DIFFERENTIAL GRADED DIVISION ALGEBRAS, THEIR MODULES, AND DG-SIMPLE ALGEBRAS

ALEXANDER ZIMMERMANN

Abstract. We give the definition of a dg-division algebra, that is a concept of a differential graded algebra which may serve as an analogue of a division algebra. We classify them completely, and show that they are either acyclic or have differential 0. Further, we prove that the graded centre of dg-simple dg-algebras is a dg-division algebra, and also the dg-endomorphism ring of a dg-simple module is a dg-division algebra. We also shall give a Jacobson-Chevalley density theorem for acyclic dg-algebras.

Introduction

Differential graded algebras (dg-algebras for short) and their differential graded modules appear in various places, mainly of geometric and topological nature. More precisely, let K be a commutative ring. A dg-algebra over K is a $\mathbb{Z}$-graded algebra with a graded K-linear endomorphism d of degree 1 with $d^2 = 0$ satisfying the Leibniz equation $d(ab) = d(a)\cdot b + (-1)^{|a|}a\cdot d(b)$ for all homogeneous $a, b \in A$. Dg-modules are defined similarly. For more precise definitions we refer to Section 1.1. Though defined by Cartan [4] in 1954 already, the ring theory of dg-algebras remained largely unexplored until quite recently. In 2002 Aldrich and Garcia Rozas characterized the dg-algebras whose dg-module category is semisimple. They obtained that this is the case if and only if the algebra is acyclic and the algebra of cycles $\ker(d)$ is semisimple as graded modules. Orlov studied in [11, 12] finite dimensional dg-algebras over a field K with a geometric motivation. Using this approach Goodbody [7] studied a version of a dg-Jacobson radical. In a sequel of papers we studied more systematically the ring theory of dg-algebras. In [15] we defined dg-Jacobson radicals in a much more general and natural setting, study dg-simple dg-algebras, dg-simple dg-modules, and the relation with dg-simple dg-algebras. We study dg-orders and define locally free dg-class groups. In [17] we define Ore localisation of dg-algebras, proving that under some hypotheses, the localised ring is again dg. Using this, we study and define dg-uniform dimension, and a dg-Goldie theorem for dg-Goldie rings in [17].

In the present paper we first define a dg-division algebra as a dg-ring without non trivial left or right dg-ideals. The question of an appropriate concept for a dg-division algebra was posed recently by Violeta Borges Marques and Julie Symons. We show in Theorem 2.4 that under a certain mild technical condition dg-division algebras are precisely those for which the ring of cycles is a $\mathbb{Z}$-graded-division algebras (cf [10, page 38] for the graded concept). For this result we need a technical assumption, namely that the set of left regular elements of the ring of cycles coincides with the set of right regular elements. This holds true, by a result of Goodearl and Stafford [8] for graded-Noetherian graded-prime rings of cycles, using a graded version of Goldie's result. As a second main result we then classify completely dg-division rings under these conditions in Theorem 2.9. Namely, they are either acyclic or else they have differential 0. The structure is then completely given by a result from Aldrich and Garcia Rozas [1] and Nastacescu and van Oystaen [10, A.1.4.3 Corollary].

Further, we prove that endomorphism complexes of dg-simple dg-modules over dg-algebras are dg-division algebras. We provide further examples. We show that the graded centre of a dg-simple algebra is a dg-division algebra.

In a final section we show a dg-version of the Jordan-Chevalley density theorem in case of acyclic dg-algebras.

Date: August 8, 2024.
2020 *Mathematics Subject Classification.* Primary: 16E45; Secondary: 12H05; 13A02; 16N60; 16W50.
Key words and phrases. differential graded algebras.

The paper is organized as follows. In Section 1 we recall the definition of dg-algebras and their modules, the constructions we use in the paper, and we define dg-division algebra, the main structure studied in our paper. In Section 2 we study the main properties of dg-division algebras with respect to the cycles, and some occurrences of dg-division algebras. In Section 3 we give additional properties of acyclic dg-algebras, which is then used in Section 4 where w prove a version of the Jacobson-Chevalley density theorem for acyclic dg-algebras.

Acknowledgement: I wish to thank Violeta Borges Marques and Julie Symons for having asked me during the Oberwolfach workshop 'Hochschild (co-)homology and applications' in April 2024 for a possible concept of a dg-field.

1. Elementary Definitions

1.1. **Differential graded algebra and their modules; definitions and notations.** In this subsection we recall notations and basic conventions, as well as the construction of a tensor product of two dg-algebras over a common graded commutative subalgebra of their graded centres. This should be well-known, but we could not find a reference, and in any case the reader may appreciate an explicit verification right in the paper.

Let K be a commutative ring. A differential graded K-algebra (dg-K-algebra for short) (A,d) is a $\mathbb{Z}$-graded algebra A with a graded K-linear endomorphism of degree 1 with $d^2=0$ and satisfying the Leibniz rule $d(a\cdot b)=d(a)\cdot b+(-1)^{|a|}a\cdot d(b)$. By definition a dg-ring is a dg-$\mathbb{Z}$-algebra.

If (A,d) is a dg-K-algebra, then define A^{op} to be the same graded additive group as A, but multiplication defined as $a\cdot_{op} b:=(-1)^{|a|\cdot|b|}b\cdot a$. Then (A^{op},d) is again a dg-K-algebra.

