DEFORMED PREPROJECTIVE ALGEBRAS OF TYPE L: KÜLSHAMMER SPACES AND DERIVED EQUIVALENCES

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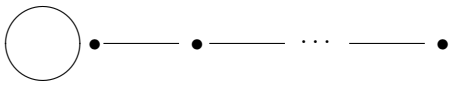
Abstract. Białkowski, Erdmann and Skowroński classified those indecomposable self-injective algebras for which the Nakayama shift of every (non-projective) simple module is isomorphic to its third syzygy. It turned out that these are precisely the deformations, in a suitable sense, of preprojective algebras associated to the simply laced ADE Dynkin diagrams and of another graph L_n, which also occurs in the Happel-Preiser-Ringel classification of subadditive but not additive functions. In this paper we study these deformed preprojective algebras of type L_n via their Külshammer spaces, for which we give precise formulae for their dimensions. These are known to be invariants of the derived module category, and even invariants under stable equivalences of Morita type. As main application of our study of Külshammer spaces we can distinguish many (but not all) deformations of the preprojective algebra of type L_n up to stable equivalence of Morita type, and hence also up to derived equivalence.

1. Introduction

Preprojective algebras have been introduced by Gelfand and Ponomarev [10] and nowadays occur prominently in various areas in mathematics. For a quiver (i.e. a finite directed graph) Q its preprojective algebra is defined by the following process: to any arrow a in Q which is not a loop introduce a new arrow $\overline{a}$ in the opposite direction; for a loop a set $\overline{a} := a$, leading to a new quiver $\overline{Q}$. Then the preprojective algebra $P(Q)$ of type Q over a field K is defined by the quiver with relations $K\overline{Q}/I$ where the ideal is generated by the relations, one for each vertex v in Q, of the form $\sum_{s(a)=v} a\overline{a}$, where $s(a)$ denotes the starting vertex of the arrow a. Note that the preprojective algebra is independent of the orientation of the quiver Q. The preprojective algebra for a quiver associated to a tree is known to be finite-dimensional if and only if the quiver Q is a disjoint union of some orientations of simply laced ADE Dynkin diagrams. The finitely generated modules of the preprojective algebras for ADE Dynkin quivers have remarkable homological properties. Namely, by a result of Schofield [22] each non-projective indecomposable module has Ω-period at most 6, where Ω denotes Heller's syzygy operator; for proofs see [1], [5], [8].

In an attempt to characterise those selfinjective finite-dimensional algebras which share these remarkable periodicity properties, Białkowski, Erdmann and Skowroński introduced in [2] deformations of the preprojective algebras of ADE Dynkin quivers and of an additional graph of type L_n of the following form



which already occurred in the Happel-Preiser-Ringel classification [11] of subadditive but not additive functions.

The deformations $P^f(Q)$ are obtained by perturbing the usual preprojective relation $\sum_{s(a)=v} a\overline{a}$ at one particular vertex by adding a certain polynomial expression f. It turns out that proper deformations occur only for the diagrams of types D, E, and L. For more details on the actual relations we refer to [2, Section 3]. The Białkowski-Erdmann-Skowroński deformations are different and should not be confused with the deformed preprojective algebras of Crawley-Boevey and Holland [6].

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The main result of Białkowski, Erdmann and Skowroński gives the following surprising classification of selfinjective algebras sharing the periodicity properties of preprojective algebras of Dynkin type.

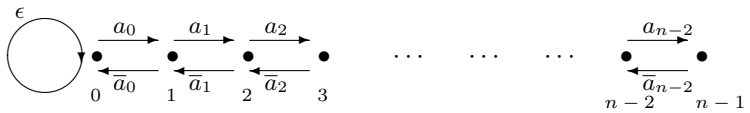
Theorem. ([2, Theorem 1.2]) *Let Λ be a basic, connected, finite-dimensional, selfinjective algebra over an algebraically closed field. Then the following statements are equivalent:*

(i) Λ is isomorphic to a deformed preprojective algebra $P^f(Q)$ for a quiver of type ADE or L.

(ii) $\Omega^3(S) \cong \nu^{-1}S$ for every non-projective simple right Λ-module S, where ν is the Nakayama transformation.

In our present paper we shall study the deformed preprojective algebras of type L_n in the Białkowski-Erdmann-Skowroński sense. Let us start by giving a precise definition of these algebras.

Let K be a field, let $p(X) \in K[X]$ be a polynomial and let $n \in \mathbb{N}$. Then let L_n^p be the K-algebra given by the following quiver with n vertices



subject to the following relations

$$a_i\overline{a}_i + \overline{a}_{i-1}a_{i-1} = 0 \text{ for all } i \in \{1, \ldots, n-2\} \ ,$$
$$\overline{a}_{n-2}a_{n-2} = 0 \ , \ \epsilon^{2n} = 0 \ , \ \epsilon^2 + a_0\overline{a}_0 + \epsilon^3 p(\epsilon) = 0.$$

These algebras are the *deformed preprojective algebras of type L_n*, in the sense of Białkowski, Erdmann and Skowroński [2]. Note that the usual preprojective relations are deformed only at the vertex 0.

More details on these algebras are collected in Section 3 below. In particular we determine their Cartan matrices and we provide an explicit K-basis of the algebra given by a set of paths in the quiver. Moreover, we determine explicitly the centre and the commutator subspace of the deformed preprojective algebras of type L_n.

An important structural property is that all deformed preprojective algebras of type L_n are symmetric algebras. This is a yet unpublished result of Białkowski, Erdmann and Skowroński [3, Corollary 4]; since we build on it in the present paper we give an independent and direct proof of this fact for the sake of completeness (cf. Section 3.2).

It is a subtle question for which deformation polynomials p the deformed preprojective algebras L_n^p become isomorphic. This problem has now been solved by Białkowski, Erdmann and Skowroński [3, Theorem 2]: over a field K of characteristic different from 2 all deformed preprojective algebras of type L_n are isomorphic; however, in characteristic 2 the algebras corresponding to the set of deformation polynomials $p(X) = X^{2j}$ for $j \in \{0, 1, 2, \ldots, n-1\}$ give a complete classification up to Morita equivalence.

We are not building on this classification in the present paper but we use it as a motivation for restricting our computations of Külshammer spaces in Section 4 to the case of deformation polynomials X^{2j}.

The Białkowski-Erdmann-Skowroński characterisation of the selfinjective algebras where for each non-projective simple module the third syzygy is isomorphic to the (inverse) Nakayama transformation can be seen as a condition on the stable module category; we therefore believe it is natural to aim at a classification of the deformed preprojective algebras up to stable equivalence or up to derived equivalence, rather than up to Morita equivalence.

Our main results in this paper provide partial answers to these problems. We are able to distinguish several of the deformed preprojective algebras $L_n^{X^{2j}}$ up to stable equivalence of Morita type and up to derived equivalence. Our main applications in this direction are summarised in the following result.

Theorem 1.1. *Let K be a perfect field of characteristic 2.*

(a) *If two deformed preprojective algebras L_n^p and L_m^q are stably equivalent of Morita type or derived equivalent, then $n = m$.*

(b) *For $n \in \mathbb{N}$, let $j, k \in \{0, 1, \ldots, n-1\}$ be different numbers such that $\{j, k\} \neq \{n-2r, n-2r-1\}$ for every $1 \leq r \leq \lceil \frac{n-2}{2} \rceil$. Then the deformed preprojective algebras $L_n^{X^{2j}}$ and $L_n^{X^{2k}}$ are not stably equivalent of Morita type, and also not derived equivalent.*

These results are obtained as a consequence of a detailed study of the Külshammer spaces for the deformed preprojective algebras of type L. These spaces have been defined by Külshammer in the 1980's for symmetric algebras over a field of positive characteristic; we recall briefly the construction and some fundamental properties from [16]. For an algebra A over a field K let $[A, A]$ be the K-vector space generated by $\{ab - ba \in A \mid a, b \in A\}$ and call this space the commutator subspace of A. Külshammer defined for a K-algebra A over a perfect field K of characteristic $p > 0$ the K-vector spaces $T_i(A) := \{x \in A \mid x^{p^i} \in [A, A]\}$ for every integer $i \geq 0$. They form an ascending series

$$[A, A] = T_0(A) \subseteq T_1(A) \subseteq T_2(A) \subseteq \ldots \subseteq T_i(A) \subseteq T_{i+1}(A) \subseteq \ldots.$$

In [27] it was shown by the second author that for symmetric algebras over a perfect field the codimension of the commutator space of A in $T_i(A)$ is invariant under derived equivalences, and in [17] Liu, Zhou and the second author showed that this codimension is an invariant under stable equivalences of Morita type. In joint work with Bessenrodt [4] we showed that the codimension of $T_i(A)$ in A is an invariant of the derived category of A for general (not necessarily symmetric) finite dimensional algebras.

The derived invariance of the various codimensions of Külshammer spaces proved already to be very useful to distinguish derived equivalence classes of symmetric algebras, see [12], [13], [14], and also to distinguish stable equivalence classes of Morita type, see [24], [25].

For obtaining our above results on deformed preprojective algebras of type L_n (over a perfect field of characteristic 2) we determine the dimension of their Külshammer spaces $T_i(L_n^{X^{2j}})$; our main result in this direction is the following.

Theorem 1.2. *Let K be a perfect field of characteristic 2. Then for every $0 \leq j < n$ we have*

(a) $\dim_K T_i(L_n^{X^{2j}}) - \dim_K [L_n^{X^{2j}}, L_n^{X^{2j}}] = n - \max\left(\left\lceil \frac{2n-(2^{i+1}-2)j-(2^{i+1}-1)}{2^{i+1}} \right\rceil, 0\right)$

(b) $\dim_K [L_n^{X^{2j}}, L_n^{X^{2j}}] = \frac{1}{3}n(n-1)(2n+5)$.

The paper is organised as follows. In Section 2 we recall some results for selfinjective algebras and we propose a method to compute the centre and the quotient of the algebra modulo the commutator space for selfinjective algebras which we believe should be useful in other situations as well. In Section 3 we study the deformed preprojective algebras L_n^p, give a K-basis, the Cartan matrix, the commutator space, and the centre of the algebras. In Section 4 we compute the Külshammer spaces $T_n(L_n^{X^{2j}})$ and deduce the main results.

2. Hochschild homology and Nakayama automorphisms of self-injective algebras

In this section we present some general methods to deal with selfinjective algebras, in particular for getting a Nakayama automorphism and related bilinear forms explicitly. Strictly speaking the results of this section are not used in this generality in the rest of this paper since the deformed preprojective algebras of type L are symmetric (we give an independent proof of this result of Białkowski, Erdmann and Skowroński in Section 3.2 below). However, symmetricity of an algebra is usually not easy to verify so that the methods of this section can be used to deal with Külshammer ideals in cases where one only has selfinjectivity; therefore the methods of this section might be of independent interest.

We need to compute rather explicitly in the degree 0 Hochschild homology of self-injective algebras. This needs some theoretical preparations in order to be able to determine a basis of the commutator subspace of the algebras we need to deal with.

2.1. **The Nakayama-twisted centre.** Let K be a field and let A be a K-algebra. We need to get alternative descriptions of the degree 0 Hochschild homology. By definition of Hochschild homology (using the standard Hochschild complex) we have $HH_0(A) \cong A/[A, A]$.

If A is symmetric, then by definition $A \simeq \mathrm{Hom}_K(A, K)$ as A-A-bimodules (i.e. as $A \otimes_K A^{op}$-module), and so we get

$$\begin{aligned} \mathrm{Hom}_K(A/[A,A], K) &\simeq \mathrm{Hom}_K(HH_0(A), K) \simeq \mathrm{Hom}_K(A \otimes_{A \otimes_K A^{op}} A, K) \\ &\simeq \mathrm{Hom}_{A \otimes_K A^{op}}(A, \mathrm{Hom}_K(A, K)) \simeq \mathrm{Hom}_{A \otimes_K A^{op}}(A, A) \\ &\simeq HH^0(A) \simeq Z(A). \end{aligned}$$

This chain of isomorphisms is one of the main tools for the proof of the main theorem in [26].

If A is only a Frobenius algebra we shall give an analogous description. So we need to understand $\mathrm{Hom}_K(A,K)$ as $A\otimes_K A^{op}$-module. If A is a Frobenius K-algebra then still $A\simeq \mathrm{Hom}_K(A,K)$ as a left A-module. Hence, $\mathrm{Hom}_K(A,K)$ is a free left A-module of rank 1. Moreover,

$$\mathrm{End}_A({}_A\,\mathrm{Hom}_K(A,K))\simeq \mathrm{End}_A({}_AA)\simeq A$$

and so $\mathrm{Hom}_K(A,K)$ is a progenerator over A with endomorphism ring isomorphic to A, hence inducing a Morita self-equivalence of A. Therefore the isomorphism class of $\mathrm{Hom}_K(A,K)$ is in the Picard group $Pic_K(A)$. Moreover, as $\mathrm{Hom}_K(A,K)$ is free of rank 1 as left-module one gets that $\mathrm{Hom}_K(A,K)$ is in the image of

$$\mathrm{Out}_K(A):=\mathrm{Aut}_K(A)/\mathrm{Inn}(A)$$

in $Pic_K(A)$, where this identification is given by sending $\alpha\in\mathrm{Aut}_K(A)$ to the invertible bimodule ${}_1A_\alpha$ which is A as vector space, on which $a\in A$ acts by multiplication on the left and by $\alpha(a)$ on the right (cf. [20, (37.16) Theorem]). Hence, there is an automorphism $\nu\in\mathrm{Aut}_K(A)$ so that

$$\mathrm{Hom}_K(A,K)\simeq {}_1A_\nu$$

as A-A-bimodules and ν is unique up to an inner automorphism.

Definition 2.1. *Let A be a Frobenius K-algebra. Then there is an automorphism ν of A so that $\mathrm{Hom}_K(A,K)\simeq {}_1A_\nu$ as A-A-bimodules. This automorphism is unique up to inner automorphisms and is called the* Nakayama automorphism.

For the dual of the degree 0 Hochschild homology we get

$$\begin{aligned}\mathrm{Hom}_K(A/[A,A],K) &\simeq \mathrm{Hom}_K(HH_0(A),K)\simeq \mathrm{Hom}_K(A\otimes_{A\otimes_K A^{op}}A,K)\\ &\simeq \mathrm{Hom}_{A\otimes_K A^{op}}(A,\mathrm{Hom}_K(A,K))\simeq \mathrm{Hom}_{A\otimes_K A^{op}}(A,\,{}_1A_\nu)\\ &\simeq \{a\in A\mid b\cdot a=a\cdot\nu(b)\text{ for all }b\in A\}\end{aligned}$$

where the last isomorphism is given by sending a homomorphism to the image of $1\in A$.

Definition 2.2. *Let A be a Frobenius K-algebra with Nakayama automorphism ν. Then the* Nakayama twisted centre *is defined to be*

$$Z_\nu(A):=\{a\in A\mid b\cdot a=a\cdot\nu(b)\text{ for all }b\in A\}.$$

Remark 2.3. (1) The automorphism ν is unique only up to an inner automorphism. If ν is inner, let $\nu(a)=u\cdot a\cdot u^{-1}$. Then

$$\begin{aligned}\{a\in A\mid b\cdot a=a\cdot\nu(b)\text{ for all }b\in A\} &= \{a\in A\mid b\cdot a=a\cdot u\cdot b\cdot u^{-1}\text{ for all }b\in A\}\\ &= \{a\in A\mid b\cdot(a\cdot u)=(a\cdot u)\cdot b\text{ for all }b\in A\}\\ &= \{a\in A\mid a\cdot u\in Z(A)\}=Z(A)\cdot u^{-1}\end{aligned}$$

and likewise the twisted centres with respect of two different Nakayama automorphisms differ by multiplication by a unit.

