DIFFERENTIAL GRADED ORDERS, THEIR CLASS GROUPS AND IDÈLES

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Abstract. For a Dedekind domain R with field of fractions K a classical R-order in a semisimple K-algebra A is an R-projective R-subalgebra Λ of A such that $K\Lambda = A$. We study differential graded K-algebras which are semisimple as K-algebras and define differential graded R-orders as a differential graded R-subalgebras, which are in addition classical R-orders in A. We give a series of examples for such differential graded algebras and orders. We show that any differential graded R-order is contained in a maximal differential graded order. We develop parts of the classical ring theory in the differential graded setting, in particular the properties of analogues of the Jacobson radical. We further define class groups of differential graded orders as subgroups of the Grothendieck group of locally free differential graded modules. We define idèles in this setting showing that these idèle groups map surjectively to the differential graded class group. Finally we give a homomorphism to the class group of the homology of the differential graded order and prove a Mayer-Vietoris like sequence for each central idempotent of A, including the analogous one for the kernel groups of these morphisms.

Introduction

Differential graded algebras were introduced by Cartan in [4]. These are, by definition, a pair (A, d) of an associative $\mathbb{Z}$-graded algebra A with an endomorphism $d : A \longrightarrow A$ of degree 1 and $d^2 = 0$, satisfying $d(a\cdot b) = a\cdot d(b) + (-1)^{|a|}a\cdot d(b)$ for all homogeneous elements $a, b \in A$. Differential graded algebras became prominent in the past few decades in order to provide a powerful tool for homological algebra.

Most research was directed in this perspective. In particular, as differential graded algebras occur naturally in the homological algebra of algebraic and differential varieties, these examples provided the guideline of reasonable assumptions. In particular, a quite systematic assumption is that differential graded algebras are connective, that is all homogeneous components in positve degrees are assumed to vanish. Often, they are algebras of infinite dimension over a base field.

We ask what happens if the algebras are more linked to classical finite dimensional algebras. Differential graded algebras are algebras at first, and so one might be tempted to understand ring theoretic properties of differential graded algebras. To our greatest surprise we could not find much research which was carried out up to now in this direction. We fist study properties such as Jacobson radicals or analogues of these, and see that the concepts in this direction in the differential graded case are different. Unlike the classical situation the intersection of all differential graded maximal left ideals is not the same as the intersection of annihilators of differential graded simple left modules. This holds however up to homotopy. Further, we prove a Nakayama's lemma for the two-sided notion given by intersection of annihilators of simple dg-modules.

We then study differential graded orders in differential graded semisimple algebras. Since in characteristic different from 2 the differential vanishes automatically on central idempotents, we study finite dimensional simple algebras, i.e. matrix algebras over skew fields by Wedderburn's theorem. We provide examples that there are various interesting differential graded structures on these matrix algebras. They bear interesting properties. In particular, their homology is not semisimple in general.

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Then, we consider differential graded orders in these algebras, naively as classical orders which are stable under the differential graded structure. Graded orders were studied by LeBruyn, van Oystaen, and van den Bergh in [15]. The differential gives an important extra structure. One may ask if it would not be reasonable to ask in addition that the homology is an order in the homology of the algebra. However, this additional condition is restrictive and it does not seem to allow a rich theory.

We prove that maximal differential graded orders always exist, that any differential order is contained in a maximal one, and give examples that not all maximal orders allow a differential structure. Clearly, our algebras are not connective. Non connective differential graded algebras may have strange properties. We mention Raedschelder and Stevenson [18] for striking examples and properties in this direction.

Finally we define class groups of locally free differential graded orders. We give a K-theoretic definition and prove its equivalence with a second definition provided by idèles. The only modification which is necessary, is that we need to consider the subalgebra of cycles, and not the entire algebra. We then show that the functor 'taking homology' then maps to the ordinary class group of the homology algebra, at least if the homology algebra of the ambiant semisimple algebra is semisimple. The kernel of the resulting map then bounds the fibre of stable quasi-isomorphism classes instead of stable isomorphism classes of locally free modules.

We finally mention that a differential graded Jordan-Zassenhaus theorem is not available for the moment. Hence I do not known if differential graded class groups are finite.