A differential graded left module (M,δ) over (A,d) is a $\mathbb{Z}$-graded A-module M with an endomorphism δ of degree 1 satisfying $\delta(a\cdot m)=d(a)\cdot m+(-1)^{|a|}a\cdot\delta(m)$ for all homogeneous $a\in A$ and $m\in M$. Occasionally we denote a differential graded module by dg-module for short.

A differential graded right module over (A,d) is a differential graded left module over (A^{op},d).

For two differential graded left modules (M,δ_M) and (N,δ_N) over (A,d) we put

$$\mathrm{Hom}^\bullet_A((M,\delta_M),(N,\delta_N)):=\{f\in\mathrm{Hom}_{graded}(M,N)\mid f(am)=(-1)^{|a|\cdot|f|}af(m)\}$$

and

$$d_{\mathrm{Hom}}(f):=\delta_N\circ f-(-1)^{|f|}f\circ\delta_M.$$

Further, $\mathrm{End}^\bullet_A(M,\delta_M):=\mathrm{Hom}^\bullet_A((M,\delta_M),(M,\delta_M))$. Then $(\mathrm{End}^\bullet_A(M,\delta_M),d_{\mathrm{Hom}})$ is a dg-K-algebra, and $(\mathrm{Hom}^\bullet_A((M,\delta_M),(N,\delta_N)),d_{\mathrm{Hom}})$ is a left dg-module over $\mathrm{End}^\bullet_A(N,\delta_N)$ and a right dg-module over $\mathrm{End}^\bullet_A(M,\delta_M)$.

We further mention that for a graded algebra A by some group G we denote be a gr-simple A-module an A-module which does not allow a G-graded submodule other than 0 or itself. Similarly, we denote by a gr-simple algebra an algebra which is gr-simple as $A\otimes A^{op}$-module, and a gr-division algebra an algebra where all homogeneous non zero elements are invertible. Occasionally we use the notion graded-simple for gr-simple, etc.

1.2. **Elements of the ring theory of differential graded algebra.**

Definition 1.1. A dg-K-algebra (A,d) is *dg-simple* if (A,d) does not contain any twosided dg-ideal other than 0 and A.

Definition 1.2. A dg-K-algebra (A,d) is a *dg-division algebra* if (A,d) does not contain any dg-left ideal nor a dg-right ideal other than 0 and A.

Remark 1.3. In case we want to stress that a dg-division algebra is commutative (or graded commutative) we shall call it a dg-field (or graded commutative dg-field).

Example 1.4. Let K ba a field. Then by [17] the dg-ring $K[X,X^{-1}]$ with differential $d(X^{2n+1})=X^{2n}$ and $d(X^{2n})=0$ for all $n\in\mathbb{Z}$ is dg-simple. Since it is commutative, it is a dg-field. Note however that the ungraded ring $K[X,X^{-1}]$ is not Artinian (though Noetherian).

Acyclic dg-algebras will play a crucial rôle. They are classified in [1].

Theorem 1.5. [1, Theorem 4.7; Proposition 4.3] *Let (A,d) be a dg-algebra over a commutative ring R. Then the following statements are equivalent:*

1. *(A,d) is acyclic*
2. *The left regular module is a projective object in the category of left dg-modules*
3. *$1 \in \mathrm{im}(d)$*
4. *Taking cycles is a right exact functor $Z(-)$ from the category of left (A,d)-dg-modules to the category of graded modules over $\ker(d)$.*
5. *$(A \otimes_{\ker(d)} - Z(-))$ are mutually inverse equivalences between the category of left (A,d)-dg-modules to the category of graded modules over $\ker(d)$.*

Further, if (A,d) is acyclic, and $d(y) = 1$ for some homogeneous element $y \in A$, then we have an isomorphism of dg-algebras between A and the following quotient of the twisted polynomial algebra

$$A \simeq \ker(d)[X;D]/(X^2 - y^2)$$

with $D(a) := (-1)^{|a|} d(yay)$, for all homogeneous $a \in \ker(d)$.

2. Relating dg-division algebras with gr-division algebra

Lemma 2.1. *Let (A,d) be a dg-ring and suppose that $\ker(d)$ is a $\mathbb{Z}$-gr division algebra. Then (A,d) is a dg-division algebra.*

Proof. Let (I,d) be a non trivial dg-left ideal of (A,d). Then $I \cap \ker(d)$ is a $\mathbb{Z}$-graded ideal of $\ker(d)$.

Since $I \neq 0$, there is $0 \neq x \in I$. If $x \in \ker(d)$, we have shown that $I \cap \ker(d) \neq 0$. If $d(x) \neq 0$, then $d(x) \in I$ since I is a dg-ideal, and $d(x) \in \ker(d)$ since $d^2 = 0$. Hence $I \cap \ker(d) \neq 0$.