(2) In general the Nakayama twisted centre will not be a ring. However, if $z\in Z(A)$ and $a\in Z_\nu(A)$ then

$$b\cdot za=zba=za\cdot\nu(b)$$

and $za\in Z_\nu(A)$. Hence $Z_\nu(A)$ is a $Z(A)$-submodule of A. The module structure does not depend on the chosen Nakayama automorphism, up to isomorphism of $Z(A)$-modules.

We summarise the above discussion in the following Lemma.

Lemma 2.4. *If A is a Frobenius K-algebra, then there is an automorphism ν of A, unique up to an inner automorphism so that $\mathrm{Hom}_K(A,K)\simeq {}_1A_\nu$ as an A-A-bimodule and $\mathrm{Hom}_K(HH_0(A),K)\simeq Z_\nu(A)$ as $Z(A)$-modules.*

The Frobenius algebra A is symmetric if and only if the Nakayama automorphism ν is inner. ■

Remark 2.5. (1) The automorphism ν is the well-known Nakayama automorphism. (The diligent reader might observe that we are dealing with left modules while originally Nakayama in [19] dealt with right modules, so our ν would be the inverse of the original Nakayama automorphism.)

(2) Using that $HH_0(A)\cong A/[A,A]$, the dimension of the commutator subspace of a selfinjective algebra A can therefore be expressed as

$$\dim_K[A,A]=\dim_K A-\dim_K Z_\nu(A).$$

2.2. **How to get the Nakayama automorphism explicitly.** Let K be a field and let A be a Frobenius K-algebra. In order to compute the Nakayama automorphism ν we need to find an explicit isomorphism $A \longrightarrow \mathrm{Hom}_K(A,K)$ as A-modules. Suppose we get two isomorphisms $\alpha_1 : A \longrightarrow \mathrm{Hom}_K(A,K)$ and $\alpha_2 : A \longrightarrow \mathrm{Hom}_K(A,K)$. Then $\alpha_2^{-1} \circ \alpha_1 : A \longrightarrow A$ is an automorphism of the regular A-module A. Hence, α_1 will differ from α_2 by multiplication by an invertible element $u \in A$. The corresponding Nakayama automorphisms ν_1 and ν_2 computed from α_1 and from α_2 will then differ by the inner automorphism given by conjugation with u. It is therefore sufficient to find one isomorphism $\alpha : A \longrightarrow \mathrm{Hom}_K(A,K)$. Given such an isomorphism α of A-modules, the form $\langle x,y\rangle_\alpha := (\alpha(y))(x)$ for $x,y \in A$ is a non degenerate associative bilinear form on A.

Let $\langle\ ,\ \rangle : A \times A \longrightarrow K$ be a non-degenerate associative K-bilinear form on A (which exists since A is Frobenius), then we get a vector space isomorphism

$$A \xrightarrow{\alpha} \mathrm{Hom}_K(A,K)\ ,\ a \mapsto \langle -, a\rangle.$$

Lemma 2.6. *A non-degenerate associative bilinear form $\langle\ ,\ \rangle : A\times A \longrightarrow K$ induces an isomorphism $A \longrightarrow \mathrm{Hom}_K(A,K)$ as left A-modules by mapping $a \in A$ to the linear form*

$$A \ni b \mapsto \langle b, a\rangle \in K.$$

Proof. By the above discussions the map is an isomorphism of vector spaces. For verifying the module homomorphism property recall the action of A on the dual space $\mathrm{Hom}_K(A,K)$; it is given by $(b\cdot\varphi)(c) = \varphi(cb)$ for all $b,c \in A$ and all $\varphi \in \mathrm{Hom}_K(A,K)$. Then, using that the bilinear form is associative we get

$$\alpha(b\cdot a)(c) = \langle c, b\cdot a\rangle = \langle c\cdot b, a\rangle = (b\cdot\alpha(a))\,(c)$$

for all $a,b,c \in A$, so the map is a homomorphism of left A-modules. □

Proposition 2.7. *Let K be a field and let A be a Frobenius K-algebra. Then the Nakayama automorphism ν of A satisfies $\langle a,b\rangle = \langle b,\nu(a)\rangle$ for all $a,b \in A$, and any automorphism satisfying this formula is a Nakayama automorphism.*

Proof. There is a non-degenerate associative bilinear form on A, which induces an isomorphism between A and the linear forms on A as A-modules by Lemma 2.6. The isomorphism gives an isomorphism of A-A-bimodules of ${}_1A_\nu$ and $\mathrm{Hom}_K(A,K)$ by

$${}_1A_\nu \xrightarrow{\varphi} \mathrm{Hom}_K(A,K)\ ,\ a \mapsto \langle -, a\rangle = \varphi(a).$$

By the twisted bimodule action on ${}_1A_\nu$ we have that $\varphi(1)\cdot a = \varphi(1\cdot a) = \varphi(\nu(a))$ and $b\cdot\varphi(1) = \varphi(b\cdot 1) = \varphi(b)$. Since for $f \in \mathrm{Hom}_K(A,K)$ the A-A-bimodule action on $\mathrm{Hom}_K(A,K)$ is given by $(fa)(b) = f(ab)$ and $(af)(b) = f(ba)$ for all $a,b \in A$, one gets

$$\langle a,b\rangle = (\varphi(b))(a) = (b\cdot\varphi(1))(a) = \varphi(1)(ab) = (\varphi(1)\cdot a)(b) = \varphi(\nu(a))(b) = \langle b,\nu(a)\rangle.$$

Hence, the Nakayama automorphism has the above property. Conversely, if an automorphism ν satisfies $\langle a,b\rangle = \langle b,\nu(a)\rangle$ for all $a,b \in A$, then the mapping $A \longrightarrow \mathrm{Hom}_K(A,K)$ given by $a \mapsto \langle -,a\rangle$ gives an isomorphism of A and $\mathrm{Hom}_K(A,K)$ as A-modules, inducing the element ${}_1A_\nu$ in the Picard group of A. □

We shall later need such a bilinear form explicitly. The following very useful result can be found in [27, Proposition 2.15]; see also [14, Proposition 3.1] for a proof in the case of weakly symmetric algebras.

Proposition 2.8. *Let $A = KQ/I$ be a Frobenius algebra given by the quiver Q and ideal of relations I, and fix a K-basis $\mathcal{B}$ of A consisting of pairwise distinct non-zero paths of the quiver Q. Assume that $\mathcal{B}$ contains a basis of the socle $soc(A)$ of A. Define a K-linear mapping ψ on the basis elements by*

$$\psi(b) = \begin{cases} 1 & \text{if } b \in soc(A)\setminus\{0\} \\ 0 & \text{otherwise} \end{cases}$$

for $b \in \mathcal{B}$. Then an associative non-degenerate K-bilinear form $\langle -,-\rangle$ for A is given by $\langle x,y\rangle := \psi(xy)$.

Remark 2.9. The above bilinear form is in general not symmetric, even if the algebra A is symmetric. For explicit examples we refer to [13, Section 4, proof of main theorem, part (3)] and [27].

Actually, this form is basically the only possible form, at least for finite dimensional basic selfinjective algebras over an algebraically closed field K.

Proposition 2.10. *Let A be a finite dimensional basic Frobenius K-algebra over an algebraically closed field K. Then for every non degenerate associative bilinear form $\langle\, .\,\rangle : A \times A \longrightarrow K$ there is a K-basis $\mathcal{B}$ containing a K-basis of the socle so that $\langle x,\ y\rangle = \psi(xy)$ where*

$$\psi(b) = \begin{cases} 1 & \text{if } b \in soc(A) \setminus \{0\} \\ 0 & \text{otherwise} \end{cases}$$

for $b \in \mathcal{B}$.

Proof. Given an associative bilinear form $\langle .,.\rangle : A \times A \longrightarrow K$ there is a linear map $\psi : A \longrightarrow K$ defined by $\psi(x) := \langle 1, x\rangle$ and for any $x, y \in A$ one gets $\langle x, y\rangle = \langle 1, xy\rangle = \psi(xy)$. Hence ψ determines the associative bilinear map and the associative bilinear map determines ψ.

The algebra is basic and so the socle of A is a direct sum of pairwise non-isomorphic one-dimensional simple A-modules. Let $\{s_1, \ldots, s_n\} \in A$ so that s_iA is simple for every $i \in \{1, 2, \ldots, n\}$ and so that $\mathrm{soc}(A) =< s_1, \ldots, s_n >_K$.

Given an associative non degenerate bilinear form $\langle .,.\rangle$ then $\langle ., s_i\rangle : A \longrightarrow K$ is a non zero linear form on A, since the bilinear form is non degenerate. Hence there is an element $a \in A$ so that $\langle a, s_i\rangle \neq 0$. Now, by the Wedderburn-Malçev theorem, there is an element $\rho \in \mathrm{rad}(A)$ so that $a = \sum_{i=1}^{n} \lambda_i e_i + \rho$ for scalars $\lambda_i \in K$, and where $e_i^2 = e_i$ is an indecomposable idempotent of A, where $e_{\nu^{-1}(i)}s_i = s_i$, and where $e_{\nu^{-1}(j)}s_i = 0$ for $j \neq i$. Hence,

$$\langle a, s_i\rangle = \langle 1, as_i\rangle = \langle 1, \lambda_{\nu^{-1}(i)}s_i\rangle = \lambda_{\nu^{-1}}(i)$$

We replace s_i by $\lambda^{-1}_{\nu^{-1}(i)}s_i$ and get $\langle 1, s_i\rangle = 1$. Take a K-basis $\mathcal{B}_i$ of $\ker(\langle ., s_i\rangle)$ in $Ae_{\nu^{-1}(i)}$. Then, since $A = \bigoplus_{j=1}^{n} Ae_j$,

$$\mathcal{B} := \bigcup_{i=1}^{n} \mathcal{B}_i \cup \{s_1, s_2, \ldots, s_n\}$$

is a K-basis of A satisfying the hypotheses of Proposition 2.8. Moreover, if $xy \in \mathcal{B}$, then there is a unique $e_i^2 = e_i$ so that $xye_i = xy$, and so

$$\langle x, y\rangle = \langle 1, xy\rangle = \sum_{i=1}^{n} \langle 1, xye_i\rangle = \begin{cases} 1 & \text{if } xy \in \mathrm{soc}(A) \\ 0 & \text{else} \end{cases}$$

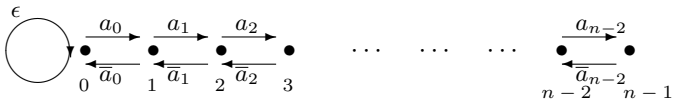
This shows the statement. □

3. Deformed preprojective algebras of type L

3.1. **K-bases of the deformed preprojective algebras of type L.** The aim of this section is to obtain an explicit vector space basis for any deformed preprojective algebra of type L and to deduce some structural properties. In particular we shall get the Cartan matrices and give an independent and constructive proof of a result of Białkowski, Erdmann and Skowroński [3, Corollary 4] that the deformed preprojective algebras of type L are symmetric algebras.

For the convenience of the reader we start by recalling from the introduction the definition of the deformed preprojective algebras of type L.

Let K be a field. For any $n \in \mathbb{N}$ and any polynomial $p(X) \in K[X]$ let L_n^p be the K-algebra given by the following quiver with n vertices $0, 1, \ldots, n-1$ of the form



subject to the following relations

$$a_s\overline{a}_s + \overline{a}_{s-1}a_{s-1} = 0 \text{ for all } s \in \{1, \ldots, n-2\}\ ,$$
$$\overline{a}_{n-2}a_{n-2} = 0\ ,\ \epsilon^{2n} = 0\ ,\ \epsilon^2 + a_0\overline{a}_0 + \epsilon^3 p(\epsilon) = 0.$$

Recall that the deformed preprojective algebras of type L are selfinjective (cf [2, Lemma 3.2]), and hence Frobenius.

Our first aim is to give a K-basis of the algebra L_n^p. We start by providing a generating set. Considering a path starting at the vertex i and ending at the vertex j, we have two cases.

Firstly suppose that the path does not contain ϵ.

If $i < j$, then using the relations $a_s\overline{a}_s + \overline{a}_{s-1}a_{s-1} = 0$ for all $s \in \{1, \dots, n-2\}$ in L_n^p we may replace the path, up to a sign, by one of the following elements of L_n^p:

- the path $a_i a_{i+1} \dots a_{j-1}$
- or the path $a_i a_{i+1} \dots a_{j-1} a_j \dots a_\ell \overline{a}_\ell \overline{a}_{\ell-1} \dots \overline{a}_j$ (for some $j \le l \le n-2$)
- or by 0.

In fact up to a sign we can order the arrows in the path so that all a_r's come first and then all the $\overline{a}_r$'s; we can do this unless we hit a subpath $\dots \overline{a}_{n-2}a_{n-2} \dots$ in which case the path becomes 0 in L_n^p.

Similarly, if $i \ge j$ we may replace the given path, up to a sign, by one of the following elements:

- the path $\overline{a}_{i-1}\overline{a}_{i-2} \dots \overline{a}_j$
- or the path $a_i a_{i+1} \dots a_\ell \overline{a}_\ell \overline{a}_{\ell-1} \dots \overline{a}_i \dots \overline{a}_j$ (for some $i \le l \le n-2$)
- or by 0.

Secondly, suppose the path contains ϵ.

Suppose such a path p contains ϵ at at least two positions, i.e. it is of the form $p = p_1\epsilon^s p_2 \epsilon^t p_3$ where p_2 is a closed path starting and ending at the vertex 0. Using the relations $a_s\overline{a}_s + \overline{a}_{s-1}a_{s-1} = 0$ for all $s \in \{1, \dots, n-2\}$ and the fact that $a_0\overline{a}_0$ commutes with ϵ (because $a_0\overline{a}_0 = -(\epsilon^2 + \epsilon^3 p(\epsilon))$) we see that p_2 commutes with ϵ and hence $p = p_1 p_2 \epsilon^{s+t} p_3$. Thus for any path containing ϵ we may suppose that it contains some power of ϵ at exactly one position.

Moreover, using the relations $\epsilon^2 + \epsilon^3 p(\epsilon) + a_0\overline{a}_0 = 0$ and $\epsilon^{2n} = 0$ we may replace any path containing powers of ϵ by a linear combination of paths containing only ϵ.

In fact, the relation above reads $\epsilon^2(1 + \epsilon p(\epsilon)) = -a_0\overline{a}_0$. Obviously, $\epsilon p(\epsilon)$ is nilpotent and therefore $1 + \epsilon p(\epsilon)$ is invertible. Let m be minimal such that $(\epsilon p(\epsilon))^{m+1} = 0$; then we get

$$\epsilon^2 = -(a_0\overline{a}_0)(1 + \epsilon p(\epsilon))^{-1} = -(a_0\overline{a}_0)\left(1 - \epsilon p(\epsilon) + (\epsilon p(\epsilon))^2 - \dots + (-1)^m(\epsilon p(\epsilon))^m\right).$$

In particular, ϵ^2 is a linear combination of (desired) paths involving ϵ at most once and a term of the form $a_0\overline{a}_0\epsilon^2\tilde{p}(\epsilon)$ for some polynomial $\tilde{p}$. The crucial observation is that in the latter term the paths involving ϵ^2 have longer length than before. But the algebra is finite-dimensional thus there is a number N so that all paths with length bigger than N represent the 0 element. So given a path $p_1\epsilon^s p_2$ with $s \ge 2$ one can successively replace ϵ^2 by the above expression until one reaches a linear combination of paths involving ϵ only once.

Using again the relations $a_s\overline{a}_s + \overline{a}_{s-1}a_{s-1} = 0$ for all $s \in \{1, \dots, n-2\}$ and the fact that $a_0\overline{a}_0$ commutes with ϵ (because $a_0\overline{a}_0 = -(\epsilon^2 + \epsilon^3 p(\epsilon))$) we can move all a_r's in the path to the right of ϵ. (Note that by combining these two reductions we can indeed guarantee that in each path occurring in the linear combination ϵ occurs only once.) Thus the given path represents the same element in L_n^p as a linear combination of paths of the following forms

- $\overline{a}_{i-1}\overline{a}_{i-2} \dots \overline{a}_0\epsilon a_0 a_1 \dots a_{j-1}$
- $\overline{a}_{i-1}\overline{a}_{i-2} \dots \overline{a}_0\epsilon a_0 a_1 \dots a_{\ell-1} a_\ell \overline{a}_\ell \overline{a}_{\ell-1} \dots \overline{a}_j$ (for some $j \le l \le n-2$).

