

GREEN CORRESPONDENCE AND RELATIVE PROJECTIVITY FOR PAIRS OF ADJOINT FUNCTORS BETWEEN TRIANGULATED CATEGORIES

ALEXANDER ZIMMERMANN

ABSTRACT. Auslander and Kleiner proved in 1994 an abstract version of Green correspondence for pairs of adjoint functors between three categories. They produce additive quotients of certain subcategories giving the classical Green correspondence in the special setting of modular representation theory. Carlson, Peng and Wheeler showed in 1998 that Green correspondence in the classical setting of modular representation theory is actually an equivalence between triangulated categories with respect to a non standard triangulated structure. In the present note we first define and study a version of relative projectivity, respectively relative injectivity with respect to pairs of adjoint functors. We then modify Auslander-Kleiner's construction such that the correspondence holds in the setting of triangulated categories.

INTRODUCTION

Green correspondence is a very classical and highly important tool in modular representation theory of finite groups. For a finite group G and a field k of finite characteristic p , we associate to every indecomposable kG -module M a p -subgroup D , called its vertex. Simplifying slightly, Green correspondence then says that for H being a subgroup of G containing $N_G(D)$, restriction and induction give a mutually inverse bijection between the indecomposable kH -modules with vertex D and the indecomposable kG -modules with vertex D . It was known for a long time that this is actually a categorical correspondence, and in case of trivial intersection Sylow p -subgroups it was known to be more precisely actually an equivalence between the triangulated stable categories. Only in 1998 Carlson, Peng and Wheeler showed in [13] that it is possible to define triangulated structures also in the general case, and again the Green correspondence is an equivalence between triangulated categories.

Auslander and Kleiner showed in [1] that Green correspondence has a vast generalisation, and actually is a property of pairs of adjoint functors between three categories

$$\begin{array}{ccccc} & S' & & S & \\ \mathcal{D} & \xrightarrow{\quad} & \mathcal{H} & \xrightarrow{\quad} & \mathcal{G} \\ & T' & & T & \end{array}$$

such that (S, T) and (S', T') are adjoint pairs and an additional mild hypothesis on the unit of the adjunction (S, T) . Auslander-Kleiner show that then there is an equivalence between certain additive quotient categories mimicking the classical Green correspondence. For more details we recall the precise statement as Theorem 1.2 and Corollary 1.3 below.

Auslander-Kleiner do not study the question whether their abstract Green correspondence will provide an equivalence between triangulated categories. The present paper aims to fill this gap. Starting with triangulated categories \mathcal{D} , \mathcal{H} , \mathcal{G} and pairs of adjoint triangle functors (S', T') and (S, T) as above, we replace the additive quotient construction by Verdier localisation modulo the thick subcategories generated by the subcategories for which Auslander and Kleiner take the additive quotient. We obtain this way triangulated quotient categories and we show the precise analogue of Theorem 1.2 for the Verdier localisations instead of the additive quotient categories. In case S is left and right adjoint to T , and if in addition the unit of the adjunction is a monomorphism and the counit is an epimorphism our result shows that the additive quotient category is actually already

Date: January 28, 2020.

2010 Mathematics Subject Classification. Primary: 16E35; Secondary: 18E30; 20C05; 16S34; 18D10 .

Key words and phrases. Green correspondence; Verdier localisation; triangulated category; relative projectivity; vertex of modules.

triangulated, and that therefore the Verdier localisation and the additive quotient coincide. This way we directly generalise the result of Carlson, Peng and Wheeler [13].

In recent years classification results of thick subcategories of various triangulated categories were obtained mainly by parameterisations with subvarieties of support varieties. However, most results use those thick subcategories which form an ideal in an additional monoidal structure, so-called tensor triangulated categories. Since many examples, such as non principal blocks of group rings actually are not quite tensor triangulated, since a unit is missing we study more general a semigroup tensor structure, which is basically the same as a monoidal structure, but without a unit object. We study properties of our triangulated Green correspondence in this setting.

We further recall the classical situation and explain how we can recover parts of the results of Wang-Zhang [33] and Benson-Wheeler [6] using our approach.

The paper is organised as follows. In Section 1 we recall the main result of Auslander-Kleiner. Generalising the case of relative projective with respect to subgroups in the case of module categories, Section 2 then introduces the notion of T -relative projective (resp. T -relatively injective) objects in categories for functors T , and characterises this property in case of T having a left (resp. right) adjoint S . Here we push further and generalise a result due to Broué [10, Theorem 6.8]. We illustrate our constructions in the case of group algebras. Section 3 then compares Verdier localisation and the additive quotient categories. We prove there as well our first main result Theorem 3.17, generalising Auslander-Kleiner's theorem to triangulated categories using Verdier localisations. In Section 4 we revisit tensor triangulated categories and study their behaviour within our setting. In particular in Subsection 4.3 we compare our results to existing results in the literature in the case of group rings, their stable and derived categories, generalising various situations in this context.

Acknowledgement. I thank the referee for very careful reading and numerous suggestions which improved greatly the paper, in particular Section 2, but also throughout. I also thank Olivier Dudas for giving me the reference [10, Theorem 6.8]

1. SUMMARY OF AUSLANDER-KLEINER'S THEORY

Let $\mathcal{D}, \mathcal{H}, \mathcal{G}$ be three additive categories and S, S', T, T' be additive functors

$$\begin{array}{ccccc} \mathcal{D} & \xrightarrow{S'} & \mathcal{H} & \xrightarrow{S} & \mathcal{G} \\ & \xleftarrow{T'} & & \xleftarrow{T} & \\ & & \mathcal{H} & & \end{array}$$

such that (S, T) and (S', T') are adjoint pairs. Let $\epsilon : \text{id}_{\mathcal{H}} \rightarrow TS$ be the unit of the adjunction (S, T) . Assume that there is an endofunctor U of \mathcal{H} such that $TS = \text{id}_{\mathcal{H}} \oplus U$, denote by $p_1 : TS \rightarrow \text{id}_{\mathcal{H}}$ the projection, and suppose that $p_1 \circ \epsilon$ is an isomorphism. If ϵ is a split monomorphism, then this is satisfied, but the condition is slightly weaker. Auslander-Kleiner [1] prove a Green correspondence result for this situation.

Notation 1.1. • For a functor $F : \mathcal{A} \rightarrow \mathcal{B}$ and a full subcategory \mathcal{V} of \mathcal{B} denote for short $F^{-1}(\mathcal{V})$ the full subcategory of \mathcal{A} consisting of objects A such that $F(A) \in \text{add}(\mathcal{V})$.

- For an additive category \mathcal{W} and an additive subcategory \mathcal{V} denote by \mathcal{W}/\mathcal{V} the category whose objects are the same objects as those of \mathcal{W} , and for any two objects X, Y of \mathcal{W} we put

$$(\mathcal{W}/\mathcal{V})(X, Y) := \mathcal{W}(X, Y) / I_{\mathcal{V}}^{\mathcal{W}}(X, Y),$$

where

$$I_{\mathcal{V}}^{\mathcal{W}}(X, Y) := \{f \in \mathcal{W}(X, Y) \mid \exists V \in \text{obj}(\mathcal{V}), g \in \mathcal{W}(V, Y), h \in \mathcal{W}(X, V) : f = g \circ h\}.$$

- If \mathcal{S} and \mathcal{R} are subcategories of a Krull-Schmidt category \mathcal{W} , then $\mathcal{R} - \mathcal{S}$ denotes the full subcategory of \mathcal{R} consisting of those objects X of \mathcal{R} such that no direct factor of X is an object of \mathcal{S} .
- Recall that a full triangulated subcategory \mathcal{U} of a triangulated category is thick (épaisse), if it is in addition closed under taking direct summands (and a fortiori under isomorphisms) in \mathcal{T} .

- Let \mathcal{U} be a thick (épaisse) subcategory of a triangulated category \mathcal{T} . Then the Verdier localisation $\mathcal{T}_{\mathcal{U}}$ (cf [31, Chapitre I, § 2, no1, no3, no4], [32]) is the category formed by the same objects as the objects of \mathcal{T} and morphisms in $\mathcal{T}_{\mathcal{U}}$ are limits of diagrams

$$X \xleftarrow{s} Z \xrightarrow{f} Y$$

where f and s are morphisms in \mathcal{T} , and where $\text{cone}(s)$ is an object in \mathcal{U} . The notation we use for the Verdier localisation is not quite standard, however in order to distinguish from the additive quotient above we decided to use this notation (cf Remark 3.4).

Theorem 1.2. [1, Theorem 1.10] *Assume the hypotheses at the beginning of the section. Let \mathcal{Y} be a full additive subcategory of \mathcal{H} and let $\mathcal{Z} := (US')^{-1}(\mathcal{Y})$. Then the following two conditions (†) are equivalent.*

- Each object of $S'T'\mathcal{Y}$ is a direct factor of an object of \mathcal{Y} and of an object of $U^{-1}(\mathcal{Y})$.
- Each object of $TSS'T'\mathcal{Y}$ is a direct factor of an object of \mathcal{Y} .

Suppose that the above conditions hold for \mathcal{Y} . Then

- (1) S and T induce functors

$$\mathcal{H}/S'T'\mathcal{Y} \xrightarrow{S} \mathcal{G}/SS'T'\mathcal{Y} \text{ and } \mathcal{G}/SS'T'\mathcal{Y} \xrightarrow{T} \mathcal{H}/\mathcal{Y}$$

- (2) For any object L of \mathcal{D} and any object B of $U^{-1}(\mathcal{Y})$ the functor S induces an isomorphism

$$\mathcal{H}/S'T'\mathcal{Y}(S'L, B) \longrightarrow \mathcal{G}/SS'T'\mathcal{Y}(SS'L, SB)$$

- (3) For any object L of $(US')^{-1}\mathcal{Y}$ and any object A of \mathcal{G} the functor T induces an isomorphism

$$\mathcal{G}/SS'T'\mathcal{Y}(SS'L, B) \longrightarrow \mathcal{H}/\mathcal{Y}(TSS'L, TA)$$

- (4) The restrictions of S

$$(\text{add}S'\mathcal{Z})/S'T'\mathcal{Y} \xrightarrow{S} (\text{add}SS'\mathcal{Z})/SS'T'\mathcal{Y}$$

and T

$$(\text{add}SS'\mathcal{Z})/SS'T'\mathcal{Y} \xrightarrow{T} (\text{add}TSS'\mathcal{Z})/\mathcal{Y}$$

are equivalences of categories, and

$$(\text{add}S'\mathcal{Z})/S'T'\mathcal{Y} \xrightarrow{TS} (\text{add}TSS'\mathcal{Z})/\mathcal{Y}$$

is isomorphic to the natural projection.

- (5) If each object of $S'T'US'\mathcal{D}$ is a direct factor of $US'\mathcal{D}$, then $\mathcal{Y} = US'\mathcal{D}$ satisfies the hypothesis of the theorem.

A main consequence is

Corollary 1.3. [1, Corollary 1.12] *Let \mathcal{Y} be a full additive subcategory of \mathcal{H} satisfying (†) of Theorem 1.2 and suppose that \mathcal{H} and \mathcal{G} are both Krull-Schmidt categories. Using the notations of Theorem 1.2 then the following hold.*

- (1) For each indecomposable object N of $(\text{add}(S'\mathcal{Z}))/S'T'\mathcal{Y}$ the object SN has a unique indecomposable direct factor $g(N)$ which is not a direct factor of an object in $SS'T'\mathcal{Y}$.
- (2) For each indecomposable object M of $(\text{add}(SS'\mathcal{Z}))/SS'T'\mathcal{Y}$ the object TM has a unique indecomposable direct factor $f(M)$ which is not a direct factor of an object in \mathcal{Y} .
- (3) $f(g(N)) = N$
- (4) $g(f(M)) = M$.

2. RELATIVE PROJECTIVITY AND INJECTIVITY WITH RESPECT TO PAIRS OF ADJOINT FUNCTORS

2.1. Relative homological algebra revisited. We shall need to revise some facts from relative homological algebra, following [5]. Recall that a full subcategory \mathcal{X} of an additive category \mathcal{S} is contravariantly finite if for any object S of \mathcal{S} there is an object X of $\text{add}\mathcal{X}$ and a morphism $f \in \mathcal{S}(X, S)$ such that for any X' in \mathcal{X} the induced map

$$\mathcal{S}(X', f) : \mathcal{S}(X', X) \longrightarrow \mathcal{S}(X', S)$$

is surjective. We call such an object X of \mathcal{S} a right \mathcal{X} -approximation of S . The dual notion, using the covariant Hom -functor leads to the notion of a covariantly finite subcategory. With this notion in mind we shall have the following

Lemma 2.1. (*Auslander-Reiten* [2, Proposition 1.2]) *If the additive functor $T : \mathcal{S} \rightarrow \mathcal{T}$ between the additive categories \mathcal{S} and \mathcal{T} admits a left adjoint S_ℓ , then $\text{add}(\text{im}(S_\ell))$ is a contravariantly finite subcategory of \mathcal{S} . If T admits a right adjoint S_r , then $\text{add}(\text{im}(S_r))$ is a covariantly finite subcategory of \mathcal{S} .*

Proof. Let $\mathcal{X} := \text{add}(\text{im}(S_\ell))$. Consider the counit

$$\eta : S_\ell T \longrightarrow \text{id}_{\mathcal{S}}$$

of the adjoint pair (S_ℓ, T) . Evaluation on any object Q of \mathcal{S} gives a morphism

$$\eta_Q : S_\ell TQ \longrightarrow Q.$$

Now, given an object $S_\ell P$ in $\text{im}(S_\ell)$, we have

$$\begin{array}{ccc} \mathcal{S}(S_\ell P, S_\ell TQ) & \xrightarrow{\mathcal{S}(S_\ell P, \eta_Q)} & \mathcal{S}(S_\ell P, Q) \\ \uparrow \simeq & & \uparrow \simeq \\ \mathcal{T}(P, TS_\ell TQ) & & \mathcal{T}(P, TQ) \end{array}$$

and the composite map is a split epimorphism by [25, IV Theorem 1.(ii)]. Since the property holds true for direct factors of an object $S_\ell P$ as well, we showed that \mathcal{X} is a contravariantly finite subcategory of \mathcal{S} .

By the dual argument, if T admits a right adjoint S_r , then $\text{add}(\text{im} S_r)$ is a covariantly subcategory of \mathcal{S} . \square

Recall from Beligiannis and Marmaridis [5] that we may produce from contravariantly finite subcategories a relative homological algebra. Let \mathcal{X} be a contravariantly finite subcategory of an additive category \mathcal{S} . Then a morphism $g \in \mathcal{S}(A, B)$ is said to be \mathcal{X} -epic if for any object X of \mathcal{X} the morphism

$$\mathcal{S}(X, g) : \mathcal{S}(X, A) \longrightarrow \mathcal{S}(X, B)$$

is surjective. By the very definition, a right \mathcal{X} -approximation is an \mathcal{X} -epic. If \mathcal{X} is contravariantly finite and if each \mathcal{X} -epic has a kernel, [5, Theorem 2.12] shows that for any object S of \mathcal{S} the choice of a right \mathcal{X} -approximation $X_S \rightarrow S$ induces a left triangulation on the stable category \mathcal{S}/\mathcal{X} . Moreover, two such choices give equivalent left triangulated categories. Hence, a contravariantly finite subcategory \mathcal{X} such that each \mathcal{X} -epic has a kernel gives rise to the relative Ext^n -group with respect to \mathcal{X} , denoted by $\text{Ext}_{\mathcal{X}}^n(A, B)$ namely the evaluation on the object B , of the n -th derived functor of $\mathcal{S}(-, B)$, obtained by a \mathcal{X} -resolution of A .

