On the Ring Theory of Differential Graded Rings

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Let R be a commutative ring. Cartan defined in [3] a differential graded R-algebra as a $\mathbb{Z}$-graded algebra A together with an R-linear endomorphism d of degree 1 with $d^2 = 0$ satisfying $d(ab) = d(a) \cdot b + (-1)^{|a|} a \cdot d(b)$ for all homogeneous elements $a, b \in A$ of degree $|a|$, resp. $|b|$. A differential graded (A, d)-module (M, δ) (or dg-module over a dg-algebra for short) is a $\mathbb{Z}$-graded A-module with an R-linear endomorphism δ of degree 1 satisfying $\delta^2 = 0$ and $\delta(a \cdot m) = d(a) \cdot m + (-1)^{|a|} a \cdot \delta(m)$ for all homogeneous $a \in A$ and $m \in M$. Similarly we define dg-right modules and dg-bimodules. A dg-submodule of a dg-module (M, δ) is a graded submodule stable under the action of δ. A dg-module is dg-simple (S, δ) if there is no dg-submodule of S other than S or 0. A dg-algebra is dg-simple if it is simple as dg-bimodule over itself.

The ring theory of dg-algebras laid unexplored until very recently. Aldrich and Garcia Rozas characterised in [1] dg-algebras with semisimple dg-module categories. Orlov [7, 8] studied finite dimensional dg-algebras over a field mainly under a geometric perspective. Goodbody [4] used Orlov's work to define dg-Jacobson radicals and shows a version of Nakayama's lemma for finite dimensional dg-R-algebras over a field R, such that the quotient modulo the ordinary Jacobson radical is separable.

We propose a more systematic concept. Consider the set of dg-left ideals of (A, d). The intersection of the maximal elements in this poset will produce a dg-left ideal. If $d = 0$ and the grading is trivial, then this is the classical Jacobson ideal, whence two-sided. However, consider the following

Example 1. [9] The endomorphism complex of a dg-module (M, δ) is a dg-algebra, as is well-known. In case of a field K with trivial grading and 0 differential we consider the complex $K \stackrel{\text{id}}{\to} K$, being a dg-K-module concentrated in degree -1 and 0. Its endomorphism complex (A, d) is then $\begin{pmatrix} K & K \\ K & K \end{pmatrix}$ with differential $d(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $d(\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}) = \begin{pmatrix} 0 & y - x \\ 0 & 0 \end{pmatrix}$, and of course the differential of the upper right corner elements being 0. Then there is only one dg-left module over A, namely the right column.

Definition 2. [9] The dg-Jacobson radical $\mathrm{dgrad}_2(A, d)$ of a dg-algebra (A, d) is the intersection of the left annihilators of the dg-simple dg-left modules.

It is not hard to see that $\mathrm{dgrad}_2(A, d)$ is a twosied dg-ideal. A dg-module is said to be dg-Noetherian (resp. dg-Artinian) if it satisfies the ascending (resp. descending) chain condition on dg-submodules.

Theorem 3. *(dg-version of Nakayama's lemma)* [9] *Let (A, d) be a dg-algebra over a commutative ring R, and let (M, δ) be a dg-Noetherian and dg-Artinian dg-module over (A, d). Then for any dg-submodule (N, δ) of (M, δ) we get*

$$N + \mathrm{dgrad}_2(A, d) \cdot M = M \Rightarrow N = M.$$

In a similar philosophy dgnil(A,d) is defined to be the sum of nilpotent twosided dg-ideals of (A, d), and $\mathrm{dgPrad}(A, d)$ is the intersection of twosided dg-prime ideals. Here a dg-ideal is said (P, d) to dg-prime if whenever (T, d) and (S, d) are twosided dg-ideals with $ST \subseteq P$, then $S \subseteq P$ or $T \subseteq P$.

Proposition 4. [10] *Let (A, d) be a left dg-Noetherian and left dg-Artinian dg-algebra. Then* $\mathrm{dgnil}(A, d) = \mathrm{dgrad}_2(A, d) = \mathrm{dgPrad}(A, d)$.

I do not know yet if any dg-Artinian algebra is dg-Noetherian, whence if Hopkins' theorem holds in the dg-version.