Suppose that $I \cap \ker(d) = \ker(d)$. Since $d(1) = 0$, we get $1 \in I \cap \ker(d)$. Hence $1 \in I$ and therefore $A = I$. This is a contradiction to the hypothesis that I is a non trivial dg-ideal. The same holds for dg-right ideals. ■

Lemma 2.2. *Let (A,d) be a dg-ring and suppose that (A,d) does not contain any non trivial dg-left ideal. If the set of homogeneous right regular element of $\ker(d)$ coincides with the set of homogeneous left regular elements of $\ker(d)$, then $\ker(d)$ is a $\mathbb{Z}$-gr-division algebra.*

Proof. Let $u \neq 0$ be a homogenous element of $\ker(d)$. Then Au is a dg-left ideal of A. Since $u \neq 0$, we get $Au \neq 0$. Suppose that $Au = A$. Then there is $b \in A$ with $1 = bu$. Then

$$0 = d(1) = d(b) \cdot u + (-1)^{|b|} b \cdot d(u) = d(b) \cdot u.$$

Since u has a left inverse in A, it is right regular. Indeed, if $ux = 0$ in A, then $x = bux = 0$. Hence, by hypothesis, u is left regular as well. Therefore, $d(b) = 0$ and hence $b \in \ker(d)$. This shows that the equation $bu = 1$ already holds in $\ker(d)$, and hence u is left invertible in $\ker(d)$. Therefore, $\ker(d)$ does not contain any non trivial graded left ideal, and by [10, page 38; Lemma 1.4.1] u is invertible in $\ker(d)$. ■

Remark 2.3. We may consider situations where the hypothesis that the set of homogeneous right regular element of $\ker(d)$ coincides with the set of homogeneous left regular elements of $\ker(d)$ holds.

- If $\ker(d)$ is finite dimensional over some field, this is true, since multiplication by a left regular element u is realised by a matrix with non zero determinant.
- If $\ker(d)$ is graded commutative, or commutative, then trivially this property holds.
- It is known that if $\ker(d)$ is left gr-Noetherian and gr-prime, this is true. Indeed, by [8] localising at the homogeneous regular elements of $\ker(d)$ yields a gr-simple algebra with all homogeneous regular elements being invertible. Now, the proof of [6, (5.8) Proposition, (5.9) Proposition] applies verbatim to graded rings and homogeneous left invertible elements.

Theorem 2.4. *Let (A,d) be a differential graded ring. Suppose that in $\ker(d)$ the set of the left regular homogeneous elements and the set of the right regular homogeneous elements coincide. Then (A,d) is a dg-division algebra if and only if $\ker(d)$ is a gr-division algebra.*

Proof. This is a direct consequence of Lemma 2.1 and Lemma 2.2. Note that the lemmas give a more general statement for the only if direction. ■

Corollary 2.5. *Let (A, d) be a differential graded ring. Suppose that in $\ker(d)$ the set of the left regular homogeneous elements and the set of the right regular homogeneous elements coincide. Then the following statements are equivalent.*

- *(A, d) admits only trivial dg-left ideals.*
- *(A, d) admits only trivial dg-right ideals.*
- *each non zero homogeneous element of $\ker(d)$ is invertible (i.e. $\ker(d)$ is a gr-division algebra).*

Proof. The first statement implies the third statement by Lemma 2.2. Similarly, the second statement implies the third statement by Lemma 2.2. The third statement implies each of the first and the second statement by Lemma 2.1. ■

We shall need to study the graded centre of a dg-algebra. Recall the definition. Let A be a $\mathbb{Z}$-graded algebra. Then the graded centre $Z_{gr}(A)$ is defined as

$$Z_{gr}(A) := \{b \in A \mid b \text{ homogeneous and } ba = (-1)^{|a||b|}ab \text{ for all homogeneous } b \in A\}$$

Lemma 2.6. *Let (A, d) be a differential graded ring. Then $(Z_{gr}(A), d)$ is a differential graded subalgebra.*

Proof. By construction, $Z_{gr}(A)$ is a subalgebra. Further, for any $a \in Z_{gr}(A)$ we get

$$\begin{array}{rcl} d(a)b &=& d(ab) - (-1)^{|a|}a \cdot d(b) \\ &=& (-1)^{|a||b|}d(ba) - (-1)^{|a|+(|a|(|b|+1))}d(b)a \\ &=& (-1)^{|a||b|}\left(d(ba) - d(b)a\right) \\ &=& (-1)^{|a||b|+|b|}b \cdot d(a) \\ &=& (-1)^{|d(a)||b|}b \cdot d(a) \end{array}$$

Hence $(Z_{gr}(A), d|_{Z_{gr}(A)})$ is a dg-algebra again. ■

Remark 2.7. Recall that in a graded commutative ring, the square of homogeneous elements of odd degree is 0, unless the characteristic of the base ring is 2. Indeed, let x be a homogeneous element of odd degree. Then, swapping the two factors yields

$$x \cdot x = (-1)^{|x|\cdot|x|}x \cdot x = -x \cdot x.$$

Hence $2 \cdot x^2 = 0$.

Corollary 2.8. *Let (D, ∂) be a dg-division algebra. Suppose that in $\ker(\partial)$ the set of the left regular homogeneous elements and the set of the right regular homogeneous elements coincide. Then $(Z_{gr}(D), \partial)$ is a dg-division algebra as well.*

If (D, ∂) is a dg-algebra, and if a homogeneous $z \in Z_{gr}(D)$ has a homogeneous inverse in $\ker(\partial) \subseteq A$, then $a \in Z_{gr}(D)$.