Lemma 2.2. *Let \mathcal{S} and \mathcal{T} be additive categories, let $T : \mathcal{S} \rightarrow \mathcal{T}$ be an additive functor admitting a left adjoint S_ℓ . Then $g \in \mathcal{S}(A, B)$ is $\text{add}(\text{im}(S_\ell))$ -epic if and only if $T(g)$ is a split epimorphism.*

Proof. Let $g \in \mathcal{S}(A, B)$ be $\text{add}(\text{im}(S_\ell))$ -epic. Then for any object C of \mathcal{T} we get

$$\mathcal{S}(S_\ell(C), g) : \mathcal{S}(S_\ell(C), A) \longrightarrow \mathcal{S}(S_\ell(C), B)$$

is surjective. Hence, for any object C of \mathcal{T} we get

$$\mathcal{T}(C, Tg) : \mathcal{T}(C, TA) \longrightarrow \mathcal{T}(C, TB)$$

is surjective. In particular, for $C = TB$ there is $f \in \mathcal{T}(TB, TA)$ with $Tg \circ f = \text{id}_{TB}$. Hence Tg is a split epimorphism.

Let $g \in \mathcal{S}(A, B)$ be such that Tg is a split epimorphism. Then there is $f \in \mathcal{T}(TB, TA)$ with $Tg \circ f = \text{id}_{TB}$. Let C be an object of \mathcal{T} and let $h \in \mathcal{S}(S_\ell(C), B)$. We need to show that there is $k \in \mathcal{S}(S_\ell C, A)$ such that $\mathcal{S}(S_\ell(C), g)(k) = g \circ k = h$, where

$$\mathcal{S}(S_\ell(C), g) : \mathcal{S}(S_\ell(C), A) \longrightarrow \mathcal{S}(S_\ell(C), B).$$

Since T is right adjoint to S_ℓ , this is equivalent to

$$\mathcal{T}(C, Tg) : \mathcal{T}(C, TA) \longrightarrow \mathcal{T}(C, TB)$$

is surjective. For $h \in \mathcal{T}(C, TB)$ we get

$$h = (Tg \circ f) \circ h = Tg \circ (f \circ h) = \mathcal{T}(C, Tg)(f \circ h) = \mathcal{S}(S_\ell C, g)(f \circ h).$$

Clearly we can pass to direct factors of $S_\ell C$. Therefore, g is $\text{add}(\text{im}(S_\ell))$ -epic. \square

Note that Lemma 2.2 has a dual version for functors T admitting right adjoint functors S_r .

Again, in the setting of [5] the $\text{add}(\text{im}(S_\ell))$ -relative projectives, are those objects Q with

$$\text{Ext}_{\text{add}(\text{im}(S_\ell))}^n(Q, B) = 0$$

for all objects B and $n > 0$. By definition, this coincides with the objects Q for which the counit of the adjunction (S_ℓ, T) splits. These are precisely the objects in $\text{add}(\text{im}(S_\ell))$.

The dual statement applies in case of T having a right adjoint S_r , and considering covariantly finite subcategories and $\text{add}(\text{im}(S_r))$ -coresolutions instead of contravariantly finite subcategories and $\text{add}(\text{im}(S_\ell))$ -resolutions.

This motivates the following definition.

Definition 2.3. Let \mathcal{T} and \mathcal{S} be triangulated categories, and let $T : \mathcal{S} \rightarrow \mathcal{T}$ be a triangle functor.

- Suppose T has a left adjoint. Then an object Q of \mathcal{S} is *T -relative projective* if the natural transformation

$$\mathcal{S}(Q, -) \rightarrow \mathcal{T}(TQ, T-)$$

induced by T is injective.

- Suppose T has a right adjoint. Then an object Q of \mathcal{S} is *T -relative injective* if the natural transformation

$$\mathcal{S}(-, Q) \rightarrow \mathcal{T}(T-, TQ)$$

induced by T is injective.

Remark 2.4. Recall that for a field k of finite characteristic $p > 0$ and a finite group G with a subgroup H , an indecomposable kG -module M is called relatively H -projective if each epimorphism $N \twoheadrightarrow M$ of kG -modules, which is known to be split as kH -module morphism, splits as kG -module morphism. This definition of relative projectivity was developed by Hochschild [23] in the situation of a ring R , a subring S of R . Hochschild declares an R -module M to be (S, R) -projective, if any short exact sequence

$$0 \longrightarrow X \longrightarrow Y \longrightarrow M \longrightarrow 0$$

which is known to be split as short exact sequence of S -modules, is automatically split as short exact sequence of R -modules. Denoting by $\text{res}_S^R : R\text{-Mod} \longrightarrow S\text{-Mod}$ the exact functor given by restriction to the subring S , this translates into slightly more modern terms into the statement that M is (S, R) -projective if and only if

$$\text{Ext}_R^1(M, X) \longrightarrow \text{Ext}_S^1(\text{res}_S^R(M), \text{res}_S^R(X))$$

is injective for any R -module X . Hence, since $\text{res}_S^R(X)[1] \simeq \text{res}_S^R(X[1])$ we get that M is (S, R) -projective if and only if

$$\text{Hom}_{D^b(R)}(M, X[1]) \longrightarrow \text{Hom}_{D^b(S)}(\text{res}_S^R(M), \text{res}_S^R(X[1]))$$

is injective for all objects X . Since each object X can be seen as an object $X = Y[-1]$, Definition 2.3 could make sense in a broader context. We will not elaborate on this here (cf [35]).

Remark 2.5. Grime [20] defines an object to be relative projective with respect to a functor F admitting a left adjoint L as those which are direct factors of an object in the image of L . This is a direct generalisation of Green's definition [19], whereas our definition is closer to Hochschild's definition [23]. However, the concepts coincide, as will be shown in Proposition 2.10 below.

2.2. Relative projectivity for triangulated categories. We shall study the concept of T -relative projectivity/injectivity from Definition 2.3 for triangle functors T between triangulated categories admitting a left adjoint S_ℓ and a right adjoint S_r . Then the concept has a very nice interpretation.

Lemma 2.6. *Let \mathcal{S} and \mathcal{T} be additive categories and let $T : \mathcal{S} \rightarrow \mathcal{T}$ be an additive functor.*

- *If T has a left adjoint S_ℓ , then an object Q is T -relative projective if and only if the evaluation on Q of the counit η of the adjunction $\eta_Q : S_\ell TQ \rightarrow Q$ is an epimorphism. Any object in $\text{add}(\text{im}(S_\ell))$ is T -relative projective.*
- *If T has a right adjoint S_r , then an object Q is T -relative injective if and only if the evaluation on Q of the unit ϵ of the adjunction $\epsilon_Q : Q \rightarrow S_r TQ$ is a monomorphism. Any object in $\text{add}(\text{im}(S_r))$ is T -relative injective.*

Proof. Suppose that T has a left adjoint S_ℓ . Then the counit $\eta_Q : S_\ell TQ \rightarrow Q$ is an epimorphism if and only if for any object A the morphism

$$\mathcal{S}(\eta_Q, A) : \mathcal{S}(Q, A) \rightarrow \mathcal{S}(S_\ell TQ, A)$$

is a monomorphism. This in turn is equivalent to the statement that the natural transformation of functors $\mathcal{S} \rightarrow \mathbb{Z} - \text{Mod}$

$$\mathcal{S}(\eta_Q, -) : \mathcal{S}(Q, -) \rightarrow \mathcal{S}(S_\ell TQ, -)$$

is a monomorphism. Using the defining property of (S_ℓ, T) being an adjoint pair, this is equivalent to

$$\mathcal{S}(\eta_Q, -) : \mathcal{S}(Q, -) \rightarrow \mathcal{T}(TQ, T-)$$

being a monomorphism. Hence, the statement is equivalent to Q being T -relative projective. Now $\eta_{S_\ell Q'}$ is a split epimorphism for any object Q' of \mathcal{T} by [25, IV Theorem 1.(ii)].

Suppose that T has a right adjoint S_r . Then the unit $\epsilon_Q : Q \rightarrow S_r TQ$ is a monomorphism if and only if

$$\mathcal{S}(A, \epsilon_Q) : \mathcal{S}(A, Q) \rightarrow \mathcal{S}(A, S_r TQ)$$

is a monomorphism. This is equivalent to

$$\mathcal{S}(-, \epsilon_Q) : \mathcal{S}(-, Q) \rightarrow \mathcal{S}(-, S_r TQ) = \mathcal{T}(T-, TQ)$$

is a monomorphism, which is equivalent to Q is T -relative injective. Now $\epsilon_{S_r Q'}$ is a split monomorphism for any object Q' of \mathcal{T} by [25, IV Theorem 1.(ii)]. \square

Remark 2.7. Note that in a triangulated category \mathcal{S} the notions of epimorphism (resp. monomorphism) and split epimorphism (resp. split monomorphism) coincide.

Proposition 2.8. *Let \mathcal{T} and \mathcal{S} be triangulated categories and let $T : \mathcal{S} \rightarrow \mathcal{T}$ be a triangle functor. Suppose that T has a left (respectively right) adjoint S . Then an object Q is T -relative projective (respectively injective) if and only if Q is in $\text{add}(\text{im}(S))$.*

Proof. By Lemma 2.6 and Remark 2.7 Q is T -relative projective (respectively T -relative injective) if and only if Q is in $\text{add}(\text{im}(S))$. \square

Corollary 2.9. *Let \mathcal{T} and \mathcal{S} be triangulated categories and let $T : \mathcal{S} \rightarrow \mathcal{T}$ be a triangle functor. Suppose that T has a left (respectively right) adjoint S , and let $\eta : ST \rightarrow \text{id}$ be the counit (respectively $\bar{\epsilon} : \text{id} \rightarrow ST$ the unit) of the adjunction. Then Q is T -relative projective (respectively injective) if and only if η_Q is a split epimorphism (respectively $\bar{\epsilon}_Q$ is a split monomorphism).*

Proof. This is precisely Proposition 2.8 in connection with Lemma 2.6 and Remark 2.7. \square

We summarize the situation to an analogue of Higman's lemma for pairs of adjoint functors between triangulated categories.

Proposition 2.10. *Let \mathcal{T} and \mathcal{S} be triangulated categories and let $T : \mathcal{S} \rightarrow \mathcal{T}$ be a triangle functor. Suppose that T has a left (respectively right) adjoint S . Let M be an object of \mathcal{T} . Then the following are equivalent:*

- (1) *M is T -relative projective (respectively injective).*
- (2) *M is in $\text{add}(\text{im}S)$.*
- (3) *M is a direct factor of some $S(L)$ for some L in \mathcal{S} .*
- (4) *M is a direct factor of some $ST(M)$.*

Proof. (1) \Leftrightarrow (2) is Proposition 2.8.

(2) \Leftrightarrow (3) is the definition of $\text{add}(\text{im}S)$.

(3) \Rightarrow (4) is trivial.

(4) \Rightarrow (1) is Corollary 2.9. □

Remark 2.11. Note that Corollary 2.9 generalises [34, Proposition 2.1.6, Proposition 2.1.8] to this more general situation.

Remark 2.12. In case T has a left adjoint S , which is also assumed to be a right adjoint, and \mathcal{T} is an abelian or triangulated category Broué defined T -relative projective and T -relative injective objects in [10, Theorem 6.8]. In this situation he showed a version of Higman's lemma as Proposition 2.10 by completely different means.

Corollary 2.13. *Let \mathcal{S} and \mathcal{T} be triangulated categories and let $T : \mathcal{S} \rightarrow \mathcal{T}$ be a triangle functor admitting a left (resp. right) adjoint S . Then all objects of \mathcal{S} are T -relative projective (resp. injective) if and only if $\mathcal{S} = \text{add}(\text{im}(S))$.*

Note that all objects of \mathcal{S} are T -relative projective (resp. injective) if and only if the global dimension of the relative homological algebra described in Section 2.1 is 0.

Example 2.14. Let G be a group, and let H be a subgroup of finite index n . Denote by \downarrow_H^G the functor given by restriction of the G -action to the H -action, and by \uparrow_H^G the functor $kG \otimes_{kH} -$ given by induction. If n is invertible in the field k , then every object M in $D^b(kG)$ is \downarrow_H^G -relative projective. Indeed, [34, Proposition 2.1.10] shows that the multiplication $kG \otimes_{kH} kG \rightarrow kG$ splits as morphism of $kG - kG$ -bimodules. The counit of the adjunction $(\uparrow_H^G, \downarrow_H^G)$ is $kG \otimes_{kH} kG \otimes_{kG} - \rightarrow kG \otimes_{kG} -$, and by hypothesis this map splits.

2.3. Revisiting the case studied by Carlson-Peng-Wheeler. The purpose of this section is to give a structural explanation of an argument in the proof of Carlson, Peng and Wheeler for the statement that the relative stable category is triangulated (cf [13, page 304; proof of Theorem 6.2]). Note that Grime gave a slightly less general structural explanation in [20, Example 3.6].

Remark 2.15. Carlson, Peng and Wheeler consider the classical case of group rings, namely let k be a field of characteristic $p > 0$, let G be a finite group, let D be a p -subgroup of G and let H be a subgroup of G containing the normalizer of D in G . They consider the additive quotient of the module category modulo the morphisms which factor through \downarrow_E^G -projective modules, for some $E \in \mathfrak{Y}$, where $\mathfrak{Y} = \{E \leq H \cap D^g \mid g \in G \setminus H\}$, and show that this produces a triangulated category. Carlson, Peng and Wheeler use the general approach given by Happel [21, Theorem I.2.6] which shows that the additive quotient of any Frobenius category modulo relative injective-projectives is triangulated. However, Carlson, Peng and Wheeler just mention that Happel's proof for Frobenius categories to have triangulated stable categories carries over to this more general situation. The purpose of this section is to show that the fact that the proof carries over has a structural reason, and uses more precisely the properties from Section 2.1 and Section 2.2.

Note that group rings are symmetric, hence the module category is Frobenius. Moreover, the functors considered in classical Green correspondence, namely restriction and induction, are left and right adjoint to each other. We note that in our general abstract situation relative injectives and relative projectives do not coincide in general. The situation changes in case S is at the same time left and right adjoint to T and the categories are already Frobenius categories.

Recall from [12, Definition 2.1] the concept of an exact category. Let \mathcal{A} be an additive category. Given three objects A_1, A_2, A_3 in \mathcal{A} and $f \in \mathcal{A}(A_1, A_2)$ and $g \in \mathcal{A}(A_2, A_3)$. Then (f, g) is a short

exact sequence, denoted by

$$0 \longrightarrow A_1 \xrightarrow{f} A_2 \xrightarrow{g} A_3 \longrightarrow 0,$$

(or occasionally by $A_1 \xhookrightarrow{f} A_2 \twoheadrightarrow^g A_3$,) if $\ker(g) = \text{im}(f)$ and $g = \text{coker}(f)$.

An exact structure on the additive category \mathcal{A} is given by a class $E_{\mathcal{A}}$ of short exact sequences, called admissible short exact sequences, satisfying the following axioms below. If

$$0 \longrightarrow A_1 \xrightarrow{f} A_2 \xrightarrow{g} A_3 \longrightarrow 0,$$

is a short exact sequence in $E_{\mathcal{A}}$, then we say that f is an admissible monomorphism and g is an admissible epimorphism.