If $i < j$ these paths are all non-zero and they will be part of the basis to be given below. However, if $i \ge j$ some of the paths of the latter type vanish, so we shall now derive a different expression for these.

To this end observe that by using the relations $a_s\overline{a}_s + \overline{a}_{s-1}a_{s-1} = 0$ we can successively move the $\overline{a}_r$'s to the left and obtain

$$\overline{a}_{i-1}\overline{a}_{i-2} \dots \overline{a}_0\epsilon a_0 a_1 \dots a_\ell \overline{a}_\ell \overline{a}_{\ell-1} \dots \overline{a}_j = \pm\, \overline{a}_{i-1}\overline{a}_{i-2} \dots \overline{a}_0\epsilon(a_0\overline{a}_0)^{\ell-j+1} a_0 a_1 \dots a_{j-1}.$$

Moreover, using that $a_0\overline{a}_0$ commutes with ϵ and then moving the a_r's to the left we get

$$\begin{aligned}
&\overline{a}_{i-1}\overline{a}_{i-2} \dots \overline{a}_0\epsilon a_0 a_1 \dots a_{\ell-1} a_\ell \overline{a}_\ell \overline{a}_{\ell-1} \dots \overline{a}_j \\
&\quad = \pm\, \overline{a}_{i-1}\overline{a}_{i-2} \dots \overline{a}_0(a_0\overline{a}_0)^{\ell-j+1}\epsilon a_0 a_1 \dots a_{j-1} \\
&\quad = \pm\, \overline{a}_{i-1}\overline{a}_{i-2} \dots (\overline{a}_0 a_0)^{\ell-j+1}\overline{a}_0\epsilon a_0 a_1 \dots a_{j-1} \\
&\quad = \pm \begin{cases} a_i a_{i+1} \dots a_{i+\ell-j}\overline{a}_{i+\ell-j}\overline{a}_{i+\ell-j-1} \dots \overline{a}_1\overline{a}_0\epsilon a_0 a_1 \dots a_{j-1} & \text{if } i+\ell-j \le n-2 \\ 0 & \text{else} \end{cases}
\end{aligned}$$

The following result provides explicit vector space bases for the deformed preprojective algebras L_n^p of type L. Note that the bases do not involve the deformation polynomial p, i.e. the bases is independent of the polynomial.

Proposition 3.1. *A K-basis of L_n^p is given by the following paths between the vertices i and j, where $i, j \in \{0, 1, \dots, n-1\}$.*

(1) $a_i a_{i+1} \dots a_{j-1}$	*for $i < j$*
(2) $a_i a_{i+1} \dots a_{j-1} a_j \dots a_\ell \overline{a}_\ell \overline{a}_{\ell-1} \dots \overline{a}_j$	*for $i < j$ and some $j \leq \ell \leq n-2$*
(3) $\overline{a}_{i-1} \overline{a}_{i-2} \dots \overline{a}_j$	*for $i \geq j$*
(4) $a_i a_{i+1} \dots a_\ell \overline{a}_\ell \overline{a}_{\ell-1} \dots \overline{a}_i \dots \overline{a}_j$	*for $i \geq j$ and some $i \leq \ell \leq n-2$*
(5) $\overline{a}_{i-1} \overline{a}_{i-2} \dots \overline{a}_0 \epsilon a_0 a_1 \dots a_{j-1}$	*for any i, j*
(6) $\overline{a}_{i-1} \overline{a}_{i-2} \dots \overline{a}_0 \epsilon a_0 a_1 \dots a_{\ell-1} a_\ell \overline{a}_\ell \overline{a}_{\ell-1} \dots \overline{a}_j$	*for $i < j$ and some $j \leq \ell \leq n-2$*
(7) $a_i a_{i+1} \dots a_\ell \overline{a}_\ell \overline{a}_{\ell-1} \dots \overline{a}_1 \overline{a}_0 \epsilon a_0 a_1 \dots a_{j-1}$	*for $i \geq j$ and some $i \leq \ell \leq n-2$*

Remark 3.2. (1) In type (3) the case $i = j$ yields an empty product which has to be interpreted as the trivial paths e_i for every vertex i.

(2) The longest paths in this basis of L_n^p are of length $2n-1$, occurring in (5) for $i = j = n-1$ and in (7) for $i = j \in \{0, 1, \dots, n-2\}$ and $l = n-2$, respectively. These elements span the socles of the projective indecomposable modules corresponding to the vertices $i \in \{0, 1, \dots, n-1\}$. Indeed, these elements are annihilated by multiplication by any arrow. This implies that they are in the socle. Since the algebras L_n^p are self-injective (cf [2, Lemma 3.2]), the socle of each projective indecomposable module is one-dimensional.

(3) The socle element of the projective indecomposable module corresponding to the vertex 0 can also be expressed in terms of powers of the loop ϵ (recall that $\epsilon^{2n} = 0$ in the algebra L_n^p). In fact it is not hard to check that we have $\epsilon^{2n-1} = a_0 a_1 \dots a_{n-2} \overline{a}_{n-2} \overline{a}_{n-3} \dots \overline{a}_1 \overline{a}_0 \epsilon$. (Note that lower powers of ϵ are not necessarily occurring as paths in the above list, but are linear combinations of these, the precise shape depending on the deformation polynomial p.)

(4) The above basis seems very suitable for making the following inductive proof work. However, later in the paper we will also use slightly different bases involving powers of the loop ϵ.

Proof. The above discussion proves that the given elements form a generating set. We need to show that these elements are linearly independent. Since the defining relations of the algebra L_n^p are relations between closed paths, we may suppose that a linear combination of paths starting at i and ending at j is 0. By symmetry we may suppose that $i \geq j$ and hence we get a linear combination

$$\begin{aligned} 0 \quad = \quad & \nu_0 \cdot \overline{a}_{i-1} \overline{a}_{i-2} \dots \overline{a}_j + \nu_1 \cdot \overline{a}_{i-1} \overline{a}_{i-2} \dots \overline{a}_0 \epsilon a_0 a_1 \dots a_{j-1} + \\ & + \sum_{\ell=i}^{n-2} \lambda_\ell \cdot a_i a_{i+1} \dots a_\ell \overline{a}_\ell \overline{a}_{\ell-1} \dots \overline{a}_{i+1} \overline{a}_i \dots \overline{a}_j + \\ & + \sum_{\ell=i}^{n-2} \mu_\ell \cdot a_i a_{i+1} \dots a_\ell \overline{a}_\ell \overline{a}_{\ell-1} \dots \overline{a}_1 \overline{a}_0 \epsilon a_0 a_1 \dots a_{j-1} \end{aligned}$$

with scalars $\nu_0, \nu_1, \lambda_\ell, \mu_\ell \in K$. Note that the paths occurring have the following lengths: length $i - j$ for the summand of type (3) with coefficient ν_0, length $i + j + 1$ for the summand of type (5) with coefficient ν_1, length $2\ell - i - j + 2$ for the summand of type (4) with coefficient λ_ℓ, and length $2\ell - i + j + 3$ for the summand of type (7) with coefficient μ_ℓ.

Denote by $J = J(L_n^p)$ the two-sided ideal of L_n^p generated by the arrows of the quiver.

From the lengths of the paths we observe that all summands in the above expression are contained in J^{i-j+1}, except the one with coefficient ν_0. So considering the above equation modulo J^{i-j+1} we can deduce that $\nu_0 = 0$.

The remaining summands are given by paths which pass through the vertex $i+1$ (recall that $\ell \geq i$), except for the summand with coefficient ν_1. So considering the above expression modulo the two-sided ideal $L_n^p e_{i+1} L_n^p$ we get that also $\nu_1 = 0$.

Hence we are left to consider the equation

$$0 = \sum_{\ell=i}^{n-2} a_i a_{i+1} \dots a_\ell \overline{a}_\ell \overline{a}_{\ell-1} \dots \overline{a}_{i+1} \overline{a}_i \dots \overline{a}_j \left(\lambda_\ell + \mu_\ell \overline{a}_{j-1} \overline{a}_{j-2} \dots \overline{a}_1 \overline{a}_0 \epsilon a_0 a_1 \dots a_{j-1} \right).$$

We shall prove by induction on ℓ that all coefficients are 0. Observe that for each $\ell = i, \dots, n-2$ the paths with coefficients λ_ℓ and μ_ℓ pass through the vertex $\ell + 1$ but not through the vertex $\ell + 2$.

For $\ell = i$ we consider the above equation modulo the two-sided ideal $L_n^p e_{i+2} L_n^p$ and obtain that

$$0 = \lambda_i \cdot a_i \overline{a}_i \overline{a}_{i-1} \dots \overline{a}_j + \mu_i \cdot a_i \overline{a}_i \overline{a}_{i-1} \dots \overline{a}_j \dots \overline{a}_0 \epsilon a_0 a_1 \dots a_{j-1}$$

Since the first path is strictly shorter than the second we again consider the equation modulo a suitable power of the ideal J and can deduce that $\lambda_i = 0$, and then also that $\mu_i = 0$.

By a completely analogous argument we can immediately deduce inductively that all coefficients λ_ℓ and μ_ℓ are 0. □

Remark 3.3. From Proposition 3.1 one can derive the Cartan matrix of the deformed preprojective algebras of type L_n. For any basic K-algebra A with primitive idempotents $e_0, e_1, \ldots, e_n$ the coefficients of the Cartan matrix are the dimensions of e_iAe_j. Hence, if we may choose an explicit basis of e_iAe_j for all $i, j \in \{0, 1, \ldots, n\}$ we can just read off the Cartan matrix by counting these elements. Since the basis found above for L_n^p is independent of the deformation polynomial p the Cartan matrices of L_n^0 and L_n^p coincide; this has already been observed by Białkowski, Erdmann and Skowroński [2, Lemma 3.2]. The Cartan matrix C_n of the deformed preprojective algebras of type L_n actually has the following form

$$C_n = 2 \cdot \begin{pmatrix} n & n-1 & \ldots & 2 & 1 \\ n-1 & n-1 & \ldots & 2 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 2 & 2 & \ldots & 2 & 1 \\ 1 & 1 & \ldots & 1 & 1 \end{pmatrix}$$

From this shape one easily computes that the determinant of the Cartan matrix is $\det C_n = 2^n$ for all $n \in \mathbb{N}$. Moreover, the vector space dimension of L_n^p is $\frac{1}{3}n(n+1)(2n+1)$ for all $n \in \mathbb{N}$.

3.2. **Deformed preprojective algebras of type L are symmetric.** The aim of this section is to show that for any deformation polynomial $p \in K[X]$ and any $n \in \mathbb{N}$ the deformed preprojective algebra L_n^p is a symmetric algebra. This is a result of Białkowski, Erdmann and Skowroński [3, Corollary 4]. For the convenience of the reader we include an independent direct proof of this fact in this section.

According to Lemma 2.4 it suffices to show that the identity is a Nakayama automorphism for the algebra L_n^p. Recall from Proposition 2.7 that a Nakayama automorphism ν for a self-injective algebra A over a field K is characterized by the property $\langle a, b\rangle = \langle b, \nu(a)\rangle$ for all $a, b \in A$ where $\langle ., .\rangle$ is a non-degenerate associative K-bilinear form on A.

The following general observation turns out to be useful when verifying that a certain automorphism is indeed a Nakayama automorphism; namely, it suffices to check the crucial property on algebra generators of A.

Lemma 3.4. *Let A be a self-injective algebra, with a non-degenerate associative K-bilinear form $\langle ., .\rangle$. If an automorphism ν of A satisfies $\langle a_i, b\rangle = \langle b, \nu(a_i)\rangle$ for a set of algebra generators $\{a_1, \ldots, a_r\}$ and all $b \in A$ then ν is a Nakayama automorphism of A.*

Proof. Every element of A can be expressed as a product of the algebra generators. We show that $\langle a, b\rangle = \langle b, \nu(a)\rangle$ for all $a, b \in A$ by induction on the length of such an expression for a. For any algebra generator a_j and $a, b \in A$ we have

$$\langle aa_j, b\rangle = \langle a, a_jb\rangle = \langle a_jb, \nu(a)\rangle = \langle a_j, b\nu(a)\rangle = \langle b\nu(a), \nu(a_j)\rangle = \langle b, \nu(a)\nu(a_j)\rangle = \langle b, \nu(aa_j)\rangle$$

where for the first, third and fifth equality we used the associativity of the form, for the second we used the induction hypotheses, for the fourth equality we used the assumption on ν for algebra generators, and the last equality holds because ν is an algebra homomorphism. □

Recall from Proposition 2.8 the construction of an associative non-degenerate bilinear form on a self-injective algebra, depending on the choice of a suitable basis. For a basis $\mathcal{B}$ consisting of non-zero distinct paths and containing a basis of the socle this bilinear form has been defined on basis elements by $\langle a, b\rangle = \psi(ab)$ where $\psi(x) = \begin{cases} 1 & \text{if } x \in \operatorname{soc}(A) \cap \mathcal{B} \\ 0 & \text{if } x \notin \operatorname{soc}(A) \cap \mathcal{B} \end{cases}$.

For our aim of proving that the identity is a Nakayama automorphism for L_n^p we shall show that the bilinear form $\langle ., .\rangle$ corresponding to the basis $\mathcal{B}$ given in Proposition 3.1 is indeed symmetric. By the previous lemma we therefore have to verify that $\langle a, b\rangle = \langle b, a\rangle$ for every algebra generator $a \in \{e_0, \ldots, e_{n-1}, \epsilon, a_0, \ldots, a_{n-2}, \overline{a}_0, \ldots, \overline{a}_{n-2}\}$, and b running through the basis of Proposition 3.1.

It is immediate from the definition of the form $\langle ., .\rangle$ that $\langle e_i, b\rangle = \langle b, e_i\rangle$; in fact, the value on either side is 1 precisely if b is a basis element from the socle, and 0 otherwise.

So it remains to deal with the cases where a is an arrow of the quiver of L_n^p.

We start with the loop ϵ. By definition the value in both $\langle \epsilon, b\rangle$ and $\langle b, \epsilon\rangle$ is 0 unless $b \in e_0 L_n^p e_0$. By Proposition 3.1, for the latter space a basis is given by the elements e_0, ϵ, $a_0 \dots a_\ell \overline{a}_\ell \dots \overline{a}_0$ and $a_0 \dots a_\ell \overline{a}_\ell \dots \overline{a}_0 \epsilon$ where $0 \leq \ell \leq n-2$. Using the defining relations $a_s\overline{a}_s + \overline{a}_{s-1}a_{s-1} = 0$ it is not difficult to see that in L_n^p we have $a_0 \dots a_\ell \overline{a}_\ell \dots \overline{a}_0 = \pm(a_0\overline{a}_0)^{\ell+1}$. From this we can deduce, by using the relation $\epsilon^2 + \epsilon^3 p(\epsilon) + a_0\overline{a}_0 = 0$, that ϵ commutes with every element of $e_0 L_n^p e_0$. But then we clearly have for all $b \in e_0 L_n^p e_0$ that

$$\langle \epsilon, b\rangle = \psi(\epsilon b) = \psi(b\epsilon) = \langle b, \epsilon\rangle.$$

We now consider the case $a = a_r$ (for some $0 \leq r \leq n-2$). Again by definition the value in both $\langle a_r, b\rangle$ and $\langle b, a_r\rangle$ is 0 unless $b \in e_{r+1} L_n^p e_r$. Moreover, for a basis element $b \in e_{r+1} L_n^p e_r$ the value in both $\langle a_r, b\rangle$ and $\langle b, a_r\rangle$ is also 0 unless $a_r b$ (resp. $b a_r$) is a nonzero element in the socle of L_n^p. According to Remark 3.2 (2) we know that $a_r b$ and $b a_r$ can only be a nonzero element in the socle if b is a path of length $2n-2$. However, it is immediately checked that the only basis element $b \in e_{r+1} L_n^p e_r$ in Proposition 3.1 of length $2n-2$ is

$$b = a_{r+1}a_{r+2} \dots a_{n-2}\overline{a}_{n-2}\overline{a}_{n-3} \dots \overline{a}_0 \epsilon a_0 a_1 \dots a_{r-1}.$$

For this element we have that $a_r b$ is the socle element in $\mathcal{B}$ (of type (7)) corresponding to the vertex r and that $b a_r$ is the socle element corresponding to the vertex $r+1$ (of type (7) if $r < n-2$ and of type (5) if $r = n-2$), i.e. we can deduce

$$\langle a_r, b\rangle = \psi(a_r b) = 1 = \psi(b a_r) = \langle b, a_r\rangle.$$

Since by the above remarks in all other cases for b both values $\langle a_r, b\rangle$ and $\langle b, a_r\rangle$ vanish we get the desired statement $\langle a_r, b\rangle = \langle b, a_r\rangle$ for all basis elements b from Proposition 3.1.

