Abstract. Białkowski, Erdmann and Skowroński classified those indecomposable self-injective algebras for which the Nakayama shift of every (non-projective) simple module is isomorphic to its third syzygy. It turned out that these are precisely the deformations, in a suitable sense, of preprojective algebras associated to the simply laced \(ADE\) Dynkin diagrams and of another graph \(L_n\), which also occurs in the Happel-Preiser-Ringel classification of subadditive but not additive functions. In this paper we study these deformed preprojective algebras of type \(L_n\) via their Külshammer spaces, for which we give precise formulae for their dimensions. These are known to be invariants of the derived module category, and even invariants under stable equivalences of Morita type. As main application of our study of Külshammer spaces we can distinguish many (but not all) deformations of the preprojective algebra of type \(L_n\) up to stable equivalence of Morita type, and hence also up to derived equivalence.

1. Introduction

Preprojective algebras have been introduced by Gelfand and Ponomarev [11] and nowadays occur prominently in various areas in mathematics. For a quiver (i.e. a finite directed graph) \(Q\) its preprojective algebra is defined by the following process: to any arrow \(a\) in \(Q\) which is not a loop introduce a new arrow \(\bar{a}\) in the opposite direction; for a loop \(a\) set \(\bar{a} := a\), leading to a new quiver \(\overline{Q}\). Then the preprojective algebra \(P(Q)\) of type \(Q\) over a field \(K\) is defined by the quiver with relations \(K\overline{Q}/I\) where the ideal is generated by the relations, one for each vertex \(v\) in \(Q\), of the form \(\sum_{s(a)=v} a\bar{a}\), where \(s(a)\) denotes the starting vertex of the arrow \(a\). Note that the preprojective algebra is independent of the orientation of the quiver \(Q\). The preprojective algebra for a quiver associated to a tree is known to be finite-dimensional if and only if the quiver \(Q\) is a disjoint union of some orientations of simply laced \(ADE\) Dynkin diagrams. The finitely generated modules of the preprojective algebras for \(ADE\) Dynkin quivers have remarkable homological properties. Namely, by a result of Schofield [23] each non-projective indecomposable module has \(\Omega\)-period at most 6, where \(\Omega\) denotes Heller’s syzygy operator; for proofs see [1], [6], [9].

In an attempt to characterise those selfinjective finite-dimensional algebras which share these remarkable periodicity properties, Białkowski, Erdmann and Skowroński introduced in [3] deformations of the preprojective algebras of \(ADE\) Dynkin quivers and of an additional graph of type \(L_n\) of the following form

\[
\begin{array}{ccccccc}
\circ & \bullet & \bullet & \cdots & \bullet & \circ
\end{array}
\]

which already occurred in the Happel-Preiser-Ringel classification [12] of subadditive but not additive functions.

The deformations \(P_f(Q)\) are obtained by perturbing the usual preprojective relation \(\sum_{s(a)=v} a\bar{a}\) at one particular vertex by adding a certain polynomial expression \(f\). It turns out that proper deformations occur only for the diagrams of types \(D\), \(E\), and \(L\). For more details on the actual relations we refer to [3, Section 3]. The Białkowski-Erdmann-Skowroński deformations are different.
and should not be confused with the deformed preprojective algebras of Crawley-Boevey and Holland [7].

The main result of Białkowski, Erdmann and Skowroński gives the following surprising classifica-
tion of selfinjective algebras sharing the periodicity properties of preprojective algebras of Dynkin
type.

**Theorem.** ([3, Theorem 1.2]) Let $\Lambda$ be a basic, connected, finite-dimensional, selfinjective algebra
over an algebraically closed field. Then the following statements are equivalent:

(i) $\Lambda$ is isomorphic to a deformed preprojective algebra $P^f(Q)$ for a quiver of type ADE or $L$.

(ii) $\Omega^3(S) \cong \nu^{-1}S$ for every non-projective simple right $\Lambda$-module $S$, where $\nu$ is the Nakayama
transformation.

In our present paper we shall study the deformed preprojective algebras of type $L_n$ in the
Białkowski-Erdmann-Skowroński sense. Let us start by giving a precise definition of these algebras.

Let $K$ be a field, let $p(X) \in K[X]$ be a polynomial and let $n \in \mathbb{N}$. Then let $L_n^p$ be the $K$-algebra
given by the following quiver with $n$ vertices

subject to the following relations

$$a_i \pi_i + \pi_{i-1} a_{i-1} = 0 \text{ for all } i \in \{1, \ldots, n-2\},$$

$$\pi_{n-2} a_{n-2} = 0, \quad \epsilon^{2n} = 0, \quad \epsilon^2 + a_0 \epsilon + \epsilon^3 p(\epsilon) = 0.$$ 

These algebras are the deformed preprojective algebras of type $L_n$, in the sense of Białkowski, Erd-
mann and Skowroński [3]. Note that the usual preprojective relations are deformed only at the vertex 0.

More details on these algebras are collected in Section 3 below. In particular we determine their
Cartan matrices and we provide an explicit $K$-basis of the algebra given by a set of paths in the
quiver. Moreover, we determine explicitly the centre and the commutator subspace of the deformed
preprojective algebras of type $L_n$.

An important structural property is that all deformed preprojective algebras of type $L_n$ are
symmetric algebras. This is a yet unpublished result of Białkowski, Erdmann and Skowroński [4];
since we build on it in the present paper we give an independent proof of this fact for the sake of
completeness (cf. Section 3.2).

It is a subtle question for which deformation polynomials $p$ the deformed preprojective algebras
$L_n^p$ become isomorphic. We have been informed by Skowroński (cf. also the talk of Białkowski
in Tokyo at the ICRA XIV) that over a field $K$ of characteristic different from 2 all deformed
preprojective algebras of type $L_n$ are isomorphic. However, in characteristic 2 the situation is
more complex; Białkowski, Erdmann and Skowroński have given a series of pairwise non-isomorphic
deformed preprojective algebras of type $L_n$ (over an algebraically closed field), see [3, Proposition
6.1]. Moreover, they have even announced [2] a complete classification of the deformed preprojective
algebras of type $L_n$ up to Morita equivalence (over an algebraically closed field of characteristic
2); namely, the algebras corresponding to the set of deformation polynomials $p(X) = X^{2j}$ for $j \in
\{0, 1, 2, \ldots, n-1\}$ give a complete classification up to Morita equivalence.

We are not building on this classification in the present paper but we use it as a motivation
for restricting our computations of Külshammer spaces in Section 4 to the case of deformation
polynomials $X^{2j}$.

The Białkowski-Erdmann-Skowroński characterisation of the selfinjective algebras where for each
non-projective simple module the third syzygy is isomorphic to the (inverse) Nakayama transfor-
mation can be seen as a condition on the stable module category; we therefore believe it is natural
to aim at a classification of the deformed preprojective algebras up to stable equivalence or up to
derived equivalence, rather than up to Morita equivalence.

Our main results in this paper provide partial answers to these problems. We are able to distin-
guish several of the deformed preprojective algebras $L_n^{X^{2j}}$ up to stable equivalence of Morita type and
up to derived equivalence. Our main applications in this direction are summarised in the following
result.
Theorem 1.1. Let $K$ be a perfect field of characteristic 2.

(a) If two deformed preprojective algebras $L^p_n$ and $L^q_n$ are stably equivalent of Morita type or derived equivalent, then $n = m$.

(b) For $n \in \mathbb{N}$ let $j, k \in \{0, 1, \ldots, n-1\}$ be different numbers such that \( \{j, k\} \neq \{n-2r, n-2r-1\} \) for every $1 \leq r \leq \left\lfloor \frac{n-2}{2} \right\rfloor$. Then the deformed preprojective algebras $L^X_{n,j}$ and $L^X_{n,k}$ are not stably equivalent of Morita type, and also not derived equivalent.

These results are obtained as a consequence of a detailed study of the Külshammer spaces for the deformed preprojective algebras of type $L$. These spaces have been defined by Külshammer in the 1980’s for symmetric algebras over a field of positive characteristic; we recall briefly the construction and some fundamental properties from [17]. For an algebra $A$ over a field $K$ let $[A, A]$ be the $K$-vector space generated by \( \{ab - ba \in A \mid a, b \in A\} \) and call this space the commutator subspace of $A$. Külshammer defined for a $K$-algebra $A$ over a perfect field $K$ of characteristic $p > 0$ the $K$-vector spaces $T_i(A) := \{ x \in A \mid x^p \in [A, A]\}$ for every integer $i \geq 0$. They form an ascending series

\[ [A, A] = T_0(A) \subseteq T_1(A) \subseteq T_2(A) \subseteq \cdots \subseteq T_i(A) \subseteq T_{i+1}(A) \subseteq \cdots. \]

In [28] it was shown by the second author that for symmetric algebras over a perfect field the codimension of the commutator space of $A$ in $T_i(A)$ is invariant under derived equivalences, and in [18] Liu, Zhou and the second author showed that this codimension is an invariant under stable equivalences of the derived category of $A$. In joint work with Bessenrodt [5] we showed that the codimension of $T_i(A)$ in $A$ is an invariant of the derived category of $A$ for general (not necessarily symmetric) finite dimensional algebras.

The derived invariance of the various codimensions of Külshammer spaces proved already to be very useful to distinguish derived equivalence classes of symmetric algebras, see [13], [14], [15], and also to distinguish stable equivalence classes of Morita type, see [25], [26].

For obtaining our above results on deformed preprojective algebras of type $L_n$ (over a perfect field of characteristic 2) we determine the dimension of their Külshammer spaces $T_i(L^X_{n,j})$; our main result in this direction is the following.

Theorem 1.2. Let $K$ be a perfect field of characteristic 2. Then for every $0 \leq j < n$ we have

(a) $\dim_K T_i(L^X_{n,j}) - \dim_K [L^X_{n,j}, L^X_{n,j}] = n - \max \left( \left\lfloor \frac{2n - (2^{i+1} - 2)(2^{i+1} - 1)}{2^{i+1}} \right\rfloor, 0 \right)$

(b) $\dim_K [L^X_{n,j}, L^X_{n,j}] = \frac{1}{2} n (n - 1)(2n + 5)$.

The paper is organised as follows. In Section 2 we recall some results for selfinjective algebras and we propose a method to compute the centre and the quotient of the algebra modulo the commutator space for selfinjective algebras which we believe should be useful in other situations as well. In Section 3 we study the deformed preprojective algebras $L^p_n$, give a $K$-basis, the Cartan matrix, the commutator space, and the centre of the algebras. In Section 4 we compute the Külshammer spaces $T_n(L^X_{n,j})$ and deduce the main results.

2. Hochschild homology and Nakayama automorphisms of self-injective algebras

In this section we present some general methods to deal with selfinjective algebras, in particular for getting a Nakayama automorphism and related bilinear forms explicitly. Strictly speaking the results of this section are not used in this generality in the rest of this paper since the deformed preprojective algebras of type $L$ are symmetric (we give an independent proof of this result of Białkowski, Erdmann and Skowroński in Section 3.2 below). However, symmetricity of an algebra is usually not easy to verify so that the methods of this section can be used to deal with Külshammer ideals in cases where one only has selfinjectivity; therefore the methods of this section might be of independent interest.

We need to compute rather explicitly in the degree 0 Hochschild homology of self-injective algebras. This needs some theoretical preparations in order to be able to determine a basis of the commutator subspace of the algebras we need to deal with.

2.1. The Nakayama-twisted centre. Let $K$ be a field and let $A$ be a $K$-algebra. We need to get alternative descriptions of the degree 0 Hochschild homology. By definition of Hochschild homology (using the standard Hochschild complex) we have $HH_0(A) \cong A/[A, A]$. 
If $A$ is symmetric, then by definition $A \simeq \text{Hom}_K(A, K)$ as $A$-$A$-bimodules (i.e. as $A \otimes_K A^{op}$-module), and so we get

$$\text{Hom}_K(A/[A, A], K) \simeq \text{Hom}_K(HH_0(A), K) \simeq \text{Hom}_K(A \otimes_{A \otimes_K A^{op}} A, K) \simeq \text{Hom}_{A \otimes_K A^{op}}(A, \text{Hom}_K(A, K)) \simeq \text{Hom}_{A \otimes_K A^{op}}(A, A) \simeq HH^K(A) \simeq Z(A).$$

This chain of isomorphisms is one of the main tools for the proof of the main theorem in [27].

If $A$ is only self-injective we shall give an analogous description. So we need to understand $\text{Hom}_K(A, K)$ as $A \otimes_K A^{op}$-module. If $A$ is a self-injective $K$-algebra then still $A \simeq \text{Hom}_K(A, K)$ as a left $A$-module. Hence, $\text{Hom}_K(A, K)$ is a free left $A$-module of rank 1. Moreover,

$$\text{End}_A(\text{Hom}_K(A, K)) \simeq \text{End}_A(A) \simeq A$$

and so $\text{Hom}_K(A, K)$ is a generator over $A$ with endomorphism ring isomorphic to $A$, hence inducing a Morita self-equivalence of $K$ and so $\text{Hom}_K(A, K)$ is a Nakayama automorphism and is called the Nakayama automorphism.

**Definition 2.2.** Let $\nu \in \text{Aut}_K(A)$ be a self-injective $K$-algebra. Then there is an automorphism $\nu$ of $A$ so that $\text{Hom}_K(A, K) \simeq 1_A \nu$ as $A$-$A$-bimodules. This automorphism is unique up to inner automorphisms and is called the Nakayama automorphism.

For the dual of the degree 0 Hochschild homology we get

$$\text{Hom}_K(A/[A, A], K) \simeq \text{Hom}_K(HH_0(A), K) \simeq \text{Hom}_K(A \otimes_{A \otimes_K A^{op}} A, K) \simeq \text{Hom}_{A \otimes_K A^{op}}(A, \text{Hom}_K(A, K)) \simeq \text{Hom}_{A \otimes_K A^{op}}(A, 1_A \nu) \simeq \{a \in A \mid b \cdot a = a \cdot \nu(b) \text{ for all } b \in A\}$$

where the last isomorphism is given by sending a homomorphism to the image of $1 \in A$.

**Definition 2.2.** Let $A$ be a self-injective $K$-algebra with Nakayama automorphism $\nu$. Then the Nakayama twisted centre is defined to be

$$Z_\nu(A) := \{a \in A \mid b \cdot a = a \cdot \nu(b) \text{ for all } b \in A\}.$$

**Remark 2.3.** (1) The automorphism $\nu$ is unique only up to an inner automorphism. If $\nu$ is inner, let $\nu(a) = u \cdot a \cdot u^{-1}$. Then

$$\{a \in A \mid b \cdot a = a \cdot \nu(b) \text{ for all } b \in A\} = \{a \in A \mid b \cdot a = a \cdot u \cdot b \cdot u^{-1} \text{ for all } b \in A\} = \{a \in A \mid b \cdot (a \cdot u) = (a \cdot u) \cdot b \text{ for all } b \in A\} = \{a \in A \mid a \cdot u \in Z(A)\} = Z(A) \cdot u^{-1}$$

and likewise the twisted centres with respect of two different Nakayama automorphisms differ by multiplication by a unit.