- For all objects A the identity on A is admissible monomorphism and admissible epimorphism.
- Admissible monomorphisms are closed under composition, and admissible epimorphisms are closed under composition.
- If $\alpha : X \rightarrow Y$ is an admissible monomorphism, and $f : X \rightarrow Z$ is any morphism, then the pushout

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & Y \\ f \downarrow & & \downarrow \tilde{f} \\ Z & \xrightarrow{\tilde{\alpha}} & U \end{array}$$

exists and $\tilde{\alpha}$ is an admissible monomorphism.

- If $\alpha : Y \rightarrow X$ is an admissible epimorphism, and $f : Z \rightarrow X$ is any morphism, then the pullback

$$\begin{array}{ccc} Y & \xrightarrow{\alpha} & X \\ \tilde{f} \uparrow & & \uparrow f \\ U & \xrightarrow{\hat{\alpha}} & Z \end{array}$$

exists and $\hat{\alpha}$ is an admissible epimorphism.

An exact category is an additive category \mathcal{A} with a class $E_{\mathcal{A}}$ of short exact sequences, stable under isomorphism and satisfying the above axioms. See [12] for an exhaustive development of exact categories.

Proposition 2.16. *Let $(\mathcal{S}, E_{\mathcal{S}})$ and $(\mathcal{T}, E_{\mathcal{T}})$ be exact categories with $E_{\mathcal{S}}$ respectively $E_{\mathcal{T}}$ being the class of admissible exact sequences. If $T : \mathcal{S} \rightarrow \mathcal{T}$ is a functor with a left adjoint S_{ℓ} and a right adjoint S_r ,*

- *then*

$$E_T := \left\{ \left(X \xhookrightarrow{f} Y \twoheadrightarrow^g Z \right) \in E_{\mathcal{S}} \mid \left(TX \xhookrightarrow{Tf} TY \twoheadrightarrow^{Tg} TZ \right) \in E_{\mathcal{T}} \right\}$$

defines an exact structure on \mathcal{S} .

- *If moreover the unit $\epsilon : \text{id}_{\mathcal{S}} \rightarrow S_r T$ is an admissible monomorphism in $E_{\mathcal{S}}$ and if the counit $\eta : S_{\ell} T \rightarrow \text{id}_{\mathcal{S}}$ is an admissible epimorphism in $E_{\mathcal{S}}$,*
 - *then (\mathcal{S}, E_T) has enough T -relative projectives and enough T -relative injectives.*
 - *Suppose now in addition that \mathcal{S} and \mathcal{T} are abelian Frobenius (i.e. an abelian category which is Frobenius with respect to the class of all exact sequences). Then the class of T -relative projectives coincides with the class of objects in $\text{add}(\text{im} S_{\ell})$ and the class of T -relative injectives coincides with the class of objects in $\text{add}(\text{im} S_r)$.*

Proof. We first show that E_T is an exact structure. $T(\text{id}_A) = \text{id}_{TA}$, which implies the first condition. T maps compositions to compositions, and hence compositions of admissible monics/epics are admissible monics/epics. Sequences are closed under isomorphisms, as T is exact and hence

maps isomorphisms to isomorphisms. Let $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z$ be an exact sequence in $E_{\mathcal{S}}$ and let $X \xrightarrow{f} X'$ be any morphism. Then, since $E_{\mathcal{S}}$ is an exact structure, we may form the pushout

$$\begin{array}{ccccc} X & \xrightarrow{\alpha} & Y & \xrightarrow{\beta} & Z \\ \downarrow f & & \downarrow g & & \parallel \\ X' & \xrightarrow{\tilde{\alpha}} & Y' & \xrightarrow{\tilde{\beta}} & Z \end{array}$$

As $E_{\mathcal{S}}$ is an exact structure, the lower row is an element of $E_{\mathcal{S}}$. The sequence

$$0 \longrightarrow X \xrightarrow{\begin{pmatrix} f \\ -\alpha \end{pmatrix}} X' \oplus Y \xrightarrow{(\tilde{\alpha}, g)} Y' \longrightarrow 0$$

is exact, since the above is a pushout and $\alpha, \tilde{\alpha}$ are monomorphisms. Since T is exact,

$$0 \longrightarrow TX \xrightarrow{\begin{pmatrix} Tf \\ -T\alpha \end{pmatrix}} TX' \oplus TY \xrightarrow{(T\tilde{\alpha}, Tg)} TY' \longrightarrow 0$$

is exact. Therefore

$$\begin{array}{ccccc} TX & \xrightarrow{T\alpha} & TY & \xrightarrow{T\beta} & TZ \\ \downarrow Tf & & \downarrow Tg & & \parallel \\ TX' & \xrightarrow{T\tilde{\alpha}} & TY' & \xrightarrow{T\tilde{\beta}} & TZ \end{array}$$

is a pushout diagram. Since the above row is in $E_{\mathcal{T}}$, and since $E_{\mathcal{T}}$ is an exact structure, also the lower row is in $E_{\mathcal{T}}$. This shows the third axiom. Dually also the fourth axiom holds.

We now assume the additional condition on the unit and the counit. The fact that $\text{add}(\text{im} S_{\ell})$ are T -relative injective objects and $\text{add}(\text{im} S_r)$ are T -relative projective objects is Lemma 2.6. The fact that we then get enough T -relative projective objects follows from the hypothesis on the counit, and the fact that we then get enough T -relative injective objects follows from the hypothesis on the unit.

The hypothesis on \mathcal{S} and \mathcal{T} being Frobenius with respect to all short exact sequences implies that the stable categories modulo projective-injective objects $\underline{\mathcal{S}}$ and $\underline{\mathcal{T}}$ are triangulated (cf Happel [21, Theorem I.2.6]). Proposition 2.8 applied to this triangulated category shows that $\text{add}(\text{im} S_{\ell})$ are precisely the T -relative projective objects and $\text{add}(\text{im} S_r)$ are precisely the T -relative injective objects of this new exact structure. \square

Remark 2.17. Note that the hypothesis of ϵ being a monomorphism and η being an epimorphism for the adjunctions involved is very strong. For an abelian category \mathcal{S} , if $T : \mathcal{S} \rightarrow \mathcal{T}$ has left and right adjoints S_{ℓ} and S_r , then T is exact. Further, Eilenberg-Moore [16, Proposition 1.5] (cf also Grime [20, Lemma 2.1]) show that the unit $\text{id} \rightarrow S_r T$, as in Proposition 2.16, is a monomorphism if and only if $TX = 0$ implies $X = 0$, if and only if the counit $S_{\ell} T \rightarrow \text{id}$ is an epimorphism. The counit $S_{\ell} T \rightarrow \text{id}$ is an epimorphism if and only if T is faithful.

In order to assure all quotient categories in Theorem 1.2 being triangulated, using Proposition 2.16 we need to assume the hypothesis for all the functors S, S', T, T' , and hence get quite a few restrictions on these functors.

Remark 2.18. The first item in Proposition 2.16 should be compared with the statement [16, Theorem II.2.1] by Eilenberg-Moore.

Remark 2.19. Let $(\mathcal{S}, E_{\mathcal{S}})$ and $(\mathcal{T}, E_{\mathcal{T}})$ be exact categories, let $T : \mathcal{S} \rightarrow \mathcal{T}$ be an exact functor admitting a left adjoint S_{ℓ} and a right adjoint S_r , and suppose the unit $\epsilon : \text{id}_{\mathcal{S}} \rightarrow S_r T$ of the adjoint property (T, S_r) is a monomorphism, and the counit $\eta : S_{\ell} T \rightarrow \text{id}_{\mathcal{S}}$ of the adjoint property (S_{ℓ}, T) is an epimorphism. Suppose moreover that $\text{add}(\text{im}(S_r)) = \text{add}(\text{im}(S_{\ell}))$. If in addition \mathcal{S} and \mathcal{T} are abelian Frobenius categories (i.e. an abelian category which is Frobenius with respect to the exact structure given by all exact sequences, then Proposition 2.16 shows that

$$E_T := \left\{ \left(X \xrightarrow{f} Y \xrightarrow{g} Z \right) \in E_{\mathcal{S}} \mid \left(TX \xrightarrow{Tf} TY \xrightarrow{Tg} TZ \right) \in E_{\mathcal{T}} \right\}$$

is a Frobenius structure on \mathcal{S} . We call this the T -relative Frobenius structure. Following Happel [21, Theorem I.2.6] the stable category $\underline{\mathcal{S}}^T$ of \mathcal{S} modulo the T -relative projectives is in this case a triangulated category. The distinguished triangles are constructed as follows. Given two objects M and N in \mathcal{S} and $f \in \mathcal{S}(M, N)$. Then we may form the pushout diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & M & \xrightarrow{\tilde{\epsilon}_M} & S_r T M & \longrightarrow & \Omega_T^{-1}(M) \longrightarrow 0 \\ & & \downarrow f & & \downarrow & & \parallel \\ 0 & \longrightarrow & N & \xrightarrow{c_1(f)} & C(f) & \xrightarrow{c_2(f)} & \Omega_T^{-1}(M) \longrightarrow 0 \end{array}$$

(or analogously the pullback diagram along $\eta_N : S_\ell T N \rightarrow N$). Then $\underline{\mathcal{S}}^T$ is a triangulated category with distinguished triangles being isomorphic to triangles of the form

$$M \xrightarrow{f} N \xrightarrow{c_1(f)} C(f) \xrightarrow{c_2(f)} \Omega_T^{-1}(M)$$

for any $f \in \mathcal{S}(M, N)$.

We recall a result implicit in Grime [20].

Proposition 2.20. (cf [20, Theorem 3.3]) *Let $(\mathcal{S}, E_{\mathcal{S}})$ and $(\mathcal{T}, E_{\mathcal{T}})$ be exact categories and let $T : \mathcal{S} \rightarrow \mathcal{T}$ be a functor which admits a left adjoint S_ℓ and a right adjoint S_r . Assume that the counit $S_\ell T \rightarrow \text{id}_{\mathcal{S}}$ of the adjoint pair (S_ℓ, T) is an admissible epimorphism in the exact category $(\mathcal{S}, E_{\mathcal{S}})$ and that the unit $\text{id}_{\mathcal{S}} \rightarrow S_r T$ of the adjoint pair (T, S_r) is an admissible monomorphism in the exact category $(\mathcal{S}, E_{\mathcal{S}})$. Putting*

$$E_T := \left\{ \left(X \xrightarrow{f} Y \xrightarrow{g} Z \right) \in E_{\mathcal{S}} \mid TX \xrightarrow{Tf} TY \xrightarrow{Tg} TZ \text{ is split exact in } \mathcal{T} \right\}$$

then (\mathcal{S}, E_T) is an exact category with enough projective and enough injective objects. The full subcategory of projective objects coincides with the full subcategory $\text{add}(\text{im}(S_\ell))$ and the full subcategory of injective objects coincides with the full subcategory $\text{add}(\text{im}(S_r))$.

Grime's proposition follows from Proposition 2.16 when it is applied to the case of the split exact structure on \mathcal{T} .

2.4. Relative projectivity for derived categories of group rings. We shall apply our concept of relative projectivity to the special case of the derived category of a block of a group ring kG . We first note that if A is a finite dimensional k -algebra over a field k , then $D^b(A)$ is a Krull-Schmidt category. Let G be a finite group, let H be a subgroup of G , let k be a field of characteristic $p > 0$. Then we consider the functors \uparrow_H^G and \downarrow_H^G . Note that both functors are exact functors between $kG\text{-mod}$ and $kH\text{-mod}$. These functors form an adjoint pair, in the sense that $(\uparrow_H^G, \downarrow_H^G)$ and $(\downarrow_H^G, \uparrow_H^G)$ are both adjoint pairs. Note that since both functors are exact, they provide functors

$$S := \uparrow_H^G : D^b(kH) \rightarrow D^b(kG)$$

and

$$T := \downarrow_H^G : D^b(kG) \rightarrow D^b(kH).$$

We define $\mathcal{G} := D^b(kG)$ and $\mathcal{H} := D^b(kH)$. Moreover, $(\uparrow_H^G, \downarrow_H^G)$ and $(\downarrow_H^G, \uparrow_H^G)$ are both adjoint pairs also between the derived categories. As for its restriction to the module categories we have

Lemma 2.21. *Let $K \leq H \leq G$ be an increasing sequence of groups. Then for the functors*

$$\begin{aligned} \uparrow_H^G & : D^b(kH) \rightarrow D^b(kG), \\ \uparrow_K^H & : D^b(kK) \rightarrow D^b(kH), \\ \downarrow_H^G & : D^b(kG) \rightarrow D^b(kH), \\ \downarrow_K^H & : D^b(kH) \rightarrow D^b(kK) \end{aligned}$$

we get

$$\uparrow_H^G \circ \uparrow_K^H = \uparrow_K^G \text{ and } \downarrow_K^H \circ \downarrow_H^G = \downarrow_K^G.$$

Proof. This follows trivially from the module case. □

Note that the notion of \downarrow_H^G -relative projectivity in $D^b(kG)$ corresponds to the similar concept of relative projectivity with respect to a subalgebra as developed in [34, Section 2.1.1]. We shall need to extend the statements from there to our more general situation.

Lemma 2.22. *Let G be a finite group, and let k be a field of characteristic $p > 0$. Let D be a minimal subgroup of G such that the bounded complex of kG -modules M is \downarrow_D^G -relative projective. Then D is a p -group.*

Proof. Let $D \in \text{Syl}_p(G)$. By Example 2.14 every object M of $D^b(kG)$ is \downarrow_D^G -relative projective since $|G : D|$ is prime to p by the definition of a Sylow subgroup. If M is \downarrow_H^G -relative projective, by Proposition 2.8 it is in $\text{add}(\text{im } \uparrow_H^G)$ and if $D' \in \text{Syl}_p(H)$, then M is also $\downarrow_{D'}^G$ -relative projective, whence in $\text{add}(\text{im } \uparrow_{D'}^G)$ by Proposition 2.8 again. Therefore M is in $\text{add}(\text{im } \uparrow_{D'}^G)$, and therefore $\downarrow_{D'}^G$ -relatively projective, again by Proposition 2.8. \square

Definition 2.23. Let G be a finite group, and let k be a field of characteristic $p > 0$. Then, an indecomposable object M of $D^b(kG)$ has vertex D if M is relatively kD -projective, and if D is minimal with this property.

Proposition 2.24. *The vertex D of an indecomposable object M of $D^b(kG)$ is a p -subgroup of G , and D is unique up to conjugacy.*

Proof. Using Lemma 2.22 we only need to show unicity up to conjugation.

The unicity part up to conjugation can be shown completely analogous to the classical case. Suppose that M is a direct summand of $L \uparrow_K^G$ and of $N \uparrow_H^G$ for two subgroups H and K of G and two indecomposable objects L in $D^b(kK)$ and N in $D^b(kH)$. By Proposition 2.10 we may suppose $L = M \downarrow_K^G$ and $N = M \downarrow_H^G$. Then M is a direct factor of

$$\begin{aligned} M \downarrow_H^G \uparrow_H^G \downarrow_K^G \uparrow_K^G &= \bigoplus_{KgH \in K \backslash G/H} {}^g M \downarrow_H^G \downarrow_{{}^g H \cap K} {}^g \uparrow_{{}^g H \cap K}^G \\ &= \bigoplus_{KgH \in K \backslash G/H} {}^g M \downarrow_{{}^g H \cap K}^G \uparrow_{{}^g H \cap K}^G \end{aligned}$$

Using the Krull-Schmidt property, M is a direct factor of ${}^g M \downarrow_{{}^g H \cap K}^G \uparrow_{{}^g H \cap K}^G$ for some g , and since K is minimal, there is $g \in G$ such that ${}^g H = K$. \square

Remark 2.25. The statements of the above results should remain true when we replace this quite specific setting by a Mackey functor with values in the functor category between triangulated categories.