Finally we consider the case where $a = \overline{a}_r$ for some $0 \leq r \leq n-2$. This is mainly analogous to the previous case but at a certain point pointed out below one has to be careful. Again by definition the value in both $\langle \overline{a}_r, b\rangle$ and $\langle b, \overline{a}_r\rangle$ is 0 unless $b \in e_r L_n^p e_{r+1}$ and $\overline{a}_r b$ and $b\overline{a}_r$ are nonzero elements in the socle. By Remark 3.2 (2) the products $\overline{a}_r b$ and $b\overline{a}_r$ can only be nonzero elements in the socle if b is a path of length $2n-2$. The only basis element $b \in e_r L_n^p e_{r+1}$ of length $2n-2$ in Proposition 3.1 occurs in type (6) (for $r < n-2$) and in type (5) (for $r = n-2$) and has the form

$$b = \overline{a}_{r-1}\overline{a}_{r-2} \dots \overline{a}_0 \epsilon a_0 \dots a_{n-2}\overline{a}_{n-2} \dots \overline{a}_{r+1}.$$

Now, when calculating the values of $\langle \overline{a}_r, b\rangle$ and $\langle b, \overline{a}_r\rangle$ one has to be careful since the products

$$\overline{a}_r b = \overline{a}_r\overline{a}_{r-1}\overline{a}_{r-2} \dots \overline{a}_0 \epsilon a_0 \dots a_{n-2}\overline{a}_{n-2} \dots \overline{a}_{r+1} \in e_{r+1} L_n^p e_{r+1}$$

and

$$b\overline{a}_r = \overline{a}_{r-1}\overline{a}_{r-2} \dots \overline{a}_0 \epsilon a_0 \dots a_{n-2}\overline{a}_{n-2} \dots \overline{a}_{r+1}\overline{a}_r \in e_r L_n^p e_r$$

are not elements of the basis $\mathcal{B}$ given in Proposition 3.1; the only exception is for $r = n-2$ where $\overline{a}_{n-2} b$ is a basis element of type (5). However, Lemma 3.6 (1) below shows how to express these in terms of the basis $\mathcal{B}$; namely for all $r \in \{0, \dots, n-2\}$ we have that $\overline{a}_r b$ and $b\overline{a}_r$ are equal (not only up to a scalar!) to the socle elements occurring in the basis $\mathcal{B}$. Hence we obtain that $\langle \overline{a}_r, b\rangle = \psi(\overline{a}_r b) = 1 = \psi(b\overline{a}_r) = \langle b, \overline{a}_r\rangle$, as desired.

Summarizing our above arguments we have now shown that the associative non-degenerate bilinear form $\langle ., . \rangle$ corresponding (in the sense of 2.8) to the basis $\mathcal{B}$ of Proposition 3.1 is symmetric. We therefore have given an independent proof of the following result, which is originally due to Białkowski, Erdmann and Skowroński [3, Corollary 4].

Theorem 3.5. *Let K be a field (of any characteristic). For all $n \in \mathbb{N}$ and every polynomial $p \in K[X]$ the deformed preprojective algebra L_n^p is a symmetric algebra.*

We complete this section by providing an auxiliary result on relations in the algebras L_n^p; the first part provides the missing calculations in the last part of the above proof, the second part will be used in later sections.

Lemma 3.6. *The following identities hold in the algebra L_n^p.*

(a) *For all $r \in \{0, \dots, n-2\}$ we have that*

$$\overline{a}_{r-1}\overline{a}_{r-2} \dots \overline{a}_0 \epsilon a_0 \dots a_{n-2}\overline{a}_{n-2} \dots \overline{a}_{r+1}\overline{a}_r = a_r a_{r+1} \dots a_{n-2}\overline{a}_{n-2}\overline{a}_{n-3} \dots \overline{a}_0 \epsilon a_0 a_1 \dots a_{r-1}$$

(b) *For all $l \in \{0, 1, \dots, n-2\}$ we have that $\overline{a}_{n-2}\overline{a}_{n-3} \dots \overline{a}_l a_l = 0$.*

Proof. (a) We shall use frequently the relations $a_s\overline{a}_s + \overline{a}_{s-1}a_{s-1} = 0$ for $s \in \{1, \dots, n-2\}$ and carefully keep track of the signs occurring.

In the expression on the left hand side of assertion (a) we start by successively moving $\overline{a}_{n-2}, \dots, \overline{a}_r$ to the left (but still right of ϵ); note that each such move gives a minus sign. Setting $c := (n-2) + (n-3) + \dots + (r+1) + r$ for abbreviation we obtain that

$$\overline{a}_{r-1}\overline{a}_{r-2}\dots\overline{a}_0\epsilon a_0\dots a_{n-2}\overline{a}_{n-2}\dots\overline{a}_{r+1}\overline{a}_r = (-1)^c\,\overline{a}_{r-1}\overline{a}_{r-2}\dots\overline{a}_0\epsilon(a_0\overline{a}_0)^{n-r-1}a_0a_1\dots a_{r-1}. \tag{1}$$

It follows directly from the defining relation $\epsilon^2 + \epsilon^3 p(\epsilon) + a_0\overline{a}_0 = 0$ that ϵ commutes with $a_0\overline{a}_0$, so the expression on the right hand side of (1) is equal to

$$(-1)^c\,\overline{a}_{r-1}\overline{a}_{r-2}\dots\overline{a}_0(a_0\overline{a}_0)^{n-r-1}\epsilon a_0a_1\dots a_{r-1}. \tag{2}$$

The part to the right of ϵ already has the desired shape. To the left of ϵ we now successively move a_0's to the left; for the first a_0 we need r such moves and obtain that the expression in (2) equals

$$(-1)^c(-1)^r a_r\overline{a}_r\overline{a}_{r-1}\dots\overline{a}_1\overline{a}_0(a_0\overline{a}_0)^{n-r-2}\epsilon a_0a_1\dots a_{r-1}. \tag{3}$$

For moving the next a_0 we need $r+1$ moves etc and eventually get another sign of $(-1)^c$; more precisely the expression in (3) is equal to

$$(-1)^c(-1)^c a_r a_{r+1}\dots a_{n-2}\overline{a}_{n-2}\overline{a}_{n-3}\dots\overline{a}_0\epsilon a_0a_1\dots a_{r-1} \tag{4}$$

where the signs cancel so that this is precisely the right hand side in the assertion of part (a) of the lemma.

(b) We show this by reverse induction on l. For $l = n-2$ this is just the relation $\overline{a}_{n-2}a_{n-2} = 0$. For $l < n-2$ we use the defining relation $\overline{a}_l a_l = a_{l+1}\overline{a}_{l+1}$ and obtain

$$\overline{a}_{n-2}\overline{a}_{n-3}\dots\overline{a}_{l+1}\overline{a}_l a_l = \overline{a}_{n-2}\overline{a}_{n-3}\dots\overline{a}_{l+1}a_{l+1}\overline{a}_{l+1}$$

where the latter is zero by induction hypothesis. □

3.3. **Linking L^p_{n+1} and L^p_n.** For proving statements about the algebras L^p_n we shall often argue by induction and then the following result will turn out to be useful. As usual we denote the trivial path of length zero corresponding to the vertex i by e_i.

Lemma 3.7. *For any $n \geq 1$, there is an algebra epimorphism $\pi_n : L^p_{n+1} \longrightarrow L^p_n$ satisfying*

$$\pi_n(e_i) = e_i \text{ for all } 0 \leq i \leq n-1\ ,\ \pi_n(e_n) = 0,\ \pi_n(\epsilon) = \epsilon,$$
$$\pi_n(a_i) = a_i,\ \pi_n(\overline{a}_i) = \overline{a}_i \text{ for all } 0 \leq i \leq n-2\ ,\ \pi_n(a_{n-1}) = 0,\ \pi_n(\overline{a}_{n-1}) = 0.$$

Moreover, π_n induces an algebra isomorphism

$$L^p_{n+1}/(L^p_{n+1}e_nL^p_{n+1}) \simeq L^p_n\ .$$

Proof. The map π_n is well-defined since the defining relations for the algebra L^p_{n+1} are clearly verified in L^p_n (perhaps the only not entirely obvious check is that $\pi_n(a_{n-1}\overline{a}_{n-1} + \overline{a}_{n-2}a_{n-2}) = \pi_n(a_{n-1})\pi_n(\overline{a}_{n-1}) + \pi_n(\overline{a}_{n-2})\pi_n(a_{n-2}) = 0 + \overline{a}_{n-2}a_{n-2}$ which is zero in L^p_n).

For the second statement we need to determine the kernel of π_n (since π_n is surjective by definition). By definition of π_n we have that $L^p_{n+1}e_nL^p_{n+1}$ is contained in the kernel. On the other hand, the dimension of the kernel is the difference of the dimensions of the algebras L^p_{n+1} and L^p_n. These are given in Remark 3.3 and we obtain

$$\dim\ker\pi_n = \frac{1}{3}(n+1)(n+2)(2n+3) - \frac{1}{3}n(n+1)(2n+1) = 2(n+1)^2.$$

However, in the basis of L^p_{n+1} provided in Proposition 3.1, there are already $2(n+1)^2$ basis elements which pass through the vertex n, i.e. are contained in $L^p_{n+1}e_nL^p_{n+1}$. (More precisely, there are n such paths of type (1), $\frac{n(n-1)}{2}$ of type (2), $n+1$ of type (3), $\frac{n(n+1)}{2}$ of type (4), $2n+1$ of type (5), $\frac{n(n-1)}{2}$ of type (6) and $\frac{n(n+1)}{2}$ of type (7), respectively.)

Since $L^p_{n+1}e_nL^p_{n+1} \subseteq \ker\pi_n$ and dimensions agree, the second claim of the lemma follows. □

Remark 3.8. Since $\epsilon^{2n} = 0$ in L^p_n, also ϵ^{2n} is in the kernel of π_n. The dimension arguments in the above proof thus show that ϵ^{2n} is contained in $L^p_{n+1}e_nL^p_{n+1}$, i.e., ϵ^{2n} is a linear combination of paths passing through the vertex n (which could also be checked directly).

3.4. **Generating the commutator subspace.** We start with some computations on the basis elements occurring in Proposition 3.1.

The basis in Proposition 3.1 of L_n^p is actually a union of bases of $e_iL_n^pe_j$ for $i,j \in \{0,1,\dots,n-1\}$. We consider the case $i=j$ in Proposition 3.1. Only the basis elements of type (3), (4), (5) and (7) admit $i=j$. Up to signs (which are not essential since we are only interested in generating sets) we have

$$\begin{aligned}
(a_i\overline{a}_i)^2 &= \pm a_ia_{i+1}\overline{a}_{i+1}\overline{a}_i\\
(a_i\overline{a}_i)^3 &= \pm a_ia_{i+1}a_{i+2}\overline{a}_{i+2}\overline{a}_{i+1}\overline{a}_i\\
\dots &\quad \dots\\
(a_i\overline{a}_i)^{\ell+1} &= \pm a_ia_{i+1}\dots a_{i+\ell}\overline{a}_{i+\ell}\dots\overline{a}_{i+1}\overline{a}_i\\
(a_i\overline{a}_i)^{\ell+2} &= \pm (a_i\overline{a}_i)(a_ia_{i+1}\dots a_{i+\ell}\overline{a}_{i+\ell}\dots\overline{a}_{i+1}\overline{a}_i)\\
&= \pm a_ia_{i+1}\dots a_{i+\ell}a_{i+\ell+1}\overline{a}_{i+\ell+1}\overline{a}_{i+\ell}\dots\overline{a}_{i+1}\overline{a}_i
\end{aligned}$$

for all ℓ. Hence the basis elements of type (4) for $i=j$ can be expressed as $\pm(a_i\overline{a}_i)^m$ for certain m. In particular, for $i=0$ one gets

$$(a_0\overline{a}_0)^m = \pm\left(\epsilon^{2m}(1+\epsilon p(\epsilon))^m\right).$$

Moreover, we see that

$$\begin{aligned}
&[a_i, a_{i+1}\dots a_\ell\overline{a}_\ell\overline{a}_{\ell-1}\dots\overline{a}_1\overline{a}_0\epsilon a_0a_1\dots a_{i-1}] =\\
&\qquad = \quad a_ia_{i+1}\dots a_\ell\overline{a}_\ell\overline{a}_{\ell-1}\dots\overline{a}_1\overline{a}_0\epsilon a_0a_1\dots a_{i-1} - a_{i+1}\dots a_\ell\overline{a}_\ell\overline{a}_{\ell-1}\dots\overline{a}_1\overline{a}_0\epsilon a_0a_1\dots a_{i-1}a_i
\end{aligned}$$

for some $i \le \ell \le n-2$. Hence, two different basis elements of type (7) for $i=j$ differ by a commutator. Therefore, modulo commutator $[L_n^p, L_n^p]$ we need to consider the basis elements of type (7) only for $i=j=0$:

$$a_0a_1\dots a_{\ell-1}a_\ell\overline{a}_\ell\overline{a}_{\ell-1}\dots\overline{a}_1\overline{a}_0\epsilon \text{ for some } 0\le\ell\le n-2$$

But now,

$$a_0a_1\dots a_{\ell-1}a_\ell\overline{a}_\ell\overline{a}_{\ell-1}\dots\overline{a}_1\overline{a}_0\epsilon = \pm(a_0\overline{a}_0)^{\ell+1}\epsilon = \pm\epsilon(\epsilon^2+\epsilon^3p(\epsilon))^{\ell+1} = \epsilon^{2\ell+3}(1+\epsilon p(\epsilon))^{\ell+1}$$

Therefore, the basis of $e_0L_n^pe_0$ in Proposition 3.1 consists of one element of type (3), one element of type (5), elements of type (7) which have the form $\epsilon^{2\ell+3}(1+\epsilon p(\epsilon))^{\ell+1}$ and elements of type (4) of the form $\epsilon^{2\ell+2}(1+\epsilon p(\epsilon))^{\ell+1}$. Hence the set

$$\{e_0\}\cup\{\epsilon\}\cup\{\epsilon^{2\ell+3}(1+\epsilon p(\epsilon))^{\ell+1} \mid 0\le\ell<n-1\}\cup\{\epsilon^{2\ell+2}(1+\epsilon p(\epsilon))^{\ell+1} \mid 0\le\ell<n-1\}$$

forms a basis of $e_0L_n^pe_0$.

Lemma 3.9. *The set $\{\epsilon^\ell \mid 0\le\ell<2n\}$ is a K-basis of $e_0L_n^pe_0$.*

Remark 3.10. Of course, we put $\epsilon^0 = e_0$ in Lemma 3.9.