Here is an outline of the paper. In Section 1 we review some basic facts on differential graded algebras and differential graded modules. In Section 2 we recall a result due to Aldrich and Garcia Rozas on semisimplicity of the category of differential graded modules. In Section 3 we provide basically two classes of algebras which are simple as algebras and differential graded. Choosing special cases provide a wealth of interesting examples for what follows. Section 4 develops classical ring theory of differential graded algebras, in particular different concepts of radicals of differential graded algebras, providing the technical tools for discussions in further sections. Section 5 then introduces our main object of study, differential graded orders and differential graded lattices. We provide a weak version and a strong version in parallel, but focus mainly on the weak version, which seems to have more and easier accessible properties. As fist property we study maximal differential graded orders in Section 6. In Section 7 we introduce class groups of differential graded orders and differential graded idèles, show the equivalence of the K-theoretic and the idèle theoretic setting. In Section 8 we prove a Mayer-Vietoris like sequence for class groups of differential graded orders.

1. Foundations of dg-algebras and dg-modules

1.1. **Generalities.** Let R be a commutative ring. Recall from Cartan [4] and Keller [12] that

- a differential graded R-algebra (or dg-algebra for short) is a $\mathbb{Z}$-graded R-algebra $A = \bigoplus_{n\in\mathbb{Z}} A_n$ together with a graded R-linear endomorphism d of square 0 of degree 1 (i.e. $d(A_n) \subseteq A_{n+1}$ and $d \circ d = 0$) and such that
$$d(ab) = d(a)b + (-1)^{|a|}ad(b)$$
for all homogeneous elements $a, b \in A$. A homomorphism of differential graded algebras $f : (A, d_A) \to (B, d_B)$ is a degree 0 homogeneous R-algebra map f such that $f \circ d_A = d_B \circ f$.
- If (A, d) is a differential graded algebra, then (A^{op}, d^{op}) is a differential graded algebra (cf e.g. [22, Definition 11.1]) with $x \cdot_{op} y := (-1)^{|x|\cdot|y|}yx$ for any homogeneous elements $x, y \in A$, and $d^{op}(x) = d(x)$. We hence write $d^{op} = d$.

- A differential graded left A-module (or dg-module for short) is then an A-module M, $\mathbb{Z}$-graded as an R-module with graded R-linear endomorphism d_M of square 0 and of degree 1, such that
$$d_M(ma) = d_M(m)a + (-1)^{|m|}md(a)$$
for all homogeneous elements $a \in A$ and $m \in M$. A differential graded (A,d)-right module is a differential graded (A^{op}, d)-left module.
- Let (A, d_A) be a differential graded R-algebra and let (M, δ_M) and (N, δ_N) be differential graded (A, d_A)-modules. Then a homomorphism of differential graded modules is an R-linear map $f : M \to N$, homogeneous of degree 0 with $f \circ \delta_M = \delta_N \circ f$, with $f(am) = af(m)$ for all $a \in A$ and $m \in M$.
- If (M, d_M) is a dg-A-module, then let $M[k]$ be the dg-module given by $(M[k])_n = M_{k+n}$ for all $n \in \mathbb{Z}$ and $d_{M[k]} = d_M$. It is easy to see that $M[k]$ is a dg-module again.
- Let (M, d_M) and (N, d_N) be differential graded (A, d)-modules. The homomorphism complex $Hom^{\bullet}_A(M, N)$ is the $\mathbb{Z}$-graded A-module given by
$$\begin{aligned}(Hom^{\bullet}_A(M,N))_n := \{f : M \to N \mid\ & f \in Hom_{\mathbb{Z}}(M,N) \text{ and} \\ & f(M_k) \subseteq N_{k+n} \text{ and} \\ & f(am) = (-1)^{|a|n} a \cdot f(m)\}.\end{aligned}$$
The elements f of $Hom^{\bullet}_A(M, N)$ are not asked to be compatible with the differentials in any way. Let $d_{Hom} : Hom^{\bullet}_A(M, N) \to Hom^{\bullet}_A(M, N)$ given by
$$d_{Hom}(f) := d_N \circ f - (-1)^{|f|} f \circ d_M.$$
Then
$$\begin{aligned} d^2_{Hom}(f) &= d_{Hom}(d_N \circ f - (-1)^{|f|} f \circ d_M) \\ &= d_N \circ (d_N \circ f - (-1)^{|f|} f \circ d_M) \\ &\quad -(-1)^{|f|-1}(d_N \circ f - (-1)^{|f|} f \circ d_M) \circ d_M \\ &= (-1)^{|f|+1}(d_N \circ f \circ d_M - d_N \circ f \circ d_M) \\ &= 0 \end{aligned}$$
Therefore $(Hom^{\bullet}_A(M, N), d_{Hom})$ is a complex of R-modules.
- The category $A - dgMod$ of dg-modules over the dg-algebra (A, d) is abelian with morphisms the degree zero cycles of $(Hom^{\bullet}_A(M, N), d_{Hom})$ (cf [22, Lemma 22.4.2]). If A is Noetherian, then the subcategory $A - dgmod$ of finitely generated dg A modules is abelian as well.
- The homotopy category $K(A - dgMod)$ is the category with objects A dg-modules and morphisms $Hom_{K(A-dgMod)}(M, N) = H_0(Hom^{\bullet}_A(M, N), d_{Hom})$.
- If A is artinian and (A, d) a differential graded algebra. Then the category $A-dgmod$ has split idempotents and is hence Krull-Schmidt. This follows from the fact that the image of a homomorphism of dg-modules is a dg-module again.