Lemma 2.26. *Let G be a finite group, and let k be a field of characteristic $p > 0$. Let M be an indecomposable object of $D^b(kG)$. If M is kH -projective, then each indecomposable direct factor of $H^n(M)$ for all $n \in \mathbb{N}$ is relatively H -projective.*

Proof. By Proposition 2.8 we see that M is relatively $D^b(kH)$ -projective if and only if M is a direct factor of $L \uparrow_H^G$ for some L in $D^b(kH)$. Since \uparrow_H^G is exact, also $H^n(L \uparrow_H^G)$ has a direct factor $H^n(M)$. However, $H^n(L \uparrow_H^G) = H^n(L) \uparrow_H^G$ and hence $H^n(M)$ is a direct factor of $H^n(L) \uparrow_H^G$. Hence, by Higman's lemma [34, Proposition 2.1.15] from modular representation theory, each direct factor of $H^n(M)$ is relatively H -projective. \square

3. LOCALISING ON TRIANGULATED SUBCATEGORIES

As in [1] we consider the situation of three triangulated categories with functors

$$\begin{array}{ccccc} & & S' & & S \\ & \swarrow & \downarrow & \searrow & \downarrow \\ \mathcal{D} & \xleftarrow{T'} & \mathcal{H} & \xleftarrow{T} & \mathcal{G} \end{array}$$

such that (S, T) and (S', T') are adjoint pairs. Let $\epsilon : \text{id}_{\mathcal{H}} \rightarrow TS$ be the unit of the adjunction (S, T) and suppose that the unit is a split monomorphism. Hence

$$\text{id}_{\mathcal{H}} \xrightarrow{\epsilon} TS \xrightarrow{\pi} U \xrightarrow{0} \text{id}_{\mathcal{H}} [1]$$

is a distinguished triangle of functors. In particular, $TS = \text{id}_{\mathcal{H}} \oplus U$.

Remark 3.1. Let \mathcal{T} and \mathcal{S} be triangulated categories, and let $F : \mathcal{S} \rightarrow \mathcal{T}$ be a triangle functor. Then with the convention of Notion 1.1 for any full triangulated subcategory \mathcal{U} of \mathcal{T} the category $F^{-1}(\mathcal{U})$ is not necessarily triangulated. However, if \mathcal{U} is in addition closed under direct summands, then this is true, as is shown in Proposition 3.2 below.

Proposition 3.2. *Let \mathcal{T} and \mathcal{S} be triangulated categories, and let $F : \mathcal{S} \rightarrow \mathcal{T}$ be a triangle functor. Then for any thick subcategory \mathcal{U} of \mathcal{T} the category $F^{-1}(\mathcal{U})$ is a triangulated subcategory of \mathcal{S} .*

Proof. Let X be an object of \mathcal{S} such that $F(X)$ is a direct factor of the object U of \mathcal{U} . Hence, $F(X) \oplus U' = U$ for some object U' of \mathcal{T} . Since \mathcal{U} is closed under direct factors, $F(X)$ and U' are actually already objects of \mathcal{U} . Let X_1 and X_2 be two objects of \mathcal{S} such that $F(X_1)$ and $F(X_2)$ are objects of \mathcal{U} . If

$$X_1 \xrightarrow{\alpha} X_2 \longrightarrow C(\alpha) \longrightarrow X_1[1]$$

is a distinguished triangle in \mathcal{S} , since F is a triangle functor, also

$$FX_1 \xrightarrow{F\alpha} FX_2 \longrightarrow FC(\alpha) \longrightarrow FX_1[1]$$

is a distinguished triangle in \mathcal{T} , and hence $F(C(\alpha)) \simeq C(F(\alpha))$. Since \mathcal{U} is triangulated $C(F(\alpha))$ is an object in \mathcal{U} , and since \mathcal{U} is closed under isomorphisms, $F(C(\alpha))$ is an object of \mathcal{U} . Hence $C(\alpha)$ is an object of $F^{-1}(\mathcal{U})$. Therefore, $F^{-1}(\mathcal{U})$ is a triangulated subcategory of \mathcal{S} . \square

Lemma 3.3. *Let \mathcal{Y} be a full triangulated subcategory of \mathcal{H} . Then $\mathcal{Z} := (US')^{-1}(\mathcal{Y})$ satisfies $S'(\mathcal{Z}) = S'(\mathcal{D}) \cap U^{-1}(\mathcal{Y})$. Moreover, \mathcal{Z} is triangulated if \mathcal{Y} is thick.*

Proof. By definition $S'(\mathcal{Z})$ is the full subcategory of \mathcal{H} formed by objects $S'M$ such that $US'M \in \text{add}(\mathcal{Y})$. Hence $S'(\mathcal{Z})$ is contained in $S'(\mathcal{D}) \cap U^{-1}(\mathcal{Y})$. Moreover, an object X in $S'(\mathcal{D}) \cap U^{-1}(\mathcal{Y})$ is an object of the form $S'M$, since $X \in S'(\mathcal{D})$ and such that $US'M \in \text{add}(\mathcal{Y})$ since $X \in U^{-1}(\mathcal{Y})$. The rest follows from Proposition 3.2. \square

Remark 3.4. We remind the reader that we have two different localisation or quotient constructions of a triangulated category \mathcal{T} by a triangulated subcategory \mathcal{U} (cf Notion 1.1).

- First we have the additive quotient, denoted traditionally \mathcal{S}/\mathcal{U} having the same objects as \mathcal{S} but we consider morphisms between two objects as residue classes of morphisms in \mathcal{T} modulo those factoring through an object of \mathcal{U} .
- Second, the Verdier localisation [31, 32] which we denote by $\mathcal{S}_{\mathcal{U}}$. In the literature the Verdier localisation is often denoted by \mathcal{S}/\mathcal{U} . In order to distinguish from the additive quotient we decided to use the symbol $\mathcal{S}_{\mathcal{U}}$, contrary to the established convention in the literature.

Lemma 3.5. *Let \mathcal{S} be a triangulated category, and let \mathcal{U} be a thick subcategory of \mathcal{S} . Then there is a unique and natural functor $\mathcal{S}/\mathcal{U} \xrightarrow{L_{\mathcal{U}}} \mathcal{S}_{\mathcal{U}}$ making the diagram*

$$\begin{array}{ccc} & \mathcal{S} & \\ Q_{\mathcal{U}} \swarrow & & \searrow V_{\mathcal{U}} \\ \mathcal{S}/\mathcal{U} & \xrightarrow{\quad L_{\mathcal{U}} \quad} & \mathcal{S}_{\mathcal{U}} \end{array}$$

commutative. Here we denote $\mathcal{S} \xrightarrow{Q_{\mathcal{U}}} \mathcal{S}/\mathcal{U}$ and $\mathcal{S} \xrightarrow{V_{\mathcal{U}}} \mathcal{S}_{\mathcal{U}}$ the canonical functors given by the respective universal properties.

Proof. The proof is implicit in [27, Proposition 1.3]. The statement follows from the well-known fact that for any additive category \mathcal{A} and any additive functor $F : \mathcal{S} \rightarrow \mathcal{A}$ such that $F(U) = 0$ for any object U of \mathcal{U} , there is a unique additive functor $F^* : \mathcal{S}/\mathcal{U} \rightarrow \mathcal{A}$ with $F = F^* \circ Q_{\mathcal{U}}$. For $\mathcal{A} = \mathcal{S}_{\mathcal{U}}$, we observe that $F = V_{\mathcal{U}} : \mathcal{S} \rightarrow \mathcal{S}_{\mathcal{U}}$ is additive with $F(U) = 0$ for any object U of \mathcal{U} . Indeed, a morphism in the localisation becomes invertible if its cone is in \mathcal{U} . Hence, for an object U of \mathcal{U} the cone of the zero morphism on U is in \mathcal{U} . Therefore the image $V_{\mathcal{U}}(0_U)$ of the zero morphism 0_U on U in the localisation is invertible in $\mathcal{S}_{\mathcal{U}}$. The only object with invertible zero endomorphism is the zero object in $\mathcal{S}_{\mathcal{U}}$. This proves the statement. \square

Remark 3.6. We need to recall from Verdier [32, Chapitre II, Section 2.1, 2.2], or alternatively from Stacks project [29, Part 1, Chapter 13, Section 13.6], some properties of Verdier localisation. If $F : \mathcal{S} \rightarrow \mathcal{T}$ is a triangle functor between triangulated categories, then the full subcategory $\ker(F)$ of \mathcal{S} generated by those objects X of \mathcal{S} such that $F(X) = 0$ is thick. If \mathcal{U} is a full triangulated subcategory of some triangulated category \mathcal{T} , then the Verdier localisation defined by inverting all morphisms f in \mathcal{T} with cone in \mathcal{U} is triangulated, and there is a canonical functor $V_{\mathcal{U}} : \mathcal{T} \rightarrow \mathcal{T}_{\mathcal{U}}$ with \mathcal{U} is a full triangulated subcategory of $\ker(V_{\mathcal{U}})$. Moreover, $\ker(V_{\mathcal{U}})$ is thick, namely the smallest thick subcategory of \mathcal{T} containing \mathcal{U} , the thickening $\text{thick}(\mathcal{U})$.

Remark 3.7. We see that even if \mathcal{Y} is a thick subcategory of the triangulated category \mathcal{C} and if H is a triangle functor $\mathcal{C} \rightarrow \mathcal{D}$ for some triangulated category \mathcal{D} , then $H(\mathcal{Y})$ is triangulated, but is not thick anymore in general. The Verdier localisation of \mathcal{D} at $H(\mathcal{Y})$ has good properties with respect to thick subcategories. Since $\ker(V_{H(\mathcal{Y})}) = \text{thick}(H(\mathcal{Y}))$, we need to consider $\text{thick}(H(\mathcal{Y}))$. Then

$$\ker(\mathcal{T} \xrightarrow{V_{\text{thick}(H(\mathcal{Y}))}} \mathcal{T}_{\text{thick}(H(\mathcal{Y}))}) = \text{thick}(H(\mathcal{Y})) = \ker(\mathcal{T} \xrightarrow{V_{H\mathcal{Y}}} \mathcal{T}_{H\mathcal{Y}}).$$

Lemma 3.8. Let \mathcal{C} and \mathcal{D} be triangulated categories, let \mathcal{Y} be a subcategory of \mathcal{C} , and let $H : \mathcal{C} \rightarrow \mathcal{D}$ be a triangle functor. Then H extends to a unique functor $\mathcal{C}/\mathcal{Y} \rightarrow \mathcal{D}_{\text{thick}H\mathcal{Y}}$, also denoted by H , such that $H \circ Q_{\mathcal{Y}} = V_{\text{thick}H\mathcal{Y}} \circ H$, i.e. making the diagram

$$\begin{array}{ccccc} \mathcal{C} & \xrightarrow{H} & \mathcal{D} & & \\ Q_{\mathcal{Y}} \downarrow & & \searrow Q_{H\mathcal{Y}} & & \downarrow V_{\text{thick}H\mathcal{Y}} \\ \mathcal{C}/\mathcal{Y} & \xrightarrow{H} & \mathcal{D}/H\mathcal{Y} & \longrightarrow & \mathcal{D}_{\text{thick}H\mathcal{Y}} \\ & \searrow H & & & \\ & & \mathcal{D}_{\text{thick}H\mathcal{Y}} & & \end{array}$$

commutative. The functor $\mathcal{D}/H\mathcal{Y} \rightarrow \mathcal{D}/\text{thick}H\mathcal{Y}$ combined with the functor $L_{\text{thick}H\mathcal{Y}} : \mathcal{D}/\text{thick}H\mathcal{Y} \rightarrow \mathcal{D}_{\text{thick}H\mathcal{Y}}$ make the right triangle of the above diagram commutative.

Proof. Consider

$$\mathcal{C} \xrightarrow{H} \mathcal{D} \xrightarrow{V_{\text{thick}H\mathcal{Y}}} \mathcal{D}_{\text{thick}H\mathcal{Y}}.$$

Then for all objects X of \mathcal{Y} we get $(V_{\text{thick}H\mathcal{Y}} \circ H)(X) = 0$. Likewise consider

$$\mathcal{C} \xrightarrow{H} \mathcal{D} \xrightarrow{Q_{H\mathcal{Y}}} \mathcal{D}/H\mathcal{Y}.$$

Again, for all objects X of \mathcal{Y} we get $(Q_{H\mathcal{Y}} \circ H)(X) = 0$. Hence there is a unique functor $\mathcal{C}/\mathcal{Y} \xrightarrow{H_1} \mathcal{D}_{\text{thick}H(\mathcal{Y})}$ satisfying $H_1 \circ Q_{\mathcal{Y}} = V_{\text{thick}H\mathcal{Y}} \circ H$ and a unique functor $\mathcal{C}/\mathcal{Y} \xrightarrow{H_2} \mathcal{D}/H(\mathcal{Y})$ satisfying $H_2 \circ Q_{\mathcal{Y}} = Q_{H\mathcal{Y}} \circ H$. Moreover, by the universal property of $\mathcal{D}/H\mathcal{Y}$ the functor H_1 factors through H_2 and through $L_{\text{thick}H\mathcal{Y}}$. \square

Lemma 3.9. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a triangle functor between triangulated categories, let \mathcal{X} be a full subcategory of \mathcal{C} , and let \mathcal{Y} be a full subcategory of \mathcal{D} . If $F(\mathcal{X}) \subseteq \mathcal{Y}$, then there exists a unique additive functor $\mathcal{C}_{\text{thick}\mathcal{X}} \rightarrow \mathcal{D}_{\text{thick}\mathcal{Y}}$, still denoted by F , making the following diagram commutative.

$$\begin{array}{ccccc} \mathcal{C}/\mathcal{X} & \longrightarrow & \mathcal{C}/\text{thick}\mathcal{X} & \xrightarrow{L_{\text{thick}\mathcal{X}}} & \mathcal{C}_{\text{thick}\mathcal{X}} \\ F \downarrow & & F \downarrow & & F \downarrow \\ \mathcal{D}/\mathcal{Y} & \longrightarrow & \mathcal{D}/\text{thick}\mathcal{Y} & \xrightarrow{L_{\text{thick}\mathcal{Y}}} & \mathcal{D}_{\text{thick}\mathcal{Y}} \end{array}$$

Proof. Since F is a triangle functor,

$$F(\text{thick}\mathcal{X}) \subseteq \text{thick}F\mathcal{X} \subseteq \text{thick}\mathcal{Y}$$

and the left square is commutative. By Lemma 3.8 the right square is commutative as well. \square

For a triangulated category \mathcal{T} and a full subcategory \mathcal{X} there is a natural functor $\mathcal{T}/\mathcal{X} \rightarrow \mathcal{T}/\text{thick}\mathcal{X}$. For simplicity the composition

$$\mathcal{T}/\mathcal{X} \longrightarrow \mathcal{T}/\text{thick}\mathcal{X} \xrightarrow{L_{\text{thick}\mathcal{X}}} \mathcal{T}_{\text{thick}\mathcal{X}}$$

is also denote by $L_{\text{thick}\mathcal{X}}$.

Proposition 3.10. *Let $\mathcal{D}, \mathcal{H}, \mathcal{G}$ be three triangulated categories and triangle functors S, S', T, T'*

$$\begin{array}{ccc} \mathcal{D} & \xrightleftharpoons[S']{S} & \mathcal{H} \\ & \xleftarrow{T'} & \xleftarrow{T} \end{array} \mathcal{G}$$

so that (S, T) and (S', T') are adjoint pairs. Let $\epsilon : \text{id}_{\mathcal{H}} \rightarrow TS$ be the unit of the adjunction (S, T) . Assume that there is an endofunctor U of \mathcal{H} such that $TS = \text{id}_{\mathcal{H}} \oplus U$, denote by $p_1 : TS \rightarrow \text{id}_{\mathcal{H}}$ the projection, and suppose that $p_1 \circ \epsilon$ is an isomorphism.