Proof. We know that the set

$$\mathcal{S} := \{e_0\}\cup\{\epsilon\}\cup\{\epsilon^{2\ell+3}(1+\epsilon p(\epsilon))^{\ell+1} \mid 0\le\ell<n-1\}\cup\{\epsilon^{2\ell+2}(1+\epsilon p(\epsilon))^{\ell+1} \mid 0\le\ell<n-1\}$$

forms a basis of $e_0L_n^pe_0$. Expressing these elements as linear combinations of the set $\{\epsilon^\ell \mid 0\le\ell<2n\}$ one obtains a square lower triangular matrix with diagonal entries 1. Hence since $\mathcal{S}$ is a basis, also $\{\epsilon^\ell \mid 0\le\ell<2n\}$ is a K-basis of $e_0L_n^pe_0$. □

Lemma 3.11. *For all $n\ge 0$ the factor space $L_{n+1}^p/[L_{n+1}^p, L_{n+1}^p]$ has a K-linear generating set*

$$\{e_0, e_1, \dots, e_n\}\cup\{\epsilon^{2m+1} \mid 0\le m\le n\}.$$

Proof. It is a general fact that every non-closed path (i.e. a path with different start and end point) is a commutator; in fact, take the commutator with the trivial path corresponding to the start point (or end point). Moreover, it is easy to see that the cosets of the trivial paths are always linearly independent modulo the commutator subspace.

In view of Lemma 3.7 we only need to detect elements outside the commutator of L_n^p which become a commutator in L_{n+1}^p and determine which elements of $L_{n+1}^pe_nL_{n+1}^p$ are commutators.

We shall proceed by induction on n. The lemma is clearly true for $n=0$. For $n>0$ we can use the list of closed paths given in Proposition 3.1. We shall start by identifying certain closed paths as being commutators.

• For all $m \in \{0, 1, \dots, n-1\}$ we have

$$\begin{aligned} a_m a_{m+1} \dots a_{n-1} \overline{a}_{n-1} \overline{a}_{n-2} \dots \overline{a}_m &= (a_m a_{m+1} \dots a_{n-1})(\overline{a}_{n-1} \overline{a}_{n-2} \dots \overline{a}_m) \\ &\quad -(\overline{a}_{n-1} \overline{a}_{n-2} \dots \overline{a}_m)(a_m a_{m+1} \dots a_{n-1}) \\ &\in [L^p_{n+1}, L^p_{n+1}] \end{aligned}$$

since $(\overline{a}_{n-1} \overline{a}_{n-2} \dots \overline{a}_m)(a_m a_{m+1} \dots a_{n-1}) = 0$ by Lemma 3.6.

• Using the defining relation $\overline{a}_{n-1} a_{n-1} = 0$ in L^p_{n+1} we have that

$$a_{n-1} \overline{a}_{n-1} = a_{n-1} \overline{a}_{n-1} - \overline{a}_{n-1} a_{n-1} = [a_{n-1}, \overline{a}_{n-1}] \in [L^p_{n+1}, L^p_{n+1}].$$

Moreover, using the relation $\overline{a}_i a_i = a_{i+1} \overline{a}_{i+1}$ for all $i \in \{0, 1, \dots, n-2\}$ we have that

$$[a_i, \overline{a}_i] = a_i \overline{a}_i - \overline{a}_i a_i = a_i \overline{a}_i - a_{i+1} \overline{a}_{i+1}.$$

Inductively we can assume that $a_{i+1} \overline{a}_{i+1} \in [L^p_{n+1}, L^p_{n+1}]$ and hence we deduce that $a_i \overline{a}_i \in [L^p_{n+1}, L^p_{n+1}]$ for all $i \in \{0, 1, \dots, n-1\}$.

Moreover, consider a power $(a_i \overline{a}_i)^m$ for some integer $m \geq 2$. Then for all $i \in \{0, 1, \dots, n-2\}$ we have

$$[a_i, (\overline{a}_i a_i)^{m-1} \overline{a}_i] = (a_i \overline{a}_i)^m - (\overline{a}_i a_i)^m = (a_i \overline{a}_i)^m - (a_{i+1} \overline{a}_{i+1})^m.$$

Inductively, we obtain that $(a_i \overline{a}_i)^m \equiv (a_{n-1} \overline{a}_{n-1})^m \mod [L^p_{n+1}, L^p_{n+1}]$; but $(a_{n-1} \overline{a}_{n-1})^m = 0$ for $m \geq 2$ (using the defining relation $\overline{a}_{n-1} a_{n-1} = 0$).

Together with the above arguments for the case $m = 1$ we can thus deduce

$$(a_i \overline{a}_i)^m \in [L^p_{n+1}, L^p_{n+1}] \text{ for all } i \in \{0, 1, \dots, n-1\} \text{ and all } m \geq 1. \tag{5}$$

• In particular the preceding arguments imply that $(\epsilon^2 + \epsilon^3 p(\epsilon))^m = -(a_0 \overline{a}_0)^m \in [L^p_{n+1}, L^p_{n+1}]$ for all integers $m \geq 1$.

In a second step after showing certain closed paths to be commutators we now examine (nontrivial) closed paths in $L^p_{n+1} e_n L^p_{n+1}$ and in particular determine the dimension of the image in the factor space $L^p_{n+1} e_n L^p_{n+1} / \left([L^p_{n+1}, L^p_{n+1}] \cap L^p_{n+1} e_n L^p_{n+1} \right)$. According to Proposition 3.1 there are two types of such paths, the long paths $a_i a_{i+1} \dots a_{n-1} \overline{a}_{n-1} \dots \overline{a}_0 \epsilon a_0 \dots a_{i-1}$ for $i \in \{0, 1, \dots, n\}$ corresponding to socle elements and the short paths $a_i a_{i+1} \dots a_{n-1} \overline{a}_{n-1} \dots \overline{a}_i$ for $i \in \{0, 1, \dots, n-1\}$.

For the latter we already observed at the beginning of the proof that they are all commutators, i.e. that $a_i a_{i+1} \dots a_{n-1} \overline{a}_{n-1} \dots \overline{a}_i \in [L^p_{n+1}, L^p_{n+1}]$ for all $i \in \{0, 1, \dots, n-1\}$.

For the former paths, corresponding to socle elements, consider for $i \in \{1, \dots, n\}$ the commutator

$$\begin{aligned} [a_i \dots a_{n-1} \overline{a}_{n-1} \dots \overline{a}_0 \epsilon, a_0 \dots a_{i-1}] &= a_i a_{i+1} \dots a_{n-1} \overline{a}_{n-1} \dots \overline{a}_0 \epsilon a_0 \dots a_{i-1} \\ &\quad -a_0 \dots a_{i-1} a_i a_{i+1} \dots a_{n-1} \overline{a}_{n-1} \dots \overline{a}_0 \epsilon \\ &= a_i a_{i+1} \dots a_{n-1} \overline{a}_{n-1} \dots \overline{a}_0 \epsilon a_0 \dots a_{i-1} - \epsilon^{2n+1} \end{aligned}$$

where for the last equation see Remark 3.2 (3). Therefore, all the long paths corresponding to socle elements are equivalent to ϵ^{2n+1} modulo the commutator space.

Therefore, the image of $L^p_{n+1} e_n L^p_{n+1}$ in $L^p_{n+1} / [L^p_{n+1}, L^p_{n+1}]$ is 2-dimensional with a basis given by the cosets of $\{e_n, \epsilon^{2n+1}\}$.

The assertion of the lemma now follows by induction, using that $L^p_{n+1} / \left(L^p_{n+1} e_n L^p_{n+1} \right) = L^p_n$ by Lemma 3.7. □

Remark 3.12. We did not yet prove that this generating set is actually a K-basis. This fact is going to be shown in Proposition 3.14.

3.5. **The centre.** The aim of this section is to have a look at the centres of the deformed preprojective algebras L^p_n of type L. The centre will be important to us by the following observation. It is not so difficult to write down quite a lot of commutators, as we have seen in Section 3.4. It is however difficult in general to show that these commutators actually generate the commutator space. By the discussion in Section 2.1, for a symmetric algebra A we get

$$\dim_K A = \dim_K [A, A] + \dim_K Z(A).$$

Since the canonical projection $\pi_n : L^p_{n+1} \longrightarrow L^p_n$ from Lemma 3.7 is a surjective algebra homomorphism, the restriction of π_n to the centre $Z(L^p_{n+1})$ induces a ring homomorphism $Z(L^p_{n+1}) \longrightarrow Z(L^p_n)$. In principle, we could use this to determine the centre of L^p_n inductively although this might become quite technical.

Fortunately, with the methods developed in this paper we shall not really need to determine the entire centre; it will turn out that it suffices to find one central element whose powers generate a large enough central subspace.

Lemma 3.13. *The following holds for the deformed preprojective algebra L_n^p.*

(1) *The element $\epsilon^2 + \epsilon^3 p(\epsilon) + \sum_{\ell=0}^{n-3}(-1)^{\ell+1}\overline{a}_\ell a_\ell$ is contained in the centre of L_n^p.*
(2) *The following subset is a K-free subset of the centre*

$$\left\{ \left(\epsilon^2 + \epsilon^3 p(\epsilon) + \sum_{\ell=0}^{n-3}(-1)^{\ell+1}\overline{a}_\ell a_\ell \right)^s \mid 0 \leq s \leq n-1 \right\} \subseteq Z(L_n^p).$$

(3) $\mathrm{soc}(L_n^p) \subseteq Z(L_n^p)$.

Proof. (1) For proving that $\lambda := \epsilon^2 + \epsilon^3 p(\epsilon) + \sum_{\ell=0}^{n-3}(-1)^{\ell+1}\overline{a}_\ell a_\ell$ is a central element it is sufficient to show that it commutes with the algebra generators of L_n^p. This is clear for the trivial paths e_i since all paths occurring in λ are closed paths. For the loop ϵ we get $\epsilon\lambda = \epsilon^3 + \epsilon^4 p(\epsilon) = \lambda\epsilon$. For the arrow a_0 we have, using the defining relation $\epsilon^2 + \epsilon^3 p(\epsilon) + a_0\overline{a}_0 = 0$, that

$$\lambda a_0 = (\epsilon^2 + \epsilon^3 p(\epsilon))a_0 = (-a_0\overline{a}_0)a_0 = a_0\lambda.$$

Similarly, for the arrow $\overline{a}_0$ have

$$\lambda\overline{a}_0 = -\overline{a}_0 a_0\overline{a}_0 = \overline{a}_0(\epsilon^2 + \epsilon^3 p(\epsilon)) = \overline{a}_0\lambda.$$

For the arrows a_i where $1 \leq i \leq n-2$, we use the relations $\overline{a}_{i-1}a_{i-1} + a_i\overline{a}_i = 0$ and get

$$\lambda a_i = (-1)^i\overline{a}_{i-1}a_{i-1}a_i = (-1)^{i+1}a_i\overline{a}_i a_i = a_i\lambda.$$

Finally, we get in a similar fashion for the arrows a_i where $1 \leq i \leq n-2$ that

$$\lambda\overline{a}_i = (-1)^{i+1}\overline{a}_i a_i\overline{a}_i = (-1)^{i+2}\overline{a}_i\overline{a}_{i-1}a_{i-1} = (-1)^i\overline{a}_i\overline{a}_{i-1}a_{i-1} = \overline{a}_i\lambda.$$

(2) The fact that the powers of $\epsilon^2 + \epsilon^3 p(\epsilon) + \sum_{\ell=0}^{n-3}(-1)^{\ell+1}\overline{a}_\ell a_\ell$ form a linearly independent set comes from the fact that the powers of ϵ form a linearly independent set (cf Lemma 3.9).

(3) This holds since multiplication from the left or right of any basis element of the socle with a path of length at least 1 gives 0. Moreover, the algebras L_n^p are weakly symmetric, i.e. the socle has a basis consisting of closed paths, so any socle element also commutes with the trivial paths. □

We are now in the position to give a basis of the commutator space $[L_n^p, L_n^p]$ of the deformed preprojective algebras L_n^p of type L. As a consequence we can strengthen the statement in Lemma 3.11; namely the cosets of $\{e_0, \ldots, e_{n-1}\} \cup \{\epsilon^{2\ell+1} \mid 0 \leq \ell \leq n-1\}$ are a basis (and not only a generating set) of the factor space $L_n^p/[L_n^p, L_n^p]$.

Proposition 3.14. *For any polynomial $p(X) \in K[X]$ and $n \geq 2$ the following holds for the deformed preprojective algebras L_n^p.*

(a) *The commutator space has dimension*

$$\dim_K [L_n^p, L_n^p] = \dim_K L_n^p - 2n = \frac{1}{3}n(n-1)(2n+5).$$

(b) *The centre has dimension* $\dim_K Z(L_n^{X^p}) = 2n$.
(c) *The cosets of $\{e_0, \ldots, e_{n-1}\} \cup \{\epsilon^{2\ell+1} \mid 0 \leq \ell \leq n-1\}$ form a basis of the factor space $L_n^p/[L_n^{X^p}, L_n^p]$.*
(d) *In terms of the basis $\mathcal{B}$ of L_n^p given in Proposition 3.1, a K-basis of $[L_n^p, L_n^p]$ is given by*
  (i) *all non-closed paths in $\mathcal{B}$ (i.e. with starting vertex different from the ending vertex),*
  (ii) *all closed paths in $\mathcal{B}$ of even length at least 2 with starting vertex different from vertex 0,*
  (iii) *the difference of two closed paths in $\mathcal{B}$ of equal odd length with consecutive starting vertices i and $i+1$ where $0 \leq i \leq n-2$,*
  (iv) *the elements $a_0 a_1 \ldots a_\ell \overline{a}_\ell \overline{a}_{\ell-1} \ldots \overline{a}_0$ where $0 \leq \ell \leq n-2$.*

Proof. (a) For every symmetric algebra A we have

$$\dim_K [A, A] = \dim_K A - \dim_K Z(A) \tag{6}$$

since $\mathrm{Hom}_K(A/[A,A], K) \simeq Z(A)$ as $Z(A)$-modules.

In our situation for $A = L_n^p$ we get from Lemma 3.11 the lower bound

$$\dim_K [A, A] \geq \dim_K L_n^p - 2n. \tag{7}$$

We shall now produce sufficiently many linear independent elements in the centre to obtain this also as an upper bound. The centre $Z(L_n^p)$ contains the K-free subset

$$\left\{ \left(\epsilon^2 + \epsilon^3 p(\epsilon) + \sum_{\ell=0}^{n-3} (-1)^{\ell+1} \overline{a}_\ell a_\ell \right)^s \mid 0 \leq s \leq n-1 \right\} \subseteq Z(L_n^p)$$

of cardinality n. The K-vector space generated by these elements intersects with $\mathrm{soc}(L_n^p)$ only in $\{0\}$ since ϵ^{2s+2} is not in the socle for $s \in \{0, 1, \ldots, n-2\}$. However, $\mathrm{soc}(L_n^p)$ belongs to the centre. Hence we get that $Z(L_n^p)$ is of dimension at least $2n$.

Altogether, we get a lower bound for the dimension of the centre, namely

$$\dim_K (Z(L_n^p)) \geq (n-1) + n + 1 = 2n.$$

Plugging this into formula (6) and combining with (7) proves the first equality in part (a) of the lemma.

The second equality then follows by a direct calculation from the formula for the dimension of L_n^p given in Remark 3.3.

(b) The statement in part (b) now follows directly from part (a) by using formula (6).

(c) Follows by combining part (a) and Lemma 3.11.

(d) For each of the types (i)-(iv) we shall first verify that all these elements are actually contained in the commutator space, and then count their number. At the end it will turn out that the total number of elements in (i)-(iv) is $(\dim_K L_n^p - 2n)$, i.e. equal to the dimension of the commutator space, cf. part (a). Since the elements are linearly independent (being part of a basis), the claim of part (d) then follows.

(i) Non-closed paths are always commutators (take the commutator with the trivial path corresponding to the starting vertex).