(2) In general the Nakayama twisted centre will not be a ring. However, if $z \in Z(A)$ and $a \in Z_\nu(A)$ then

$$b \cdot za = zba = za \cdot \nu(b)$$

and $za \in Z_\nu(A)$. Hence $Z_\nu(A)$ is a $Z(A)$-submodule of $A$. The module structure does not depend on the chosen Nakayama automorphism, up to isomorphism of $Z(A)$-modules.

We summarise the above discussion in the following Lemma.

**Lemma 2.4.** If $A$ is a self-injective $K$-algebra, then there is an automorphism $\nu$ of $A$, unique up to an inner automorphism so that $\text{Hom}_K(A, K) \simeq 1_A \nu$ as an $A$-$A$-bimodule and $\text{Hom}_K(HH_0(A), K) \simeq Z_\nu(A)$ as $Z(A)$-modules.

The selfinjective algebra $A$ is symmetric if and only if the Nakayama automorphism $\nu$ is inner.
Remark 2.5. (1) The automorphism \( \nu \) is the well-known Nakayama automorphism. (The diligent reader might observe that we are dealing with left modules while originally Nakayama in [20] dealt with right modules, so our \( \nu \) would be the inverse of the original Nakayama automorphism.)

(2) Using that \( HH_0(A) \cong A/[A, A] \), the dimension of the commutator subspace of a selfinjective algebra \( A \) can therefore be expressed as

\[
\dim_K[A, A] = \dim_K A - \dim_K Z_{\nu}(A).
\]

2.2. How to get the Nakayama automorphism explicitly. Let \( K \) be a field and let \( A \) be a self-injective \( K \)-algebra. In order to compute the Nakayama automorphism \( \nu \) we need to find an explicit isomorphism \( A \to \text{Hom}_K(A, K) \) as \( A \)-modules. Suppose we get two isomorphisms \( \alpha_1 : A \to \text{Hom}_K(A, K) \) and \( \alpha_2 : A \to \text{Hom}_K(A, K) \). Then \( \alpha_2^{-1} \circ \alpha_1 : A \to A \) is an automorphism of the regular \( A \)-module \( A \). Hence, \( \alpha_1 \) will differ from \( \alpha_2 \) by multiplication by an invertible element \( u \in A \). The corresponding Nakayama automorphisms \( \nu_1 \) and \( \nu_2 \) computed from \( \alpha_1 \) and from \( \alpha_2 \) will then differ by the inner automorphism given by conjugation with \( u \). It is therefore sufficient to find one isomorphism \( \alpha : A \to \text{Hom}_K(A, K) \). Given such an isomorphism \( \alpha \) of \( A \)-modules, the form \( \langle x, y \rangle_\alpha := \langle \alpha(y))(x) \rangle \) for \( x, y \in A \) is a non-degenerate associative bilinear form on \( A \).

Let \( \langle , \rangle : A \times A \to K \) be a non-degenerate associative \( K \)-bilinear form on \( A \) (which exists since \( A \) is self-injective), then we get a vector space isomorphism

\[
A \overset{\alpha}{\to} \text{Hom}_K(A, K), \quad a \mapsto \langle - , a \rangle.
\]

Lemma 2.6. A non-degenerate associative bilinear form \( \langle , \rangle : A \times A \to K \) induces an isomorphism \( A \to \text{Hom}_K(A, K) \) as \( A \)-modules by mapping \( a \in A \) to the linear form \( A \ni b \mapsto \langle b, a \rangle \in K \).

Proof. By the above discussions the map is an isomorphism of vector spaces. For verifying the module homomorphism property recall the action of \( A \) on the dual space \( \text{Hom}_K(A, K) \); it is given by \( (b \cdot \varphi)(c) = \varphi(cb) \) for all \( b, c \in A \) and all \( \varphi \in \text{Hom}_K(A, K) \). Then, using that the bilinear form is associative we get

\[
\alpha(b \cdot a)(c) = \langle c, b \cdot a \rangle = \langle c, b \rangle a = (b \cdot \alpha(a))(c)
\]

for all \( a, b, c \in A \), so the map is a homomorphism of left \( A \)-modules.

Proposition 2.7. Let \( K \) be a field and let \( A \) be a self-injective \( K \)-algebra. Then the Nakayama automorphism \( \nu \) of \( A \) satisfies \( \langle a, b \rangle = \langle b, \nu(a) \rangle \) for all \( a, b \in A \), and any automorphism satisfying this formula is a Nakayama automorphism.

Proof. There is a non-degenerate associative bilinear form on \( A \), which induces an isomorphism between \( A \) and the linear forms on \( A \) as \( A \)-modules by Lemma 2.6. The isomorphism gives an isomorphism of \( A \)-\( A \)-bimodules of \( 1_A \) and \( \text{Hom}_K(A, K) \) by

\[
1_A \overset{\varphi}{\to} \text{Hom}_K(A, K), \quad a \mapsto \langle - , a \rangle = \varphi(a).
\]

By the twisted bimodule action on \( 1_A \), we have that \( \varphi(1) \cdot a = \varphi(1 \cdot a) = \varphi(a) \) and \( b \cdot \varphi(1) = \varphi(b \cdot 1) = \varphi(b) \). Since for \( f \in \text{Hom}_K(A, K) \) the \( A \)-\( A \)-bimodule action on \( \text{Hom}_K(A, K) \) is given by \( (fa)(b) = f(ab) \) and \( (af)(b) = f(ba) \) for all \( a, b \in A \), one gets

\[
\langle a, b \rangle = (\varphi(b))(a) = (b \cdot \varphi(1))(a) = \varphi(1)(ab) = (\varphi(1) \cdot a)(b) = \varphi(\nu(a))(b) = \langle b, \nu(a) \rangle.
\]

Hence, the Nakayama automorphism has the above property. Conversely, if an automorphism \( \nu \) satisfies \( \langle a, b \rangle = \langle b, \nu(a) \rangle \) for all \( a, b \in A \), then the mapping \( A \to \text{Hom}_K(A, K) \) given by \( a \mapsto \langle - , a \rangle \) gives an isomorphism of \( A \) and \( \text{Hom}_K(A, K) \) as \( A \)-modules, inducing the element \( 1_A \nu \) in the Picard group of \( A \).

We shall later need such a bilinear form explicitly. The following very useful result can be found in [28, Proposition 2.15]; see also [15, Proposition 3.1] for a proof in the case of weakly symmetric algebras.

Proposition 2.8. Let \( A = KQ/I \) be a self-injective algebra given by the quiver \( Q \) and ideal of relations \( I \), and fix a \( K \)-basis \( B \) of \( A \) consisting of pairwise distinct non-zero paths of the quiver \( Q \). Assume that \( B \) contains a basis of the socle \( \text{soc}(A) \) of \( A \). Define a \( K \)-linear mapping \( \psi \) on the basis elements by

\[
\psi(b) = \begin{cases} 
1 & \text{if } b \in \text{soc}(A) \setminus \{0\} \\
0 & \text{otherwise}
\end{cases}
\]
for $b \in \mathcal{B}$. Then an associative non-degenerate $K$-bilinear form $\langle \cdot, \cdot \rangle$ for $A$ is given by $\langle x, y \rangle := \psi(xy)$.

**Remark 2.9.** The above bilinear form is in general not symmetric, even if the algebra $A$ is symmetric. For explicit examples we refer to [14, Section 4, proof of main theorem, part (3)] and [28].

Actually, this form is basically the only possible form, at least for finite dimensional basic selfinjective algebras over an algebraically closed field $K$.

**Proposition 2.10.** Let $A$ be a finite dimensional basic selfinjective $K$-algebra over an algebraically closed field $K$. Then for every non degenerate associative bilinear form $\langle \cdot, \cdot \rangle : A \times A \rightarrow K$ there is a $K$-basis $\mathcal{B}$ containing a $K$-basis of the socle so that $\langle x, y \rangle = \psi(xy)$ where

$$
\psi(b) = \begin{cases} 
1 & \text{if } b \in \text{soc}(A) \setminus \{0\} \\
0 & \text{otherwise}
\end{cases}
$$

for $b \in \mathcal{B}$.

**Proof.** Given an associative bilinear form $\langle \cdot, \cdot \rangle : A \times A \rightarrow K$ there is a linear map $\psi : A \rightarrow K$ defined by $\psi(x) := \langle 1, x \rangle$ and for any $x, y \in A$ one gets $\langle x, y \rangle = \langle 1, xy \rangle = \psi(xy)$. Hence $\psi$ determines the associative bilinear map and the associative bilinear map determines $\psi$.

The algebra is basic and so the socle of $A$ is a direct sum of pairwise non-isomorphic one-dimensional simple $A$-modules. Let $\{s_1, \ldots, s_n\} \in A$ so that $s_iA$ is simple for every $i \in \{1, 2, \ldots, n\}$ and so that $\text{soc}(A) = \langle s_1, \ldots, s_n \rangle_k$.

Given an associative non degenerate bilinear form $\langle \cdot, \cdot \rangle$ then $\langle \cdot, s_i \rangle : A \rightarrow K$ is a non zero linear form on $A$, since the bilinear form is non degenerate. Hence there is an element $a \in A$ so that $\langle a, s_i \rangle \neq 0$. Now, by the Wedderburn-Malcev theorem, there is an element $\rho \in \text{rad}(A)$ so that $a = \sum_{i=1}^{n} \lambda_i e_i + \rho$ for scalars $\lambda_i \in K$, and where $e_i^2 = e_i$ is an indecomposable idempotent of $A$, where $e_{\nu^{-1}(i)} s_i = s_i$, and where $e_{\nu^{-1}(i)} s_i = 0$ for $j \neq i$. Hence,

$$
\langle a, s_i \rangle = \langle 1, a s_i \rangle = \langle 1, \lambda_{\nu^{-1}(i)} s_i \rangle = \lambda_{\nu^{-1}(i)}
$$

We replace $s_i$ by $\lambda_{\nu^{-1}(i)} s_i$ and get $\langle 1, s_i \rangle = 1$. Take a $K$-basis $\mathcal{B}$ of $\ker(\langle \cdot, s_i \rangle)$ in $A e_{\nu^{-1}(i)}$. Then, since $A = \bigoplus_{j=1}^{n} Ae_j$,

$$
\mathcal{B} := \bigcup_{i=1}^{n} \mathcal{B}_i \cup \{s_1, s_2, \ldots, s_n\}
$$

is a $K$-basis of $A$ satisfying the hypotheses of Proposition 2.8. Moreover, if $xy \in \mathcal{B}$, then there is a unique $e_i^2 = e_i$ so that $xye_i = xy$, and so

$$
\langle x, y \rangle = \langle 1, xy \rangle = \sum_{i=1}^{n} \langle 1, xy e_i \rangle = \begin{cases} 
1 & \text{if } xy \in \text{soc}(A) \\
0 & \text{else}
\end{cases}
$$

This shows the statement. \qed

3. **Deformed preprojective algebras of type $L$**

3.1. **$K$-bases of the deformed preprojective algebras of type $L$** The aim of this section is to obtain an explicit vector space basis for any deformed preprojective algebra of type $L$ and to deduce some structural properties. In particular we shall get the Cartan matrices and provide an independent proof of a result of Białkowski, Erdmann and Skowroński [4] that the deformed preprojective algebras of type $L$ are symmetric algebras.

For the convenience of the reader we start by recalling from the introduction the definition of the deformed preprojective algebras of type $L$.

Let $K$ be a field. For any $n \in \mathbb{N}$ and any polynomial $p(X) \in K[X]$ let $L_n^p$ be the $K$-algebra given by the following quiver with $n$ vertices $0, 1, \ldots, n-1$ of the form

```
0 — 1 — 2 — 3 — ... — n-2 — n-1
```

where $\epsilon$ and $\rho$ are such that $\epsilon_0 = a_{00}, \epsilon_1 = a_{11}, \epsilon_2 = a_{22}, \ldots, \epsilon_{n-2} = a_{n-2,n-2}, \epsilon_{n-1} = a_{n-1,n-1}$ and $\rho_0 = a_{01}, \rho_1 = a_{12}, \rho_2 = a_{23}, \ldots, \rho_{n-2} = a_{n-2,n-1}, \rho_{n-1} = a_{n-1,n-2}$.
subject to the following relations

\[ a_s \pi_s + \pi_{s-1}a_{s-1} = 0 \text{ for all } s \in \{1, \ldots, n-2\} \]

\[ \pi_{n-2}a_{n-2} = 0, \quad \epsilon^{2n} = 0, \quad \epsilon^2 + a_0 \pi_0 + \epsilon^3 p(\epsilon) = 0. \]

Our first aim is to give a \( K \)-basis of the algebra \( L^p_n \). We start by providing a generating set. Considering a path starting at the vertex \( i \) and ending at the vertex \( j \), we have two cases.

Firstly suppose that the path does not contain \( \epsilon \).

If \( i < j \), then using the relations \( a_s \pi_s + \pi_{s-1}a_{s-1} = 0 \) for all \( s \in \{1, \ldots, n-2\} \) in \( L^p_n \) we may replace the path, up to a sign, by one of the following elements of \( L^p_n \):

- the path \( a_i a_{i+1} \ldots a_{j-1} \)
- or the path \( a_i a_{i+1} \ldots a_{j-1}a_j \ldots a_j \pi_{\ell-1} \ldots \pi_j \) (for some \( j \leq \ell \leq n-2 \))
- or by 0.

In fact up to a sign we can order the arrows in the path so that all \( a_i \)'s come first and then all the \( \pi_r \)'s; we can do this unless we hit a subpath \( \pi_{n-2}a_{n-2} \ldots \pi_0 \) in which case the path becomes 0 in \( L^p_n \).

Similarly, if \( i \geq j \) we may replace the given path, up to a sign, by one of the following elements:

- the path \( \pi_i \pi_{i-2} \ldots \pi_j \)
- or the path \( a_i a_{i+1} \ldots a_{j} \pi_{\ell-1} \ldots \pi_j \) (for some \( i \leq \ell \leq n-2 \))
- or by 0.

Secondly, suppose the path contains \( \epsilon \).