Let \mathcal{Y} be a thick subcategory of \mathcal{H} , and suppose that each object of $TSS'T'\mathcal{Y}$ is a direct factor of an object of \mathcal{Y} . Then S and T induce triangle functors

$$\mathcal{H}_{\text{thick}S'T'\mathcal{Y}} \xrightarrow{S} \mathcal{G}_{\text{thick}SS'T'\mathcal{Y}} \text{ and } \mathcal{G}_{\text{thick}SS'T'\mathcal{Y}} \xrightarrow{T} \mathcal{H}_{\mathcal{Y}}$$

making the diagram

$$\begin{array}{ccccc} \mathcal{H}/\text{thick}S'T'\mathcal{Y} & \xrightarrow{S} & \mathcal{G}/\text{thick}SS'T'\mathcal{Y} & \xrightarrow{T} & \mathcal{H}/\mathcal{Y} \\ L_{\text{thick}S'T'\mathcal{Y}} \downarrow & & \downarrow L_{\text{thick}SS'T'\mathcal{Y}} & & \downarrow L_{\mathcal{Y}} \\ \mathcal{H}_{\text{thick}S'T'\mathcal{Y}} & \xrightarrow{S} & \mathcal{G}_{\text{thick}SS'T'\mathcal{Y}} & \xrightarrow{T} & \mathcal{H}_{\mathcal{Y}} \end{array}$$

commutative.

Proof. The existence of the functors in the left square and the commutativity of the left square follow from Lemma 3.8. From Lemma 3.8 we get natural functors giving a commutative diagram

$$\begin{array}{ccccc} \mathcal{H}/S'T'\mathcal{Y} & \xrightarrow{S} & \mathcal{G}/SS'T'\mathcal{Y} & \xrightarrow{T} & \mathcal{H}/TSS'T'\mathcal{Y} \\ L_{\text{thick}S'T'\mathcal{Y}} \downarrow & & \downarrow L_{\text{thick}SS'T'\mathcal{Y}} & & \downarrow L_{\text{thick}SS'T'\mathcal{Y}} \\ \mathcal{H}_{\text{thick}S'T'\mathcal{Y}} & \xrightarrow{S} & \mathcal{G}_{\text{thick}SS'T'\mathcal{Y}} & \xrightarrow{T} & \mathcal{H}_{\text{thick}TSS'T'\mathcal{Y}} \end{array}$$

For $\mathcal{X} = SS'T'\mathcal{Y}$, Theorem 1.2 shows that $T(\mathcal{X}) \subseteq \mathcal{Y}$. Using Lemma 3.9 and the fact that \mathcal{Y} is thick, and therefore $\text{thick}\mathcal{Y} = \mathcal{Y}$, we obtain a commutative diagram

$$\begin{array}{ccccc} \mathcal{H}/S'T'\mathcal{Y} & \xrightarrow{S} & \mathcal{G}/SS'T'\mathcal{Y} & \xrightarrow{T} & \mathcal{H}/\mathcal{Y} \\ L_{\text{thick}S'T'\mathcal{Y}} \downarrow & & \downarrow L_{\text{thick}SS'T'\mathcal{Y}} & & \downarrow L_{\mathcal{Y}} \\ \mathcal{H}_{\text{thick}S'T'\mathcal{Y}} & \xrightarrow{S} & \mathcal{G}_{\text{thick}SS'T'\mathcal{Y}} & \xrightarrow{T} & \mathcal{H}_{\mathcal{Y}} \end{array}$$

as requested.

The fact that

$$\mathcal{H}_{\text{thick}S'T'\mathcal{Y}} \xrightarrow{S} \mathcal{G}_{\text{thick}SS'T'\mathcal{Y}} \text{ and } \mathcal{G}_{\text{thick}SS'T'\mathcal{Y}} \xrightarrow{T} \mathcal{H}_{\mathcal{Y}}$$

are triangle functors comes from the universal property of the Verdier localisation (cf [32, Chapitre II, Théorème 2.2.6]). \square

Corollary 3.11. *Let $\mathcal{D}, \mathcal{H}, \mathcal{G}$ be three triangulated categories and triangle functors S, S', T, T'*

$$\begin{array}{ccc} \mathcal{D} & \xrightleftharpoons[S']{S} & \mathcal{H} \\ & \xleftarrow{T'} & \xleftarrow{T} \end{array} \mathcal{G}$$

so that (S, T) and (S', T') are adjoint pairs. Let $\epsilon : \text{id}_{\mathcal{H}} \rightarrow TS$ be the unit of the adjunction (S, T) . Assume that there is an endofunctor U of \mathcal{H} such that $TS = \text{id}_{\mathcal{H}} \oplus U$, denote by $p_1 : TS \rightarrow \text{id}_{\mathcal{H}}$ the projection, and suppose that $p_1 \circ \epsilon$ is an isomorphism.

Let \mathcal{Y} be a thick subcategory of \mathcal{H} , and suppose that each object of $TSS'T'\mathcal{Y}$ is a direct factor of an object of \mathcal{Y} . Then we have a commutative diagram

$$\begin{array}{ccccc}
 \mathcal{H}_{\text{thick}S'T'\mathcal{Y}} & \xrightarrow{S} & \mathcal{G}_{\text{thick}SS'T'\mathcal{Y}} & & \\
 \downarrow \text{can} & \swarrow T & \searrow L_{\text{thick}SS'T'\mathcal{Y}} & & \\
 \mathcal{H}_{\mathcal{Y}} & & & & \\
 & \swarrow L_{\text{thick}S'T'\mathcal{Y}} & \searrow L_{\mathcal{Y}} & & \\
 & & \mathcal{H}/S'T'\mathcal{Y} & \xrightarrow{S} & \mathcal{G}/SS'T'\mathcal{Y} \\
 & & \downarrow \text{can} & \swarrow T & \\
 & & \mathcal{H}/\mathcal{Y} & &
 \end{array}$$

Proof. Indeed, since each object of $TSS'T'\mathcal{Y}$ is a direct factor of an object of \mathcal{Y} , each object of $S'T'\mathcal{Y}$ is a direct factor of an object of \mathcal{Y} . Hence, there is a natural functor can as indicated. The rest of the statement is an immediate consequence of Proposition 3.10. \square

Corollary 3.12. Let \mathcal{D} , \mathcal{H} , \mathcal{G} be three triangulated categories and triangle functors S, S', T, T'

$$\begin{array}{ccccc}
 \mathcal{D} & \xrightarrow{S'} & \mathcal{H} & \xrightarrow{S} & \mathcal{G} \\
 & \swarrow T' & \swarrow T & & \\
 & & \mathcal{H} & &
 \end{array}$$

so that (S, T) and (S', T') are adjoint pairs. Let $\epsilon : \text{id}_{\mathcal{H}} \rightarrow TS$ be the unit of the adjunction (S, T) . Assume that there is an endofunctor U of \mathcal{H} such that $TS = \text{id}_{\mathcal{H}} \oplus U$, denote by $p_1 : TS \rightarrow \text{id}_{\mathcal{H}}$ the projection, and suppose that $p_1 \circ \epsilon$ is an isomorphism.

Let \mathcal{Y} be a thick subcategory of \mathcal{H} , put $\mathcal{Z} := (U S')^{-1}(\mathcal{Y})$, and suppose that each object of $TSS'T'\mathcal{Y}$ is a direct factor of an object of \mathcal{Y} . Then the restriction of S to the subcategory $\text{add}S'\mathcal{Z}/S'T'\mathcal{Y}$ and the restriction of T to the subcategory $\text{add}SS'\mathcal{Z}/SS'T'\mathcal{Y}$ are equivalences and gives a commutative diagram

$$\begin{array}{ccccc}
 \text{thick}S'\mathcal{Z}_{\text{thick}S'T'\mathcal{Y}} & \xrightarrow{S} & \text{thick}SS'\mathcal{Z}_{\text{thick}SS'T'\mathcal{Y}} & & \\
 \downarrow \text{can} & \swarrow T & \searrow L_{\text{thick}SS'T'\mathcal{Y}} & & \\
 \text{thick}S'\mathcal{Z}_{\mathcal{Y}} & & & & \\
 & \swarrow L_{\text{thick}S'T'\mathcal{Y}} & \searrow L_{\mathcal{Y}} & & \\
 & & \text{add}S'\mathcal{Z}/S'T'\mathcal{Y} & \xrightarrow{S} & \text{add}SS'\mathcal{Z}/SS'T'\mathcal{Y} \\
 & & \downarrow \text{can} & \swarrow T & \\
 & & \text{add}TSS'\mathcal{Z}/\mathcal{Y} & &
 \end{array}$$

where the lower triangle consists of equivalences.

Proof. The fact that the lower triangle exists and is commutative follows from Theorem 1.2. Since for any subcategory \mathcal{X} of \mathcal{T} we get that $\text{add}\mathcal{X}$ is a full subcategory of $\text{thick}\mathcal{X}$, we have a commutative

diagram

$$\begin{array}{ccc}
 (\text{thick}S'Z)/S'T'\mathcal{Y} & \xrightarrow{s} & (\text{thick}SS'Z)/SS'T'\mathcal{Y} \\
 \downarrow \text{can} & \nwarrow T & \nwarrow \\
 (\text{thick}S'Z)/\mathcal{Y} & & \text{add}S'Z/S'T'\mathcal{Y} \xrightarrow{s} \text{add}SS'Z/SS'T'\mathcal{Y} \\
 & \nwarrow & \downarrow \text{can} \\
 & & \text{add}TSS'Z/\mathcal{Y}
 \end{array}$$

By Lemma 3.8 we obtain a commutative diagram

$$\begin{array}{ccc}
 \text{thick}S'Z_{\text{thick}S'T'\mathcal{Y}} & \xrightarrow{s} & \text{thick}SS'Z_{\text{thick}SS'T'\mathcal{Y}} \\
 \downarrow \text{can} & \nwarrow T & \nwarrow L_{\text{thick}S'T'\mathcal{Y}} \\
 \text{thick}S'Z_{\mathcal{Y}} & & \text{thick}S'Z_{\text{thick}SS'T'\mathcal{Y}} \\
 & \nwarrow L_{\mathcal{Y}} & \nwarrow L_{\text{thick}SS'T'\mathcal{Y}} \\
 & & (\text{thick}S'Z)/S'T'\mathcal{Y} \xrightarrow{s} (\text{thick}SS'Z)/SS'T'\mathcal{Y} \\
 & & \downarrow \text{can} \\
 & & (\text{thick}S'Z)/\mathcal{Y}
 \end{array}$$

Composition of the two diagrams yields the statement. \square

Proposition 3.13. *Let \mathcal{T} and \mathcal{U} be two triangulated categories, let $F : \mathcal{T} \rightarrow \mathcal{U}$ be a triangle functor, let \mathcal{X} be a full additive subcategory of \mathcal{T} and let \mathcal{Y} be a full additive subcategory of \mathcal{X} . Then the restriction of F to $\mathcal{X}/\mathcal{Y} \xrightarrow{F_{\mathcal{X}}} (\text{add}F\mathcal{X})/F\mathcal{Y}$ extends to a functor $(\text{thick}\mathcal{X})/\mathcal{Y} \xrightarrow{F_{\text{thick}\mathcal{X}}} (\text{thick}F\mathcal{X})/F\mathcal{Y}$ such that $F_{\text{thick}\mathcal{X}}$ coincides with $F_{\mathcal{X}}$ on the subcategory \mathcal{X}/\mathcal{Y} .*

Proof. Let \mathcal{A} and \mathcal{B} be full subcategories of a triangulated category \mathcal{V} , then as in [4] we denote by $\mathcal{A} * \mathcal{B}$ the full subcategory of \mathcal{V} generated by $C(t)[-1]$ where

$$A \xrightarrow{f} C(t)[-1] \xrightarrow{s} B \xrightarrow{t} A[1]$$

is a distinguished triangle, and where A is an object of \mathcal{A} , B is an object of \mathcal{B} , and $t \in \mathcal{T}(B, A[1])$. Since $F : \mathcal{T} \rightarrow \mathcal{U}$ is a triangle functor, F sends distinguished triangles to distinguished triangle. Therefore $F(\mathcal{A} * \mathcal{B})$ is a subcategory of $F(\mathcal{A}) * F(\mathcal{B})$. Hence F induces a functor

$$\mathcal{A} * \mathcal{B} \xrightarrow{F} (F\mathcal{A}) * (F\mathcal{B}).$$

Let $X_1 \in \text{add}F\mathcal{A}$ and $X_2 \in \text{add}F\mathcal{B}$. Then for any $t \in \mathcal{U}(X_2, X_1[1])$ we get

$$C(t)[-1] \in \text{add}(F(\mathcal{A}) * F(\mathcal{B})).$$

Indeed, denote by

$$X_1 \xrightarrow{f} C(t)[-1] \xrightarrow{s} X_2 \xrightarrow{t} X_1[1]$$

the distinguished triangle given by t . Let X'_1 and X'_2 be objects of \mathcal{U} such that $X_1 \oplus X'_1 \in F(\mathcal{A})$ and $X_2 \oplus X'_2 \in F(\mathcal{B})$. Then

$$X_1 \oplus X'_1 \xrightarrow{\begin{pmatrix} f & 0 \\ 0 & \text{id}_{X'_1} \\ 0 & 0 \end{pmatrix}} C(t)[-1] \oplus X'_1 \oplus X'_2 \xrightarrow{\begin{pmatrix} s & 0 & 0 \\ 0 & 0 & \text{id}_{X'_2} \end{pmatrix}} X_2 \oplus X'_2 \xrightarrow{\begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix}} (X_1 \oplus X'_1)[1]$$

is a distinguished triangle. Hence $C(t)[-1] \in \text{add}(F(\mathcal{A}) * F(\mathcal{B}))$. This shows

$$\text{add}(F(\mathcal{A})) * \text{add}(F(\mathcal{B})) \subseteq \text{add}(F(\mathcal{A}) * F(\mathcal{B})),$$

and therefore

$$\text{add}(\text{add}(F(\mathcal{A})) * \text{add}(F(\mathcal{B}))) = \text{add}(F(\mathcal{A}) * F(\mathcal{B})).$$

If we define $(\mathcal{Z})_n := (\mathcal{Z})_{n-1} * \mathcal{Z}$ for any subcategory \mathcal{Z} of \mathcal{U} , and $(\mathcal{Z})_1 := \mathcal{Z}$, then

$$\text{thick}(F(\mathcal{X})) = \bigcup_{n \in \mathbb{N}} \text{add}((\text{add}(F(\mathcal{X})))_n) = \bigcup_{n \in \mathbb{N}} \text{add}((F(\mathcal{X}))_n) = \text{add}(\bigcup_{n \in \mathbb{N}} (F(\mathcal{X}))_n).$$

Now,

$$\text{thick}\mathcal{X} = \bigcup_{n \in \mathbb{N}} \text{add}\mathcal{X}_n = \text{add}\left(\bigcup_{n \in \mathbb{N}} \mathcal{X}_n\right)$$

and $F(\mathcal{X}_n) \subseteq (F\mathcal{X})_n$. Hence

$$\begin{aligned} F(\text{thick}\mathcal{X})/F\mathcal{Y} &= F(\text{add}\left(\bigcup_{n \in \mathbb{N}} \mathcal{X}_n\right))/F\mathcal{Y} \\ &\subseteq \text{add}\left(\bigcup_{n \in \mathbb{N}} F(\mathcal{X}_n)\right)/F\mathcal{Y} \\ &\subseteq \text{add}\left(\bigcup_{n \in \mathbb{N}} (F(\mathcal{X}))_n\right)/F\mathcal{Y} \\ &= \text{thick}(F(\mathcal{X}))/F\mathcal{Y} \end{aligned}$$

Therefore F extends to a functor

$$\text{thick}(\mathcal{X})/\mathcal{Y} \xrightarrow{F_{\text{thick}\mathcal{X}}} \text{thick}(F(\mathcal{X}))/F\mathcal{Y}.$$

By construction the restriction of $F_{\text{thick}\mathcal{X}}$ to \mathcal{X}/\mathcal{Y} coincides with $F_{\mathcal{X}}$. \square

Remark 3.14. If in Proposition 3.13 the functor F induces an equivalence $\mathcal{X}/\mathcal{Y} \rightarrow (\text{add}F\mathcal{X})/F\mathcal{Y}$, then there is no reason why this should imply an equivalence

$$\text{thick}(\mathcal{X})/\mathcal{Y} \rightarrow \text{thick}(F(\mathcal{X}))/F\mathcal{Y}.$$

Lemma 3.15. *Let \mathcal{Y} be a subcategory of \mathcal{X} admitting finite direct sums. Then the natural projection $\mathcal{X}/\mathcal{Y} \rightarrow \mathcal{X}/(\text{add}\mathcal{Y})$ is an equivalence of categories.*

Proof. Since \mathcal{Y} is a subcategory of $\text{add}\mathcal{Y}$, if a morphism f factors through an object of \mathcal{Y} , it factors also through an object of $\text{add}\mathcal{Y}$. Hence, the natural projection is well-defined and full. If f factors through an object X of $\text{add}\mathcal{Y}$, then there is an object X' of $\text{add}\mathcal{Y}$, such that $X \oplus X'$ is an object of \mathcal{Y} . Extending by the zero morphism to and from X' , hence f factors also through the object $X \oplus X'$ of \mathcal{Y} . This shows that the natural projection is faithful as well.