Lemma 1.1. *Let (A, d) be a differential graded R-algebra, and let (M, d_M) and (N, d_N) be differential graded (A, d)-modules.*

- *Then $Z_0(Hom^{\bullet}_A(M, N), d_{Hom}) = (\ker(d_{Hom})_0$ consists the homomorphisms of complexes, and $H_0(Hom^{\bullet}_A(M, N), d_{Hom})$ is the morphisms $M \to N$ in the homotopy category.*
- *If $M = N$, then $End^{\bullet}_A(M)$ is a dg-algebra and composition of graded morphisms yields a differential graded $End^{\bullet}_A(M)$-module structure on $Hom^{\bullet}_A(M, N)$.*

Proof: [12] ■

Lemma 1.2. $d_A(1) = 0$.

Proof. $d_A(1) = 1 \cdot d_A(1) + d_A(1) \cdot 1$, which implies $d_A(1) = 0$. ■

Corollary 1.3. *Let R be a commutative ring and let (A, d) be a differential graded R-algebra. Then* $\ker(d)$ *is a graded subalgebra of A.*

Proof. Since d is R-linear, $\ker(d)$ is an R-module. Since d is homogeneous, $\ker(d)$ is graded. By Lemma 1.2 $\ker(d)$ contains 1. Further, $d(xy) = d(x)y \pm xd(y)$ for any homogeneous elements x, y shows that $x, y \in \ker(d) \Rightarrow xy \in \ker(d)$. ■

Remark 1.4. We note that Example 3.6 shows that in general $d(u) \neq 0$ for a unit u in A. Further, units are not necessarily in degree 0, as is shown by Example 3.6.

Lemma 1.5. *Let (A, d) be a differential graded algebra. Then, the algebra structure of A induces a graded algebra structure on $H(A, d)$ (and hence differential graded with differential 0).*

Proof. Let $a, b \in \ker(d)$ be homogeneous. Then $d(a \cdot b) = d(a) \cdot b + (-1)^{|a|} a \cdot d(b) = 0$. Hence $a \cdot b \in \ker(d)$. Let $a = d(x)$ and $c \in \ker(d)$ be homogeneous elements. Then

$$ac = d(x)c = d(xc) - (-1)^{|b|} xd(c) = d(xc) \in \mathrm{im}(d).$$

Likewise $ca \in \mathrm{im}(d)$. Hence, the multiplication is well-defined. The fact that the additive law is well-defined and combines to a ring structure is trivial. ■

Lemma 1.6. *Let (A, d) be a differential graded algebra and let (M, d_M) be a differential graded right (resp. left) A-module. Then the A-module structure on M induces a right (resp. left) $H(A, d)$-module structure on $H(M, d_M)$.*

Proof. We only need to show that the induced action is well-defined. We shall give the proof for a right module. The left module case is analogous.

Let $a \in \ker(d)$ be a homogeneous element, and let $m \in \ker(d_M)$ be a homogeneous element. Then

$$d_M(m \cdot a) = d_M(m) \cdot a + (-1)^{|m|} m \cdot d(a) = 0$$

Hence $m \cdot a \in \ker(d_M)$ again. Moreover,

$$\begin{aligned} d_M(n) \cdot (a + d(b)) &= d_M(n)a + d_M(n)d(b) \\ &= d_M(na) - (-1)^{|a|} nd(a) + d_M(nd(b)) - (-1)^{|n|} nd^2(b) \\ &= d_M(na) + d_M(nd(b)) - (-1)^{|a|} nd(a) \\ &= d_M(na) + d_M(nd(b)) \end{aligned}$$

is in $\mathrm{im}(d_M)$, since $a \in \ker(d)$ and hence $nd(a) = 0$. Therefore, the multiplication of $H(A)$ on $H(M)$ is well-defined. ■

2. Semisimplicity in the category of differential graded modules

We recall a theorem of Tempest and Garcia-Rozas [1]. A differential graded A-module X is simple if the only differential graded A-submodules are 0 and X. An abelian category is semisimple if all subobjects admit a complement.