Using the relations \( \epsilon^2 + \epsilon^3 p(\epsilon) + a_0 \pi_0 = 0 \) and \( \epsilon^{2n} = 0 \) we may replace any path containing powers of \( \epsilon \) by a linear combination of paths containing only \( \epsilon \). Moreover, using the relations \( a_s \pi_s + \pi_{s-1}a_{s-1} = 0 \) for all \( s \in \{1, \ldots, n-2\} \) and the fact that \( a_0 \pi_0 \) commutes with \( \epsilon \) (because \( a_0 \pi_0 = -(\epsilon^2 + \epsilon^3 p(\epsilon)) \)) we can move all \( a_i \)'s in the path to the right of \( \epsilon \). (Note that by combining these two reductions we can indeed guarantee that in each path occurring in the linear combination \( \epsilon \) occurs only once.) Thus the given path represents the same element in \( L^p_n \) as a linear combination of paths of the following forms

- \( \pi_{i-1} \pi_{i-2} \ldots \pi_0 \epsilon a_0 a_1 \ldots a_j \)
- \( \pi_{i-1} \pi_{i-2} \ldots \pi_0 \epsilon a_0 a_1 \ldots a_{j-1} a_j \pi_{\ell-1} \ldots \pi_j \) (for some \( j \leq \ell \leq n-2 \)).

If \( i < j \) these paths are all non-zero and they will be part of the basis to be given below. However, if \( i \geq j \) some of the paths of the latter type vanish, so we shall now derive a different expression for these.

To this end observe that by using the relations \( a_s \pi_s + \pi_{s-1}a_{s-1} = 0 \) we can successively move the \( \pi_r \)'s to the left and obtain

\[ \pi_{i-1} \pi_{i-2} \ldots \pi_0 \epsilon a_0 a_1 \ldots a_i \pi_{\ell-1} \ldots \pi_j = \pm \pi_{i-1} \pi_{i-2} \ldots \pi_0 \epsilon (a_0 \pi_0)^{\ell-j+1} a_0 a_1 \ldots a_{j-1}. \]

Moreover, using that \( a_0 \pi_0 \) commutes with \( \epsilon \) and then moving the \( a_r \)'s to the left we get

\[
\begin{align*}
\pi_{i-1} \pi_{i-2} \ldots \pi_0 \epsilon a_0 a_1 & \ldots a_{\ell-1} a_\ell \pi_{\ell-1} \ldots \pi_j \\
= \pm & \pi_{i-1} \pi_{i-2} \ldots \pi_0 (a_0 \pi_0)^{\ell-j+1} \epsilon a_0 a_1 \ldots a_{j-1} \\
= \pm & \pi_{i-1} \pi_{i-2} \ldots \pi_0 (a_0 \pi_0)^{\ell-j+1} \epsilon a_0 a_1 \ldots a_{j-1} \\
\end{align*}
\]

\[
= \begin{cases} 
\pi_{i-1} \pi_{i-2} \ldots \pi_0 \epsilon a_0 a_1 \ldots a_i \pi_{\ell-1} \ldots \pi_j 
& \text{if } i + \ell - j \leq n - 2 \\
0 
& \text{else}
\end{cases}
\]

The following result provides explicit vector space bases for the deformed preprojective algebras \( L^p_n \) of type \( L \). Note that the bases do not involve the deformation polynomial \( p \), i.e. the bases is independent of the polynomial.

**Proposition 3.1.** A \( K \)-basis of \( L^p_n \) is given by the following paths between the vertices \( i \) and \( j \), where \( i, j \in \{0,1,\ldots,n-1\} \).

1. \( a_i a_{i+1} \ldots a_{j-1} \) for \( i < j \)
2. \( a_i a_{i+1} \ldots a_{j-1} a_j \ldots a_j \pi_{\ell-1} \ldots \pi_j \) for \( i < j \) and some \( j \leq \ell \leq n-2 \)
3. \( \pi_{i-1} \pi_{i-2} \ldots \pi_j \) for \( i \geq j \)
4. \( a_i a_{i+1} \ldots a_i \pi_{\ell-1} \ldots \pi_{j-1} \ldots \pi_j \) for \( i \geq j \) and some \( i \leq \ell \leq n-2 \)
5. \( \pi_{i-1} \pi_{i-2} \ldots \pi_0 \epsilon a_0 a_1 \ldots a_{j-1} \) for any \( i, j \)
6. \( \pi_{i-1} \pi_{i-2} \ldots \pi_0 \epsilon a_0 a_1 \ldots a_{j-1} a_j \pi_{\ell-1} \ldots \pi_j \) for \( i < j \) and some \( j \leq \ell \leq n-2 \)
7. \( a_i a_{i+1} \ldots a_i \pi_{\ell-1} \ldots \pi_0 \epsilon a_0 a_1 \ldots a_{j-1} \) for \( i \geq j \) and some \( i \leq \ell \leq n-2 \)
Remark 3.2. (1) In type (3) the case $i = j$ yields an empty product which has to be interpreted as the trivial paths $e_i$ for every vertex $i$.

(2) The longest paths in this basis of $L^p_n$ are of length $2n - 1$, occurring in (5) for $i = j = n - 1$ and in (7) for $i = j \in \{0, 1, \ldots, n - 2\}$ and $l = n - 2$, respectively. These elements span the socles of the projective indecomposable modules corresponding to the vertices $i \in \{0, 1, \ldots, n - 1\}$.

(3) The socle element of the projective indecomposable module corresponding to the vertex $0$ can also be expressed in terms of powers of the loop $\epsilon$ (recall that $\epsilon^{2n} = 0$ in the algebra $L^p_n$). In fact it is not hard to check that we have $\epsilon^{2n-1} = a_0a_1 \ldots a_{n-2} \bar{a}_{n-2} \bar{a}_{n-3} \ldots \bar{a} \bar{a}$. (Note that lower powers of $\epsilon$ are not necessarily occurring as paths in the above list, but are linear combinations of these, the precise shape depending on the deformation polynomial $p$.) In particular, the socle elements are precisely the basis elements having length $2n - 1$.

(4) The above basis seems very suitable for making the following inductive proof work. However, later in the paper we will also use slightly different bases involving powers of the loop $\epsilon$.

Proof. The above discussion proves that the given elements form a generating set. We need to show that these elements are linearly independent. Since the defining relations of the algebra $L^p_n$ are relations between closed paths, we may suppose that a linear combination of paths starting at $i$ and ending at $j$ is 0. By symmetry we may suppose that $i \geq j$ and hence we get a linear combination

$$0 = \nu_0 \cdot \bar{a}_{i-1} \bar{a}_{i-2} \ldots \bar{a}_j + \nu_1 \cdot \bar{a}_{i-1} \bar{a}_{i-2} \ldots \bar{a}_0 \epsilon a_0 a_1 \ldots a_{j-1} +$$

$$+ \sum_{\ell=1}^{n-2} \lambda_\ell \cdot a_i a_{i+1} \ldots a_{\ell+1} \bar{a} \bar{a}_1 \ldots \bar{a}_i \epsilon a_0 a_1 \ldots a_{j-1}$$

with scalars $\nu_0, \nu_1, \lambda_\ell, \mu_\ell \in K$. Note that the paths occurring have the following lengths: length $i - j$ for the summand of type (3) with coefficient $\nu_0$, length $i + j + 1$ for the summand of type (5) with coefficient $\nu_1$, length $2\ell - i - j + 2$ for the summand of type (4) with coefficient $\lambda_\ell$, and length $2\ell - i + j + 3$ for the summand of type (7) with coefficient $\mu_\ell$.

Denote by $J = J(L^p_n)$ the two-sided ideal of $L^p_n$ generated by the arrows of the quiver.

From the lengths of the paths we observe that all summands in the above expression are contained in $J^{i-j+1}$, except the one with coefficient $\nu_0$. So considering the above equation modulo $J^{i-j+1}$ we can deduce that $\nu_0 = 0$.

The remaining summands are given by paths which pass through the vertex $i + 1$ (recall that $\ell \geq i$), except for the summand with coefficient $\nu_1$. So considering the above equation modulo the two-sided ideal $L^p_{\ell e_{i+1} L^p_n}$ we get that also $\nu_1 = 0$.

Hence we are left to consider the equation

$$0 = \sum_{\ell=1}^{n-2} a_i a_{i+1} \ldots a_{\ell+1} \bar{a}_1 a_0 a_1 \ldots a_{j-1} \left(\lambda_\ell + \mu_\ell \bar{a}_j \bar{a}_{j-2} \ldots \bar{a} a_0 a_1 \ldots a_{j-1}\right).$$

We shall prove by induction on $\ell$ that all coefficients are 0. Observe that for each $\ell = i, \ldots, n - 2$ the paths with coefficients $\lambda_\ell$ and $\mu_\ell$ pass through the vertex $\ell + 1$ but not through the vertex $\ell + 2$.

For $\ell = i$ we consider the above equation modulo the two-sided ideal $L^p_{\ell e_{i+2} L^p_n}$ and obtain that

$$0 = \lambda_i \cdot a_i \bar{a}_j \bar{a}_{j-2} \ldots \bar{a} a_0 a_1 \ldots a_{j-1} + \mu_i \cdot a_i \bar{a} a_{i-1} \ldots \bar{a}_j \ldots \bar{a} \bar{a} a_0 a_1 \ldots a_{j-1}$$

Since the first path is strictly shorter than the second we again consider the equation modulo a suitable power of the ideal $J$ and can deduce that $\lambda_i = 0$, and then also that $\mu_i = 0$.

By a completely analogous argument we can immediately deduce inductively that all coefficients $\lambda_\ell$ and $\mu_\ell$ are 0.

Remark 3.3. From Proposition 3.1 one can derive the Cartan matrix of the deformed preprojective algebras of type $L_n$. Since the basis is independent of the deformation polynomial $p$ the Cartan matrix of $L^p_n$ and $L^p_n$ coincide; this has already been observed by Białkowski, Erdmann and Skowroński [3, Lemma 3.2]. The Cartan matrix $C_n$ of the deformed preprojective algebras of type $L_n$ actually has
the following form

\[
C_n = 2 \cdot \begin{pmatrix}
  n & n-1 & \ldots & 2 & 1 \\
  n-1 & n-1 & \ldots & 2 & 1 \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  2 & 2 & \ldots & 2 & 1 \\
  1 & 1 & \ldots & 1 & 1
\end{pmatrix}
\]

From this shape one easily computes that the determinant of the Cartan matrix is \(\det C_n = 2^n\) for all \(n \in \mathbb{N}\). Moreover, the vector space dimension of \(L_n^p\) is \(\frac{1}{2}n(n+1)(2n+1)\) for all \(n \in \mathbb{N}\).

3.2. Deformed preprojective algebras of type L are symmetric. The aim of this section is to show that for any deformation polynomial \(p \in K[X]\) and any \(n \in \mathbb{N}\) the deformed preprojective algebra \(L_n^p\) is a symmetric algebra. This is a result of Białkowski, Erdmann and Skowroński [4], as announced in [2]. Since this result is not yet available in the literature we include an independent proof in this section for the convenience of the reader.

According to Lemma 2.4 it suffices to show that the identity is a Nakayama automorphism for the algebra \(L_n^p\). Recall from Proposition 2.7 that a Nakayama automorphism \(\nu\) for a self-injective algebra \(A\) over a field \(K\) is characterized by the property \(\langle a, b \rangle = \langle b, \nu(a) \rangle\) for all \(a, b \in A\) where \(\langle \ldots \rangle\) is a non-degenerate associative \(K\)-bilinear form on \(A\).

The following general observation turns out to be useful when verifying that a certain automorphism is indeed a Nakayama automorphism; namely, it suffices to check the crucial property on algebra generators of \(A\).

**Lemma 3.4.** Let \(A\) be a self-injective algebra, with a non-degenerate associative \(K\)-bilinear form \(\langle \ldots \rangle\). If an automorphism \(\nu\) of \(A\) satisfies \(\langle a, b \rangle = \langle b, \nu(a) \rangle\) for a set of algebra generators \(\{a_1, \ldots, a_r\}\) and all \(b \in A\) then \(\nu\) is a Nakayama automorphism of \(A\).

**Proof.** Every element of \(A\) can be expressed as a product of the algebra generators. We show that \(\langle a, b \rangle = \langle b, \nu(a) \rangle\) for all \(a, b \in A\) by induction on the length of such an expression for \(a\). For any algebra generator \(a_j\) and \(a, b \in A\) we have

\[
\langle aa_j, b \rangle = \langle a, a_j b \rangle = \langle a_j b, \nu(a) \rangle = \langle a_j b, \nu(a_j) \rangle = \langle b, \nu(a_j a) \rangle
\]

where for the first, third and fifth equality we used the associativity of the form, for the second we used the induction hypotheses, for the fourth equality we used the assumption on \(\nu\) for algebra generators, and the last equality holds because \(\nu\) is an algebra homomorphism. \(\square\)

Recall from Proposition 2.8 the construction of an associative non-degenerate bilinear form on a self-injective algebra, depending on the choice of a suitable basis. For a basis \(B\) consisting of non-zero distinct paths and containing a basis of the socle this bilinear form has been defined on basis elements by \(\langle a, b \rangle = \psi(ab)\) where \(\psi(x) = \begin{cases} 1 & \text{if } x \in \text{soc}(A) \cap B \\ 0 & \text{if } x \notin \text{soc}(A) \cap B \end{cases}\).

For our aim of proving that the identity is a Nakayama automorphism for \(L_n^p\) we shall show that the bilinear form \(\langle \ldots \rangle\) corresponding to the basis \(B\) given in Proposition 3.1 is indeed symmetric. By the previous lemma we therefore have to verify that \(\langle a, b \rangle = \langle b, a \rangle\) for every algebra generator \(a \in \{e_0, \ldots, e_{n-1}, e, a_0, \ldots, a_{n-2}, \bar{a}_0, \ldots, \bar{a}_{n-2}\}\) and \(b\) running through the basis of Proposition 3.1.

It is immediate from the definition of the form \(\langle \ldots \rangle\) that \(\langle e, b \rangle = \langle b, e \rangle\); in fact, the value on either side is 1 precisely if \(b\) is a basis element from the socle, and 0 otherwise.

So it remains to deal with the cases where \(a\) is an arrow of the quiver of \(L_n^p\).