From the above it also follows that the natural projection is dense, since the objects of both quotient categories coincide, and the natural projection is the identity on objects. \square

Proposition 3.16. *Let \mathcal{T} and \mathcal{U} be triangulated categories, let \mathcal{Y} be a subcategory of \mathcal{T} , and let $F : \mathcal{T} \rightarrow \mathcal{U}$ be a triangle functor. Suppose that F induces an equivalence*

$$F_Q : \mathcal{T}/(\text{thick}\mathcal{Y}) \rightarrow \mathcal{U}/(\text{thick}F(\mathcal{Y})).$$

Then F induces a dense and full triangle functor

$$F_V : \mathcal{T}_{(\text{thick}\mathcal{Y})} \rightarrow \mathcal{U}_{(\text{thick}F(\mathcal{Y}))}.$$

If in addition $F(\text{thick}\mathcal{Y})$ is thick in \mathcal{U} then F_V is a triangle equivalence.

Proof. The functor F_V exists by the universal property of the Verdier localisation [32, Chapitre II, Corollaire 2.2.11.c].

We shall now show that F_V is dense. The objects of $\mathcal{U}/(\text{thick}F(\mathcal{Y}))$ coincides with the objects of $\mathcal{U}_{(\text{thick}F(\mathcal{Y}))}$, since they both coincides with the objects of \mathcal{U} . By hypothesis, for every object U of \mathcal{U} there is an object T of \mathcal{T} , and $f \in \mathcal{U}(FT, U)$ as well as $g \in \mathcal{U}(U, FT)$ such that $g \circ f - \text{id}_{FT}$ factors through an object Y' of $\text{thick}(F\mathcal{Y})$ and $f \circ g - \text{id}_U$ factors through an object Y of $\text{thick}(F\mathcal{Y})$. Hence, applying $L_{\text{thick}F\mathcal{Y}}$ to these equations, and observing that $L_{\text{thick}F\mathcal{Y}}(Y) = 0$,

respectively $L_{\text{thick}F\mathcal{Y}}(Y') = 0$ for all objects Y in $\text{thick}(F\mathcal{Y})$, respectively all objects Y' in $\text{thick}(F\mathcal{Y})$, we get that the image of f in the Verdier localisation is an isomorphism. Hence F_V is dense.

We will show now that F_V is full.

First step: Let $f \in \mathcal{U}(FZ, FX)$. Since F_Q is full, there is $f' \in \mathcal{T}(Z, X)$ such that $f - Ff'$ factors through an object M of $\text{thick}(F\mathcal{Y})$. Hence there is $g \in \mathcal{U}(M, X)$ and $h \in \mathcal{U}(Z, M)$ with $f - Ff' = g \circ h$ in \mathcal{U} . We denote by $(1, f)$ the morphism represented by the diagram $FZ \xleftarrow{\text{id}_{FZ}} FZ \xrightarrow{f} FX$. Then $(1, f - Ff') = (1, g) \circ (1, h)$ in $\mathcal{U}_{(\text{thick}F(\mathcal{Y}))}$. Since $M \simeq 0$ in $\mathcal{U}_{(\text{thick}F(\mathcal{Y}))}$, we get

$$(1, f) - (1, Ff') = (1, f - Ff') = 0$$

in $\mathcal{U}_{(\text{thick}F(\mathcal{Y}))}$ and therefore

$$(1, f) = (1, Ff') = F_V(1, f').$$

Hence $f = F_Q f'$ in $\mathcal{U}/\text{thick}F\mathcal{Y}$ implies $(1, f) = F_V(1, f')$ in $\mathcal{U}_{(\text{thick}F(\mathcal{Y}))}$.

Second step: Let $FX \xleftarrow{s} \hat{Z} \xrightarrow{f} FY$ represent a morphism in $\mathcal{U}_{(\text{thick}F(\mathcal{Y}))}(F_V X, F_V Y)$. Since F_V is dense, we may suppose that $\hat{Z} = FZ$ for some object Z of \mathcal{U} . Then $s = F_Q s'$ for some $s' \in \mathcal{T}(Z, X)$, and $f = F_Q f'$ for some $f' \in \mathcal{T}(Z, Y)$, giving $(1, s) = F_V(1, s')$, and $(1, f) = F_V(1, f')$ by the first step. Now,

$$F(\text{cone}(s')) \simeq \text{cone}(F(s')) \simeq \text{cone}(s) \in \text{thick}(F(\mathcal{Y}))$$

and therefore $F(\text{cone}(s')) = 0$ in $\mathcal{U}/(\text{thick}F\mathcal{Y})$. Since F induces an equivalence

$$\mathcal{U}/(\text{thick}F\mathcal{Y}) \simeq \mathcal{T}/(\text{thick}\mathcal{Y}),$$

we get $\text{cone}(s') = 0$ in $\mathcal{T}/(\text{thick}\mathcal{Y})$, which shows that $\text{cone}(s') \in \text{thick}\mathcal{Y}$. Hence $X \xleftarrow{s'} Z \xrightarrow{f'} Y$ maps to $FX \xleftarrow{s} FZ \xrightarrow{f} FY$. Therefore F_V is full.

We now assume that in addition $F(\text{thick}\mathcal{Y})$ is thick in \mathcal{U} .

We need to show that F_V is faithful. Since $F(\text{thick}\mathcal{Y})$ is thick in \mathcal{U} , we get $\text{thick}(F(\mathcal{Y})) = F(\text{thick}\mathcal{Y})$. Hence, using the notation from [28], and using that F_Q is an equivalence, $\ker(F_V) = 0$. By a result of Rickard [26, first paragraph on page 446] or Ringel and Zhang [28, Proposition 3.1, Proposition 3.3, Theorem 1.1] we see that F_V is faithful. This finishes the proof. \square

Theorem 3.17. (*Green correspondence for triangulated categories*) Let $\mathcal{D}, \mathcal{H}, \mathcal{G}$ be three triangulated categories and let S, S', T, T' be triangle functors

$$\begin{array}{ccccc} \mathcal{D} & \xrightleftharpoons[S']{} & \mathcal{H} & \xrightleftharpoons[T]{} & \mathcal{G} \end{array}$$

such that (S, T) and (S', T') are adjoint pairs. Let $\epsilon : \text{id}_{\mathcal{H}} \rightarrow TS$ be the unit of the adjunction (S, T) . Assume that there is an endofunctor U of \mathcal{H} such that $TS = \text{id}_{\mathcal{H}} \oplus U$, denote by $p_1 : TS \rightarrow \text{id}_{\mathcal{H}}$ the projection, and suppose that $p_1 \circ \epsilon$ is an isomorphism.

Let \mathcal{Y} be a thick subcategory of \mathcal{H} , put $\mathcal{Z} := (US')^{-1}(\mathcal{Y})$, and suppose that each object of $TSS'T'\mathcal{Y}$ is a direct factor of an object of \mathcal{Y} .

(1) Then S and T induce triangle functors S_Z and T_Z fitting into the commutative diagram

$$\begin{array}{ccc} (\text{thick}(S'\mathcal{Z}))_{(\text{thick}(S'T'\mathcal{Y}))} & \xrightarrow{S_Z} & (\text{thick}(SS'\mathcal{Z}))_{(\text{thick}(SS'T'\mathcal{Y}))} \\ \text{can} \downarrow & \swarrow T_Z & \\ (\text{thick}(S'\mathcal{Z}))_{\mathcal{Y}} & & \end{array}$$

of Verdier localisations.

- (2) There is an additive functor S_{thick} , induced by S , and an additive functor T_{thick} induced by T , making the diagram

$$\begin{array}{ccc}
 (S'\mathcal{Z})/(S'T'\mathcal{Y}) & \xrightarrow{\pi_1} & (\text{thick}(S'\mathcal{Z}))/\text{thick}(S'T'\mathcal{Y}) \\
 \downarrow S & & \downarrow S_{\text{thick}} \\
 (SS'\mathcal{Z})/(SS'T'\mathcal{Y}) & \xrightarrow{\pi_2} & (\text{thick}(SS'\mathcal{Z}))/\text{thick}(SS'T'\mathcal{Y}) \\
 \downarrow T & & \downarrow T_{\text{thick}} \\
 S'\mathcal{Z}/\mathcal{Y} & \xrightarrow{\pi_3} & \text{thick}(S'\mathcal{Z})/\text{thick}(\mathcal{Y})
 \end{array}$$

commutative. Moreover, the restriction to the respective images of π_1 , respectively π_2 , respectively π_3 of functors S_{thick} and T_{thick} on the right is an equivalence.

- (3) S and T induce equivalences S_L and T_L of additive categories fitting into the commutative diagram

$$\begin{array}{ccc}
 (\text{thick}(S'\mathcal{Z}))_{(\text{thick}(S'T'\mathcal{Y}))} & \xrightarrow{S_Z} & (\text{thick}(SS'\mathcal{Z}))_{(\text{thick}(SS'T'\mathcal{Y}))} \\
 \downarrow L_{S'T'\mathcal{Y}}((S'\mathcal{Z})/(S'T'\mathcal{Y})) \xrightarrow{S_L} L_{SS'T'\mathcal{Y}}((SS'\mathcal{Z})/(SS'T'\mathcal{Y})) & & \downarrow \\
 \downarrow \text{can} & \swarrow T_L & \downarrow \\
 L_{\mathcal{Y}}((S'\mathcal{Z})/\mathcal{Y}) & & \downarrow T_Z \\
 \downarrow & \swarrow & \downarrow \\
 (\text{thick}(S'\mathcal{Z}))_{\mathcal{Y}} & & \downarrow
 \end{array}$$

where the outer triangle consists of triangulated categories and triangle functors, and the inner triangle are full additive subcategories.

- (4) If S and T induce equivalences of additive categories

$$\begin{array}{ccc}
 (\text{thick}(S'\mathcal{Z}))/\text{thick}(S'T'\mathcal{Y}) & \xrightarrow{S_{\text{thick}}} & (\text{thick}(SS'\mathcal{Z}))/\text{thick}(SS'T'\mathcal{Y}) \\
 \downarrow \text{can} & \swarrow T_{\text{thick}} & \downarrow \\
 (\text{thick}(S'\mathcal{Z}))/\text{thick}(\mathcal{Y}) & & \downarrow
 \end{array}$$

then the restriction S_Z of S to the triangulated category $(\text{thick}(S'\mathcal{Z}))_{\text{thick}(S'T'\mathcal{Y})}$ and the restriction T_Z of T to the triangulated category $(\text{thick}(SS'\mathcal{Z}))_{\text{thick}(SS'T'\mathcal{Y})}$ are equivalences of triangulated categories, making the diagram

$$\begin{array}{ccc}
 (\text{thick}(S'\mathcal{Z}))_{(\text{thick}(S'T'\mathcal{Y}))} & \xrightarrow{S_Z} & (\text{thick}(SS'\mathcal{Z}))_{(\text{thick}(SS'T'\mathcal{Y}))} \\
 \downarrow \text{can} & \swarrow T_Z & \downarrow \\
 (\text{thick}(S'\mathcal{Z}))_{\mathcal{Y}} & & \downarrow
 \end{array}$$

commutative.

Proof. We first recall from Lemma 3.3 that \mathcal{Z} is triangulated. By Corollary 3.12 the functors coming from Theorem 1.2 extend to functors on the localisations. Now S and T are equivalences on the additive quotient constructions. Using Proposition 3.10 and Proposition 3.13, the functors extend to triangle functors

$$\begin{array}{ccc}
 (\text{thick}(S'\mathcal{Z}))_{(\text{thick}(S'T'\mathcal{Y}))} & \xrightarrow{S_Z} & (\text{thick}(SS'\mathcal{Z}))_{(\text{thick}(SS'T'\mathcal{Y}))} \\
 \downarrow \text{can} & \swarrow T_Z & \downarrow \\
 (\text{thick}(S'\mathcal{Z}))_{(\mathcal{Y})} & & \downarrow
 \end{array}$$

The functors S and T are triangle functors on the ambient categories, and hence they induce functors S_{thick} and T_{thick} as required.

Since Theorem 1.2 shows that S is an equivalence with quasi-inverse T on the above subcategories, the functor T_{thick} is also a quasi-inverse to S_{thick} on the images under π_1 , π_2 and π_3 . Corollary 3.12 shows item (3).

Suppose now that S and T induce equivalences

$$\begin{array}{ccc} (\text{thick}(S'\mathcal{Z})) / (\text{thick}(S'T'\mathcal{Y})) & \xrightarrow{S_{\text{thick}}} & (\text{thick}(SS'\mathcal{Z})) / (\text{thick}(SS'T'\mathcal{Y})) \\ \text{can} \downarrow & & \nwarrow T_{\text{thick}} \\ (\text{thick}(S'\mathcal{Z})) / \text{thick}\mathcal{Y} & & \end{array}$$

Since by hypothesis each object of $TSS'T'\mathcal{Y}$ is a direct factor of an object of \mathcal{Y} , the right vertical functor can be the identity. The restriction of the functors S and T in the statement of item (4) are full and dense by Proposition 3.16. Since their composition TS is the identity, the functors are also faithful. The statement follows. \square

Remark 3.18. Note that in Theorem 3.17.(1) and 3.17.(3) the functor T maps from the localisation at the thick subcategory of images under S to the localisation at the thick subcategory of images under T . Note that by Proposition 2.10 we get a functor from the localisation at the thick subcategory generated by T -relative injective objects to the localisation at the thick subcategory generated by S -relative projective objects.