Theorem 2.1.
- [1, Theorem 4.7] *Let (A, d) be a differential graded algebra. Then the following are equivalent:*
  1. *The regular differential graded A-module A is projective in the category of dg-modules*
  2. *A is acyclic.*
  3. *$1_A \in im(d)$*
  4. *any left dg-module is acyclic.*
  5. *The functor $A \otimes_{Z(A)} -$ from graded $Z(A)$-modules to differential graded A-modules is an equivalence with quasi-inverse $Z(-)$.*

- [1, Theorem 5.3] *The category of differential graded modules over (A, d) is semisimple precisely when the category of graded modules over the graded algebra of cycles $Z(A) = \ker(d)$ is semisimple and A is an acyclic complex. This in turn is equivalent with the left regular differential graded (A, d)-module (A, d) is semisimple as a differential graded module, in the sense that all differential graded ideals have a complement.*

Note that Tempest and Garcia-Rozas show [1, Lemma 4.2] that if $1_A = d(z)$, then $A = Z(A) \oplus Z(A)y$.

Remark 2.2. Let K be a commutative ring such that 2 is a regular element in K, and let (A, d) be a differential graded K-algebra with centre $Z(A)$, and let $e^2 = e \in Z(A)$ be homogeneous. Then, copying the argument of [6, Theorem 1.4 and Corollary 1.5], since $e^2 = e \in Z(A)$, we get that e is necessarily in A_0. Moreover,

$$d(e) = d(e^2) = d(e)e + ed(e) = 2ed(e) = 4ed(e)$$

Hence $ed(e) = 0$. Moreover $(1-e)d(e) = (1-e)d(e)e + (1-e)ed(e) = 0$, and hence $d(e) = 0$. Therefore, if $A = A_1 \times A_2$ as algebra, then $(A, d) = (A_1, d) \times (A_2, d)$ as differential graded algebras.

Corollary 2.3. *As we have seen in Remark 2.2 for a differential graded algebra A which is semisimple artinian as algebra over a field of characteristic different from 2, by Wedderburn's theorem we have that A is a direct sum of differential graded algebras, each of which is a matrix algebra of a skew field.*

3. Some examples

We first give an example for a $\mathbb{Z}$-grading of matrix algebras.

Example 3.1. (1) Let $A = Mat_{n\times n}(D)$ for some skew-field D, finite dimensional over K, and $n > 1$ an integer. Then A is graded by putting A_0 the subalgebra given by the main diagonal entries

$$\begin{pmatrix} * & 0 & \dots & \dots & 0 \\ 0 & * & 0 & & \vdots \\ \vdots & \ddots & * & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & * \end{pmatrix}$$

Then upper one diagonal coefficients $(a_{i,i+1})_i$ give the degree 1

$$\begin{pmatrix} 0 & * & 0 & \dots & 0 \\ 0 & 0 & * & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & 0 & * \\ 0 & \dots & \dots & 0 & 0 \end{pmatrix}$$

the upper 2 diagonal coefficients $(a_{i,i+2})_i$ give the degree 2, etc. The lower one diagonal coefficients $(a_{i+1,i})_i$ give the degree -1, the lower 2 diagonal coefficients $(a_{i+2,i})_i$ give the degree -2, etc. Elementary matrix multiplication proves the definition for a graded algebra. Moreover, A is semisimple as an algebra.

(2) Let $A = \bigoplus_{n\in\mathbb{Z}} A_n$ be a graded semisimple algebra, then for any $u \in A^\times$ we see that $A^u := \bigoplus_{n\in\mathbb{Z}} uA_nu^{-1}$ is again a graded semisimple algebra.

We produce a series of examples showing that our concept is non trivial and allows interesting phenomena.

Remark 3.2. Note that Example 3.1 provides gradings such that $A \cdot e \not\subseteq A \cdot d(e)$ for any primitive idempotent e of degree 0 and any differential d of degree 1. Also for $End_K^\bullet(L)$ for a bounded complex L of K-vector spaces there is such an idempotent. Indeed, in this case the degree 0 component is formed by matrix algebras along the diagonal, and the degree 1 component is formed by matrices right up to these.