We start with the loop \(e\). By definition the value in both \(\langle e, b \rangle\) and \(\langle b, e \rangle\) is 0 unless \(b \in e_0 L_n^p e_0\). By Proposition 3.1, for the latter space a basis is given by the elements \(e_0, e, e_0 a_0 \ldots a_{\ell-1} \bar{a}_0 \ldots \bar{a}_0\) and \(e_0 a_0 \ldots a_{\ell-2} \bar{a}_0 \ldots \bar{a}_0\) where \(0 \leq \ell \leq n - 2\). Using the defining relations \(a_0 \bar{a}_0 = 0\) and \(a_0 a_0 \ldots a_{\ell-1} \bar{a}_0 = - (a_0 \bar{a}_0)^{\ell+1}\) it is not difficult to see that in \(L_n^p\) we have \(a_0 \ldots a_{\ell} \bar{a}_0 \ldots \bar{a}_0 = \pm (a_0 \bar{a}_0)^{\ell+1}\). From this we can deduce, by using the relation \(\epsilon^2 + \epsilon^3 p(\epsilon) + a_0 \bar{a}_0 = 0\), that \(\epsilon\) commutes with every element of \(e_0 L_n^p e_0\). But then we clearly have for all \(b \in e_0 L_n^p e_0\) that

\[
\langle e, b \rangle = \psi(eb) = \psi(be) = \langle b, e \rangle.
\]

We now consider the case \(a = a_r\) (for some \(0 \leq r \leq n - 2\)). Again by definition the value in both \(\langle a_r, b \rangle\) and \(\langle b, a_r \rangle\) is 0 unless \(b \in e_{r+1} L_n^p e_r\). Moreover, for a basis element \(b \in e_{r+1} L_n^p e_r\) the value in both \(\langle a_r, b \rangle\) and \(\langle b, a_r \rangle\) is also 0 unless \(a_r b\) (resp. \(b a_r\)) is a nonzero element in the socle of \(L_n^p\). According to Remark 3.2(2) we know that \(a_r b\) and \(b a_r\) can only be a nonzero element in the socle
if $b$ is a path of length $2n-2$. However, it is immediately checked that the only basis element $b \in e_{r+1}L^p_ne_r$ in Proposition 3.1 of length $2n-2$ is

$$b = a_{r-1}a_{r-2} \ldots a_{n-2}a_{n-3} \ldots a_0 \epsilon a_1 \ldots a_{r-1}.$$  

For this element we have that $a_r b$ is the socle element in $\mathcal{B}$ (of type (7)) corresponding to vertex $r$ and that $b a_r$ is the socle element corresponding to vertex $r+1$ (of type (7) if $r < n-2$ and of type (5) if $r = n-2$), i.e. we can deduce

$$\langle a_r, b \rangle = \psi(a_r b) = 1 = \psi(b a_r) = \langle b, a_r \rangle.$$  

Since by the above remarks in all other cases for $b$ both values $\langle a_r, b \rangle$ and $\langle b, a_r \rangle$ vanish we get the desired statement $\langle a_r, b \rangle = \langle b, a_r \rangle$ for all basis elements $b$ from Proposition 3.1.

Finally we consider the case where $a = \bar{a}_r$ for some $0 \leq r \leq n-2$. This is mainly analogous to the previous case but at a certain point pointed out below one has to be careful. Again by definition the value in both $\langle \bar{a}_r, b \rangle$ and $\langle b, \bar{a}_r \rangle$ is 0 unless $b \in e_rL^p_ne_{r+1}$ and $\bar{a}_r b$ and $b \bar{a}_r$ are nonzero elements in the socle. By Remark 3.2 (2) the products $\bar{a}_r b$ and $b \bar{a}_r$ can only be nonzero elements in the socle if $b$ is a path of length $2n-2$. The only basis element $b \in e_rL^p_ne_{r+1}$ of length $2n-2$ in Proposition 3.1 occurs in type (6) (for $r < n-2$) and in type (5) (for $r = n-2$) and has the form

$$b = \bar{a}_{r-1} \bar{a}_{r-2} \ldots \bar{a}_0 \epsilon a_0 \ldots a_{n-2} \bar{a}_{n-2} \ldots \bar{a}_{r+1}.$$  

Now, when calculating the values of $\langle \bar{a}_r, b \rangle$ and $\langle b, \bar{a}_r \rangle$ one has to be careful since the products

$$\bar{a}_r b = \bar{a}_r \bar{a}_{r-1} \bar{a}_{r-2} \ldots \bar{a}_0 \epsilon a_0 \ldots a_{n-2} \bar{a}_{n-2} \ldots \bar{a}_{r+1} \in e_rL^p_ne_{r+1}$$

and

$$b \bar{a}_r = \bar{a}_{r-1} \bar{a}_{r-2} \ldots \bar{a}_0 \epsilon a_0 \ldots a_{n-2} \bar{a}_{n-2} \ldots \bar{a}_{r+1} \epsilon a_r \in e_rL^p_ne_r$$

are not elements of the basis $\mathcal{B}$ given in Proposition 3.1; the only exception is for $r = n-2$ where $\bar{a}_{n-2}b$ is a basis element of type (5). However, Lemma 3.6 (1) below shows how to express these in terms of the basis $\mathcal{B}$; namely for all $r \in \{0, \ldots, n-2\}$ we have that $\bar{a}_r b$ and $b \bar{a}_r$ are equal (not only up to a scalar!) to the socle elements occurring in the basis $\mathcal{B}$. Hence we obtain that

$$\langle \bar{a}_r, b \rangle = \psi(\bar{a}_r b) = 1 = \psi(b \bar{a}_r) = \langle b, \bar{a}_r \rangle,$$

as desired.

Summarizing our above arguments we have now shown that the associative non-degenerate bilinear form $\langle \ldots \rangle$ corresponding (in the sense of 2.8) to the basis $\mathcal{B}$ of Proposition 3.1 is symmetric. We therefore have given an independent proof of the following result, which is due to Białkowski, Erdmann and Skowroński [4].

**Theorem 3.5.** Let $K$ be a field (of any characteristic). For all $n \in \mathbb{N}$ and every polynomial $p \in K[X]$ the deformed preprojective algebra $L^p_n$ is a symmetric algebra.

We complete this section by providing an auxiliary result on relations in the algebras $L^p_n$; the first part provides the missing calculations in the last part of the above proof, the second part will be used in later sections.

**Lemma 3.6.** The following identities hold in the algebra $L^p_n$.

(a) For all $r \in \{0, \ldots, n-2\}$ we have that

$$\bar{a}_r \bar{a}_{r-1} \bar{a}_{r-2} \ldots \bar{a}_0 \epsilon a_0 \ldots a_{n-2} \bar{a}_{n-2} \ldots \bar{a}_{r+1} = a_r a_{r+1} \ldots a_{n-2} \bar{a}_{n-2} \ldots \bar{a}_{r+1} \bar{a}_r a_0 \ldots \bar{a}_1.$$  

(b) For all $l \in \{0,1,\ldots,n-2\}$ we have that $\bar{a}_{n-2} \bar{a}_{n-3} \ldots \bar{a}_l a_1 = 0$.

**Proof.** (a) We shall use frequently the relations $a_i \bar{a}_s + \bar{a}_{s-1} a_{s-1} = 0$ for $s \in \{1,\ldots,n-2\}$ and carefully keep track of the signs occurring.

In the expression on the left hand side of assertion (a) we start by successively moving $\bar{a}_{n-2}, \ldots, \bar{a}_r$ to the left (but still right of $\epsilon$); note that each such move the gives a minus sign. Setting $c := (n-2)+(n-3)+\ldots+(r+1)+r$ for abbreviation we obtain that

$$\bar{a}_r \bar{a}_{r-1} \bar{a}_{r-2} \ldots \bar{a}_0 \epsilon a_0 \ldots a_{n-2} \bar{a}_{n-2} \ldots \bar{a}_{r+1} = (-1)^c \bar{a}_r \bar{a}_{r-1} \bar{a}_{r-2} \ldots \bar{a}_0 \epsilon (a_0 \bar{a}_0)^{n-r-1} a_0 a_1 \ldots a_{r-1}.$$  

It follows directly from the defining relation $c^2 + c^3 p(\epsilon) + a_0 \bar{a}_0 = 0$ that $\epsilon$ commutes with $a_0 \bar{a}_0$, so the expression on the right hand side of (1) is equal to

$$(-1)^c \bar{a}_r \bar{a}_{r-1} \bar{a}_{r-2} \ldots \bar{a}_0 (a_0 \bar{a}_0)^{n-r-1} \epsilon a_0 a_1 \ldots a_{r-1}.$$  

(2)
The part to the right of $\epsilon$ already has the desired shape. To the left of $\epsilon$ we now successively move the $a_0$’s to the left; for the first $a_0$ we need $r$ such moves and obtain that the expression in (2) equals
\[(3) \quad (\pi_i a_0 a_1 \ldots a_r - a_0 a_1 \ldots a_r) = 0.
\]
For moving the next $a_0$ we need $r + 1$ moves etc and eventually get another sign of $(-1)^e$; more precisely the expression in (3) is equal to
\[(4) \quad (\pi_i a_0 a_1 \ldots a_{n-2} a_{n-3} \ldots a_{n-2} \pi_{n-3} \ldots \pi_{n-1} a_{n-1} a_1 \ldots a_r)\]
where the signs cancel so that this is precisely the right hand side in the assertion of part (a) of the lemma.

(b) We show this by reverse induction on $l$. For $l = n - 2$ this is just the relation $\pi_{n-2} a_{n-2} = 0$. For $l < n - 2$ we use the defining relation $\pi_i a_l = a_{l+1} \pi_{l+1}$ and obtain
\[\pi_{n-2} \pi_{n-3} \ldots \pi_{l+1} a_l = \pi_{n-2} \pi_{n-3} \ldots \pi_{l+1} a_{l+1} \pi_{l+1}\]
where the latter is zero by induction hypothesis. \hfill $\square$

3.3. Linking $L_n^{P_{n+1}}$ and $L_n^{P_n}$. For proving statements about the algebras $L_n^{P_n}$ we shall often argue by induction and then the following result will turn out to be useful. As usual we denote the trivial path of length zero corresponding to the vertex $i$ by $e_i$.

**Lemma 3.7.** For any $n \geq 1$, there is an algebra epimorphism $\pi_n : L_n^{P_{n+1}} \rightarrow L_n^{P_n}$ satisfying
\[\pi_n(e_i) = e_i \text{ for all } 0 \leq i \leq n - 1, \quad \pi_n(a_n) = 0, \quad \pi_n(\epsilon) = \epsilon,\]
\[\pi_n(a_i) = a_i, \quad \pi_n(\pi_i) = \pi_i \text{ for all } 0 \leq i \leq n - 2, \quad \pi_n(a_{n-1}) = 0, \quad \pi_n(\pi_{n-1}) = 0.\]
Moreover, $\pi_n$ induces an algebra isomorphism
\[L_n^{P_{n+1}}/(L_n^{P_{n+1}} e_n L_n^{P_{n+1}}) \simeq L_n^{P_n}.
\]

**Proof.** The map $\pi_n$ is well-defined since the defining relations for the algebra $L_n^{P_{n+1}}$ are clearly verified in $L_n^{P_n}$ (perhaps the only not entirely obvious check is that $\pi_n(a_{n-1} \pi_{n-1} + \pi_{n-2} a_{n-2}) = \pi_n(a_{n-1}) \pi_n(\pi_{n-1}) + \pi_n(\pi_{n-2}) a_{n-2} = 0 + \pi_n a_{n-2}$ which is zero in $L_n^{P_n}$).

For the second statement we need to determine the kernel of $\pi_n$ (since $\pi_n$ is surjective by definition). By definition of $\pi_n$ we have that $L_n^{P_{n+1}} e_n L_n^{P_{n+1}}$ is contained in the kernel. On the other hand, the dimension of the kernel is the difference of the dimensions of the algebras $L_n^{P_{n+1}}$ and $L_n^{P_n}$. These are given in Remark 3.3 and we obtain
\[\dim \ker \pi_n = \frac{1}{3} n(n+1)(n+2)(2n+3) - \frac{1}{3} n(n+1)(2n+1) = 2(n+1)^2.
\]

However, in the basis of $L_n^{P_{n+1}}$ provided in Proposition 3.1, there are already $2(n+1)^2$ basis elements which pass through the vertex $n$, i.e. are contained in $L_n^{P_{n+1}} e_n L_n^{P_{n+1}}$. (More precisely, there are $n$ such paths of type (1), $\frac{n(n-1)}{2}$ of type (2), $n + 1$ of type (3), $\frac{n(n+1)}{2}$ of type (4), $2n + 1$ of type (5), $\frac{n(n-1)}{2}$ of type (6) and $\frac{n(n+1)}{2}$ of type (7), respectively.)

Since $L_n^{P_{n+1}} e_n L_n^{P_{n+1}} \subseteq \ker \pi_n$ and dimensions agree, the second claim of the lemma follows. \hfill $\square$

**Remark 3.8.** Since $\epsilon^h = 0$ in $L_n^{P_n}$, also $\epsilon^{2n}$ is in the kernel of $\pi_n$. The dimension arguments in the above proof thus show that $\epsilon^{2n}$ is contained in $L_n^{P_{n+1}} e_n L_n^{P_{n+1}}$, i.e., $\epsilon^{2n}$ is a linear combination of paths passing through the vertex $n$ (which could also be checked directly).

3.4. Generating the commutator subspace. We start with some computations on the basis elements occurring in Proposition 3.1.

The basis in Lemma 3.1 of $L_n^{P_n}$ is actually a union of bases of $e_i L_n^{P_n} e_j$ for $i, j \in \{0, 1, \ldots, n - 1\}$. We consider the case $i = j$ in Lemma 3.1. Only the basis elements of type (3), (4), (5) and (7) admit $i = j$. Up to signs (which are not essential since we are only interested in generating sets) we have
\[(a_i \pi_i)^2 = \pm a_i a_{i+1} \pi_{i+1} \pi_i
\]
\[(a_i \pi_i)^3 = \pm a_i a_{i+1} a_{i+2} \pi_{i+2} \pi_{i+1} \pi_i
\]
\[\ldots
\]
\[(a_i \pi_i)^{2r+1} = \pm a_i a_{i+1} \ldots a_{i+r} \pi_{i+r} \ldots \pi_{i+1} \pi_i
\]
\[(a_i \pi_i)^{2r+2} = \pm (a_i \pi_i)(a_i a_{i+1} \ldots a_{i+r} \pi_{i+r} \ldots \pi_{i+1} \pi_i)
\]
\[= \pm a_i a_{i+1} \ldots a_{i+r} \pi_{i+r} \ldots \pi_{i+1} \pi_i
\]

We start with some computations on the basis elements occurring in Proposition 3.1.