The number of non-closed paths in $\mathcal{B}$ can be read off from the Cartan matrix of L_n^p given in Remark 3.3. Namely as the dimension of L_n^p minus the trace of the Cartan matrix, i.e. we get $\dim_K L_n^p - n(n+1)$ non-closed paths in $\mathcal{B}$.

(ii) Such paths of even length only occur in type (4) of Proposition 3.1 and are of the form $a_i a_{i+1} \ldots a_\ell \overline{a}_\ell \overline{a}_{\ell-1} \ldots \overline{a}_i$ where $i \neq 0$ and $i \leq \ell \leq n-2$. Up to a sign, these paths are equal to $(a_i \overline{a}_i)^l$ (using the relations $\overline{a}_r a_r + a_{r+1} \overline{a}_{r+1}$) and these have been shown to be in the commutator space in the proof of Lemma 3.11.

Summing over the possibilities for the various $i \neq 0$ there are $\frac{(n-1)(n-2)}{2}$ such paths.

(iii) Such a difference of closed paths with starting vertices i and $i+1$ occurs as a difference of a path of type (7) for vertex i with a path of type (5) or (7) for vertex $i+1$. More precisely, these differences are of the form

$$a_i a_{i+1} \ldots a_\ell \overline{a}_\ell \overline{a}_{\ell-1} \ldots \overline{a}_0 \epsilon a_0 \ldots a_{i-1} - a_{i+1} \ldots a_\ell \overline{a}_\ell \overline{a}_{\ell-1} \ldots \overline{a}_0 \epsilon a_0 \ldots a_i,$$

where $i \leq \ell \leq n-2$. This element is a commutator, namely $[a_i, a_{i+1} \ldots a_\ell \overline{a}_\ell \overline{a}_{\ell-1} \ldots \overline{a}_0 \epsilon a_0 \ldots a_{i-1}]$.

Summing over the possibilities for the various i there are $\frac{n(n-1)}{2}$ such differences.

(iv) Up to a sign, these paths are equal to $(a_i \overline{a}_i)^l$ (using the relations $\overline{a}_r a_r + a_{r+1} \overline{a}_{r+1}$) and these have been shown to be in the commutator space in the proof of Lemma 3.11. Obviously, there are $n-1$ such paths.

The total number of elements in (i)-(iv) is easily computed to be $\dim_K L_n^{X^{2j}} - 2n$, which is the dimension of the commutator space by part (a). Thus part (d) follows. □

4. The Külshammer spaces and the main result

We now restrict to the case where the deformation polynomial p has the form X^{2j} for some integer $j \geq 0$. It has been shown in [2, Proposition 6.1] that the deformed algebras L_n^p for the polynomials $p = X^{2j}$ where $j \in \{0, 1, \ldots, n-1\}$ form a family of pairwise non-isomorphic deformed preprojective algebras of type L. Note that for all $j \geq n-1$ the algebra $L_n^{X^{2j}}$ is the (undeformed) preprojective algebra of type L; in fact, the only relation involving the polynomial p reads

$$\epsilon^2 + a_0 \overline{a}_0 + \epsilon^3 p(\epsilon) = \epsilon^2 + a_0 \overline{a}_0 + \epsilon^{2j+3} = \epsilon^2 + a_0 \overline{a}_0$$

because $\epsilon^{2n} = 0$ in L_n^p.

Moreover, it is shown in [3, Theorem 2] that the algebras $L_n^{X^{2j}}$ for $j \in \{0, 1, \ldots, n-1\}$ actually form a complete list of representatives of the isomorphism classes of deformed preprojective algebras of type L. Therefore, focussing on the case of deformation polynomials $p = X^{2j}$ is not a restriction.

We continue to consider the deformed preprojective algebras $A_n^j = L_n^{X^{2j}}$ over a field of characteristic 2. The Külshammer spaces are defined as $T_r(A_n^j) = \{x \in A_n^j \mid x^{2^r} \in [A_n^j, A_n^j]\}$ for any integer $r \geq 0$ (cf. the introduction).

In this section we shall derive the main results of the paper. Firstly, we shall give formulae for the dimensions of the Külshammer spaces $T_r(A_n^j)$, see Theorem 4.1 below. Secondly, as an application we can distinguish certain of the deformed preprojective algebras of type L_n (over a perfect field of characteristic 2) up to derived equivalence, see Theorem 4.2 below.

The crucial link to distinguish algebras up to derived equivalence by means of Külshammer spaces has been provided by the second author in [26]. There it is shown that for K being a perfect field of characteristic $p > 0$ and for Λ_1 and Λ_2 being finite dimensional K-algebras which are derived equivalent the codimensions of the Külshammer spaces are an invariant, i.e. for all $r \geq 0$ one has

$$\dim_K \Lambda_1 - \dim_K T_r(\Lambda_1) = \dim_K \Lambda_2 - \dim_K T_r(\Lambda_2). \tag{8}$$

In [17] Liu, Zhou and the second author showed that for any field K of characteristic $p > 0$ and any two finite dimensional K-algebras Λ_1 and Λ_2, if Λ_1 and Λ_2 are stably equivalent of Morita type, then

$$\dim_K T_r(\Lambda_1) - \dim_K [\Lambda_1, \Lambda_1] = \dim_K T_r(\Lambda_2) - \dim_K [\Lambda_2, \Lambda_2] \tag{9}$$

for all $r \geq 0$.

4.1. **Dimensions of Külshammer spaces.** In this section we shall prove the main result on the dimensions of the Külshammer spaces $T_i(L_n^{X^{2j}})$ for the deformed preprojective algebras. Before embarking on the general proof we shall give some explicit examples which hopefully help the reader later by illustrating the technicalities of the general arguments.

An example: the case $n = 2$. Let us look at the algebras $A_2^j = L_2^{X^{2j}}$ as an illustration. These algebras are given by a quiver with two vertices and relations $\epsilon^4 = 0$, $\overline{a}_0 a_0 = 0$ and $\epsilon^2 + \epsilon^{2j+3} + a_0\overline{a}_0 = 0$. Note that for $j \geq 1$ we get the undeformed algebra A_2^1 with relation $\epsilon^2 + a_0\overline{a}_0 = 0$, whereas for $j = 0$ we get a deformed preprojective algebra A_2^0 with relation $\epsilon^2 + \epsilon^3 + a_0\overline{a}_0 = 0$.

According to Proposition 3.1 and Remark 3.3 the algebras A_2^j are 10-dimensional with a basis given by the paths

$$e_0, e_1, \epsilon, a_0, \overline{a}_0, a_0\overline{a}_0, \epsilon a_0, \overline{a}_0\epsilon, a_0\overline{a}_0\epsilon, \overline{a}_0\epsilon a_0.$$

By Proposition 3.14 the commutator spaces $[A_2^j, A_2^j]$ are of dimension 6 and have a basis consisting of the elements

$$a_0, \overline{a}_0, a_0\overline{a}_0, \epsilon a_0, \overline{a}_0\epsilon, a_0\overline{a}_0\epsilon - \overline{a}_0\epsilon a_0.$$

Note that all these bases are independent of j.

Now we consider the first Külshammer space $T_1(A_2^j) = \{x \in A_2^j \mid x^2 \in [A_2^j, A_2^j]\}$. For any $j \geq 0$ it is immediate from the relations that the following seven basis elements of A_2^j are contained in the first Külshammer space

$$\{a_0, \overline{a}_0, \epsilon a_0, \overline{a}_0\epsilon, a_0\overline{a}_0, a_0\overline{a}_0\epsilon, \overline{a}_0\epsilon a_0\} \subset T_1(A_2^j).$$

On the other hand, it is a general observation that the trivial paths e_0, e_1 can not be summands of an element in a Külshammer space (since trivial paths can't occur as summands in an element from the commutator space). This leave us with the remaining basis element ϵ. Here the situation changes for different j.

In the undeformed case $j \geq 1$ we have that $\epsilon^2 = a_0\overline{a}_0 = [a_0, \overline{a}_0] \in [A_2^j, A_2^j]$ and hence $\epsilon \in T_1(A_2^j)$.

On the other hand, in the deformed case $j = 0$ we have the relation $\epsilon^2 = \epsilon^3 + a_0\overline{a}_0$ where $a_0\overline{a}_0$ is a commutator but $\epsilon^3 = a_0\overline{a}_0\epsilon \notin [A_2^0, A_2^0]$. Therefore, $\epsilon \notin T_1(A_2^0)$

In summary we have $\dim_K T_1(A_2^0) = 7$ whereas $\dim_K T_1(A_2^j) = 8$ for all $j \geq 1$.

Using the result from [26] quoted above in (8) we can deduce that the undeformed preprojective algebra $A_2^1 = L_2^{X^2}$ and the deformed preprojective algebra $A_2^0 = L_2^{X^0}$ are not derived equivalent. Even in this small case $n = 2$ this seems to be a nontrivial fact.

Another example: Külshammer spaces for $n = 3$. The algebras A_3^j have dimension 28, and their commutator spaces have dimension 22. There are many basis elements which are obviously in each of the Külshammer ideals $T_r(A_3^j)$, for $r \geq 1$, namely

• all non-closed paths in $\mathcal{B}$, giving 16 basis elements (since they square to zero)

• closed paths of length ≥ 3 (since the algebras have radical length 6 they also square to zero); so another six such basis elements are $a_0\overline{a}_0\epsilon$, $\overline{a}_0\epsilon a_0$, $a_0a_1\overline{a}_1\overline{a}_0\epsilon$, $a_1\overline{a}_1\overline{a}_0\epsilon a_0$, $\overline{a}_1\overline{a}_0\epsilon a_0a_1$, $a_0a_1\overline{a}_1\overline{a}_0$.

• the two basis elements $a_0\overline{a}_0$ and $a_1\overline{a}_1$ (since $(a_1\overline{a}_1)^2 = 0$ and $(a_0\overline{a}_0)^2$ is in the commutator space by the proof of Lemma 3.11).

Hence, $\dim_K T_r(A_3^j) \geq 24$ for all $r \geq 1$ and all j.

Given that the three trivial paths are not involved in any element of the Külshammer space, there is only one remaining basis element to consider, namely ϵ.

We start with the first Külshammer space $T_1(A_3^j)$.

For $j = 2$ we have $\epsilon^2 = a_0\overline{a}_0 \in [A_3^2, A_3^2]$, i.e. $\epsilon \in T_1(A_3^j)$.

For $j = 1$ we get the following congruences modulo the commutator space

$$\epsilon^2 = \epsilon^5 + a_0\overline{a}_0 \equiv \epsilon^5 = \epsilon^8 + a_0\overline{a}_0\epsilon^3 = a_0\overline{a}_0(\epsilon^6 + a_0\overline{a}_0\epsilon) = a_0\overline{a}_0a_0\overline{a}_0\epsilon \not\equiv 0$$

i.e. $\epsilon \notin T_1(A_3^1)$.

For $j = 0$ we similarly get the following congruences modulo the commutator space

$$\epsilon^2 = \epsilon^3 + a_0\overline{a}_0 \equiv \epsilon^3 = \epsilon^4 + a_0\overline{a}_0\epsilon = \epsilon^5 + a_0\overline{a}_0\epsilon^2 + a_0\overline{a}_0\epsilon = \epsilon^6 + a_0\overline{a}_0\epsilon^3 + a_0\overline{a}_0\epsilon^2 + a_0\overline{a}_0\epsilon$$
$$= a_0\overline{a}_0\epsilon^3 + a_0\overline{a}_0(\epsilon^3 + a_0\overline{a}_0) + a_0\overline{a}_0\epsilon \equiv a_0\overline{a}_0\epsilon \not\equiv 0$$

i.e. $\epsilon \notin T_1(A_3^0)$. Altogether we get $\dim_K T_1(A_3^j) = \begin{cases} 24 & if\ j = 0, 1 \\ 25 & if\ j = 2 \end{cases}$

Now we consider the second Külshammer space $T_2(A_3^j) = \{x \in A_3^j \mid x^4 \in [A_3^j, A_3^j]\}$. Again it only remains to consider the basis element ϵ.

For $j = 2$ there is nothing to check since $T_2(A_3^2)$ already attained the maximal possible dimension 25.

For $j = 1$ we get the following congruences (modulo commutator space)

$$\epsilon^4 = \epsilon^7 + a_0\overline{a}_0\epsilon^2 = a_0\overline{a}_0\epsilon^5 + a_0\overline{a}_0a_0\overline{a}_0 \equiv 0$$

i.e. $\epsilon \in T_2(A_3^1)$.

Similarly we get for $j = 0$ (modulo commutator space)

$$\epsilon^4 = \epsilon^5 + a_0\overline{a}_0\epsilon^2 = \epsilon^6 + a_0\overline{a}_0\epsilon^3 + a_0\overline{a}_0\epsilon^3 + a_0\overline{a}_0a_0\overline{a}_0 \equiv 0$$

i.e. $\epsilon \in T_2(A_3^0)$.

Altogether we get $\dim_K T_2(A_3^j) = 25$ for all j.

We now formulate the main result of this section.

Theorem 4.1. *Let K be a perfect field of characteristic 2. Then for all $0 \leq j < n$ we have*

$$\dim_K T_i(L_n^{X^{2j}}) - \dim_K [L_n^{X^{2j}}, L_n^{X^{2j}}] \quad = \quad n - \max\left(\left\lceil \frac{2n - (2^{i+1} - 2)j - (2^{i+1} - 1)}{2^{i+1}} \right\rceil,\ 0\right).$$

Proof. Lemma 3.11 provided a set of coset generators of the commutator space $[L_n^{X^{2j}}, L_n^{X^{2j}}]$ in $L_n^{X^{2j}}$, namely $\{e_0, \ldots, e_{n-1}\} \cup \{\epsilon^{(2k+1)} |\ 0 \leq k \leq n-1\}$. For our purpose of determining the Külshammer ideals we can discard the trivial paths since they can never be involved in any element of a Külshammer ideal. Therefore in order to compute $T_i(L_n^{X^{2j}})$ we need to see when a 2^i-th power of a linear combination of elements $\{\epsilon^{(2k+1)} |\ 0 \leq k \leq n-1\}$ lies in the commutator space $[L_n^{X^{2j}}, L_n^{X^{2j}}]$.

Note that $\epsilon \in e_0L_n^{X^{2j}}e_0$, and that by Proposition 3.14 a basis of the intersection $e_0L_n^{X^{2j}}e_0 \cap [L_n^{X^{2j}}, L_n^{X^{2j}}]$ is given by the paths $a_0a_1 \ldots a_\ell\overline{a}_\ell\overline{a}_{\ell-1} \ldots \overline{a}_0$ where $0 \leq \ell \leq n-2$. Moreover, we have that

$$a_0a_1 \ldots a_\ell\overline{a}_\ell\overline{a}_{\ell-1} \ldots \overline{a}_0 = (a_0\overline{a}_0)^{\ell+1} = (\epsilon^2 + \epsilon^{2j+3})^{\ell+1}$$

(no signs occurring since we are in characteristic 2). This means that in order to obtain the desired formula for the dimension of $T_i(L_n^{X^{2j}})$ one needs to consider the K-vector space

$$\tilde{T}_{n,j}(i) := \left\{ \sum_{k=0}^{n-1} b_k\epsilon^{2k+1} \;\middle|\; \left(\sum_{k=0}^{n-1} b_k\epsilon^{2k+1}\right)^{2^i} \in \left\langle (\epsilon^2 + \epsilon^{2j+3})^m |\ 1 \leq m \leq n-1 \right\rangle_K \right\}$$

whose dimension is equal to the dimension of the factor space $T_i(L_n^{X^{2j}})/[L_n^{X^{2j}}, L_n^{X^{2j}}]$. Indeed, the remarks at the beginning of the proof imply that as vector spaces we have a decomposition

$$T_i(L_n^{X^{2j}}) = [L_n^{X^{2j}}, L_n^{X^{2j}}] \oplus \tilde{T}_{n,j}(i).$$

In order to determine this dimension we therefore have to express an element $\left(\sum_{k=0}^{n-1} b_k\epsilon^{2k+1}\right)^{2^i}$ as a linear combination of the form $\sum_{m=1}^{n-1} c_m(\epsilon^2+\epsilon^{2j+3})^m$ for $c_m \in K$.