Proposition 3.3. *Let K be a field and let (A, d) be a finite dimensional differential graded algebra. Suppose that A is a split simple K-algebra. Suppose that there is a primitive idempotent e of A such that $A \cdot e \not\subseteq A \cdot d(e)$. Then there is a bounded complex L of K-modules such that $A \simeq Hom_K^\bullet(L, L)$ as differential graded algebras. Conversely, $A = Hom_K^\bullet(L, L)$ is differential graded, finite dimensional simple as algebra, such that there is a primitive idempotent e with $A \cdot e \not\subseteq A \cdot d(e)$.*

Proof. If L is a bounded complex of K-vector spaces, then $Hom_K^\bullet(L, L)$ is a full matrix ring over K, as ungraded algebra, and hence simple as algebra. Further, $Hom_K^\bullet(L, L)$ is a differential graded algebra by Lemma 1.1.

Conversely, let K be a field and let (A, d) be a finite dimensional differential graded algebra. Suppose that A is a split simple K-algebra. By Wedderburn's theorem, A is a full matrix algebra over K. Let e be a primitive idempotent of A satisfying $Ae \not\subseteq Ad(e)$. Then

$$M := A \cdot e + A \cdot d(e) \text{ and } N := A \cdot d(e)$$

are differential graded (A, d)-modules. Further $N < M$ and

$$L := M/N \neq 0$$

is a differential graded (A, d)-module. As A-module, we see that $L \simeq Ae$ is a progenerator. Hence L is a natural differential graded $(A, d) - (End^\bullet(L), d_{Hom})$ bimodule. Now, for any homogeneous $a \in A$, left multiplication by a gives a homogeneous element $\varphi(a) \in End^\bullet(L)$. Further, φ is additive, sends $1 \in A$ to the identity on L, and induces a ring homomorphism

$$\varphi : A \longrightarrow End^\bullet(L).$$

Since L is a progenerator, φ is injective. Since $\dim_K(A) = \dim_K(End^\bullet(L))$, we get that φ is an isomorphism of algebras. Now, for any homogeneous $a, b \in A$, we have

$$d(a)b = d(ab) - (-1)^{|a|} ad(b)$$

we get

$$\varphi(d(a)) = d \circ \varphi(a) - (-1)^{|a|} \varphi(a) \circ d = d_{Hom}(\varphi(a))$$

and therefore φ is an isomorphism of differential graded algebras. Further, by Remark 3.2 there is a primitive idempotent e of degree 0, by degree considerations we get $Ae \not\subseteq Ad(e)$.
■

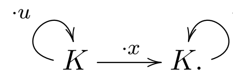
Remark 3.4. The hypothesis that there is a primitive idempotent e of A such that $A \cdot e \not\subseteq A \cdot d(e)$ is superfluous. Indeed, [6, Corollary 1.5] shows that A is isomorphic, as a graded algebra, to a matrix algebra with the main diagonal in degree 0. Transporting the differential structure via this isomorphism, we obtain the existence of such an idempotent.

Remark 3.5. After having finished and submitted the manuscript I discovered that D. Orlov defined our notion of simple differential graded algebra earlier in [17], and called it abstractly simple. Moreover, he proved the statement of Proposition 3.3 by completely different means. His proof uses scheme theoretic arguments. However, he has to assume that the primitive central idempotents are in degree 0. Our approach however gives that this can be assumed to be automatically satisfied using [6, Corollary 1.5].

Example 3.6. We consider a field K and the algebra of 2×2 matrices over K. We shall use Proposition 3.3. Except the stalk complex and complexes with differential 0, up to shift, the only possible complex realising this dg-algebra is

$$\cdots \longrightarrow 0 \longrightarrow K \xrightarrow{\cdot x} K \longrightarrow 0 \longrightarrow \cdots ,$$

concentrated in degree 0 and 1. We denote by δ the differential on the complex. Endomorphisms of degree 0 are given by two scalars u and v



A morphism of degree 1 will map the degree 0 homogeneous component to the degree 1 homogeneous component, i.e. is given by multiplication by a scalar w. A morphism of degree -1 will map the degree 1 homogeneous component to the degree 0 homogeneous component, i.e. is given by multiplication by a scalar z.

degree 1 morphism degree -1 morphism

Then, these morphism correspond to a matrix

$$\begin{pmatrix} u & w \\ z & v \end{pmatrix} \in Mat_{2\times 2}(K)$$

with the names of the variables chosen as in the above maps indicating the identification of the endomorphisms with the matrices, and the grading given as in Example 3.1.1. Here, in order to respect the usual multiplication of matrices and endomorphisms of the complex, we need to write the maps on the right, and compose them accordingly.