The basis in Lemma 3.1 of $L_n^{P_n}$ is actually a union of bases of $e_i L_n^{P_n} e_j$ for $i, j \in \{0, 1, \ldots, n - 1\}$. We consider the case $i = j$ in Lemma 3.1. Only the basis elements of type (3), (4), (5) and (7) admit $i = j$. Up to signs (which are not essential since we are only interested in generating sets) we have
for all \( \ell \). Hence the basis elements of type (4) for \( i = j \) can be expressed as \( \pm (a_i a_i) \) for certain \( m \). In particular, for \( i = 0 \) one gets

\[
(a_0 a_i)^m = \pm \left( e^{2m(1 + ep(e))} \right).
\]

Moreover, we see that

\[
[a_i, a_{i+1} \cdots a_{i+\ell} a_{i-1} \cdots a_1 a_0, a_{i+1} \cdots a_{i+\ell} a_{i-1} \cdots a_1 a_0] =
\]

\[
a_i a_{i+1} \cdots a_{i+\ell} a_{i-1} \cdots a_1 a_0 - a_{i+1} \cdots a_{i+\ell} a_{i-1} \cdots a_1 a_0 a_i a_{i+1} \cdots a_{i+\ell} a_{i-1} \cdots a_1 a_0
\]

for some \( i \leq \ell \leq n - 2 \). Hence, two different basis elements of type (7) for \( i = j \) differ by a commutator. Therefore, modulo commutator \([L^n_0, L^n_0]\) we need to consider the basis elements of type (7) only for \( i = j = 0 \):

\[
(a_0 a_1 \cdots a_{\ell-1} a_i a_{\ell-1} \cdots a_1 a_0 \epsilon)^{m} \text{ for some } 0 \leq \ell \leq n - 2
\]

But now,

\[
a_0 a_1 \cdots a_{\ell-1} a_i a_{\ell-1} \cdots a_1 a_0 \epsilon = \pm (a_0 a_i)^{\ell+1} \epsilon = \pm \epsilon (\epsilon^2 + \epsilon^3 p(e))^{\ell+1} = e^{2\ell+3(1 + ep(e))^{\ell+1}}
\]

Therefore, the basis of \( e_0 L^n_0 e_0 \) in Proposition 3.1 consists of one element of type (3), one element of type (5), elements of type (7) which have the form \( e^{2\ell+3(1 + ep(e))^{\ell+1}} \) and elements of type (4) of the form \( e^{2\ell+2(1 + ep(e))^{\ell+1}} \). Hence the set

\[
\{e_0 \cup \{e \} \cup \{e^{2\ell+3(1 + ep(e))^{\ell+1}} | 0 \leq \ell < n - 1 \} \cup \{e^{2\ell+2(1 + ep(e))^{\ell+1}} | 0 \leq \ell < n - 1 \}
\]

forms a basis of \( e_0 L^n_0 e_0 \).

**Lemma 3.9.** The set \( \{e^{\ell} | 0 \leq \ell < 2n \} \) is a \( K \)-basis of \( e_0 L^n_0 e_0 \).

**Remark 3.10.** Of course, we put \( e^0 = e_0 \) in Lemma 3.9.

**Proof.** We know that the set

\[
S := \{e_0 \cup \{e \} \cup \{e^{2\ell+3(1 + ep(e))^{\ell+1}} | 0 \leq \ell < n - 1 \} \cup \{e^{2\ell+2(1 + ep(e))^{\ell+1}} | 0 \leq \ell < n - 1 \}
\]

forms a basis of \( e_0 L^n_0 e_0 \). Expressing these elements as linear combinations of the set \( \{e^{\ell} | 0 \leq \ell < 2n \} \) one obtains a square upper triangular matrix with diagonal entries 1. Hence since \( S \) is a basis, also \( \{e^{\ell} | 0 \leq \ell < 2n \} \) is a \( K \)-basis of \( e_0 L^n_0 e_0 \). \( \square \)

**Lemma 3.11.** For all \( n \geq 0 \) the factor space \( L_{n+1}^-/[L_{n+1}^p, L_{n+1}^-] \) has a \( K \)-linear generating set

\[
\{e_0, e_1, \ldots, e_n \} \cup \{e^{2m+1} | 0 \leq m \leq n \}.
\]

**Proof.** It is a general fact that every non-closed path (i.e. a path with different start and end point) is a commutator; in fact, take the commutator with the trivial path corresponding to the start point (or end point). Moreover, it is easy to see that the cosets of the trivial paths are always linearly independent modulo the commutator subspace.

In view of Lemma 3.7 we only need to detect elements outside the commutator of \( L^n_0 \) which become a commutator in \( L^n_{n+1} \) and determine which elements of \( L^n_{n+1} e_0 L^n_{n+1} \) are commutators.

We shall proceed by induction on \( n \). The lemma is clearly true for \( n = 0 \). For \( n > 0 \) we can use the list of closed paths given in Proposition 3.1. We shall start by identifying certain closed paths as being commutators.

- For all \( m \in \{0, 1, \ldots, n - 1 \} \) we have

\[
(a_m a_{m+1} \cdots a_{n-1} a_{n-1} a_{n-2} \cdots a_{m}) = (a_m a_{m+1} \cdots a_{n-1})(a_{n-1} a_{n-2} \cdots a_{m})
\]

\[
- (a_{n-1} a_{n-2} \cdots a_{m})(a_m a_{m+1} \cdots a_{n-1})
\]

\[
\in [L_{n+1}^p, L_{n+1}^p]
\]

since \((a_{n-1} a_{n-2} \cdots a_{m})(a_m a_{m+1} \cdots a_{n-1}) = 0 \) by Lemma 3.6.

- Using the defining relation \( a_{n-1} a_{n-1} = 0 \) in \( L_{n+1}^p \) we have that

\[
a_{n-1} a_{n-1} = a_{n-1} \bar{a}_{n-1} - \bar{a}_{n-1} a_{n-1} = [a_{n-1}, \bar{a}_{n-1}] \in [L_{n+1}^p, L_{n+1}^p].
\]

Moreover, using the relation \( \bar{a}_i a_{i+1} = a_{i+1} \bar{a}_i + 1 \) for all \( i \in \{0, 1, \ldots, n - 2 \} \) we have that

\[
[a_i, \bar{a}_i] = a_i \bar{a}_i - \bar{a}_i a_i = a_i \bar{a}_i - a_{i+1} \bar{a}_{i+1}.
\]

Inductively we can assume that \( a_{i+1} \bar{a}_i \in [L_{n+1}^p, L_{n+1}^p] \) and hence we deduce that \( a_i \bar{a}_i \in [L_{n+1}^p, L_{n+1}^p] \) for all \( i \in \{0, 1, \ldots, n - 1 \} \).
Moreover, consider a power \((a_i, \pi_i)^m\) for some integer \(m \geq 2\). Then for all \(i \in \{0, 1, \ldots, n - 2\}\) we have
\[
[a_i, (a_i, \pi_i)^{-1}] = (a_i, \pi_i)^m - (a_i, \pi_i)^m - (a_i, \pi_{i+1})^m.
\]
Inductively, we obtain that \((a_i, \pi_i)^m \equiv (a_{n-1}, \pi_{n-1})^m \mod \{L^p_{n+1}, L^p_{n+1}\}\); but \((a_{n-1}, \pi_{n-1})^m = 0\) for \(m \geq 2\) (using the defining relation \(a_{n-1}a_{n-1} = 0\)).

Together with the above arguments for the case \(m = 1\) we can thus deduce
\[
(a_i, \pi_i)^m \in \{L^p_{n+1}, L^p_{n+1}\} \quad \text{for all} \quad i \in \{0, 1, \ldots, n - 1\}\]
\(\text{and all} \quad m \geq 1.\)

- In particular the preceding arguments imply that \((e^2 + e^3p(e))^m = -(a_0, \pi_0)^m \in \{L^p_{n+1}, L^p_{n+1}\}\) for all integers \(m \geq 1.\)

In a second step after showing certain closed paths to be commutators we now examine (nontrivial) closed paths in \(L^p_{n+1} e_n L^p_{n+1}\) and in particular determine the dimension of the image in the factor space \(L^p_{n+1} e_n L^p_{n+1}/[L^p_{n+1}, L^p_{n+1}]\). According to Proposition 3.1 there are two types of such paths, the long paths \(a_i a_{i+1} \ldots a_{n-1} \pi_{n-1} \ldots \pi_0 e a_0 \ldots a_{i-1}\) for \(i \in \{0, 1, \ldots, n\}\) corresponding to socle elements and the short paths \(a_i a_{i+1} \ldots a_{n-1} \pi_{n-1} \ldots \pi_i\) for \(i \in \{0, 1, \ldots, n - 1\}\).

For the latter we already observed at the beginning of the proof that they are all commutators, i.e. that \(a_i a_{i+1} \ldots a_{n-1} \pi_{n-1} \ldots \pi_i \in \{L^p_{n+1}, L^p_{n+1}\}\) for all \(i \in \{0, 1, \ldots, n - 1\}\).

For the former paths, corresponding to socle elements, consider for \(i \in \{1, \ldots, n\}\) the commutator
\[
[a_i \ldots a_{n-1} \pi_{n-1} \ldots \pi_0 e, a_0 \ldots a_{i-1}] = a_i a_{i+1} \ldots a_{n-1} \pi_{n-1} \ldots \pi_0 e a_0 \ldots a_{i-1} - a_0 \ldots a_{i-1} a_i a_{i+1} \ldots a_{n-1} \pi_{n-1} \ldots \pi_0 e = a_i a_{i+1} \ldots a_{n-1} \pi_{n-1} \ldots \pi_0 e a_0 \ldots a_{i-1} - e^{2n+1}
\]

where for the last equation see Remark 3.2 (3). Therefore, all the long paths corresponding to socle elements are equivalent to \(e^{2n+1}\) modulo the commutator space.

Therefore, the image of \(L^p_{n+1} e_n L^p_{n+1}\) in \(L^p_{n+1}/[L^p_{n+1}, L^p_{n+1}]\) is 2-dimensional with a basis given by the cosets of \(\{e_n, e^{2n+1}\}\).

The assertion of the lemma now follows by induction, using that \(L^p_{n+1}/(L^p_{n+1} e_n L^p_{n+1}) = L^p_n\) by Lemma 3.7.

**Remark 3.12.** We did not yet prove that this generating set is actually a \(K\)-basis. This fact is going to be shown in Proposition 3.14.

### 3.5. The centre
The aim of this section is to have a look at the centres of the deformed preprojective algebras \(L^p_n\) of type \(L\). The centre will be important to us by the following observation. It is not so difficult to write down quite a lot of commutators, as we have seen in Section 3.4. It is however difficult in general to show that these commutators actually generate the commutator space. By the discussion in Section 2.1, for a symmetric algebra \(A\) we get
\[
\dim_K A = \dim_K [A, A] + \dim_K Z(A).
\]

Since the canonical projection \(\pi_n : L^p_{n+1} \longrightarrow L^p_n\) from Lemma 3.7 is a surjective algebra homomorphism, the restriction of \(\pi_n\) to the centre \(Z(L^p_{n+1})\) induces a ring homomorphism \(Z(L^p_{n+1}) \longrightarrow Z(L^p_n)\). In principle, we could use this to determine the centre of \(L^p_n\) inductively although this might become quite technical.

Fortunately, with the methods developed in this paper we shall not really need to determine the entire centre; it will turn out that it suffices to find one central element whose powers generate a large enough central subspace.

**Lemma 3.13.** The following holds for the deformed preprojective algebra \(L^p_n\).

1. The element \(e^2 + e^3p(e) + \sum_{\ell=0}^{n-3}(-1)^{\ell+1}a_\ell\) is contained in the centre of \(L^p_n\).
2. The following subset is a \(K\)-free subset of the centre
\[
\left\{ \left( e^2 + e^3p(e) + \sum_{\ell=0}^{n-3}(-1)^{\ell+1}a_\ell \right)^s \mid 0 \leq s \leq n - 1 \right\} \subseteq Z(L^p_n).
\]
3. \(\text{soc}(L^p_n) \subseteq Z(L^p_n)\).
Proof. (1) For proving that \( \lambda := e^2 + e^3(p(e) + \sum_{\ell=0}^{n-3}(-1)^{\ell+1}a_{\ell}a_{\ell} \) is a central element it is sufficient to show that it commutes with the algebra generators of \( L^p_n \). This is clear for the trivial paths \( e \), since all paths occurring in \( \lambda \) are closed paths. For the loop \( \epsilon \) we get \( \epsilon \lambda = e^3 + e\epsilon (p(e) = \lambda \epsilon \). For the arrow \( a_0 \) we have, using the defining relation \( e^2 + e^3p(e) + a_0a_0 = 0 \), that

\[
\lambda a_0 = (e^2 + e^3p(e))a_0 = (-a_0a_0)a_0 = a_0\lambda.
\]

Similarly, for the arrow \( e_0 \) have

\[
\lambda e_0 = -\pi_0a_0e_0 = \pi_0(e^2 + e^3p(e)) = \pi_0\lambda.
\]

For the arrows \( a_i \) where \( 1 \leq i \leq n-2 \), we use the relations \( a_{i-1}a_{i+1} + a_i = 0 \) and get

\[
\lambda a_i = (-1)^i\pi_{i-1}a_{i+1}a_i = (-1)^{i+1}a_i\pi_{i-1}a_i = a_i\lambda.
\]

Finally, we get in a similar fashion for the arrows \( a_i \) where \( 1 \leq i \leq n-2 \) that

\[
\lambda a_i = (-1)^{i+1}a_i\pi_{a_i} = (-1)^{i+2}\pi_{a_{i-1}}a_i - (-1)^{i+1}\pi_{a_{i-1}}a_i = a_i\lambda.
\]

(2) The fact that the powers of \( e^2 + e^3p(e) + \sum_{\ell=0}^{n-3}(-1)^{\ell+1}a_{\ell}a_{\ell} \) form a linearly independent set comes from the fact that the powers of \( e \) form a linearly independent set (cf Lemma 3.9).

(3) This holds since multiplication from the left or right of any basis element of the socle with a path of length at least 1 gives 0. Moreover, the algebras \( L^p_n \) are weakly symmetric, i.e. the socle has a basis consisting of closed paths, so any socle element also commutes with the trivial paths. \( \square \)

We are now in the position to give a basis of the commutator space \( [L^p_n, L^p_n] \) of the deformed preprojective algebras \( L^p_n \) of type \( L \). As a consequence we can strengthen the statement in Lemma 3.11: namely the cosets of \( \{e_0, \ldots, e_{n-1}\} \cup \{\epsilon^{2\ell+1} \mid 0 \leq \ell \leq n-1\} \) are a basis (and not only a generating set) of the factor space \( L^p_n/[L^p_n, L^p_n] \).

Proposition 3.14. For any polynomial \( p(X) \in \mathbb{K}[X] \) and \( n \geq 2 \) the following holds for the deformed preprojective algebras \( L^p_n \):

(a) The commutator space has dimension

\[
\dim_K [L^p_n, L^p_n] = \dim_K L^p_n - 2n = \frac{1}{3}n(n-1)(2n+5).
\]

(b) The centre has dimension \( \dim_K Z(L^p_n) = 2n \).

(c) The cosets of \( \{e_0, \ldots, e_{n-1}\} \cup \{\epsilon^{2\ell+1} \mid 0 \leq \ell \leq n-1\} \) form a basis of the factor space \( L^p_n/[L^p_n, L^p_n] \).