In particular, if the characteristic of D is different from 2, then $Z_{gr}(D) \cap \ker(\partial)$ is concentrated in even degrees, and in any case $Z_{gr}(D) \cap \ker(\partial)$ is commutative and either isomorphic to a Laurent polynomial ring or concentrated in degree 0.

Proof. By Lemma 2.6 we see that $Z_{gr}(D)$ is a dg-subalgebra of (D, ∂). Further, all elements in $Z_{gr}(D) \cap \ker(\partial)$ are invertible in $\ker(d)$. However, if $z \in Z_{gr}(D)$ and $a \in \ker(\partial)$ are homogeneous elements with $az = za = 1$ (using that in a group the left and the right inverse of a fixed element coincide), then for all homogeneous $b \in D$

$$z(ab - (-1)^{|a||b|}ba) = zab - (-1)^{|a||b|}zba = b - (-1)^{|a||b|+|z||b|}bza = b - b = 0$$

using that $|a| = -|z|$. Hence $a \in Z_{gr}(D)$ as well. Using Corollary 2.5 and Remark 2.7 we proved the statement. The classification of commutative $\mathbb{Z}$-graded gr-division algebras is given in [10, Section A.1.4; Section B.1.1], and it is shown that $Z_{gr}(D) \cap \ker(\partial)$ is isomorphic to a Laurent polynomial ring or concentrated in degree 0. ■

Theorem 2.9. *Let (A,d) be a dg-division algebra and suppose that the set of homogeneous left regular elements of* $\ker(d)$ *coincides with the set of homogeneous right regular elements of* $\ker(d)$. *Then either (A,d) is acyclic or $d=0$. Further,* $\ker(d) =: R$ *is either concentrated in degree* 0*, or else* $\ker(d)$ *is a twisted Laurent polynomial ring* $\ker(d) = R_0[T,T^{-1};\sigma]$ *for some automorphism σ of R_0 with $Xr = \sigma(r)X$ for all $r \in R_0$.*

Proof. Indeed, since $H(A,d)$ is a dg-algebra and since $\ker(d)/\mathrm{im}(d) = H(A,d)$ as an algebra, $\mathrm{im}(d)$ is a twosided ideal of $\ker(d)$. However, as $\ker(d)$ is a gr-division algebra, every homogeneous non zero element of $\ker(d)$ is invertible in $\ker(d)$. Therefore, either $\ker(d) = \mathrm{im}(d)$, or $\mathrm{im}(d)$ does not contain any non zero homogeneous element of $\ker(d)$. Hence, either $H(A,d) = 0$ in the first case, or in the second case $d = 0$. The structure of $\ker(d)$ follows from Nastacescu-van Oystaen [10, A.1.4.3 Corollary]. ∎

Example 2.10. Both cases mentioned in Theorem 2.9 do occur. Let $A = K[X,X^{-1}]$ the Laurent polynomial ring over a field K. Then the differential 0 yields a gr-field, whence also a dg-field. However, also $d(X^{2n+1}) := X^{2n}$ and $d(X^{2n}) = 0$ for all $n \in \mathbb{N}$ yields a dg-algebra, which is acyclic, and a dg-field, since it is already a gr-field. But also the cycles is a Laurent polynomial algebra over K in all even degrees, which is again a gr-field.

Corollary 2.11. *Let (A,d) be a dg-division algebra, and suppose that the set of homogeneous left regular elements of* $\ker(d)$ *coincides with the set of homogeneous right regular elements of* $\ker(d)$. *Then, either (A,d) is acyclic, or* $\ker(d) = H(A,d)$ *is a gr-division algebra.*

Proof. By Theorem 2.4 we see that $\ker(d)$ is a gr-division algebra, whence any homogeneous non zero element is invertible. Therefore, also in $H(A,d) = \ker(d)/\mathrm{im}(d)$ every homogeneous non zero element is invertible. This shows the statement. ∎

Further, we recall

Lemma 2.12. [1, Lemma 4.1, Lemma 4.2] *Let (A,d) be a dg-algebra. Then (A,d) is acyclic if and only if* $1 \in \mathrm{im}(d)$.

Let (A,d) be an acyclic dg-algebra over a commutative ring R. Then, as an R-module $A = \ker(d) \oplus \ker(d)\cdot y = \ker(d) \oplus y\cdot\ker(d)$ *for any homogeneous element $y \in A$ with $d(y) = 1$.*

Remark 2.13. Note that using Lemma 2.12, if (A,d) is acyclic, then [1, Proposition 4.3] gives more explicitly $A \simeq \ker(d)[X;D]/(X^2-y^2)$, with $d(y) = 1$ and $D(a) = -(-1)^{|a|}d(yay)$ for any homogeneous $a \in \ker(d)$. Furthermore, $A = \ker(d) \oplus \ker(d)y$, and the isomorphism is $\Phi : \ker(d)[X;D] \longrightarrow A$ by $\Phi(X) = y$ and $\Phi(a) = a$ for any $a \in \ker(d)$. For any homogeneous $a,b \in \ker(d)$ we get $d(b+ay) = a$. Moreover, $D(a) = Xa - (-1)^{|a|}aX$ for any homogeneous $a \in \ker(d)$. Recall that a dg-division algebra has either differential 0 or (A,d) is acyclic. The cycles are still given by $\ker(d)$, which need to be a gr-division ring.