Compute the differential. Consider first a degree -1 morphism α_z. Then, using that the maps apply on the right and hence $d(\gamma) = \gamma d - (-1)^{|\gamma|} d\gamma$ as mappings acting on the complex. Hence

$$d(\alpha_z) = \alpha_z\delta - (-1)^{|\alpha_z|}\delta\alpha_z = \begin{pmatrix} xz & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & xz \end{pmatrix} = x \cdot \mathrm{id}.$$

Then, for the differential of a degree 0 morphism $\beta_{(u,v)}$ acting as u on the degree 0 component and as v on the degree 1-component we get

$$\begin{aligned} d(\beta_{u,v}) &= \beta_{u,v}\delta - (-1)^{|\beta_{u,v}|}\delta\beta_{u,v} \\ &= \beta_{u,v}\delta - \delta\beta_{u,v} \\ &= \begin{pmatrix} 0 & -xu \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & xv \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & x(v-u) \\ 0 & 0 \end{pmatrix} \end{aligned}$$

Note that this result is a consequence of the identities

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

together with Leibniz formula and the result of $d(\alpha_z)$. This gives a dg-algebra and any differential is of this form for some $x \in K$. If $x \neq 0$, the homology is 0.

Note that if $x \neq 0$, then the kernel of the differential is

$$\ker(d) = \{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in K\} \simeq K[\epsilon]/\epsilon^2$$

with ϵ in degree 1. This is not semisimple, and hence we do not get a semisimple differential graded algebra in the sense of Aldrich and Garcia-Rozas [1]. However, for x invertible we have $z=\frac{1}{x}\begin{pmatrix}0&0\\1&0\end{pmatrix}$ is a preimage of 1, as required in Aldrich and Garcia-Rozas [1] structure theorem.

We further observe that $\begin{pmatrix}0&1\\1&0\end{pmatrix}$ is invertible, non homogeneous with a summand of degree -1 and another summand of degree 1. However,

$$d(\begin{pmatrix}0&1\\1&0\end{pmatrix})=d(\begin{pmatrix}0&1\\0&0\end{pmatrix})+d(\begin{pmatrix}0&0\\1&0\end{pmatrix})=\begin{pmatrix}x&0\\0&x\end{pmatrix}$$

is non zero for $x\neq 0$. Hence, the differential of invertible elements are not necessarily 0. Nevertheless, $d(1)=0$.

As any differential graded (A,d)-module (M,δ) is at first an A-module, it is useful to consider the possible left dg-module structures on K^2 for the above dg-algebra $(Mat_2(K),d_x)$. So, let $\delta(\begin{pmatrix}0\\1\end{pmatrix})=\begin{pmatrix}u\\v\end{pmatrix}$ for some $u,v\in K$ and $\delta(\begin{pmatrix}1\\0\end{pmatrix})=\begin{pmatrix}s\\t\end{pmatrix}$ for some $s,t\in K$.