(d) In terms of the basis \( \mathcal{B} \) of \( L^p_n \) given in Proposition 3.1, a \( \mathbb{K} \)-basis of \( [L^p_n, L^p_n] \) is given by

(i) all non-closed paths in \( \mathcal{B} \) (i.e. with starting vertex different from the ending vertex),

(ii) all closed paths in \( \mathcal{B} \) of even length at least 2 with starting vertex different from vertex 0,

(iii) the difference of two closed paths in \( \mathcal{B} \) of equal odd length with starting vertices \( i \) and \( i+1 \) where \( 0 \leq i \leq n-2 \),

(iv) the elements \( a_0a_1 \ldots a_{n-1} \pi_{a_{n-1}} \pi_{a_{n-2}} \ldots \pi_0 \) where \( 0 \leq \ell \leq n-2 \).

Proof. (a) For every symmetric algebra \( A \) we have

\[
\dim_K [A, A] = \dim_K A - \dim_K Z(A)
\]

since \( \text{Hom}_K(A/[A, A], K) \simeq Z(A) \) as \( Z(A) \)-modules.

In our situation for \( A = L^p_n \) we get from Lemma 3.11 the lower bound

\[
\dim_K [A, A] \geq \dim_K L^p_n - 2n.
\]

We shall now produce sufficiently many linear independent elements in the centre to obtain this also as an upper bound. The centre \( Z(L^p_n) \) contains the \( K \)-free subset

\[
\left\{ \left( \epsilon^2 + e^3p(e) + \sum_{\ell=0}^{n-3}(-1)^{\ell+1}a_{\ell}a_{\ell} \right)^{s} \mid 0 \leq s \leq n-1 \right\} \subseteq Z(L^p_n)
\]

of cardinality \( n \). The \( K \)-vector space generated by these elements intersects with \( \text{soc}(L^p_n) \) only in \( \{0\} \) since \( \epsilon^{2s+2} \) is not in the socle for \( s \in \{0, 1, \ldots, n-2\} \). However, \( \text{soc}(L^p_n) \) belongs to the centre. Hence we get that \( Z(L^p_n) \) is of dimension at least \( 2n \).

Altogether, we get a lower bound for the dimension of the centre, namely

\[
\dim_K Z(L^p_n) \geq (n-1) + n + 1 = 2n.
\]
Plugging this into formula (6) and combining with (7) proves the first equality in part (a) of the lemma.

The second equality then follows by a direct calculation from the formula for the dimension of $L_n^p$ given in Remark 3.3.

(b) The statement in part (b) now follows directly from part (a) by using formula (6).

(c) Follows by combining part (a) and Lemma 3.11.

(d) For each of the types (i)-(iv) we shall first verify that all these elements are actually contained in the commutator space, and then count their number. At the end it will turn out that the total number of elements in (i)-(iv) is \((\dim_K L_n^p - 2n)\), i.e. equal to the dimension of the commutator space, cf. part (a). Since the elements are linearly independent (being part of a basis), the claim of part (d) then follows.

(i) Non-closed paths are always commutators (take the commutator with the trivial path corresponding to the starting vertex).

The number of non-closed paths in $\mathcal{B}$ can be read off from the Cartan matrix of $L_n^p$ given in Remark 3.3. Namely as the dimension of $L_n^p$ minus the trace of the Cartan matrix, i.e. we get $\dim_K L_n^p - n(n + 1)$ non-closed paths in $\mathcal{B}$.

(ii) Such paths of even length only occur in type (4) of Proposition 3.1 and are of the form $a_1a_{i+1}\ldots a_\ell \overline{a}_\ell \ldots \overline{a}_{i-1} = a_1a_{i+1}\ldots a_\ell \overline{a}_\ell \ldots \overline{a}_{i-1} a_{i-1}$, where $\ell \leq n - 2$. Up to a sign, these paths are equal to \((a_1a_\ell)\ell\) (using the relations $\overline{a}_i a_\ell + a_{\ell+1} \overline{a}_{\ell+1}$) and these have been shown to be in the commutator space in the proof of Lemma 3.11.

Summing over the possibilities for the various $i \neq 0$ there are \(\frac{(n-1)(n-2)}{2}\) such paths.

(iii) Such a difference of closed paths with starting vertices $i$ and $i + 1$ occurs as a difference of a path of type (7) for vertex $i$ with a path of type (5) or (7) for vertex $i + 1$. More precisely, these differences are of the form

$$a_1a_{i+1}\ldots a_\ell \overline{a}_\ell \ldots \overline{a}_{i-1}0 a_0 \ldots a_{i-1} - a_1a_{i+1}\ldots a_\ell \overline{a}_\ell \ldots \overline{a}_{i-1}0 a_0 \ldots a_{i-1},$$

where $\ell \leq n - 2$. This element is a commutator, namely $[a_1a_{i+1}\ldots a_\ell \overline{a}_\ell \ldots \overline{a}_{i-1}0 a_0 \ldots a_{i-1}].$

Summing over the possibilities for the various $i$ there are \(\frac{n(n-1)}{2}\) such differences.

(iv) Up to a sign, these paths are equal to \((a_1a_\ell)\ell\) (using the relations $\overline{a}_i a_\ell + a_{\ell+1} \overline{a}_{\ell+1}$) and these have been shown to be in the commutator space in the proof of Lemma 3.11. Obviously, there are $n - 1$ such paths.

The total number of elements in (i)-(iv) is easily computed to be $\dim_K L_n^{X^{2j}} - 2n$, which is the dimension of the commutator space by part (a). Thus part (d) follows.

\[\square\]

4. The Kulshammer spaces and the main result

We now restrict to the case where the deformation polynomial $p$ has the form $X^{2j}$ for some integer $j \geq 0$. It has been shown in [3, Proposition 6.1] that the deformed algebras $L_n^p$ for the polynomials $p = X^{2j}$ where $j \in \{0, 1, \ldots, n - 1\}$ form a family of pairwise non-isomorphic deformed preprojective algebras of type $L$. Note that for all $j \geq n - 1$ the algebra $L_n^{X^{2j}}$ is the (undeformed) preprojective algebra of type $L$; in fact, the only relation involving the polynomial $p$ reads

$$c^2 + a_0 \overline{a}_0 + c^3 p(c) = c^2 + a_0 \overline{a}_0 + c^{2j+3} = c^2 + a_0 \overline{a}_0$$

because $c^{2n} = 0$ in $L_n^p$.

Moreover, it has been announced [2] that the algebras $L_n^{X^{2j}}$ for $j \in \{0, 1, \ldots, n - 1\}$ actually form a complete list of representatives of the isomorphism classes of deformed preprojective algebras of type $L$; details should appear in the forthcoming paper [4].

For the above reasons, focussing on the case of deformation polynomials $p = X^{2j}$ is not really a restriction.

We continue to consider the deformed preprojective algebras $A_n^r = L_n^{X^{2j}}$ over a field of characteristic 2. The Kulshammer spaces are defined as $T_r(A_n^r) = \{x \in A_n^r \mid x^{2^r} \in [A_n^r, A_n^r]\}$ for any integer $r \geq 0$ (cf. the introduction).

In this section we shall derive the main results of the paper. Firstly, we shall give formulae for the dimensions of the Kulshammer spaces $T_r(A_n^r)$, see Theorem 4.1 below. Secondly, as an application we can distinguish certain of the deformed preprojective algebras of type $L_n$ (over a perfect field of characteristic 2) up to derived equivalence, see Theorem 4.2 below.
The crucial link to distinguish algebras up to derived equivalence by means of Külschammer spaces has been provided by the second author in [27]. There it is shown that for $K$ being a perfect field of characteristic $p > 0$ and for $\Lambda_1$ and $\Lambda_2$ being finite dimensional $K$-algebras which are derived equivalent the codimensions of the Külschammer spaces are an invariant, i.e. for all $r \geq 0$ one has
\[
\dim_K \Lambda_1 - \dim_K T_r(\Lambda_1) = \dim_K \Lambda_2 - \dim_K T_r(\Lambda_2).
\]

In [18] Liu, Zhou and the second author showed that for any field $K$ of characteristic $p > 0$ and any two finite dimensional $K$-algebras $\Lambda_1$ and $\Lambda_2$, if $\Lambda_1$ and $\Lambda_2$ are stably equivalent of Morita type, then
\[
\dim_K T_r(\Lambda_1) - \dim_K [\Lambda_1, \Lambda_1] = \dim_K T_r(\Lambda_2) - \dim_K [\Lambda_2, \Lambda_2]
\]
for all $r \geq 0$.

### 4.1. Dimensions of Külschammer spaces

In this section we shall prove the main result on the dimensions of the Külschammer spaces $T_1(L^X_n)$ for the deformed preprojective algebras. Before embarking on the general proof we shall give some explicit examples which hopefully help the reader later by illustrating the technicalities of the general arguments.

**An example: the case $n = 2$.** Let us look at the algebras $A^2_1 = L^X_2$ as an illustration. These algebras are given by a quiver with two vertices and relations $\epsilon^2 = 0$, $\alpha_0 a_0 = 0$ and $\epsilon^2 + \epsilon^{2j+3} + a_0 \alpha_0 = 0$. Note that for $j \geq 1$ we get the undeformed algebra $A^2_2$ with relation $\epsilon^2 + a_0 \alpha_0 = 0$, whereas for $j = 0$ we get a deformed preprojective algebra $A^0_2$ with relation $\epsilon^2 + \epsilon^3 + a_0 \alpha_0 = 0$.

According to Proposition 3.1 and Remark 3.3 the algebras $A^2_2$ are 10-dimensional with a basis given by the paths
\[
e_0, e_1, \epsilon, a_0, \alpha_0, a_0 \alpha_0, \epsilon a_0, \alpha_0 \epsilon, a_0 \alpha_0 \epsilon, \alpha_0 \epsilon a_0, \alpha_0 \epsilon a_0.
\]
By Proposition 3.14 the commutator spaces $[A^2_1, A^2_2]$ are of dimension 6 and have a basis consisting of the elements
\[
a_0, \alpha_0, a_0 \alpha_0, \epsilon a_0, \alpha_0 \epsilon, a_0 \alpha_0 \epsilon - \alpha_0 \epsilon a_0.
\]
Note that all these bases are independent of $j$.

Now we consider the first Külschammer space $T_1(A^2_1) = \{ x \in A^2_1 \mid x^2 \in [A^2_1, A^2_2] \}$. For any $j \geq 0$ it is immediate from the relations that the following seven basis elements of $A^2_2$ are contained in the first Külschammer space
\[
\{ a_0, \alpha_0, a_0 \alpha_0, \epsilon a_0, \alpha_0 \epsilon, a_0 \alpha_0 \epsilon, \alpha_0 \epsilon a_0 \} \subset T_1(A^2_1).
\]
On the other hand, it is a general observation that the trivial paths $e_0, e_1$ can not be summands of an element in a Külschammer space (since trivial paths can’t occur as summands in an element from the commutator space). This leaves us with the remaining basis element $\epsilon$. Here the situation changes for different $j$.

In the undeformed case $j \geq 1$ we have that $\epsilon^2 = a_0 \alpha_0 = [a_0, \alpha_0] \in [A^2_2, A^2_2]$ and hence $\epsilon \in T_1(A^2_2)$.

On the other, in the deformed case $j = 0$ we have the relation $\epsilon^2 = \epsilon^3 + a_0 \alpha_0$ where $a_0 \alpha_0$ is a commutator but $\epsilon^3 = a_0 \alpha_0 \epsilon \not\in [A^0_2, A^0_2]$. Therefore, $\epsilon \not\in T_1(A^0_2)$.

In summary we have $\dim_K T_1(A^2_1) = 7$ whereas $\dim_K T_1(A^0_2) = 8$ for all $j \geq 1$.

Using the result from [27] quoted above in (8) we can deduce that the undeformed preprojective algebra $A^2_2 = L^X_2$ and the deformed preprojective algebra $A^0_2 = L^X_2$ are not derived equivalent. Even in this small case $n = 2$ this seems to be a nontrivial fact.

**Another example: Külschammer spaces for $n = 3$.** The algebras $A^3_2$ have dimension 28, and their commutator spaces have dimension 22. There are many basis elements which are obviously in each of the Külschammer ideals $T_r(A^3_2)$, for $r \geq 1$, namely

- all non-closed paths in $B$, giving 16 basis elements (since they square to zero)
- closed paths of length $\geq 3$ (since the algebras have radical length 6 they also square to zero); so another six such basis elements are $a_0 \alpha_0 \epsilon, \alpha_0 \epsilon a_0, a_0 \alpha_0 \alpha_1 \alpha_1 \epsilon, a_1 \alpha_1 \epsilon a_0, a_0 \alpha_0 \alpha_1 \alpha_1 \epsilon, a_0 \alpha_0 \alpha_1 \alpha_1 \epsilon$
- the two basis elements $a_0 \alpha_0$ and $a_1 \alpha_1$ (since $(a_1 \alpha_1)^2 = 0$ and $(a_0 \alpha_0)^2$ is in the commutator space by the proof of Lemma 9).

Hence, $\dim_K T_r(A^3_1) \geq 24$ for all $r \geq 1$ and all $j$.

Given that the three trivial paths are not involved in any element of the Külschammer space, there is only one remaining basis element to consider, namely $\epsilon$.

We start with the first Külschammer space $T_1(A^3_2)$.

For $j = 2$ we have $\epsilon^2 = a_0 \alpha_0 \in [A^3_2, A^3_2]$, i.e. $\epsilon \in T_1(A^3_2)$. 

For $j = 1$ we get the following congruences modulo the commutator space
\[
\epsilon^2 = \epsilon^3 + a_0\overline{\alpha}_0 \equiv \epsilon^5 + a_0\overline{\alpha}_0 \equiv \epsilon^6 + a_0\overline{\alpha}_0 + a_0\overline{\alpha}_0 \epsilon \neq 0
\]
i.e. $\epsilon \not\in T_1(A^1_1)$.

For $j = 0$ we similarly get the following congruences modulo the commutator space
\[
\epsilon^2 = \epsilon^3 + a_0\overline{\alpha}_0 \equiv \epsilon^5 + a_0\overline{\alpha}_0 \equiv \epsilon^6 + a_0\overline{\alpha}_0 + a_0\overline{\alpha}_0 \epsilon \neq 0
\]
i.e. $\epsilon \not\in T_1(A^1_0)$.

Altogether we get $\dim_K T_1(A^2_0) = \begin{cases} 24 & \text{if } j = 0, 1 \\ 25 & \text{if } j = 2 \end{cases}$

Now we consider the second Kulshammer space $T_2(A^1_0) = \{ x \in A^1_0 \mid x^4 \in [A^1_1, A^1_0] \}$. Again it only remains to consider the basis element $\epsilon$.

For $j = 2$ there is nothing to check since $T_2(A^1_0)$ already attained the maximal possible dimension 25.