Proposition 2.14. *Let K be a field, let (A,d) be a differential graded algebra and let (S,δ) be a dg-simple left dg-module over (A,d). Then $(End^\bullet_A(S,\delta), d_{\mathrm{Hom}})$ is a dg-division algebra. Moreover, the set of homogeneous left regular elements of* $\ker(d_{\mathrm{Hom}})$ *coincides with the set of homogeneous right regular elements of* $\ker(d_{\mathrm{Hom}})$.

Proof. We shall use Corollary 2.5 and we shall see that the hypotheses are satisfied..

Now, let $f \neq 0$ be homogeneous with $f \in \ker(d_{\mathrm{Hom}})$. Then this is equivalent with

$$0 = d_{\mathrm{Hom}}(f) = \delta \circ f - (-1)^{|f|} f \circ \delta$$

and hence

$$f \circ \delta = (-1)^{|f|}\delta \circ f.$$

We claim that $\ker(f)$ is a dg-submodule of (S,δ). Let $x \in \ker(f)$ be homogeneous.

$$0 = \delta(f(x)) = (-1)^{|f|} f(\delta(x))$$

and hence $\delta(x) \in \ker(f)$. Moreover, for any homogeneous $a \in A$

$$f(ax) = (-1)^{|a|\cdot|f|} a \cdot f(x) = 0$$

and hence $ax \in \ker(f)$ again.

Since (S,δ) is dg-simple, either $f=0$ or $\ker(f)=0$.

Also $\mathrm{im}(f)$ is a dg-submodule of (S,δ). If $x=f(y)$, then

$$\delta(x)=\delta(f(y))=(-1)^{|f|}f(\delta(y))\in \mathrm{im}(f)$$

Further,

$$a\cdot f(x)=(-1)^{|a||f|}f(a\cdot x)\in \mathrm{im}(f)$$

again. Since (S,δ) is dg-simple, and since $\mathrm{im}(f)$ is a dg-submodule, either $\mathrm{im}(f)=0$ (which is equivalent with $f=0$ and this was excluded) or $\mathrm{im}(f)=S$. Therefore f is surjective as well.

Hence f is an isomorphism between (S,δ) and a shifted copy, whence invertible, and this shows that $\ker(d_{\mathrm{Hom}})$ is a gr-division algebra.

We want to apply Corollary 2.5. If f is a left invertible non invertible dg-endomorphism of (S,δ), then (S,δ) has a non trivial direct factor, which is contradictory to (S,δ) being simple. Likewise a right invertible non invertible dg-endomorphism leads to a contradiction.

Hence $(End_A^\bullet(S,\delta),d_{\mathrm{Hom}})$ is a dg-division algebra by Corollary 2.5. ■

We summarize the main result of this section.

Theorem 2.15. *Let (A,d) be a dg-algebra. Suppose that the set of left regular homogeneous elements of $\ker(d)$ coincides with the set of right regular homogeneous elements of $\ker(d)$.*

Then

- *(A,d) is a dg-division algebra if and only if $\ker(d)$ is a $\mathbb{Z}$-gr-division algebra (cf [10]).*
- *If (A,d) is a dg-division algebra,*
  - *then,*
    - *either $d=0$ and $\ker(d)$ is a skew field concentrated in degree 0,*
    - *or $H(A,d)=0$ and there is a skew field R_0 such that $\ker(d)\simeq R_0[X,X^{-1};\phi]$ for an automorphism ϕ of R_0 and $Xr=\phi(r)X$ for any $r\in R_0$.*
  - *If $H(A,d)=0$, then also $A\simeq \ker(d)[T;D]/(T^2-y^2)$, with $d(y)=1$ and $D(a)=-(-1)^{|a|}d(yay)$ for any homogeneous $a\in\ker(d)$. Furthermore, $A=\ker(d)\oplus\ker(d)y$, and the isomorphism is $\Phi:\ker(d)[T;D]\longrightarrow A$ by $\Phi(T)=y$ and $\Phi(a)=a$ for any $a\in\ker(d)$. For any homogeneous $a,b\in\ker(d)$ we get $d(b+ay)=a$. Moreover, $D(a)=Ta-(-1)^{|a|}aT$ for any homogeneous $a\in\ker(d)$.*

Corollary 2.16. *Let (A,d) be a dg-algebra. Suppose that the set of left regular homogeneous elements of $\ker(d)$ coincides with the set of right regular homogeneous elements of $\ker(d)$. Suppose that (A,d) is a graded commutative dg-division algebra. Then either $\ker(d)$ is a field concentrated in degree 0, or else $\ker(d)=K[X,X^{-1}]$ for some field K and, in case K is of characteristic different from 2, then X is in even degree.*

Indeed, since (A,d) is a dg-division algebra, we get that either $\ker(d)$ is concentrated in degree 0, or there is a skew field K such that $\ker(d)\simeq K[X,X^{-1};\phi]$ for an automorphism ϕ of K and $Xr=\phi(r)X$ for any $r\in K$. Now, since $r\in K$ is of degree 0, and since A is graded commutative,

$$\phi(r)X=Xr=(-1)^{|r|\cdot|X|}rX=rX$$

and hence $\Phi(r)=r$ for all $r\in K$. Since A is graded commutative,

$$X\cdot X=(-1)^{|X|\cdot|X|}X\cdot X$$

interchanging the two factors, and hence $|X|$ has to be even. ■

Lemma 2.17. *Let (A,d) be a dg-simple dg-algebra. Then $Z_{gr}(A,d)$ is a dg-division algebra.*