First Step. We shall reduce the problem to the case of K being the prime field of characteristic 2.

As is described in the remarks preceding the statement of the theorem we have to give the dimension of the K-vector space

$$\left\{ \sum_{k=0}^{n-1} b_k\epsilon^{2k+1} \in K[\epsilon] \;\middle|\; \left(\sum_{k=0}^{n-1} b_k\epsilon^{2k+1}\right)^{2^i} \in \left\langle (\epsilon^2+\epsilon^{2j+3})^m \middle|\; m\in\mathbb{N}\right\rangle_K \right\}$$

Let

$$U := \left\langle (\epsilon^2+\epsilon^{2j+3})^m \middle|\; m\in\mathbb{N}\right\rangle_K \subseteq K[\epsilon]/\epsilon^{2n}.$$

Then let

$$V := \left\langle \epsilon^{2k+1} \right\rangle_K \subseteq K[\epsilon]/\epsilon^{2n}$$

and let $\mu : K[\epsilon]/\epsilon^{2n} \longrightarrow K[\epsilon]/\epsilon^{2n}$ given by $\mu(x) := x^2$. Then

$$\left\{ \sum_{k=0}^{n-1} b_k\epsilon^{2k+1} \in K[\epsilon] \;\middle|\; \left(\sum_{k=0}^{n-1} b_k\epsilon^{2k+1}\right)^{2^i} \in \left\langle (\epsilon^2+\epsilon^{2j+3})^m \middle|\; m\in\mathbb{N}\right\rangle_K \right\} = V \cap \left(\mu^i\right)^{-1}(U).$$

Now, $U = U_0 \otimes_{\mathbb{F}_2} K$ and $V := V_0 \otimes_{\mathbb{F}_2} K$ for U_0 and V_0 being defined as U and V, but with $\mathbb{F}_2$ as base field. If K is perfect, then

$$V \cap \left(\mu^i\right)^{-1}(U) = (V_0 \otimes_{\mathbb{F}_2} K) \cap \left(\mu^i\right)^{-1}(U_0 \otimes_{\mathbb{F}_2} K) = \left(V_0 \cap \left(\mu^i\right)^{-1}(U_0)\right) \otimes_{\mathbb{F}_2} K$$

Hence the dimension of the vector space can be computed in $\mathbb{F}_2$. We hence may assume that $K = \mathbb{F}_2$.

Since K is assumed to be the prime field, we get $b^2 = b$ for all $b \in K$, and so we need to find coefficients $c_m \in K$ so that

$$\left(\sum_{k=0}^{n-1} b_k\epsilon^{2^i(2k+1)}\right) = \left(\sum_{k=0}^{n-1} b_k\epsilon^{(2k+1)}\right)^{2^i} = \sum_{m=1}^{n-1} c_m\left(\epsilon^2+\epsilon^{2j+3}\right)^m.$$

Second step. In the course of the proof we shall need to know whether certain binomial coefficients are even or odd. More precisely, write a natural number as $2^a v$ with v odd and $a \in \mathbb{N}\cup\{0\}$. Then we have that

$$\left\{ u\in\mathbb{N}\setminus\{0\} \;\middle|\; \binom{2^a v}{u} \text{ odd} \right\} \subseteq 2^a\mathbb{Z} \quad \text{and} \quad \min\left\{ u\in\mathbb{N}\setminus\{0\} \;\middle|\; \binom{2^a v}{u} \text{ odd} \right\} = 2^a.$$

In fact, both statements follow easily from the following well-known result on binomial coefficients, going back to Lucas [18]; for a proof see e.g. [9]: *Let p be a prime, and let natural numbers $M = \sum M_i p^i$ and $N = \sum N_i p^i$ be given in their p-adic expansion. Then $\binom{M}{N} \equiv \prod_i \binom{M_i}{N_i} \mod p$.*

We remark further that binomial coefficients are integers. Hence, seen in K they actually belong to the prime field. If K is of characteristic 2, then a binomial coefficient can only have values 0 or 1.

Third step. We need to study for which $b_0,\ldots,b_{n-1}$ given, there exist coefficients $c_m \in K$ so that

$$\sum_{k=0}^{n-1} b_k\epsilon^{2^i(2k+1)} = \sum_{m=1}^{n-1} c_m(\epsilon^2+\epsilon^{2j+3})^m. \tag{10}$$

We first determine a lower bound for the indices of the non-vanishing coefficients b_k. Denote by k_0 the smallest integer k so that $b_k \neq 0$. Then formula (10) reads

$$\sum_{k=k_0}^{n-1} b_k\epsilon^{2^i(2k+1)} = \sum_{m=1}^{n-1} c_m(\epsilon^2+\epsilon^{2j+3})^m. \tag{11}$$

Comparing the smallest powers of ϵ occurring on either side of equation (11) we can deduce that $c_m = 0$ for $m < 2^{i-1}(2k_0+1)$ and $c_{2^{i-1}(2k_0+1)} \neq 0$. Hence equation (11) now reads

$$\sum_{k=k_0}^{n-1} b_k \epsilon^{2^i(2k+1)} = \sum_{m=2^{i-1}(2k_0+1)}^{n-1} c_m(\epsilon^2+\epsilon^{2j+3})^m. \tag{12}$$

Using the statements on the parity of binomial coefficients from the second step and the fact that the base field is of characteristic 2 we have that

$$\begin{aligned}(\epsilon^2+\epsilon^{2j+3})^{2^{i-1}(2k_0+1)} &= \epsilon^{2^i(2k_0+1)} + \binom{2^{i-1}(2k_0+1)}{2^{i-1}} \cdot \left(\epsilon^2\right)^{2^{i-1}(2k_0+1)-2^{i-1}} \cdot \left(\epsilon^{2j+3}\right)^{2^{i-1}} + \\ &\quad + \text{ higher powers of } \epsilon^{2^{i-1}} \\ &= \epsilon^{2^i(2k_0+1)} + \epsilon^{2^{i-1}(4k_0+2j+3)} + \text{ higher powers of } \epsilon^{2^{i-1}}.\end{aligned}$$

Hence as long as $2^{i-1}(4k_0+2j+3) < 2n$ (i.e. $\epsilon^{2^{i-1}(4k_0+2j+3)}$ does not vanish), a non-zero scalar multiple of $\epsilon^{2^{i-1}(4k_0+2j+3)}$ occurs on the right hand side of equation (12). However, it can not occur on the left hand side of equation (12) since $2^{i-1}(4k_0+2j+3)$ is not divisible by 2^i. So $\epsilon^{2^{i-1}(4k_0+2j+3)}$ would also have to be a term of some other summand in $\sum_{m=2^{i-1}(2k_0+1)}^{n-1} c_m(\epsilon^2+\epsilon^{2j+3})^m$ (so that the terms can cancel out).

For $i = 1$ this is impossible, since for $m > 2^{i-1}(2k_0+1)$ the smallest possible odd exponent in $(\epsilon^2+\epsilon^{2j+3})^m$ is already larger than $2^{i-1}(4k_0+2j+3)$. Hence for $i = 1$ we must have that $4k_0+2j+3 \geq 2n$ which implies that $k_0 = \max\left(\left\lceil \frac{2n-2j-3}{4} \right\rceil, 0\right)$. Note that we indeed have to take the maximum with 0 here since the index k_0 is non-negative by definition.

Suppose now that $i \geq 2$. Then we claim that the only possibility to cancel the above term $\epsilon^{2^{i-1}(4k_0+2j+3)}$ is to put

$$c_{2^{i-2}(4k_0+2j+3)} = c_{2^{i-1}(2k_0+1)} \neq 0.$$

In fact, on the one hand we have that

$$(\epsilon^2+\epsilon^{2j+3})^{2^{i-2}(4k_0+2j+3)} = \epsilon^{2^{i-1}(4k_0+2j+3)} + \text{ higher powers of } \epsilon^{2^{i-2}}$$

so that the desired term cancels; on the other hand, it could not cancel for a smaller index m since this would have to satisfy $m \geq 2^{i-1}(2k_0+3)$ (note that the exponents on the left hand side of equation (10) are divisible by 2^i) and then by Lucas' theorem above (cf. second step) the second term in $(\epsilon^2+\epsilon^{2j+3})^m$ already has exponent

$$2^i(2k_0+2)+(2j+3)2^{i-1} = 2^{i-1}(4k_0+2j+7) > 2^{i-1}(4k_0+2j+3).$$

In a similar way, again using the second step and that the base field is of characteristic 2, we further get

$$\begin{aligned}(\epsilon^2+\epsilon^{2j+3})^{2^{i-2}(4k_0+2j+3)} &= \epsilon^{2^{i-1}(4k_0+2j+3)} + \left(\epsilon^2\right)^{2^{i-2}(4k_0+2j+3)-2^{i-2}} \cdot \left(\epsilon^{2j+3}\right)^{2^{i-2}} + \\ &\quad + \text{ higher powers of } \epsilon^{2^{i-2}} \\ &= \epsilon^{2^{i-1}(4k_0+2j+3)} + \epsilon^{2^{i-2}(8k_0+6j+7)} + \text{ higher powers of } \epsilon^{2^{i-2}}.\end{aligned}$$

Completely analogous to the case $i = 1$ above we can deduce that for $i = 2$ we have $8k_0+6j+7 \geq 2n$ and therefore $k_0 = \max\left(\left\lceil \frac{2n-6j-7}{8} \right\rceil, 0\right)$ in the case $i = 2$. This is the second correction step.

We shall show by induction on s, that the lowest power of ϵ appearing in the sum on the right hand side of equation (12) after s corrections is

$$\epsilon^{2^{i-s}(2^s k_0+(2^s-2)\cdot j+(2^s-1))}.$$

The cases $s \in \{1,2\}$ have been treated above. Suppose the formula is shown for some $s < i$. Then we shall show the formula for $s+1$: We shall need to correct with $c_{2^{i-s-1}(2^s k_0+(2^s-2)j+(2^s-1))} \neq 0$

and get higher error terms as follows:

$$\begin{aligned}
\left(\epsilon^2+\epsilon^{2j+3}\right)^{2^{i-s-1}(2^sk_0+(2^s-2)j+(2^s-1))} &= \epsilon^{2^{i-s}(2^sk_0+(2^s-2)j+(2^s-1))}+\\
&\quad +\left(\epsilon^2\right)^{2^{i-s-1}(2^sk_0+(2^s-2)j+(2^s-1))-2^{i-s-1}}\cdot\left(\epsilon^{2j+3}\right)^{2^{i-s-1}}+\\
&\quad +\text{ higher powers of }\epsilon^{2^{i-s-1}}\\
&= \epsilon^{2^{i-s}(2^sk_0+(2^s-2)j+(2^s-1))}+\\
&\quad +\epsilon^{2^{i-s-1}(2^{s+1}k_0+2\cdot(2^s-2)j+2\cdot(2^s-1-1))+2^{i-s-1}(2j+3)}+\\
&\quad +\text{ higher powers of }\epsilon^{2^{i-s-1}}\\
&= \epsilon^{2^{i-s}(2^sk_0+(2^s-2)j+(2^s-1))}+\\
&\quad +\epsilon^{2^{i-s-1}(2^{s+1}k_0+(2^{s+1}-4+2)j+(2^{s+1}-4+3))}+\\
&\quad +\text{ higher powers of }\epsilon^{2^{i-s-1}}\\
&= \epsilon^{2^{i-s}(2^sk_0+(2^s-2)j+(2^s-1))}+\\
&\quad +\epsilon^{2^{i-s-1}(2^{s+1}k_0+(2^{s+1}-2)j+(2^{s+1}-1))}+\\
&\quad +\text{ higher powers of }\epsilon^{2^{i-s-1}}
\end{aligned}$$

which shows the formula for $s+1$.

Hence, we may correct the error terms by successively choosing appropriate c_m for higher and higher m, as long as $s<i$. If $s=i$ then the error term cannot be annihilated, and therefore it must be 0. Therefore

$$2^{i+1}k_0+(2^{i+1}-2)\cdot j+(2^{i+1}-1)\geq 2n$$

which means

$$k_0\geq\frac{2n-(2^{i+1}-2)j-(2^{i+1}-1)}{2^{i+1}}$$

and therefore

$$k_0=\max\left(\left\lceil\frac{2n-(2^{i+1}-2)j-(2^{i+1}-1)}{2^{i+1}}\right\rceil,\ \ 0\right).$$

Fourth step. Suppose

$$k\geq k_0=\max\left(\left\lceil\frac{2n-(2^{i+1}-2)j-(2^{i+1}-1)}{2^{i+1}}\right\rceil\right).$$

We shall prove that then

$$\epsilon^{2^i(2k+1)}\in\left\langle(\epsilon^2+\epsilon^{2j+3})^m|\ m\in\mathbb{N}\right\rangle_K.$$

To this end we put $c_{2^{i-1}(2k+1)}=1$ and get by the second step that

$$\begin{aligned}
\left(\epsilon^2+\epsilon^{2j+3}\right)^{2^{i-1}(2k+1)}-\epsilon^{2^i(2k+1)} &= \left(\epsilon^2\right)^{2^{i-1}(2k+1)-2^{i-1}}\cdot\left(\epsilon^{2j+3}\right)^{2^{i-1}}+\text{ higher order powers of }\epsilon^{2^{i-1}}\\
&= \epsilon^{2^{i-1}(4k+2j+3)}+\text{ higher order powers of }\epsilon^{2^{i-1}}
\end{aligned}$$

Hence, we can choose coefficients $c_{2^{i-1}m}$ for certain m so that

$$\epsilon^{2^i(2k+1)}-\sum c_{2^{i-1}m}(\epsilon^2+\epsilon^{2j+3})^{2^{i-1}m}$$

is a direct sum of terms $\epsilon^{2^{i-1}\ell}$ where $\ell\geq 4k+2j+3$.

If $i=1$, we are done since then $\epsilon^{2^{i-1}\ell}=0$ since ℓ was chosen in a way that

$$\ell\geq 4k+2j+3\geq 4k_0+2j+3\geq 2n.$$

If $i\geq 2$, put $c_{2^{i-2}\ell}=1$ for all terms $\epsilon^{2^{i-1}\ell}$ of the powers of $\epsilon^{2^{i-1}}$ occurring in the above difference

$$\epsilon^{2^i(2k+1)}-\sum c_{2^{i-1}m}(\epsilon^2+\epsilon^{2j+3})^{2^{i-1}m}.$$

We know that each of these ℓ satisfies $\ell\geq 4k+2j+3$, so that all the coefficients $2^{i-2}\ell$ are bigger than $2^{i-2}(4k_0+2j+3)$.

We compute

$$\begin{aligned}\left(\epsilon^2+\epsilon^{2j+3}\right)^{2^{i-2}\ell} &= \epsilon^{2^{i-1}\ell}+\binom{2^{i-2}\ell}{2^{i-2}}\left(\epsilon^2\right)^{2^{i-2}\ell-2^{i-2}}\cdot\left(\epsilon^{2j+3}\right)^{2^{i-2}}+\\ &\quad+\text{ higher order powers of }\epsilon^{2^{i-2}}\\ &= \epsilon^{2^{i-1}\ell}+\binom{2^{i-2}\ell}{2^{i-2}}\epsilon^{2^{i-2}(2\ell+2j+1)}+\text{ higher order powers of }\epsilon^{2^{i-2}}\end{aligned}$$

Again, if $i=2$ we are done since

$$2\ell+2j+1\geq 2(4k_0+2j+3)+2j+1=8k_0+6j+7\geq 2n$$

and hence $\epsilon^{2^{i-2}(2\ell+2j+1)}=0$ for all ℓ which may occur by definition of k_0.