Remark 3.19. Consider the special situation when G is a finite group and k is a field of characteristic $p > 0$. Then, following Carlson, Peng and Wheeler [13] the classical Green correspondence is an equivalence of full additive subcategories of triangulated categories.

More precisely, let D be a p -subgroup of G and let H be a subgroup of G containing $N_G(D)$, the normalizer of D in G .

Consider $\mathcal{G} = kG - \underline{\text{mod}}$, $\mathcal{H} = kN_G(D) - \underline{\text{mod}}$, and $\mathcal{D} = kD - \underline{\text{mod}}$, the stable categories of kG -modules, kH -modules, and kD -modules. Here the stable categories are taken modulo morphisms factoring through projective modules. Let

$$S = kG \otimes_{kH} - = \text{ind}_H^G : kH - \underline{\text{mod}} \longrightarrow kG - \underline{\text{mod}}$$

and

$$S' = kH \otimes_{kD} - = \text{ind}_D^H : kD - \underline{\text{mod}} \longrightarrow kH - \underline{\text{mod}}$$

be the induction functors. These have left and right adjoints, namely the restriction

$$T := \text{Hom}_{kH}(kG, -) = \text{res}_H^G : kG - \underline{\text{mod}} \longrightarrow kH - \underline{\text{mod}}$$

is left and right adjoint to S . Similarly,

$$T' := \text{Hom}_{kH}(kG, -) = \text{res}_H^G : kH - \underline{\text{mod}} \longrightarrow kD - \underline{\text{mod}}$$

is left and right adjoint to S' .

Since group algebras are symmetric, following Remark 2.19 the stable categories $\mathcal{G} = kG - \underline{\text{mod}}$, $\mathcal{H} = kN_G(D) - \underline{\text{mod}}$, and $\mathcal{D} = kD - \underline{\text{mod}}$ are triangulated and moreover, the functors $\text{ind}_H^G, \text{ind}_D^H, \text{res}_H^G, \text{res}_D^H$ come from exact functors of the corresponding module categories, and hence are triangle functors. Further,

$$\text{res}_H^G \text{ind}_H^G = \text{id}_{kH - \underline{\text{mod}}} \oplus U$$

for $U = \bigoplus_{HgH \in H \backslash G / H \setminus \{H\}} kHgH \otimes_{kH} -$.

Therefore Theorem 3.17 applies for appropriate choices of \mathcal{Y} . Following [1, page 311, Section 3] we fix a collection \mathfrak{Y} of subgroups of H , closed under H -conjugation and under taking subgroups, we consider \mathcal{Y} the full subcategory of \mathcal{H} given by $\text{ind}_{\mathfrak{Y}}^H$, i.e. those kH -modules induced from kY -modules for some $Y \in \mathfrak{Y}$. By [1, Corollary 3.4 (a) and (b)] we may put $\mathfrak{Y} := \{Y \mid Y \leq H \cap gDg^{-1} ; g \in G \setminus H\}$ which satisfies the hypotheses of Theorem 3.17. Moreover, the functors S_L and T_L in item (3) of Theorem 3.17 implies the classical Green correspondence. Furthermore, the bijection of indecomposable kG -modules and kH -modules with vertex D is the restriction of a triangle functor between triangulated categories, namely the Verdier localisation of triangulated subcategories.

However, if D is TI, i.e. $D \cap D^g \in \{1, D\}$ for all $g \in G$, the stable categories involved in the theorem are the usual stable categories modulo projectives, which are already triangulated, and by the universal property of the Verdier localisation ([31, §2, no 3] or [32, Chapitre II, Corollaire 2.2.11.c]) there is an inverse functor to L (which was introduced in Lemma 3.5).

By the same argument, for general D , the Verdier localisation in item (3) of Theorem 3.17 is the W -stable category from Carlson-Peng-Wheeler [13] (cf also Grime [20, Example 3.6]).

4. TENSOR TRIANGULATED CATEGORIES—GREEN CORRESPONDENCE ABSTRACTLY AND FOR GROUP RINGS

We had to deal with thick subcategories of triangulated categories. Our main model was the case of versions of derived or stable categories of group rings. Classification results are known in this case, but mainly in presence of an additional monoidal structure.

4.1. Recall Balmer's results. We first recall some results from Balmer [3].

- A tensor triangulated category \mathcal{K} is an
 - essentially small
 - triangulated category \mathcal{K} together with a
 - symmetric monoidal structure $(\mathcal{K}, \otimes, 1)$,
 - such that the functor $\otimes : \mathcal{K} \times \mathcal{K} \rightarrow \mathcal{K}$ is assumed to be exact in each variable.
- A tensor triangulated functor is an exact functor between tensor triangulated categories sending the identity object to the identity object and respecting the monoidal structures.
- A \otimes -ideal \mathcal{P} of \mathcal{K} is a
 - thick triangulated subcategory
 - such that if an object M is in \mathcal{P} and X is an object in \mathcal{K} , then $M \otimes X$ is in \mathcal{P} .
- An ideal \mathcal{P} is prime if $A \otimes B$ being an object in \mathcal{P} if and only if A is an object in \mathcal{P} or B is an object in \mathcal{P} .
- The spectrum $\text{Spec}(\mathcal{K})$ is defined to be the set (!) of prime ideals of \mathcal{K} .
- The support of an object M of \mathcal{K} is

$$\text{supp}_{\mathcal{K}}(M) := \{\mathcal{P} \in \text{Spec}(\mathcal{K}) \mid M \text{ is not an object of } \mathcal{P}\}.$$

- For any family of objects \mathcal{S} of \mathcal{K} let $Z(\mathcal{S}) := \{\mathcal{P} \in \text{Spec}(\mathcal{K}) \mid \mathcal{S} \cap \mathcal{P} = \emptyset\}$. The sets $Z(\mathcal{S})$ form the closed sets of a topology, the Zariski topology on $\text{Spec}(\mathcal{K})$.
- The radical $\sqrt{\mathcal{P}}$ of an ideal \mathcal{P} is the class of objects M in \mathcal{K} such that there is $n \in \mathbb{N}$ so that $M^{\otimes n}$ is an object of \mathcal{P} . An ideal \mathcal{P} is called radical if $\sqrt{\mathcal{P}} = \mathcal{P}$.

One of the main results of [3] is

Theorem 4.1. [3, Theorem 4.10] *Let $\mathfrak{S}(\mathcal{K})$ denote the subsets $Y \subseteq \text{Spec}(\mathcal{K})$ such that $Y = \bigcup_{i \in I} Y_i$ with all Y_i closed and $\text{Spec}(\mathcal{K}) \setminus Y_i$ quasicompact. Let $\mathfrak{R}(\mathcal{K})$ be the set of radical thick \otimes -ideals of \mathcal{K} . Then the following maps are mutually inverse bijections*

$$\begin{aligned} \mathfrak{S}(\mathcal{K}) &\longleftrightarrow \mathfrak{R}(\mathcal{K}) \\ Y &\mapsto \mathcal{K}_Y := \{M \in \mathcal{K} \mid \text{supp}(M) \subseteq Y\} \\ \bigcup_{M \in \mathcal{J}} \text{supp}(M) =: \text{supp}(\mathcal{J}) &\leftarrow \mathcal{J} \end{aligned}$$

4.2. Green correspondence of the spectrum in a tensor triangulated category. Recall that, following [17] a tensor subcategory of a tensor category still has a unit element. We shall need to define a concept without this restriction since for our natural examples we do not necessarily have a unit element. Note that a semigroup is a set with a binary associative structure, and a monoid is a semigroup with a unit. We transport this vocabulary to the world of tensor categories under the name of semigroup category (cf [9]).

Definition 4.2.

- A semigroup category is a category \mathcal{C} with a symmetric binary operation $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ satisfying the associative pentagon axiom.
- A triangulated semigroup category is
 - an essentially small triangulated category \mathcal{C} ,
 - which is in addition a semigroup category (\mathcal{C}, \otimes)

- such that $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ is exact in each variable.
- A \otimes -ideal \mathcal{P} of a triangulated semigroup category \mathcal{C} is a
 - thick triangulated subcategory
 - such that if an object M is in \mathcal{P} and X is an object in \mathcal{C} , then $M \otimes X$ is in \mathcal{P} .
- Let (\mathcal{C}, \otimes) and (\mathcal{D}, \otimes) be semigroup categories.
 - A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is called semi-tensor functor if F allows a natural equivalence $J : F(V \otimes W) \rightarrow F(V) \otimes F(W)$ which satisfies the associahedron diagram [17, Diagram 2.23].
 - If (\mathcal{C}, \otimes) and (\mathcal{D}, \otimes) are triangulated semigroup categories, then a semi-tensor functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is called triangle semi-tensor functor if F is in addition a triangle functor.

Lemma 4.3. *If \mathcal{Y} is a thick semigroup triangulated subcategory (respectively \otimes -ideal) of a triangulated semigroup category \mathcal{H} , and if $F : \mathcal{D} \rightarrow \mathcal{H}$ is a triangle semi-tensor functor, then $\mathcal{Z} := F^{-1}(\mathcal{Y})$ is again a thick semigroup triangulated subcategory (resp. \otimes -ideal) of \mathcal{D} .*

Proof. By Proposition 3.3 \mathcal{Z} is triangulated. By definition, \mathcal{Z} is thick. We need to show that \mathcal{Z} is a semigroup tensor category. Let X be an object of \mathcal{Z} and Y be an object of \mathcal{Z} (resp. \mathcal{H}). Then there are objects X' and Y' such that $F(X) \oplus X'$ and $F(Y) \oplus Y'$ are objects of \mathcal{Y} . But then

$$\begin{aligned}
 (F(X) \oplus X') \otimes (F(Y) \oplus Y') &\simeq (F(X) \otimes F(Y)) \\
 &\quad \oplus (X' \otimes F(Y)) \oplus (F(X) \otimes Y') \oplus (X' \otimes Y') \\
 &\simeq F(X \otimes Y) \\
 &\quad \oplus (X' \otimes F(Y)) \oplus (F(X) \otimes Y') \oplus (X' \otimes Y')
 \end{aligned}$$

Since \mathcal{Y} is tensor triangulated, $F(X \otimes Y)$ is a direct factor of the object $(F(X) \oplus X') \otimes (F(Y) \oplus Y')$ of \mathcal{Y} . Therefore $X \otimes Y$ is an object of \mathcal{Z} . \square

Lemma 4.4. *If \mathcal{H} is a tensor triangulated category, if \mathcal{Y} is a \otimes -ideal in \mathcal{H} , then the tensor triangulated structure on \mathcal{H} induces a tensor triangulated structure on $\mathcal{H}_{\mathcal{Y}}$. Moreover, the natural functor $\nu : \mathcal{H} \rightarrow \mathcal{H}_{\mathcal{Y}}$ is a functor of tensor triangulated categories.*

Proof. The objects of \mathcal{H} coincides with the objects of $\mathcal{H}_{\mathcal{Y}}$. We need to define a tensor product $\bar{\otimes}$ on $\mathcal{H}_{\mathcal{Y}}$. Denote by $\nu : \mathcal{H} \rightarrow \mathcal{H}_{\mathcal{Y}}$ the natural functor. We define for any two objects M, N in $\mathcal{H}_{\mathcal{Y}}$ the object $M \bar{\otimes} N := \nu(M \otimes N)$ in $\mathcal{H}_{\mathcal{Y}}$.

Since \mathcal{Y} is an ideal, this construction is also well-defined on morphisms. Since $\text{id}_{\mathcal{H}}$ is the neutral element of \otimes , we get $\nu(\text{id}_{\mathcal{H}})$ is the neutral element of $\bar{\otimes}$. Since \otimes is monoidal symmetric, also $\bar{\otimes}$ is monoidal symmetric. The functor is tensor triangulated by construction. \square

Recall that every thick subcategory is a full triangulated subcategory, but a full triangulated subcategory is thick only if it is in addition closed under taking direct summands. A full triangulated subcategory \mathcal{A} of \mathcal{D} is strict if any object of \mathcal{D} which is isomorphic in \mathcal{D} to an object in \mathcal{A} is also an object of \mathcal{A} .

Proposition 4.5. *Let $(\mathcal{T}, \otimes, 1)$ be a tensor triangulated category and let \mathcal{P} and \mathcal{Q} be \otimes -ideals of \mathcal{T} . Suppose moreover that \mathcal{Q} is a full triangulated subcategory of \mathcal{P} . Suppose that \mathcal{P} is strictly full in \mathcal{T} . Then the following hold.*

- The tensor triangulated structure on \mathcal{T} induces a tensor triangulated structure $\bar{\otimes}$ on the Verdier localisation $\mathcal{T}_{\mathcal{Q}}$.
- Furthermore, consider the natural functor $\nu : \mathcal{T} \rightarrow \mathcal{T}_{\mathcal{Q}}$. Let ν' be the restriction of ν to \mathcal{P} , as indicated in the commutative diagram

$$\begin{array}{ccc}
 \mathcal{T} & \xrightarrow{\nu} & \mathcal{T}_{\mathcal{Q}} \\
 \uparrow & & \uparrow \\
 \mathcal{P} & \xrightarrow{\nu'} & \nu(\mathcal{P})
 \end{array}$$

Denote by $\mathcal{P}_{(\mathcal{Q})}$ the image $\nu(\mathcal{P})$ of \mathcal{P} in $\mathcal{T}_{\mathcal{Q}}$ under ν , and denote by $\mathcal{P}_{\mathcal{Q}}$ the Verdier localisation of \mathcal{P} at \mathcal{Q} . Then

$$\mathcal{P}_{(\mathcal{Q})} = \mathcal{P}_{\mathcal{Q}}.$$

- $\mathcal{P}_{\mathcal{Q}}$ is a $\overline{\otimes}$ -ideal of $\mathcal{T}_{\mathcal{Q}}$.

Proof. Lemma 4.4 is precisely the first statement.

Denote by $\iota : \mathcal{P} \rightarrow \mathcal{T}$ the inclusion functor. As for the second statement we have the Verdier localisation $\mathcal{P}_{\mathcal{Q}}$ of \mathcal{P} at \mathcal{Q} . Denote by $\mu : \mathcal{P} \rightarrow \mathcal{P}_{\mathcal{Q}}$ the natural functor. Then, the universal property of Verdier localisations ([31, §2, no 3] or [32, Chapitre II, Corollaire 2.2.11.c]) induces a unique functor $\sigma : \mathcal{P}_{\mathcal{Q}} \rightarrow \mathcal{P}_{(\mathcal{Q})}$ such that $\sigma \circ \mu = \nu \circ \iota$. This shows that the functor σ is dense since μ, ν, ι are the identity on objects.

We need to show that σ is fully faithful. Let Z be an object of \mathcal{T} , let P_1 and P_2 be objects of \mathcal{P} , and a diagram of morphisms of \mathcal{T}

$$\begin{array}{ccc} & Z & \\ \gamma \swarrow & & \searrow \alpha \\ P_1 & & P_2 \end{array}$$

representing a morphism ω in $\mathcal{T}_{\mathcal{Q}}(P_1, P_2)$. If γ has cone in \mathcal{Q} , since \mathcal{P} is triangulated, and since \mathcal{Q} is a triangulated subcategory of \mathcal{P} , also Z is isomorphic to an object of \mathcal{P} , and since \mathcal{P} is strictly full in \mathcal{T} , the object Z is actually an object of \mathcal{P} . Hence σ is full.