Proposition 5. [9] *If (A, d) is dg-Noetherian and dg-Artinian then* $\mathrm{dgrad}_2(A, d)$ *is the smallest twosided dg-ideal such that the quotient is a finite direct product of dg-simple dg-algebras. Hence the left and the right version of* $\mathrm{dgrad}_2(A, d)$ *coincide.*

So, how to produce dg-simple dg-algebras? Of course, simple algebras which happen to be dg-algebras, such as Example 1, are dg-simple. Orlov [7] calls these algebras formally dg-simple. However, there are more, such as the algebra of dual numbers $K[X]/X^2$ with $|X| = -1$ and $d(X) = 1$. Note that Example 1 is dg-simple, but by [1] the dg-module category is not semisimple.

In order to do produce more dg-simple algebras, we first study Ore localisation for dg-algebras.

Theorem 6. [10] *Let (R, d) be a dg-ring, and let S be a multiplicative set of homogeneous elements. Let $\mathrm{ass}_\ell(S) := \{r \in R \mid \exists s \in S : \; sr = 0\}$. Assume that either S consists of regular elements, or else $S \subseteq \ker(d)$ is a left Ore set and the image of S in $R/\mathrm{ass}_\ell(S)$ consists of regular elements of $R/\mathrm{ass}_\ell(S)$. Then*

$$d_S(b, s) := (-1)^{|s|+1}(d(s), s) \cdot (b, s) + (-1)^{|s|}(d(b), s)$$

defines a differential graded structure on R_S, and the natural homomorphism is a dg ring homomorphism $\lambda : (R, d) \to (R_S, d_S)$ such that $\lambda(S) \in R_S^\times$, the group of invertible elements of R_S, and such that for any $q \in R_S$ there exists $s \in S$ with $\lambda(s) \cdot q \in \mathrm{im}(\lambda)$. Similar statements hold for the right version.

Note that Braun-Chuang-Lazarev [2] gave a very abstract construction for lifting an Ore localisation of the homology algebra of a dg-algebra at a multiplicative Ore set $\overline{S}$ to an Ore localisation of the dg-algebra. We give here an explicit version, for an Ore set S formed by elements of $\ker(d)$, giving then by reduction the set $\overline{S}$.

We now use this result to construct dg-simple dg-algebras. Recall that a graded algebra is gr-prime if the zero ideal is a gr-prime ideal (i.e. dg-prime for the zero differential). A graded ring is called graded left Goldie if it does not allow an infinite direct sum of graded left ideals, and in addition it satisfies the ascending chain condition on left annihilators.

Theorem 7. [10] *Let R be a commutative ring and let (A, d) be a differential graded R-algebra. Suppose that* $\ker(d)$ *is a gr-prime ring and suppose that* $\ker(d)$ *is left gr-Goldie.*

- *If (A,d) is dg-Noetherian as bimodule, then the localisation of A at the homogeneous regular elements S_A of A exists and is dg-simple.*
- *If the homogeneous regular elements $S_{\ker(d)}$ of $\ker(d)$ form a left Ore set in A,*
  - *then the left Ore localisation of (A,d) at $S_{\ker(d)}$ is a dg-simple differential graded R-algebra.*
  - *Further, $S_{\ker(d)} \subseteq S_A$ and hence in case S_A is left Ore as well, A_{S_A} and $A_{S_{\ker(d)}}$ both exist, are dg-simple rings, and the natural homomorphism $A_{S_{\ker(d)}} \to A_{S_A}$ is injective.*

Recall that the classical Goldie theorem only asks for a semiprime ring obtaining then a semisimple Artinian algebra. However, Goodearl and Stafford [5] show that for $A = K[X,Y]/XY$ where K is a field, X is of degree 1 and Y is of degree 0 is graded Goldie, graded semiprime, but not graded semisimple. Though, the only non invertible homogeneous regular elements are the elements of K. Hence, the assumption of being prime is necessary. Goodearl and Stafford show in [5] a Goldie's theorem of group graded graded prime rings, graded by an abelian group. We use their result in an essential way.

We finally mention that we are able to prove in [11] a differential graded version of Posner's theorem using Karasik's result [6]. This then gives again dg-simple algebras using the theory of polynomial identity algebras.

We note that in [11] again, as in Theorem 7, and as in [1], the hypothesis of the classical theorem we want to generalise is assumed in the graded version for $\ker(d)$, and under a few additional technical assumptions we show the dg-version of the classical results. We suspect that this is a general pattern.

References

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