Proof. Let $u\in\ker(d)\cap Z_{gr}(A)$ be a homogeneous non zero element. Then Au is a twosided dg-ideal of A, using that u is in the graded centre. Since (A,d) is dg-simple, there is a homogeneous $a\in A$ with $au=1$. Then

$$0=d(1)=d(au)=d(a)u+(-1)^{|a|}ad(u)=d(a)u.$$

Since $au=1$, and since u is in the graded centre, also $ua=\pm1$, and hence multiplying by a from the right, we get $a\in\ker(d)$. Further, $a\in Z_{gr}(A)$, using Corollary 2.8. Hence, by Lemma 2.1 we get that $Z_{gr}(A)$ is a dg-division algebra. ■

Let (A,d) be a dg-K-algebra. Then the graded centre $Z_{gr}(A,d)$ is a dg-subalgebra of (A,d) (cf Lemma 2.6).

If (A,d_A) and (B,d_B) are dg-K-algebras, then $((A\otimes_K B),d_{A\otimes_K B})$ is a dg-K-algebra again with

$$d_{A\otimes B}=d_A\otimes \mathrm{id}_B+\mathrm{id}_A\otimes d_B$$

and

$$(a_1\otimes b_1)\cdot(a_2\otimes b_2)=(-1)^{|b_1||a_2|}(a_1a_2\otimes b_1b_2).$$

Lemma 2.18. *Let (A,d_A) and (B,d_B) be dg-K-algebras, and let $Z=(Z,d)$ be a common graded commutative dg-subalgebra of the graded centre of A and of B. Then there is a differential $d_{A\otimes_Z B}$ on $A\otimes_Z B$ such that $(A\otimes_Z B,d_{A\otimes_Z B})$ is a dg-algebra again.*

Proof. First, we consider the ordinary tensor product $A\otimes_Z B$. This is graded by posing the degree n component as $\bigoplus_{k\in\mathbb{Z}}A_k\otimes_Z B_{n-k}$. This is well-defined since Z is graded. Indeed, if $a\in A_k$ is in degree $k-\ell$, $b\in B_{n-k}$ is in degree $n-k$, and $z\in Z_\ell$ is in degree ℓ, then $az\otimes b=a\otimes zb$ and the term on both sides is in degree n. Further, put again

$$(a_1\otimes b_1)\cdot(a_2\otimes b_2)=(-1)^{|b_1||a_2|}(a_1a_2\otimes b_1b_2).$$

This is well-defined again, as is shown by the following computation

$$\begin{array}{rcl}(a_1z_1\otimes b_1)\cdot(a_2\otimes z_2b_2)&=&(-1)^{|b_1||a_2|}(a_1z_1a_2\otimes b_1z_2b_2)\\ &=&(-1)^{|b_1||a_2|+|z_1||a_2|+|b_1||z_2|}(a_1a_2z_1\otimes z_2b_1b_2)\\ &=&(-1)^{|b_1||a_2|+|z_1||a_2|+|b_1||z_2|}(a_1a_2\otimes z_1z_2b_1b_2)\\ &=&(-1)^{|b_1||a_2|+|z_1||a_2|+|b_1||z_2|+|z_1||z_2|}(a_1a_2\otimes z_2z_1b_1b_2)\\ &=&(-1)^{|b_1||a_2|+|z_1||a_2|+|b_1||z_2|+|z_1||z_2|}(a_1a_2z_2\otimes z_1b_1b_2)\\ &=&(-1)^{(|b_1|+|z_1|)(|a_2|+|z_2|)}(a_1a_2z_2\otimes z_1b_1b_2)\\ &=&(a_1\otimes z_1b_1)\cdot(a_2z_2\otimes b_2)\end{array}$$

for $a_1,a_2\in A$, $b_1,b_2\in B$, $z_1,z_2\in Z$ all homogeneous. Then,

$$d_{A\otimes B}=d_A\otimes \mathrm{id}_B+\mathrm{id}_A\otimes d_B$$

is well-defined again. Indeed, since $d_A(z)=d_B(z)$ for all homogeneous $z\in Z$, we get

$$\begin{array}{rcl}d_{A\otimes B}(az\otimes b)&=&d_A(az)\otimes b+(-1)^{|az|}az\otimes d_B(b)\\ &=&d_A(a)z\otimes b+(-1)^{|a|}ad_A(z)\otimes b+(-1)^{|az|}az\otimes d_B(b)\\ &=&d_A(a)\otimes zb+(-1)^{|a|}a\otimes d_A(z)b+(-1)^{|az|}a\otimes zd_B(b)\\ &=&d_A(a)\otimes zb+(-1)^{|a|}(a\otimes d_B(z)b+(-1)^{|z|}a\otimes zd_B(b))\\ &=&d_A(a)\otimes zb+(-1)^{|a|}(a\otimes d_B(zb))\\ &=&d_{A\otimes B}(a\otimes zb)\end{array}$$

for all homogeneous $a\in A$ and $b\in B$. This shows that $d_{A\otimes_Z B}$ is well-defined. The fact that $d^2_{A\otimes_Z B}=0$ is trivial, and actually follows from the classical case, such as the fact that it verifies Leibniz' rule. ∎