If λ is represented by

$$\begin{array}{ccc} & Z & \\ \gamma \swarrow & & \searrow \alpha \\ P_1 & & P_2 \end{array}$$

for some object Z of \mathcal{P} , and if $\sigma(\lambda) = 0$ in $\mathcal{P}_{\mathcal{Q}}$, then there is an object Z' of \mathcal{T} and a morphism $\delta : Z' \rightarrow Z$ with cone(δ) in \mathcal{Q} , and with $\alpha \circ \delta = 0$. But, again \mathcal{Q} is a triangulated subcategory of \mathcal{P} , and \mathcal{P} being strictly full triangulated subcategory of \mathcal{T} implies Z' is an object of \mathcal{P} . Since \mathcal{P} is a full subcategory of \mathcal{T} , the morphism δ is actually already in \mathcal{P} . Hence $\lambda = 0$. This shows that σ is faithful. Altogether we get the second statement.

Since \mathcal{P} is a \otimes -ideal, for any P in \mathcal{P} , and any X in \mathcal{T} we get $P \otimes X$ is in \mathcal{P} . Hence

$$\nu(P) \overline{\otimes} \nu(X) = \nu(P \otimes X)$$

is an object of $\mathcal{P}_{(\mathcal{Q})} = \mathcal{P}_{\mathcal{Q}}$. This proves the third statement. \square

4.3. Thick tensor triangulated categories and tensor ideals in the special case of group rings. Various results are known for classification of thick subcategories of various triangulated categories (cf e.g. [30, 7, 8, 15, 18]), giving mostly a parametrisation with certain subsets of support varieties. For a fixed, essentially small triangulated category \mathcal{D} a general result describing the relation between full triangulated essentially small subcategories \mathcal{A} and $\text{thick}(\mathcal{A}) = \mathcal{D}$ is given by Thomason.

Theorem 4.6. (Thomason [30, Theorem 2.1]) *Let \mathcal{D} be an essentially small triangulated category. Consider the set \mathfrak{A} of strictly full triangulated subcategories \mathcal{A} in \mathcal{D} , having the property that each object of \mathcal{D} is isomorphic to a direct summand of an object in \mathcal{A} . Then \mathfrak{A} is in bijection with the set of subgroups of the Grothendieck group $K_0(\mathcal{D})$. The isomorphism is given by mapping \mathcal{A} to the subgroup $K_0(\mathcal{A})$ of $K_0(\mathcal{D})$.*

Thomason also gives [30, Theorem 3.15] a classification of tensor triangulated thick subcategories of the derived category of perfect complexes over a quasi-compact quasi-separated scheme.

We focus on those dealing with group rings. Let k be an algebraically closed field of characteristic $p > 0$ and let G be a finite group with order divisible by p . Let $H^\bullet(G)$ be $\bigoplus_{i \geq 0} H^{2i}(G, k)$ if p is odd, and $H^\bullet(G) = H^*(G, k)$ if $p = 2$. Then $H^\bullet(G)$ is a graded commutative algebra, and $\text{Ext}_{kG}^*(M, M)$ is a finitely generated $H^\bullet(G)$ -module. Let $V_G(k)$ be the maximal ideal spectrum of $H^\bullet(G)$. A set \mathcal{X} of closed subvarieties of $V_G(k)$ is said to be closed under specialisation if whenever $W \in \mathcal{X}$ and $W' \subseteq W$, then we also get $W' \in \mathcal{X}$. For a set \mathcal{X} of closed subvarieties of $V_G(k)$ which is closed under specialisation we let $\mathcal{C}(\mathcal{X})$ be the thick subcategory of kG - mod consisting of modules M with

$$V_G(M) := \{ \mathfrak{m} \in V_G(k) \mid \text{Ann}_{H^\bullet(G)}(\text{Ext}_{kG}^*(M, M)) \subseteq \mathfrak{m} \} \in \mathcal{X}.$$

Benson, Carlson and Rickard showed in [7] the following.

Theorem 4.7. [7, Theorem 3.4] *Let k be an algebraically closed field, and let G be a finite group. Let $V_G(k)$ be the maximal ideal spectrum of $H^\bullet(G)$. Then the thick tensor ideals I in $kG - \underline{\text{mod}}$ are of the form $\mathcal{C}(\mathcal{X})$ for some non empty set \mathcal{X} of homogeneous subvarieties of $V_G(k)$, closed under specialisation and finite unions.*

Carlson and Iyengar [15] determined the thick subcategories of the derived category of the group algebra of a finite group. For each object M of $D^b(kG)$ there is a morphism of k -algebras $H^*(G, k) \rightarrow \text{Ext}_{kG}^*(M, M)$. Again $\text{Ext}_{kG}^*(M, M)$ is a finitely generated $H^\bullet(G)$ -module. Then

$$V_{D^b(kG)}(M) := \text{Supp}_{H^\bullet(G)}(M) := \{\wp \in \text{Spec}(H^\bullet(G)) \mid H(M_\wp) \neq 0\} \subseteq \text{Spec}(H^\bullet(G)).$$

Theorem 4.8. [15, Theorem 6.6 and Corollary 6.7] *For an algebraically closed field k of characteristic $p > 0$ and a finite group G with order divisible by p , and two objects M and N in $D^b(kG)$ with $V_{D^b(kG)}(M) \subseteq V_{D^b(kG)}(N)$ we have that M is in the thick tensor ideal generated by N . In particular, if \mathcal{C} is a thick tensor ideal of $D^b(kG)$, then there is a specialisation closed subset V of $V_{D^b(kG)}(k)$ such that \mathcal{C} equals the subcategory obtained by all those M in $D^b(kG)$ with $V_{D^b(kG)}(M) \subseteq V$.*

Carlson [14] studied thick subcategories of what he calls relatively stable categories of group rings. Let \mathfrak{H} be a set of subgroups of G . A kG -module M is called \mathfrak{H} -projective if M is \downarrow_H^G -relative projective for all $H \in \mathfrak{H}$. It is classical that a module M is \mathfrak{H} -projective if and only if M is a direct summand of modules which are induced from modules over elements of \mathfrak{H} . The category $kG - \underline{\text{mod}}_{\mathfrak{H}}$ has the same objects as $kG - \text{mod}$. However, the set of morphisms from M to N is the set of equivalence classes of kG -module morphisms modulo those factoring through \mathfrak{H} -projective modules.

Carlson, Peng and Wheeler [13, Theorem 6.2] show that $kG - \underline{\text{mod}}_{\mathfrak{H}}$ is actually a triangulated category. Moreover, an immediate consequence is that Green correspondence is the restriction of a functor between triangulated categories to certain subcategories, namely those full subcategories generated by indecomposable modules with a specific vertex. This restriction is then an equivalence of categories.

Benson and Wheeler extend the concept to infinitely generated modules, and show in [6, Proposition 2.3] that we get again triangulated categories and a Green correspondence, which is an equivalence between triangulated categories.

These results are special cases of our more general approach, when applied to bounded derived categories of modules over the respective group rings and appropriate choices for \mathcal{Y} , as is shown by the following.

Proposition 4.9. *Let D be a p -subgroup of G , let H be a subgroup of G containing $N_G(D)$, and let $\mathfrak{Y} := \{S \leq H \cap {}^g D \mid g \in G \setminus H\}$ as well as $\mathfrak{X} := \{S \leq D \cap {}^g D \mid g \in G \setminus H\}$.*

Then for \mathcal{Y} being the class of complexes having indecomposable factors with vertex in \mathfrak{Y} , the natural functors

$$L_{SS'T'\mathcal{Y}} : kG - \underline{\text{mod}}_{\mathfrak{X}} \rightarrow D^b(kG)_{\text{thick}(SS'T'\mathcal{Y})}$$

and

$$L_{\mathcal{Y}} : kH - \underline{\text{mod}}_{\mathfrak{Y}} \rightarrow D^b(kH)_{\text{thick}(\mathcal{Y})}$$

are equivalences of triangulated categories.

Proof. Using [33, Lemma 4.1], Lemma 3.5 defines $L_{SS'T'\mathcal{Y}}$ and $L_{\mathcal{Y}}$. Since the subcategory of bounded complexes of finitely generated projectives is a subcategory of \mathcal{Y} , the category $D^b(kG)_{\text{thick}(SS'T'\mathcal{Y})}$ is a localisation of the singularity category $D_{sg}(kG)$ of kG . The singularity category of a self-injective algebra is just the stable category of the algebra modulo projective-injectives (cf Keller-Vossieck [24], Rickard [27], Buchweitz [11]). Likewise $D^b(kH)_{\text{thick}(\mathcal{Y})}$ is a localisation of the stable category of kH . Since the categories $kG - \underline{\text{mod}}_{\mathfrak{X}}$ and $kH - \underline{\text{mod}}_{\mathfrak{Y}}$ are triangulated, the universal property of the Verdier localisation ([31, §2, no 3] or [32, Chapitre II, Corollaire 2.2.11.c]) gives the quasi-inverse functors to $L_{SS'T'\mathcal{Y}}$ and $L_{\mathcal{Y}}$ respectively. \square

Observe that now Theorem 3.17 item (3) gives Carlson, Peng and Wheeler's theorem.

Moreover, Harris in [22] and independently Wang and Zhang in [33] give a blockwise version of the Green correspondence.

Localising subcategories are a vast variety in this setting. Carlson [14] shows for example that for $p = 2$ and a collection \mathcal{C} of subgroups H of G all of which containing an elementary abelian subgroup of rank at least 2, then the spectrum of the relatively \mathcal{C} -stable category is not Noetherian.

Note that for a non principal block we do not get a monoidal category but only a semigroup category in the sense of Definition 4.2. Indeed, the unit element is the trivial module, which belongs to the principal block.

REFERENCES

- [1] Maurice Auslander and Mark Kleiner, *Adjoint functors and an Extension of the Green Correspondence for Group Representations*, Advances in Mathematics **104** (1994) 297-314.
- [2] Maurice Auslander and Idun Reiten, *Homologically finite subcategories*, In H. Tachikawa and S. Brenner (Eds.), Representations of Algebras and Related Topics (London Mathematical Society Lecture Note Series **168**, 1-42).
- [3] Paul Balmer, *The spectrum of prime ideals in tensor triangulated categories*, Journal für die reine und angewandte Mathematik **588** (2005), 149-168.
- [4] Alexander A. Beilinson, Joseph Bernstein, Pierre Deligne, *Faisceaux pervers*. Astérisque 100, vol. 1. Société Mathématique de France (1982).
- [5] Apostolidis Beligiannis and Nikolaos Marmaridis, *Left triangulated categories arising from contravariantly finite subcategories*, Communications in Algebra **22** (12), (1994) 5021-5036.
- [6] David J. Benson and Wayne Wheeler, *The Green correspondence for infinitely generated modules*, Journal of the London Mathematical Society **63** (2001) 69-82.
- [7] David J. Benson, Jon F. Carlson and Jeremy Rickard, *Thick subcategories of the stable module category*, Fundamenta Mathematicae **153** (1997) 59-80.
- [8] David J. Benson, Srikanth Iyengar and Henning Krause, *Stratifying modular representations of finite groups*, Annals of Mathematics **174** (2011) 1643-1684.
- [9] Mitya Boyarchenko, *Associative constraints in monoidal categories*, unpublished manuscript, University of Chicago 2011.
- [10] Michel Broué, *Higman's criterion revisited*, Michigan Mathematical Journal **58** (2009), no.1, 125-179.
- [11] Ragnar Buchweitz, *Maximal Cohen-Macaulay modules and Tate-cohomology over Gorenstein rings*, unpublished manuscript, Hannover 1987.
- [12] Theo Bühler, *Exact categories*, Expositiones Math. **28** (2010) 1-69.
- [13] Jon F. Carlson, Chuang Peng, and Wayne Wheeler, *Transfer maps and Virtual Projectivity*, Journal of Algebra **204** (1998) 286-311.
- [14] Jon F. Carlson, *Thick subcategories of the relative stable category*, in 'Geometric and Topological Aspects of the Representation Theory of Finite Groups, PIMS Summer School and Workshop, July 27-August 5, 2016; Springer Proceedings in Mathematics and Statistics 2018, pages 25-49.
- [15] Jon F. Carlson and Srikanth Iyengar, *Thick subcategories of the bounded derived category of a finite group*, Transactions of the American Mathematical Society **367** (2015) 2703-2717.
- [16] Samuel Eilenberg and J. C. Moore, *Foundations of relative homological algebra* Memoirs of the American Mathematical Society **55** (1966).
- [17] Pavel Etingof, Shlomo Gelaki, Dmitri Nikshych, and Victor Ostrik, *Tensor Categories*, American Mathematical Society Surveys and Monographs 205, Providence Rhode Island 2015.
- [18] Eric Friedlander and Julia Pevtsova, *II-supports for finite group schemes over a field*, Duke Mathematical Journal **139** (2007) 317-368.
- [19] James Alexander Green, *On the indecomposable representations of a finite group*, Mathematische Zeitschrift **70** (1959) 430-445.
- [20] Matthew Grime, *Adjoint functors and triangulated categories*, Communications in Algebra **36** (2008) 3589-3607.
- [21] Dieter Happel, *Triangulated categories in the representation theory of finite dimensional algebras*, London Mathematical Society Lecture Note Series **119** (1988).
- [22] Morton E. Harris, *A block extension of categorical results in the Green correspondence*, Journal of group theory **17** (2014) 1117-1131.
- [23] Gerhard Hochschild, *Relative Homological Algebra*, Transactions of the American Mathematical Society **82** (1956) 246-269.
- [24] Bernhard Keller and Dieter Vossieck, *Sous les catégories dérivées*, Comptes Rendus de l'Académie des Sciences Paris **305** (1987) 225-228.
- [25] Saunders MacLane, *CATEGORIES FOR THE WORKING MATHEMATICIAN*, Second edition, Springer Verlag 1978.
- [26] Jeremy Rickard, *Morita theory for derived categories*, Journal of the London Mathematical Society (2) **39** (1989) 436-456.
- [27] Jeremy Rickard, *Derived categories and stable equivalences*, Journal of Pure and Applied Algebra **61** (1989) 303-317.
- [28] Claus M. Ringel and Pu Zhang, *Objective triangle functors*, Science China Mathematics, February 2015 Vol. 58, no2; 221-232
- [29] The Stacks project authors, *The Stacks Project*, <https://stacks.math.columbia.edu>, year: 2019.
- [30] Robert W. Thomason, *The classification of triangulated subcategories*, Compositio Mathematica **105** (1997) 1-27.

- [31] Jean-Louis Verdier, *Catégories dérivées: Quelques résultats (état 0)*, in SGA $4\frac{1}{2}$ -Cohomologie étale (Pierre Deligne) Springer Lecture notes **569** (1977) 262-311.
- [32] Jean-Louis Verdier, DES CATÉGORIES DÉRIVÉES DES CATÉGORIES ABÉLIENNES, Astérisque **239** (1996), Société Mathématique de France.
- [33] Lizhong Wang and Jiping Zhang, *Relatively stable equivalences of Morita type for blocks*, Journal of pure and applied Algebra **222** (2018) no 9, 2703-2717.
- [34] Alexander Zimmermann, REPRESENTATION THEORY: A HOMOLOGICAL ALGEBRA POINT OF VIEW, Springer Verlag Cham, 2014.
- [35] Alexander Zimmermann, *Remarks on a triangulated version of Auslander-Kleiner's Green correspondence*, preprint 2020.

UNIVERSITÉ DE PICARDIE,
DÉPARTEMENT DE MATHÉMATIQUES ET LAMFA (UMR 7352 DU CNRS),
33 RUE ST LEU,
F-80039 AMIENS CEDEX 1,
FRANCE
Email address: alexander.zimmermann@u-picardie.fr