For $j = 1$ we get the following congruences (modulo commutator space)
\[
\epsilon^4 = \epsilon^7 + a_0\overline{\alpha}_0 \epsilon^2 = a_0\overline{\alpha}_0 \epsilon^5 + a_0\overline{\alpha}_0 a_0\overline{\alpha}_0 \epsilon \equiv 0
\]
i.e. $\epsilon \in T_2(A^1_0)$.

Similarly we get for $j = 0$ (modulo commutator space)
\[
\epsilon^4 = \epsilon^5 + a_0\overline{\alpha}_0 \epsilon^2 = a_0\overline{\alpha}_0 \epsilon^3 + a_0\overline{\alpha}_0 a_0\overline{\alpha}_0 \epsilon \equiv 0
\]
i.e. $\epsilon \in T_2(A^2_0)$.

Altogether we get $\dim_K T_2(A^1_0) = 25$ for all $j$.

We now formulate the main result of this section.

**Theorem 4.1.** Let $K$ be a perfect field of characteristic 2. Then for all $0 \leq j < n$ we have
\[
\dim_K T_i(L^X_n) - \dim_K [L^X_n, L^X_n] = n - \max \left( \left\lfloor \frac{2n - (2^{j+1} - 2)j - (2^{j+1} - 1)}{2^{j+1}} \right\rfloor, 0 \right).
\]

**Proof.** Lemma 3.11 provided a set of coset generators of the commutator space $[L^X_n, L^X_n]$ in $L^X_n$, namely $\{e_0, \ldots, e_{n-1}\} \cup \{\epsilon^{(2k+1)} \mid 0 \leq k \leq n - 1\}$. For our purpose of determining the Kulshammer ideals we can discard the trivial paths since they can never be involved in any element of a Kulshammer ideal. Therefore in order to compute $T_i(L^X_n)$ we need to see when a $2^j$-th power of a linear combination of elements $\{\epsilon^{(2k+1)} \mid 0 \leq k \leq n - 1\}$ lies in the commutator space $[L^X_n, L^X_n]$.

Note that $\epsilon \in e_0 L^X_n e_0$, and that by Proposition 3.14 a basis of the intersection $e_0 L^X_n e_0 \cap [L^X_n, L^X_n]$ is given by the paths $a_0 a_1 \ldots a_{n-1} \overline{\alpha}_0 \overline{\alpha}_{n-1} \ldots \overline{\alpha}_0$ where $0 \leq \ell \leq n - 2$. Moreover, we have that
\[
a_0 a_1 \ldots a_{n-1} \overline{\alpha}_0 \overline{\alpha}_{n-1} \ldots \overline{\alpha}_0 \in (\epsilon^{(2k+1)})^{\ell+1} = (\epsilon^2 + \epsilon^{2j+3})^{\ell+1}
\]
(no signs occurring since we are in characteristic 2). This means that in order to obtain the desired formula for the dimension of $T_i(L^X_n)$ one needs to consider the $K$-vector space
\[
\tilde{T}_n,j(i) := \left\{ \left( \sum_{k=0}^{n-1} b_k \epsilon^{2k+1} \right) \left( \sum_{k=0}^{n-1} b_k e^{2k+1} + \epsilon^{2j+3} \right)^m \in \langle \epsilon^2 + \epsilon^{2j+3} \rangle_K \mid 1 \leq m \leq n - 1 \right\}_K
\]
whose dimension is equal to the dimension of the factor space $T_i(L^X_n)/[L^X_n, L^X_n]$.

In order to determine this dimension we therefore have to express an element $(\sum_{k=0}^{n-1} b_k \epsilon^{2k+1})^{2^j}$ as a linear combination of the form $\sum_{m=1}^{n-1} c_m (\epsilon^2 + \epsilon^{2j+3})^m$ for $c_m \in K$. 

First Step. We shall reduce the problem to the case of $K$ being the prime field of characteristic 2.

As is described in the remarks preceding the statement of the theorem we have to give the dimension of the $K$-vector space

$$\left\{ \sum_{k=0}^{n-1} b_k \epsilon^{2k+1} \in K[\epsilon] \left| \left( \sum_{k=0}^{n-1} b_k \epsilon^{2k+1} \right)^{2^i} \in \langle (\epsilon^2 + \epsilon^{2j+3})^m \mid m \in \mathbb{N} \rangle_K \right. \right\}$$

Let

$$U := \langle (\epsilon^2 + \epsilon^{2j+3})^m \mid m \in \mathbb{N} \rangle_K \subseteq K[\epsilon]/\epsilon^{2n}.$$ 

Then let

$$V := \langle \epsilon^{2k+1} \rangle_K \subseteq K[\epsilon]/\epsilon^{2n}$$

and let $\mu : K[\epsilon]/\epsilon^{2n} \to K[\epsilon]/\epsilon^{2n}$ given by $\mu(x) := x^2$. Then

$$\left\{ \sum_{k=0}^{n-1} b_k \epsilon^{2k+1} \in K[\epsilon] \left| \left( \sum_{k=0}^{n-1} b_k \epsilon^{2k+1} \right)^{2^i} \in \langle (\epsilon^2 + \epsilon^{2j+3})^m \mid m \in \mathbb{N} \rangle_K \right. \right\} = V \cap (\mu')^{-1}(U).$$

Now, $U = U_0 \otimes_{\mathbb{F}_2} K$ and $V := V_0 \otimes_{\mathbb{F}_2} K$ for $U_0$ and $V_0$ being defined as $U$ and $V$, but with $\mathbb{F}_2$ as base field. If $K$ is perfect, then

$$V \cap (\mu')^{-1}(U) = (V_0 \otimes_{\mathbb{F}_2} K) \cap (\mu')^{-1}(U_0) = (V_0 \cap (\mu')^{-1}(U_0)) \otimes_{\mathbb{F}_2} K.$$

Hence the dimension of the vector space can be computed in $\mathbb{F}_2$. We hence may assume that $K = \mathbb{F}_2$.

Since $K$ is assumed to be the prime field, we get $b^2 = b$ for all $b \in K$, and so we need to find coefficients $c_m \in K$ so that

$$\left( \sum_{k=0}^{n-1} b_k \epsilon^{2(2k+1)} \right)^{2^i} = \sum_{m=1}^{n-1} c_m (\epsilon^2 + \epsilon^{2j+3})^m.$$

Second step. In the course of the proof we shall need to know whether certain binomial coefficients are even or odd. More precisely, write a natural number as $2^a v$ with $v$ odd and $a \in \mathbb{N} \cup \{0\}$. Then we have that

$$\left\{ u \in \mathbb{N} \setminus \{0\} \mid \left( \frac{2^a u}{v} \right) \text{ odd} \right\} \subseteq 2^a \mathbb{Z} \quad \text{and} \quad \min \left\{ u \in \mathbb{N} \setminus \{0\} \mid \left( \frac{2^a u}{v} \right) \text{ odd} \right\} = 2^a.$$

In fact, both statements follow easily from the following well-known result on binomial coefficients, going back to Lucas [19]: for a proof see e.g. [10]: Let $p$ be a prime, and let natural numbers $M = \sum M_i p^i$ and $N = \sum N_i p^i$ be given in their $p$-adic expansion. Then $(\frac{M}{N}) \equiv \prod_i (\frac{M_i}{N_i}) \mod p$.

We remark further that binomial coefficients are integers. Hence, seen in $K$ they actually belong to the prime field. If $K$ is of characteristic 2, then a binomial coefficient can only have values 0 or 1.

Third step. We need to study for which $b_0, \ldots, b_{n-1}$ given, there exist coefficients $c_m \in K$ so that

$$\sum_{k=0}^{n-1} b_k \epsilon^{2(2k+1)} = \sum_{m=1}^{n-1} c_m (\epsilon^2 + \epsilon^{2j+3})^m. \quad (10)$$

We first determine a lower bound for the indices of the non-vanishing coefficients $b_k$. Denote by $k_0$ the smallest integer $k$ so that $b_k \neq 0$. Then formula (10) reads

$$\sum_{k=k_0}^{n-1} b_k \epsilon^{2(2k+1)} = \sum_{m=1}^{n-1} c_m (\epsilon^2 + \epsilon^{2j+3})^m. \quad (11)$$

Comparing the smallest powers of $\epsilon$ occurring on either side of equation (11) we can deduce that $c_m = 0$ for $m < 2^{2i-1}(2k_0+1)$ and $c_{2^{2i-1}(2k_0+1)} \neq 0$. Hence equation (11) now reads

$$\sum_{k=k_0}^{n-1} b_k \epsilon^{2(2k+1)} = \sum_{m=2^{2i-1}(2k_0+1)}^{n-1} c_m (\epsilon^2 + \epsilon^{2j+3})^m. \quad (12)$$
Using the statements on the parity of binomial coefficients from the second step and the fact that the base field is of characteristic 2 we have that

\[
(\epsilon^2 + \epsilon^{2j+3})^{2i-1}(2k_0+1) = \epsilon^{2i}(2k_0+1) + \left(\epsilon^{2i-1}(2k_0+1)\right) \cdot \left(\epsilon^2\right)^{2i-1}(2k_0+1) - 2^{i-1} \cdot \left(\epsilon^{2j+3}\right)^{2i-1} + \\
+ \text{higher powers of } \epsilon^{2i-1}
\]

Hence as long as \(2^{i-1}(4k_0 + 2j + 3) < 2n\) (i.e. \(\epsilon^{2i-1}(4k_0+2j+3)\) does not vanish), a non-zero scalar multiple of \(\epsilon^{2i-1}(4k_0+2j+3)\) occurs on the right hand side of equation (12). However, it can not occur on the left hand side of equation (12) since \(2^{i-1}(4k_0 + 2j + 3)\) is not divisible by \(2^i\). So \(\epsilon^{2i-1}(4k_0+2j+3)\) would also have to be a term of some other summand in \(\sum_{m=2^{-1}(2k_0+1)} \epsilon m (\epsilon^2 + \epsilon^{2j+3})^m\) (so that the terms can cancel out).

For \(i = 1\) this is impossible, since for \(m > 2^{i-1}(2k_0 + 1)\) the smallest possible odd exponent in \((\epsilon^2 + \epsilon^{2j+3})^m\) is already larger than \(2^{i-1}(4k_0 + 2j + 3)\). Hence for \(i = 1\) we must have that \(4k_0 + 2j + 3 \geq 2n\) which implies that \(k_0 = \max\left(\left\lfloor \frac{2n-2j-3}{2} \right\rfloor, 0\right)\). Note that we indeed have to take the maximum with 0 here since the index \(k_0\) is non-negative by definition.

Suppose now that \(i \geq 2\). Then we claim that the only possibility to cancel the above term \(\epsilon^{2i-1}(4k_0+2j+3)\) is to put

\[
c_{2^{i-2}(4k_0+2j+3)} = c_{2^{i-1}(2k_0+1)} \neq 0.
\]

In fact, on the one hand we have that

\[
(\epsilon^2 + \epsilon^{2j+3})^{2i-2}(4k_0+2j+3) = \epsilon^{2i-1}(4k_0+2j+3) + \text{higher powers of } \epsilon^{2i-2}
\]

so that the desired term cancels; on the other hand, it could not cancel for a smaller index \(m\) since this would have to satisfy \(m \geq 2^{i-1}(2k_0 + 3)\) (note that the exponents on the left hand side of equation (10) are divisible by \(2^i\)) and then by Lucas’ theorem above (cf. second step) the second term in \((\epsilon^2 + \epsilon^{2j+3})^m\) already has exponent

\[
2^i(2k_0 + 2) + (2j + 3)2^{i-1} = 2^{i-1}(4k_0 + 2j + 7) > 2^{i-1}(4k_0 + 2j + 3).
\]

In a similar way, again using the second step and that the base field is of characteristic 2, we further get

\[
(\epsilon^2 + \epsilon^{2j+3})^{2i-2}(4k_0+2j+3) = \epsilon^{2i-1}(4k_0+2j+3) + \left(\epsilon^2\right)^{2i-2}(4k_0+2j+3) - 2^{i-2} \cdot \left(\epsilon^{2j+3}\right)^{2i-2} + \\
+ \text{higher powers of } \epsilon^{2i-2}
\]

Completely analogous to the case \(i = 1\) above we can deduce that for \(i = 2\) we have \(8k_0 + 6j + 7 \geq 2n\) and therefore \(k_0 = \max\left(\left\lfloor \frac{2n-6j-7}{8} \right\rfloor, 0\right)\) in the case \(i = 2\). This is the second correction step.

We shall show by induction on \(s\), that the lowest power of \(\epsilon\) appearing in the sum on the right hand side of equation (12) after \(s\) corrections is

\[
\epsilon^{2^{s-i-1}(4k_0+(2s-2)j+(2s-1))}.
\]

The cases \(s \in \{1, 2\}\) have been treated above. Suppose the formula is shown for some \(s < i\). Then we shall show the formula for \(s+1\): We shall need to correct with \(\epsilon^{2^{s-i-1}(4k_0+(2s-2)j+(2s-1))} \neq 0\)
and get higher error terms as follows:

\[
(\epsilon^2 + \epsilon^{2j+3})^{2^{s-i-1}(2^s k_0 + (2^s-2)j + (2^s-1))} = \epsilon^{2^{s-i}(2^s k_0 + (2^s-2)j + (2^s-1))} + \\
\quad + \epsilon^{2^{s-i-1}(2^s k_0 + (2^s-2)j + (2^s-1)) - 2^{s-i-1}} \cdot (\epsilon^{2j+3})^{2^{s-i-1}} + \\
\quad + \text{higher powers of } \epsilon^{2^{s-i-1}} \\
= \epsilon^{2^{s-i}(2^s k_0 + (2^s-2)j + (2^s-1))} + \\
\quad + \epsilon^{2^{s-i-1}(2^s k_0 + (2^s-2)j + (2^s-1)) + 2(2^s-1)(2^s+3)} + \\
\quad + \text{higher powers of } \epsilon^{2^{s-i-1}} \\
= \epsilon^{2^{s-i}(2^s k_0 + (2^s-2)j + (2^s-1))} + \\
\quad + \epsilon^{2^{s-i-1}(2^s k_0 + (2^s-2)j + (2^s-1)) + 2(2^s-1)(4^s+2)} + \\
\quad + \text{higher powers of } \epsilon^{2^{s-i-1}} \\
= \epsilon^{2^{s-i}(2^s k_0 + (2^s-2)j + (2^s-1))} + \\
\quad + \epsilon^{2^{s-i-1}(2^s k_0 + (2^s-2)j + (2^s-1)) + 2(2^s-1)+3} + \\
\quad + \text{higher powers of } \epsilon^{2^{s-i-1}}
\]

which shows the formula for \( s + 1 \).