$$\begin{array}{rcl}
0&=&\delta(\begin{pmatrix}0&0\\1&0\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix})=\begin{pmatrix}x&0\\0&x\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}-\begin{pmatrix}0&0\\1&0\end{pmatrix}\cdot\begin{pmatrix}u\\v\end{pmatrix}=\begin{pmatrix}0\\x-u\end{pmatrix}\\
0&=&\delta(\begin{pmatrix}1&0\\0&0\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix})=\begin{pmatrix}0&-x\\0&0\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}+\begin{pmatrix}1&0\\0&0\end{pmatrix}\cdot\begin{pmatrix}u\\v\end{pmatrix}=\begin{pmatrix}u-x\\0\end{pmatrix}\\
\begin{pmatrix}u\\v\end{pmatrix}&=&\delta(\begin{pmatrix}0&0\\0&1\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix})=\begin{pmatrix}0&x\\0&0\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix}+\begin{pmatrix}0&0\\0&1\end{pmatrix}\cdot\begin{pmatrix}u\\v\end{pmatrix}=\begin{pmatrix}x\\v\end{pmatrix}\\
\begin{pmatrix}s\\t\end{pmatrix}&=&\delta(\begin{pmatrix}0&1\\0&0\end{pmatrix}\cdot\begin{pmatrix}0\\1\end{pmatrix})=-\begin{pmatrix}0&1\\0&0\end{pmatrix}\cdot\begin{pmatrix}u\\v\end{pmatrix}=\begin{pmatrix}-v\\0\end{pmatrix}\\
\begin{pmatrix}u\\v\end{pmatrix}&=&\delta(\begin{pmatrix}0&0\\1&0\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix})=\begin{pmatrix}x&0\\0&x\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}-\begin{pmatrix}0&0\\1&0\end{pmatrix}\cdot\begin{pmatrix}s\\t\end{pmatrix}=\begin{pmatrix}x\\-s\end{pmatrix}\\
\begin{pmatrix}as\\0\end{pmatrix}&=&\delta(\begin{pmatrix}a&0\\0&b\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix})=\begin{pmatrix}0&x(b-a)\\0&0\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}a&0\\0&b\end{pmatrix}\cdot\begin{pmatrix}s\\t\end{pmatrix}=\begin{pmatrix}as\\bt\end{pmatrix}\\
0&=&\delta(\begin{pmatrix}0&1\\0&0\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix})=-\begin{pmatrix}0&1\\0&0\end{pmatrix}\cdot\begin{pmatrix}s\\t\end{pmatrix}=\begin{pmatrix}-t\\0\end{pmatrix}\\
0&=&\delta^2(\begin{pmatrix}0\\1\end{pmatrix})=\delta(\begin{pmatrix}x\\v\end{pmatrix})=\begin{pmatrix}vx\\v^2\end{pmatrix}+\begin{pmatrix}vx\\0\end{pmatrix}=\begin{pmatrix}2vx\\v^2\end{pmatrix}
\end{array}$$

Hence $s=v=t=0$, $x=u$. Since δ is of degree 1, we need to have $\begin{pmatrix}1\\0\end{pmatrix}$ is of degree n and $\begin{pmatrix}0\\1\end{pmatrix}$ is of degree $n-1$ for some integer n. Up to shift there is a unique dg-module structure given by $\delta(\begin{pmatrix}p\\q\end{pmatrix})=\begin{pmatrix}xq\\0\end{pmatrix}$. Of course, this is a simple differential graded module.

Let $(C^*,\widehat{\delta})$ be a bounded complex of finitely generated $Mat_{2\times 2}(K)$-modules. Each homogeneous component C^n is isomorphic to a a direct sum of differential graded modules (K^2,δ) as above. Then, the total complex is a differential graded module. Using the above explicit computation can be used to show that this gives again the only possible differential graded structures on the corresponding graded module.

Remark 3.7. I wish to thank Bernhard Keller who noted[1] that the above differential graded algebra is the differential graded endomorphism algebra $Hom_K^\bullet(M,M)$ for M being the complex

$$\cdots\longrightarrow 0\longrightarrow K\xrightarrow{\cdot x}K\longrightarrow 0\longrightarrow\cdots.$$

[1] email from December 24, 2022 to the author

Example 3.8. We consider the 3×3 matrix ring over a commutative ring R. We will use an alternative grading coming from the fact that this ring is Morita equivalent to the 2 matrix ring over R.

$$Mat_3(R) = \{ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad | \quad deg(a_{31}) = deg(a_{32}) = -1;$$

$$deg(a_{11}) = deg(a_{12}) = deg(a_{21}) = deg(a_{22}) = deg(a_{33}) = 0;$$
$$deg(a_{13}) = deg(a_{23}) = 1\}$$

This grading comes from the complex L

$$\cdots \longrightarrow 0 \longrightarrow R \xrightarrow{\binom{a_{11}}{a_{21}}} R^2 \longrightarrow 0 \longrightarrow \cdots$$

We make explicit the differentials.

We get

$$d(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x & y & 0 \end{pmatrix}) = \begin{pmatrix} a_{11}x & a_{11}y & 0 \\ a_{21}x & a_{21}y & 0 \\ 0 & 0 & a_{11}x + a_{21}y \end{pmatrix}$$

and

$$d(\begin{pmatrix} \lambda & \sigma & 0 \\ \tau & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix}) = \begin{pmatrix} 0 & 0 & a_{11}(\nu - \lambda) - a_{21}\sigma \\ 0 & 0 & a_{21}(\nu - \mu) - a_{11}\tau \\ 0 & 0 & 0 \end{pmatrix}$$

We may study the homology of this differential graded algebra. Suppose that R is an integral domain. If $a_{11} \neq 0 \neq a_{21}$, then the degree 0 cycles is given by the set of matrices

$$\begin{pmatrix} \lambda & \frac{a_{11}}{a_{21}}(\nu - \lambda) & 0 \\ \frac{a_{21}}{a_{11}}(\nu - \mu) & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix}$$

for $\lambda, \mu, \nu \in R$, such that all coefficients are in R. This forms a 3-dimensional affine variety. The boundaries in degree 0 forms a 2-dimensional subvariety. Hence, the degree 0 homology has an R-torsion free part of rank 1. Degree -1 cycles are trivial, since R has no 0 divisors. The degree 1 cycles is R^2, by definition. Clearly, the degree 1 homology is $(R/(a_{11}R + a_{21}R))^2$.

