G \times G$
- $s(\gamma) = \gamma^{-1}\gamma = e$
- $r(\gamma) = \gamma\gamma^{-1} = e$
- $G^{(0)} = \{e\}$

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Lemma

The following are equivalent:

1. *G is a group;*
2. *G is a groupoid and $G^{(0)}$ is a singleton;*
3. *G is a groupoid and $G^{(2)} = G \times G$.*

Examples

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- $G^{(2)} = \{((h_2, h_1 \cdot x), (h_1, x))\}$
- Composition: $(h_2, h_1 \cdot x)(h_1, x) = (h_2 h_1, x)$
- Inverse: $(h, x)^{-1} = (h^{-1}, h \cdot x)$
- $G^{(0)} = \{e\} \times X \equiv X$

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- For $(x, y) \in Q$,
  - $s((x, y)) = (y, x)(x, y) = (y, y) \in Q$
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Examples

There are also groupoids associated to:

- directed graphs, (lecture 2)
- higher-rank graphs, (lecture 2)
- topological graphs,

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- directed graphs, (lecture 2)
- higher-rank graphs, (lecture 2)
- topological graphs,
- categories of paths,
- group actions,
- partial group actions,
- self-similar groups,
- groups acting on graphs and
- inverse semigroups
- inverse semigroup actions.

Topological motivation: discrete groups

- Recall that if H is a group with identity e, then for each $h \in H$ we have

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- In a discrete group, each singleton $\{h\}$ is a compact open set.

- The collection of all singletons is a basis for the discrete topology.

Ample Groupoids

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2. We a say a topological groupoid is ample if the topology of G has a basis of compact open bisections.

The key propoerty in condition (1) is that the source is injective on B.

Any groupoid with the discrete topology is ample.

- We always assume the unit space is Hausdorff.
- A Hausdorff topological space that has a basis of compact open sets is a (trivial) groupoid that consists only of its unit space.
- We do not necessarily assume the topology outside of the unit space is Hausdorff.

Example: the two-headed snake

- Consider the group bundle

$$G = \bigsqcup_{\alpha \in I} G_\alpha \quad \text{where}$$

$I \subseteq [0, 1]$ is the Cantor set,

G_α is the trivial group for each nonzero α and

$G_0 = \mathbb{Z}_2$.

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- G is ample, $G^{(0)}$ is Hausdorff but G is not.

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Examples (groupoids that are NOT ample)

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- a topological space that does not have a basis of compact open sets

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Examples (ample groupoids)

- discrete space
- Cantor set
- discrete group
- discrete groupoid
- a discrete group acting on a totally disoncted space
- the path groupoid G_E associated to a directed graph E (Lecture 2)
- ...and many more

Isotropy

- Let G be a groupoid.
- For each $u \in G^{(0)}$, define $uGu := \{\gamma \in G : s(\gamma) = r(\gamma) = u.\}$
- Each uGu is a group with identity u called the isotropy group at u.
    - All elements in uGu are composable and since uGu is closed under the operation, we have a binary operation.

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  - For any $\gamma \in uGu$, $u\gamma = r(\gamma)\gamma = \gamma = \gamma s(\gamma) = \gamma u$.
  - If $\gamma \in uGu$, the $\gamma^{-1} \in uGu$.

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- If G is the groupoid associated to a group H acting on a set X, then $G^{(0)} \equiv X$. For $x \in X$, xGx is the stability subgroup at x.

- If G is an equivalence relation on X, then for every $x \in X$ we have $xGx = \{(x, x)\}$.

Ample groupoids and inverse semigroups

- A S set with an associative binary operation is called a semigroup.
- Furthermore, if for every $s \in S$ there exists a unique $s^* \in S$ such that $ss^*s = s$ and $s^*ss^* = s^*$, then S is an inverse semigroup.

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Proposition

Let G be an ample groupoid and write $\mathcal{B}$ for the collection of all compact open bisections. For $B, D \in \mathcal{B}$ define

$$BD := \{bd : b \in B, d \in D \textit{ and } s(b) = r(d)\}.$$

Then $\mathcal{B}$ is an inverse semigroup where $D^* := D^{-1}$.

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One can check that the operation is well-defined. (Uses that s is a local homeomorphism.) Associativity follows from the associativity in G.

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Fix $D \in \mathcal{B}$. We have

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\begin{aligned}
&DD^{-1}D \\
&= \{ab^{-1}c : a, b, c \in D, s(a) = r(b^{-1}) = s(b), s(b^{-1}) = r(b) = r(c)\} \\
&= \{aa^{-1}a : a \in D\} \text{ because } s \text{ and } r \text{ are injective on } D \\
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Lastly, we must show uniqueness: if there is a compact open bisection M such that $DMD = D$ and $MDM = M$, then $M = D^{-1}$.

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Lastly, we must show uniqueness: if there is a compact open bisection M such that $DMD = D$ and $MDM = M$, then $M = D^{-1}$. EXERCISE

□

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- Inverse semigroups come with a natural partial order and using this we can construct the groupoid of filters, the groupoid of ultrafilters and the groupoid of tight filters.
- Sometimes constructed as groupoids of germs but the filter and germ approaches are equivalent.
- By duality, we mean we have the following for ample groupoids:

$$G \implies \text{inverse semigroup } \mathcal{B} \implies \text{tight filter groupoid} \cong G$$

The End