Hence, we may correct the error terms by successively choosing appropriate \( c_m \) for higher and higher \( m \), as long as \( s < i \). If \( s = i \) then the error term cannot be annihilated, and therefore it must be 0. Therefore

\[
2^i k_0 + (2^i - 2) \cdot j + (2^i - 1) \geq 2n
\]

which means

\[
k_0 \geq \frac{2n - (2^i - 2) j - (2^i - 1)}{2^i - 1}
\]

and therefore

\[
k_0 = \max \left( \left\lfloor \frac{2n - (2^i - 2) j - (2^i - 1)}{2^i - 1} \right\rfloor, 0 \right).
\]

**Fourth step.** Suppose

\[
k \geq k_0 = \max \left( \left\lfloor \frac{2n - (2^i + 2) j - (2^i - 1)}{2^i - 1} \right\rfloor \right).
\]

We shall prove that then

\[
\epsilon^{2^{i-1}(2k+1)} \in \langle (\epsilon^2 + \epsilon^{2j+3})^m \mid m \in \mathbb{N} \rangle_K.
\]

To this end we put \( c_{2^{i-1}(2k+1)} = 1 \) and get by the second step that

\[
(\epsilon^2 + \epsilon^{2j+3})^{2^{s-i-1}(2k+1)} - \epsilon^{2^i(2k+1)} = (\epsilon^2)^{2^{s-i-1}(2k+1)-2^{s-i}} \cdot (\epsilon^{2j+3})^{2^{s-i}} + \text{higher order powers of } \epsilon^{2^{s-i-1}}
\]

Hence, we can choose coefficients \( c_{2^{i-1}m} \) for certain \( m \) so that

\[
\epsilon^{2^{i-1}(2k+1)} - \sum c_{2^{i-1}m} (\epsilon^2 + \epsilon^{2j+3})^{2^{s-i-1}m}
\]

is a direct sum of terms \( \epsilon^{2^{i-1} \ell} \) where \( \ell \geq 4k + 2j + 3 \).

If \( i = 1 \), we are done since then \( \epsilon^{2^{i-1} \ell} = 0 \) since \( \ell \) was chosen in a way that

\[
\ell \geq 4k + 2j + 3 \geq 4k_0 + 2j + 3 \geq 2n.
\]

If \( i \geq 2 \), put \( c_{2^{i-2} \ell} = 1 \) for all terms \( \epsilon^{2^{i-1} \ell} \) of the powers of \( \epsilon^{2^{s-i}} \) occurring in the above difference

\[
\epsilon^{2^i(2k+1)} - \sum c_{2^{i-1}m} (\epsilon^2 + \epsilon^{2j+3})^{2^{s-i-1}m}.
\]

We know that each of these \( \ell \) satisfies \( \ell \geq 4k + 2j + 3 \), so that all the coefficients \( 2^{i-2} \ell \) are bigger than \( 2^{i-2}(4k_0 + 2j + 3) \).
We compute
\[ (e^2 + e^{2j+3})^{2i-2} \ell = e^{2i-1}\ell + \left(\frac{2^{i-2}\ell}{2i-2}\right) (e^2)^{2i-2}\ell - 2i - 2 \cdot (e^{2j+3})^{2i-2} + \]
+ higher order powers of \(e^{2i-2}\ell\)
\[ = e^{2i-1}\ell + \left(\frac{2^{i-2}\ell}{2i-2}\right) e^{2i-2}(2i+2j+1) + \text{ higher order powers of } e^{2i-2}\]

Again, if \(i = 2\) we are done since
\[ 2\ell + 2j + 1 \geq 2(4k_0 + 2j + 3) + 2j + 1 = 8k_0 + 6j + 7 \geq 2n \]
and hence \(e^{2i-2}(2i+2j+1) = 0\) for all \(\ell\) which may occur by definition of \(k_0\).

We use induction on \(s\) on the statement that we may choose \(c_{2i-1}\) so that only powers \(e^{2i-1}\ell\)
occur in the difference
\[ e^{2i-(2k+1)} - \sum_{m=1}^{n-1} c_{2i-1-m}(e^2 + e^{2j+3})^{2i-1-m} \]

for all \(\ell \geq (2^s k_0 + (2^s - 2) \cdot j + (2^s - 1))\).

The statement is true for \(s = 1\) and \(s = 2\) by the above discussion. Suppose it is true for \(s \leq i\).
We shall prove it for \(s + 1\).

For every term \(e^{2i-1}\ell\) which occurs as a summand in
\[ e^{2i-(2k+1)} - \sum_{m=1}^{n-1} c_{2i-1-m}(e^2 + e^{2j+3})^{2i-1-m} \]
we put \(c_{2i-1-\ell} = 1\) and then we compute
\[ (e^2 + e^{2j+3})^{2i-1-\ell} = e^{2i-1}\ell + \left(\frac{2^{i-1-\ell}}{2i-1-\ell}\right) (e^2)^{2i-1-\ell} - 2i - 2 \cdot (e^{2j+3})^{2i-1-\ell} + \]
+ higher order powers of \(e^{2i-1-\ell}\)
\[ = e^{2i-1}\ell + \left(\frac{2^{i-1-\ell}}{2i-1-\ell}\right) e^{2i-1-(2i+2j+1)} + \text{ higher order powers of } e^{2i-2} \]

Now, by induction hypothesis \(\ell \geq (2^s k_0 + (2^s - 2) \cdot j + (2^s - 1))\). Hence
\[ 2\ell + 2j + 1 \geq 2 \cdot (2^s k_0 + (2^s - 2) \cdot j + (2^s - 1)) + 2j + 1 = 2^{s+1}k_0 + (2^{s+1} - 2)j + (2^{s+1} - 1) \]
which is the statement for \(s + 1\).

But now, finally for \(s = i\) we get that the error terms are \(\ell^s\) where
\[ \ell \geq 2^{s+1}k_0 + (2^{s+1} - 2)j + (2^{s+1} - 1) \geq 2n \]
by definition of \(k_0\) and hence the error terms are 0.

Therefore,
\[ e^{2i(2k+1)} \in \left\{ \sum_{k=0}^{n-1} b_k \epsilon^{2k+1} \in K[\epsilon] \mid \left(\sum_{k=0}^{n-1} b_k \epsilon^{2k+1}\right)^{2i} \in \langle (e^2 + e^{2j+3})^n \rangle_K \right\} \text{ for all } k \geq k_0. \]

**Fifth step.** Now we are able to compute the dimension of
\[ \tilde{T}_{n,j}(i) = \left\{ \sum_{k=0}^{n-1} b_k \epsilon^{2k+1} \in K[\epsilon] \mid \left(\sum_{k=0}^{n-1} b_k \epsilon^{2k+1}\right)^{2i} \in \langle (e^2 + e^{2j+3})^n \rangle_K \right\} \]
We know by the third and fourth step that
\[ \tilde{T}_{n,j}(i) = \left\{ \sum_{k=0}^{n-1} b_k \epsilon^{2k+1} \in K[\epsilon] \mid b_k = 0 \text{ for } k < \left[ \frac{2n - (2^{i+1} - 2)j - (2^{i+1} - 1)}{2^{i+1}} \right] \right\} \]
The dimension of this space is therefore
\[ \dim \left( T_i(L_n^{X_{2j}})[L_n^{X_{2j}}, L_n^{X_{2j}}] \right) = \dim \tilde{T}_{n,j}(i) = n - \left[ \frac{2n - (2^{i+1} - 2)j - (2^{i+1} - 1)}{2^{i+1}} \right] \]
This finishes the proof. \(\square\)
4.2. Consequences for derived equivalence and stable equivalence of Morita type. In this section we shall address the main motivational question for this paper, namely when deformed preprojective algebras of type $L$ are derived equivalent or stably equivalent of Morita type. As main application of our results on Külshammer spaces we can obtain partial answers to these problems.

For both notions of equivalence it is in general a difficult question to decide whether two algebras are equivalent or not.

According to [2], for the deformed preprojective algebras of type $L_n$, Białkowski, Erdmann and Skowroński are going to show in [4] that a for an algebraically closed field $K$ the set of algebras $\{L_n^X \mid 0 \leq j \leq n-1\}$ gives a complete set of representatives for the Morita equivalence classes.

As an application of our result on Külshammer spaces we can now distinguish several of these algebras up to derived equivalence, and up to stable equivalence of Morita type.

**Theorem 4.2.** Let $K$ be a perfect field of characteristic 2.

(a) If two deformed preprojective algebras $L_n^p$ and $L_n^q$ are stably equivalent of Morita type or derived equivalent, then $n = m$.

(b) For $n \in \mathbb{N}$ let $j, k \in \{0, 1, \ldots, n-1\}$ be different numbers such that $\{j, k\} \neq \{n-2r, n-2r-1\}$ for every $1 \leq r \leq \left\lceil \frac{n-2}{2} \right\rceil$. Then the deformed preprojective algebras $L_n^{X_{2j}}$ and $L_n^{X_{2k}}$ are not stably equivalent of Morita type, and also not derived equivalent.

**Proof.** (a) It is well-known that the number of simple modules is a derived invariant.

Moreover, by a result of C. Xi [24, Proposition 5.1], the absolute value of the determinant of the Cartan matrix of an algebra is invariant under stable equivalence of Morita type. For the deformed preprojective algebras $L_n^p$, the Cartan determinant is $2^n$, see Remark 3.3, so the result follows.

(b) We use the first Külshammer space or more precisely the following difference occurring in Theorem 4.1 for the case $i = 1$,

$$
\dim_K \left( T_1(L_n^{X_{2j}}) \right) - \dim_K \left( [L_n^{X_{2j}}, L_n^{X_{2j}}] \right) = n - \max \left( \frac{2n - 2j - 3}{4}, 0 \right).
$$

By a result of Liu, Zhou and the second author [18, Corollary 7.5] this number is invariant under stable equivalences of Morita type. Since the deformed preprojective algebras of type $L_n^p$ the Cartan determinant is $2^n$, see Remark 3.3, so the result follows.

Note that for all the values $j \in \{0, \ldots, n-1\}$ this number is non-negative, so that equation (13) reads

$$
\dim_K \left( T_1(L_n^{X_{2j}}) \right) - \dim_K \left( [L_n^{X_{2j}}, L_n^{X_{2j}}] \right) = n - \left\lceil \frac{n - j - 1}{2} \right\rceil.
$$

For fixed $n$, this invariant becomes equal for two different values $j, k \in \{0, \ldots, n-1\}$ precisely when $\{j, k\} = \{n-2r, n-2r-1\}$ for some $1 \leq r \leq \left\lceil \frac{n-2}{2} \right\rceil$. This proves the assertion on stable equivalence of Morita type.

The statement on derived equivalence follows immediately by using a result by Rickard [22] and Keller and Vossieck [16] saying that for selfinjective algebras (recall that our algebras $L_n^p$ are even symmetric by Theorem 3.5) any derived equivalence induces a stable equivalence of Morita type.

**Remark 4.3.** (1) In the above theorem we have for simplicity only exploited the first Külshammer space, but of course one could also use higher Külshammer spaces for distinguishing algebras up to derived equivalence, or up stable equivalence of Morita type. For explicit examples of deformed preprojective algebras of type $L$ see Example 4.4 below.

(2) Note that part (b) of the above theorem in particular applies whenever $|j - k| \geq 2$.

(3) For any $n \in \mathbb{N}$ the (undeformed) preprojective algebra $L_n^{X_{2(n-1)}}$ is not stably equivalent of Morita type, and also not derived equivalent, to any of the algebras $L_n^{X_{2j}}$ for $j \in \{0, \ldots, n-2\}$. In fact, by the preceding remark it suffices to distinguish the algebras $L_n^{X_{2(n-1)}}$ and $L_n^{X_{2(n-2)}}$; but $j = n - 1$ and $k = n - 2$ are not of the form $\{n-2r, n-2r-1\}$ for some $1 \leq r \leq \left\lceil \frac{n-2}{2} \right\rceil$.

**Example 4.4.** (1) The case $n = 2$ revisited. Up to Morita equivalence there are two deformed preprojective algebras of type $L_2$, namely $L_2^x$ and $L_2^y$. They are not stably equivalent of Morita type (and hence not derived equivalent) by Theorem 4.2 (5). So we have a
complete classification of deformed preprojective algebras of type \( L_2 \), up to stable equivalence of Morita type (and up to derived equivalence)

(2) **The case \( n = 3 \) revisited.** There are three deformed preprojective algebras of type \( L_3 \), namely \( L_{3}^{X_0} \), \( L_{3}^{X_2} \) and \( L_{3}^{X_4} \). The algebra \( L_{3}^{X_3} \) is not stably equivalent of Morita type (and hence not derived equivalent) to the other two algebras.

But with the Külshammer spaces we can not distinguish the algebras \( L_{3}^{X_0} \) and \( L_{3}^{X_2} \). We don’t know whether these are stably equivalent of Morita type (or derived equivalent), or not.

(3) **The case \( n = 5 \).** For the five algebras (up to Morita equivalence) \( L_{5}^{X_2^j} \) where \( j \in \{0, 1, 2, 3, 4\} \) we get the following numbers for the differences

\[
\dim_K \left( T_i(L_{5}^{X_2^j}) \right) - \dim_K \left( (L_{5}^{X_2^j}, L_{5}^{X_2^{j+1}}) \right) = 5 - \max \left( \left\lfloor \frac{10 - (2^{j+1} - 2)j - (2^{j+1} - 1)}{2^{j+1}} \right\rfloor, 0 \right)
\]

which are invariants under derived equivalence and under stable equivalence of Morita type.

\[
\begin{array}{cccccc}
\hline
i & j & 0 & 1 & 2 & 3 & 4 \\
\hline
1 & 3 & 3 & 4 & 4 & 5 \\
2 & 4 & 5 & 5 & 5 & 5 \\
\geq 3 & 5 & 5 & 5 & 5 & 5 \\
\hline
\end{array}
\]

Therefore the algebras \( L_{5}^{X_0}, L_{5}^{X_2}, L_{5}^{X_4} \) and \( L_{5}^{X_5} \) are pairwise not stably equivalent of Morita type (and hence pairwise not derived equivalent). Note that \( L_{5}^{X_0} \) and \( L_{5}^{X_2} \) can only be distinguished by the second Külshammer spaces.

It remains open whether \( L_{5}^{X_4} \) and \( L_{5}^{X_5} \) are stably equivalent of Morita type (or derived equivalent), or not.

---

**References**


Thorsten Holm
Institut für Algebra, Zahlentheorie und Diskrete Mathematik, Leibniz Universität Hannover, Welfengarten 1, 30167 Hannover, Germany
E-mail address: holm@math.uni-hannover.de
URL: http://www.iazd.uni-hannover.de/~tholm

Alexander Zimmermann
Université de Picardie et LAMFA (UMR 6140 du CNRS), 33 rue St Leu, 80039 Amiens CEDEX 1, France
E-mail address: alexander.zimmermann@u-picardie.fr
URL: http://www.mathinfo.u-picardie.fr/alex/azimengl.html