We use induction on s on the statement that we may choose $c_{2^{i-s}m}$ so that only powers $\epsilon^{2^{i-s}\ell}$ occur in the difference

$$\epsilon^{2^i(2k+1)}-\sum_{m=1}^{n-1}c_{2^{i-s}m}\left(\epsilon^2+\epsilon^{2j+3}\right)^{2^{i-s}m}$$

with $\ell\geq(2^sk_0+(2^s-2)\cdot j+(2^s-1))$.

The statement is true for $s=1$ and $s=2$ by the above discussion. Suppose it is true for $s\leq i$. We shall prove it for $s+1$.

For every term $\epsilon^{2^{i-s}\ell}$ which occurs as a summand in

$$\epsilon^{2^i(2k+1)}-\sum_{m=1}^{n-1}c_{2^{i-s}m}\left(\epsilon^2+\epsilon^{2j+3}\right)^{2^{i-s}m}$$

we put $c_{2^{i-s-1}\ell}=1$ and then we compute

$$\begin{aligned}\left(\epsilon^2+\epsilon^{2j+3}\right)^{2^{i-s-1}\ell} &= \epsilon^{2^{i-s}\ell}+\binom{2^{i-s-1}\ell}{2^{i-s-1}}\left(\epsilon^2\right)^{2^{i-s-1}\ell-2^{i-s-1}}\cdot\left(\epsilon^{2j+3}\right)^{2^{i-s-1}}+\\ &\quad+\text{ higher order powers of }\epsilon^{2^{i-s-1}}\\ &= \epsilon^{2^{i-s}\ell}+\binom{2^{i-s-1}\ell}{2^{i-s-1}}\epsilon^{2^{i-s-1}(2\ell+2j+1)}+\text{ higher order powers of }\epsilon^{2^{i-2}}\end{aligned}$$

Now, by induction hypothesis $\ell\geq(2^sk_0+(2^s-2)\cdot j+(2^s-1))$. Hence

$$2\ell+2j+1\geq 2\cdot(2^sk_0+(2^s-2)\cdot j+(2^s-1))+2j+1=2^{s+1}k_0+(2^{s+1}-2)j+(2^{s+1}-1)$$

which is the statement for $s+1$.

But now, finally for $s=i$ we get that the error terms are ϵ^t where

$$t\geq 2^{i+1}k_0+(2^{i+1}-2)j+(2^{i+1}-1)\geq 2n$$

by definition of k_0 and hence the error terms are 0.

Therefore,

$$\epsilon^{2^i(2k+1)}\in\left\{\sum_{k=0}^{n-1}b_k\epsilon^{2k+1}\in K[\epsilon]\;\middle|\;\left(\sum_{k=0}^{n-1}b_k\epsilon^{2k+1}\right)^{2^i}\in\left\langle(\epsilon^2+\epsilon^{2j+3})^m\middle|\;m\in\mathbb{N}\right\rangle_K\right\}\quad\text{for all }k\geq k_0.$$

Fifth step. Now we are able to compute the dimension of

$$\tilde{T}_{n,j}(i)=\left\{\sum_{k=0}^{n-1}b_k\epsilon^{2k+1}\in K[\epsilon]\;\middle|\;\left(\sum_{k=0}^{n-1}b_k\epsilon^{2k+1}\right)^{2^i}\in\left\langle(\epsilon^2+\epsilon^{2j+3})^m\middle|\;m\in\mathbb{N}\right\rangle_K\right\}$$

We know by the third and fourth step that

$$\tilde{T}_{n,j}(i)=\left\{\sum_{k=0}^{n-1}b_k\epsilon^{2k+1}\in K[\epsilon]\;\middle|\;b_k=0\text{ for }k<\left\lceil\frac{2n-(2^{i+1}-2)j-(2^{i+1}-1)}{2^{i+1}}\right\rceil\right\}$$

The dimension of this space is therefore

$$\dim\left(T_i(L_n^{X^{2j}})/[L_n^{X^{2j}},L_n^{X^{2j}}]\right)=\dim\tilde{T}_{n,j}(i)=n-\left\lceil\frac{2n-(2^{i+1}-2)j-(2^{i+1}-1)}{2^{i+1}}\right\rceil$$

This finishes the proof. □

4.2. **Consequences for derived equivalence and stable equivalence of Morita type.** In this section we shall address the main motivational question for this paper, namely when deformed preprojective algebras of type L are derived equivalent or stably equivalent of Morita type. As main application of our results on Külshammer spaces we can obtain partial answers to these problems.

For both notions of equivalence it is in general a difficult question to decide whether two algebras are equivalent or not.

For the deformed preprojective algebras of type L_n, Białkowski, Erdmann and Skowroński show in [3, Theorem 2] that for an algebraically closed field K the set of algebras $\{L_n^{X^{2j}} \mid 0 \leq j \leq n-1\}$ gives a complete set of representatives for the Morita equivalence classes.

As an application of our result on Külshammer spaces we can now distinguish several of these algebras up to derived equivalence, and up to stable equivalence of Morita type.

Theorem 4.2. *Let K be a perfect field of characteristic 2.*

(a) *If two deformed preprojective algebras L_n^p and L_m^q are stably equivalent of Morita type or derived equivalent, then $n = m$.*

(b) *For $n \in \mathbb{N}$, let $j, k \in \{0, 1, \ldots, n-1\}$ be different numbers such that $\{j, k\} \neq \{n-2r, n-2r-1\}$ for every $1 \leq r \leq \lceil \frac{n-2}{2} \rceil$. Then the deformed preprojective algebras $L_n^{X^{2j}}$ and $L_n^{X^{2k}}$ are not stably equivalent of Morita type, and also not derived equivalent.*

Proof. (a) It is well-known that the number of simple modules is a derived invariant.

Moreover, by a result of C. Xi [23, Proposition 5.1], the absolute value of the determinant of the Cartan matrix of an algebra is invariant under stable equivalence of Morita type. For the deformed preprojective algebras L_n^p the Cartan determinant is 2^n, see Remark 3.3, so the result follows.

(b) We use the first Külshammer space or more precisely the following difference occurring in Theorem 4.1 for the case $i = 1$,

$$\dim_K \left(T_1(L_n^{X^{2j}})\right) - \dim_K \left([L_n^{X^{2j}}, L_n^{X^{2j}}]\right) \quad = \quad n - \max\left(\left\lceil \frac{2n-2j-3}{4} \right\rceil, 0\right). \tag{13}$$

By a result of Liu, Zhou and the second author [17, Corollary 7.5] this number is invariant under stable equivalences of Morita type. Since the numerator $2n-2j-3$ is congruent to 1 or 3 modulo 4 we have

$$\left\lceil \frac{2n-2j-3}{4} \right\rceil = \left\lceil \frac{2n-2j-2}{4} \right\rceil = \left\lceil \frac{n-j-1}{2} \right\rceil.$$

Note that for all the values $j \in \{0, \ldots, n-1\}$ this number is non-negative, so that equation (13) reads

$$\dim_K \left(T_1(L_n^{X^{2j}})\right) - \dim_K \left([L_n^{X^{2j}}, L_n^{X^{2j}}]\right) \quad = \quad n - \left\lceil \frac{n-j-1}{2} \right\rceil.$$

For fixed n, this invariant becomes equal for two different values $j, k \in \{0, \ldots, n-1\}$ precisely when $\{j, k\} = \{n-2r, n-2r-1\}$ for some $1 \leq r \leq \lceil \frac{n-2}{2} \rceil$. This proves the assertion on stable equivalence of Morita type.

The statement on derived equivalence follows immediately by using a result by Rickard [21] and Keller and Vossieck [15] saying that for selfinjective algebras (recall that our algebras L_n^p are even symmetric by Theorem 3.5) any derived equivalence induces a stable equivalence of Morita type. □

Remark 4.3. (1) In the above theorem we have for simplicity only exploited the first Külshammer space, but of course one could also use higher Külshammer spaces for distinguishing algebras up to derived equivalence, or up stable equivalence of Morita type. For explicit examples of deformed preprojective algebras of type L see Example 4.4 below.

(2) Note that part (b) of the above theorem in particular applies whenever $|j-k| \geq 2$.

(3) For any $n \in \mathbb{N}$ the (undeformed) preprojective algebra $L_n^{X^{2(n-1)}}$ is not stably equivalent of Morita type, and also not derived equivalent, to any of the algebras $L_n^{X^{2j}}$ for $j \in \{0, \ldots, n-2\}$. In fact, by the preceding remark it suffices to distinguish the algebras $L_n^{X^{2(n-1)}}$ and $L_n^{X^{2(n-2)}}$; but $j = n-1$ and $k = n-2$ are not of the form $\{n-2r, n-2r-1\}$ for some $1 \leq r \leq \lceil \frac{n-2}{2} \rceil$.

Example 4.4. (1) **The case $n = 2$ revisited.** Up to Morita equivalence there are two deformed preprojective algebras of type L_2, namely $L_2^{X^0}$ and $L_2^{X^2}$. They are not stably equivalent of Morita type (and hence not derived equivalent) by the Example in Section 4.1. So

we have a complete classification of deformed preprojective algebras of type L_2, up to stable equivalence of Morita type (and up to derived equivalence)

(2) **The case $n = 3$ revisited.** There are three deformed preprojective algebras of type L_3, namely $L_3^{X^0}$, $L_3^{X^2}$ and $L_3^{X^4}$. The algebra $L_3^{X^4}$ is not stably equivalent of Morita type (and hence not derived equivalent) to the other two algebras.

But with the Külshammer spaces we can not distinguish the algebras $L_3^{X^0}$ and $L_3^{X^2}$. We don't know whether these are stably equivalent of Morita type (or derived equivalent), or not.

(3) **The case $n = 5$.** For the five algebras (up to Morita equivalence) $L_5^{X^{2j}}$ where $j \in \{0, 1, 2, 3, 4\}$ we get the following numbers for the differences

$$\dim_K \left(T_i(L_5^{X^{2j}})\right) - \dim_K \left([L_5^{X^{2j}}, L_5^{X^{2j}}]\right) \quad = \quad 5 - \max\left(\left\lceil \frac{10 - (2^{i+1} - 2)j - (2^{i+1} - 1)}{2^{i+1}} \right\rceil, 0\right)$$

which are invariants under derived equivalence and under stable equivalence of Morita type.

$i \setminus j$	0	1	2	3	4
1	3	3	4	4	5
2	4	5	5	5	5
≥ 3	5	5	5	5	5

Therefore the algebras $L_5^{X^0}$, $L_5^{X^2}$, $L_5^{X^4}$ and $L_5^{X^8}$ are pairwise not stably equivalent of Morita type (and hence pairwise not derived equivalent). Note that $L_5^{X^0}$ and $L_5^{X^2}$ can only be distinguished by the second Külshammer spaces.

It remains open whether $L_5^{X^4}$ and $L_5^{X^6}$ are stably equivalent of Morita type (or derived equivalent), or not.

References

[1] M. Auslander, I. Reiten, *DTr-periodic modules and functors.* Representation theory of algebras (Cocoyoc, 1994), 39-50, CMS Conf. Proc., 18, Amer. Math. Soc., Providence, RI, 1996.

[2] J. Białkowski, K. Erdmann and A. Skowroński, *Deformed preprojective algebras of generalised Dynkin type.* Trans. Amer. Math. Soc., **359** (2007) 2625-2650.

[3] J. Białkowski, K. Erdmann, A. Skowroński, *Deformed preprojective algebras of generalised Dynkin type $\mathbb{L}_n$: classification and symmetricity.* J. Algebra, to appear.

[4] C. Bessenrodt, T. Holm, A. Zimmermann, *Generalised Reynolds ideals for non-symmetric algebras.* J. Algebra **312** (2007) 985-994.

[5] R.-O. Buchweitz, *Finite representation type and periodic Hochschild (co-)homology.* Trends in the representation theory of finite-dimensional algebras (Seattle, WA, 1997), 81–109, Contemp. Math., 229, Amer. Math. Soc., Providence, RI, 1998.

[6] W. Crawley-Boevey, M. P. Holland, *Noncommutative deformations of Kleinian singularities.* Duke Math. J. 92 (1998), no. 3, 605-635.

[7] K. Erdmann, A. Skowroński, *Periodic Algebras.* Trends in representation theory of algebras and related topics, 201–251, EMS Ser. Congr. Rep., Eur. Math. Soc., Zrich, 2008.

[8] K. Erdmann, N. Snashall, *On Hochschild cohomology of preprojective algebras. I, II.* J. Algebra 205 (1998), no. 2, 391-412, 413-434.

[9] N. J. Fine, *Binomial coefficients modulo a prime.* Amer. Math. Monthly 54 (1947), 589-592.

[10] I. M. Gel'fand, V. A. Ponomarev, *Model algebras and representations of graphs.* (Russian) Funktsional. Anal. i Prilozhen. 13 (1979), no. 3, 1-12. English translation: Functional Anal. Appl. 13 (1979), no. 3, 157-165 (1980).

[11] D. Happel, U. Preiser, C. M. Ringel, *Vinberg's characterization of Dynkin diagrams using subadditive functions with application to DTr-periodic modules.* Representation theory, II (Proc. Second Internat. Conf., Carleton Univ., Ottawa, Ont., 1979), pp. 280–294, Lecture Notes in Math., 832, Springer, Berlin, 1980.

[12] T. Holm, A. Skowroński, *Derived equivalence classification of symmetric algebras of domestic type.* J. Math. Soc. Japan **58** (2006), no.4, 1133-1149.

[13] T. Holm, A. Skowroński, *Derived equivalence classification of symmetric algebras of polynomial growth.* Glasgow Math. J., **53** (2011), 277-291.

[14] T. Holm, A. Zimmermann, *Generalized Reynolds ideals and the scalar problem for derived equivalences of tame blocks of finite groups.* J. Algebra **320** (2008) 3425-3437.

[15] B. Keller, D. Vossieck, *Sous les catégories dérivées.* C. R. Acad. Sci. Paris Sr. I Math. 305 (1987), no. 6, 225-228.

[16] B. Külshammer, *Bemerkungen über die Gruppenalgebra als symmetrische Algebra I, II, III, IV.* J. Algebra **72** (1981), 1–7; J. Algebra **75** (1982), 59–69; J. Algebra **88** (1984), 279–291; J. Algebra **93** (1985), 310–323.

[17] Y. Liu, G. Zhou, A. Zimmermann, *Higman ideal, stable Hochschild homology and Auslander-Reiten conjecture.* Math. Z., to appear.

[18] E. Lucas, *Théorie des Nombres, Tome I.* Librairie Scientifique et Technique Albert Blanchard, Paris 1961.

[19] T. Nakayama, *On Frobeniusean algebras. II.* Ann. Math. 42 (1941), 1-21.

[20] I. Reiner, Maximal Orders, Academic Press London-New York-San Francisco, 1975.

[21] J. Rickard, *Derived equivalences as derived functors.* J. London Math. Soc. 43 (1991), no. 1, 37-48.
[22] A. Schofield, *Wild algebras with periodic Auslander-Reiten translate.* Unpublished manuscript.
[23] C. Xi, *Stable equivalences of adjoint type.* Forum Math. **20** (2008), 81-97.
[24] G. Zhou, A. Zimmermann, *Classifying tame blocks and related algebras up to stable equivalences of Morita type.* J. Pure Appl. Algebra **215** (2011), 2969-2986.
[25] G. Zhou, A. Zimmermann, *Auslander-Reiten conjecture for symmetric algebras of polynomial growth.* Beiträge zur Algebra und Geometrie, to appear.
[26] A. Zimmermann, *Invariance of generalized Reynolds ideals under derived equivalences.* Mathematical Proceedings of the Royal Irish Academy **107**A (1) (2007) 1-9.
[27] A. Zimmermann, *On the use of Külshammer type invariants in representation theory.* Bulletin of the Iranian Mathematical Society, to appear (2011), `arXiv:1010.6240`.

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