Lemma 2.19. *Let (A,d_A) and (B,d_B) be two dg-simple dg-algebras with $Z_A=Z_{gr}(A,d_A)$ and $Z_B=Z_{gr}(B,d_B)$. Let Z be a common dg-subsalgebra of Z_A and Z_B. Then $Z_A\otimes_Z Z_B$ is in the graded centre of $(A\otimes_Z B,d_{A\otimes_Z B})$.*

Proof. Indeed, $Z_A\otimes_Z Z_B$ is in the graded center, as

$$\begin{array}{rcl}(z_1\otimes z_2)\cdot(a\otimes b)&=&(-1)^{|a||z_2|}(z_1a\otimes z_2b)\\ &=&(-1)^{|a||z_2|+|z_1||a|+|z_2||b|}(az_1\otimes bz_2)\\ &=&(-1)^{|a||z_2|+|z_1||a|+|z_2||b|+|b||z_1|}(a\otimes b)(z_1\otimes z_2)\\ &=&(-1)^{|z_1\otimes z_2||a\otimes b|}(a\otimes b)(z_1\otimes z_2)\end{array}$$

This shows the lemma. ∎

3. Remarks on acyclic dg-algebras

Lemma 3.1. *Let (A,d) and (B,∂) be dg-algebras and suppose that there is a dg-algebra homomorphism $\varphi:(B,\partial)\longrightarrow(A,d)$. If (B,∂) is acyclic, then (A,d) is acyclic as well.*

Proof. Recall from Theorem 1.5 that a dg-algebra (C,δ) is acyclic if and only if $1_C\in\mathrm{im}(\delta)$. Let hence $b\in B$ be homogeneous with $\partial(b)=1_B$. Then

$$1_A=\varphi(1_B)=\varphi(\partial(b))=d(\varphi(b))\in\mathrm{im}(d)$$

and hence (A,d) is acyclic as well. (We may also argue that (A,d) becomes a left (B,∂)-dg-module, and Theorem 1.5 applies.) ■

Corollary 3.2. *Let (Z,δ) be an acyclic dg-field. Then any dg-algebra (A,d) containing (Z,δ) as a subalgebra is acyclic as well.*

Indeed, Lemma 3.1 applies immediately. ■

Remark 3.3. Let (A,d) be a dg-algebra and let (S_A,δ_A) be a dg-simple dg-module over (A,d). Then by Proposition 2.14 the dg-algebra

$$(\mathrm{End}^\bullet_A(S_A,\delta_A),d_{\mathrm{Hom}})=:(D_A,\partial_A)$$

is a dg-division algebra, and by Theorem 2.15 we get that either (D_A,∂_A) is acyclic or $\partial_A=0$.

If $\partial_A=0$, then every graded endomorphism of S_A automatically graded-commutes with the differential of S_A.

Lemma 3.4. *Let (A,d) be a dg-algebra and let (S_A,δ_A) be a dg-simple dg-module over (A,d). Then (S_A,δ_A) is acyclic if the dg-division algebra $(\mathrm{End}^\bullet_A(S_A,\delta_A),d_{\mathrm{Hom}})$ is acyclic.*

Proof. Put

$$(\mathrm{End}^\bullet_A(S_A,\delta_A),d_{\mathrm{Hom}})=:(D_A,\partial_A)$$

If (D_A,∂_A) is acyclic, we get that $1_{D_A}\in\mathrm{im}(\partial_A)$. Hence the identity endomorphism of S_A is a cycle. But this is equivalent with the fact that the identity is homotopic to 0 (cf e.g. [14, Definition 3.4.6]). This then shows that actually (S_A,δ_A) is acyclic as well. ■

Example 3.5. The converse of Lemma 3.4 is false. Our favorite example

$$A=\begin{pmatrix}K&K\\K&K\end{pmatrix}$$

with differential

$$d(\begin{pmatrix}x&u\\z&y\end{pmatrix})=\begin{pmatrix}z&y-x\\0&z\end{pmatrix}$$

is a counterexample. The differential is non zero, obviously. The graded centre is just the scalar multiples of the identity.

Also, there is a unique dg-simple dg-left module (S,δ) over (A,d), namely the right matrix column. The differential on this dg-simple is non zero. Actually, (S,δ) is acyclic. The endomorphism ring of this dg-simple (S,δ) is just K, the graded centre. However, the differential on the graded centre is 0.

Remark 3.6. There is an alternative proof of Lemma 3.4. Suppose that (D_A,∂_A) is acyclic. Since (S_A,δ_A) is a dg-right module over (D_A,∂_A) (cf [13, Lemma 22.13.3] or [14, Proposition 3.3.17]), and since by Theorem 1.5 we get that all dg-modules over acyclic dg-algebras are acyclic, also (S_A,δ_A) is acyclic.

Remark 3.7. If we had that any dg-module over a dg-division ring admits a basis in the cycles, then we also had that any dg-module over a dg-division ring is either acyclic or has differential 0. Counterexamples are obiquitous.

Our favorite example Example 3.5 is a counterexample. The differential is non zero, obviously. The graded centre is just the scalar multiples of the identity. If the statement was correct, then the algebra would be isomorphic to the matrix ring with zero differential, which is obviously false.

Lemma 3.8. *Let (A, d_A) and (B, d_B) be dg-algebras with $Z_{gr}(A, d_A) \supseteq Z$ and $Z_{gr}(B, d_B) \supseteq Z$ for some graded commutative dg-algebras (Z, ∂). Suppose that (A, d_A) or (B, d_B) is acyclic. Then $(A \otimes_Z B, d_{A\otimes_Z B})$ is acyclic as well.*

Proof. Indeed, by Lemma 2.12 we need to show that $1_{A\otimes_Z B} \in \mathrm{im}(d_{A\otimes_Z B})$. Since (A, d_A) or (B, d_B) is acyclic, there is a homogeneous element $z_A \in A$ or $z_B \in B$ with $d_A(z_A) = 1_A$ or $d_B(z_B) = 1_B$. However, $d_{A\otimes_Z B} = d_A \otimes_Z \mathrm{id}_B + \mathrm{id}_A \otimes_Z d_B$. Suppose that (A, d_A) is acyclic. But then

$$d_{A\otimes_Z B}(z_A \otimes 1_B) = d_A(z_A) \otimes 1_B = 1_A \otimes_Z 1_B$$

This shows the statement. ∎

4. A differential graded Jacobson-Chevalley density theorem

A very basic result in ring theory is the Chevalley-Jacobson density theorem. A graded version of the Chevalley-Jacobson density theorem can be found in Chen et al. [5]. The result actually follows from earlier work of Liu, Beattie and Fang [9].

Theorem 4.1. [5, 9] *Let A be a group graded algebra. Let M be a gr-simple graded A-module and let $D = \mathrm{End}_{A-graded}(M)$. Let $x_1, \ldots, x_k$ be homogeneous D-independent elements of M, and let $y_1, \ldots, y_k$ be any elements of M. Then there is $a \in A$ with $ax_i = y_i$ for any $i \in \{1, \ldots, k\}$.*

We shall need to show that if (A, d) is a dg-algebra and (M, δ) is a dg-simple dg-module over (A, d), then $\ker(\delta)$ is a $\mathbb{Z}$-gr-simple $\ker(d)$-module. In the special case of an acyclic (A, d)-algebra, this follows from Theorem 1.5, due to Aldrich and Garcia-Rozas [1].

As a consequence, if (A, d) is acyclic, then a dg-module (M, δ) over (A, d) is dg-simple if and only if $\ker(\delta)$ is a graded-simple $\ker(d)$-module. We hence may apply Theorem 4.1 to dg-simple dg-modules in this case.

Remark 4.2.
- An unpublished result [3] due to G.M.Bergman shows that for $\mathbb{Z}$-graded rings, the Jacobson radical (ungraded version) is homogeneous. This means that any simple module is automatically graded.
- Recall that Bahturin, Zaicev and Sehgal classified in [2] finite-dimensional simple G-graded K-algebras A for a group G and an algebraically closed field K, subject to some hypotheses with respect to G and to the base field. In particular, if either K is of characteristic 0 or the order of any finite subgroup of G is coprime to the characteristic of K, then A is a matrix algebra over a graded skew-field $K^\alpha H$ for some finite subgroup H of G and α a 2-cocycle with values in $K^\times$.

  If $G = \mathbb{Z}$, then there is no non trivial finite subgroup. Hence, considering finite dimensional graded simple algebras we are left with gradings on full matrix algebras over F.
- Note that $\ker(d_{A\otimes_Z B})$ is in general strictly bigger that $\ker(d_A) \otimes_Z B + A \otimes_Z \ker(d_B)$. Hence, the equivalence of categories in Theorem 1.5 does not behave well with respect to tensor products.

Theorem 4.3. *Let (A, d) be an acyclic dg-algebra, i.e. a dg-algebra with $H(A, d) = 0$. Let (M, δ) be a dg-simple dg-module over (A, d) and let*

$$(D, \partial) := \mathrm{End}^\bullet_A((M, \delta), d_{\mathrm{Hom}}).$$

Then (D, ∂) is a dg-division algebra, and also a $\mathbb{Z}$-gr-division algebra. Moreover, for each family $x_1, \ldots, x_k$ of D-independent elements of $\ker(\delta)$ and each family $y_1, \ldots, y_k$ of elements of $\ker(\delta)$, there is an element $a \in \ker(d)$ with $ax_i = y_i$ for all $i \in \{1, \ldots, k\}$.

Proof. Let $N := \ker(\delta)$ and $B := \ker(d)$ to shorten the notation. As (M, δ) is dg-simple over (A, d), by the equivalence of categories in Theorem 1.5.(5) we get that N is $\mathbb{Z}$-graded simple as graded B-module. Further, $\mathrm{End}_{B-graded}(N) \simeq D = \mathrm{End}^\bullet_A((M, \delta), d_{\mathrm{Hom}})$ again by the equivalence of categories in Theorem 1.5.(5). Hence, D is also a $\mathbb{Z}$-gr-division algebra. The statement now follows directly from Theorem 4.1. ∎

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Université de Picardie,
Département de Mathématiques et LAMFA (UMR 7352 du CNRS),
33 rue St Leu,
F-80039 Amiens Cedex 1,
France

Email address: `alexander.zimmermann@u-picardie.fr`