4. Dg-algebras with respect to Semisimplicity and Noetherian ring theoretical properties

As we shall need to consider semisimplicity of differential graded algebras, it makes sense to consider differential graded versions of the Jacobson radicals. Further, we shall consider semisimplicity of the homology algebra.

4.1. **Semisimplicity of the homology.**

Example 4.1. Consider Example 3.8. The special case $a_{11} = 0 \neq a_{21}$ produces an interesting dg-algebra A, which is semisimple as an algebra. Then

$$d(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x & y & 0 \end{pmatrix}) = a_{21} \cdot \begin{pmatrix} 0 & 0 & 0 \\ x & y & 0 \\ 0 & 0 & y \end{pmatrix}$$

and

$$d(\begin{pmatrix} \lambda & \sigma & 0 \\ \tau & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix}) = a_{21} \cdot \begin{pmatrix} 0 & 0 & -\sigma \\ 0 & 0 & \nu - \mu \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence,

$$\ker(d) = \{\begin{pmatrix} \lambda & 0 & s \\ \tau & \mu & t \\ 0 & 0 & \mu \end{pmatrix} \mid u, v, s, t, x \in K\},$$

which is isomorphic to the semidirect product of the 2×2 lower matrix algebra acting on its natural 2-dimensional representation: $\begin{pmatrix} K \\ K \end{pmatrix} \rtimes \begin{pmatrix} K & 0 \\ K & K \end{pmatrix}$. Clearly, this algebra is not semisimple. However, $H(A) = K$.

In this context is may be worth to mention the following criterion in this context. We shall need the statement later.

Lemma 4.2. *Let K be a field and let (A, d) be a finite dimensional differential graded K-algebra. Then $(\ker(d), d)$ is a subalgebra of (A, d). If $\ker(d)$ is semisimple, then $H(A, d)$ is semisimple as well. Moreover, the restriction of the epimorphism $\ker(d) \twoheadrightarrow H(A, d)$ to $\ker(d)^\times$ yields a surjective group homomorphism $\ker(d)^\times \longrightarrow H(A, d)^\times$.*

Proof. By Corollary 1.3 $\ker(d)$ is a graded subalgebra of A. By hypothesis, $\ker(d)$ is a semisimple K-algebra. Now, by Lemma 1.5 there is an epimorphism $\ker(d) \xrightarrow{\pi} H(A, d)$ of algebras, and the preimage of a twosided ideal in $H(A, d)$ is a twosided ideal of $\ker(d)$. Therefore $H(A, d) \times \ker(\pi) = \ker(d)$ as algebras, and as a consequence any unit $H(A, d)$ lifts to a unit of $\ker(d)$. ■

Remark 4.3. Let (A, d) be a semisimple artinian dg-algebra and let (L, d_L) and (M, d_M) be differential graded (A, d)-modules. Then $\ker((d_{Hom})_0) = End_{(A,d)}(L, d)$ is not semisimple in general since not every homomorphism $\alpha : (L, d_L) \to (M, d_M)$ of complexes decomposes as $\alpha = \gamma \circ \delta$ for a split epimorphism δ of complexes and a split monomorphism γ of complexes.

We recall the easy argument. We consider the special case when A is concentrated in degree 0 and L is finitely generated in each degree.

Since A is semisimple, L is a semisimple A-module. We claim that $L = L_c \oplus H(L)$ is the direct sum of contractible 2-term complexes L_c and its homology $H(L)$. Indeed, since there is no non zero A-module homomorphism between two non isomorphic simple A-modules, we may assume, multiplying by a specific central idempotent if necessary, that A actually is simple. By Morita equivalence we can therefore assume that A is a skew field.

Choose a basis of $\mathrm{im}(d_{-1})$, extend it to a basis of $\ker(d_0)$, and then the so-obtained basis to a basis of L_0. For each basis element of $\mathrm{im}(d_{-1})$ choose a preimage in L_{-1}. The vector space generated by these elements intersects $\mathrm{im}(d_{-2})$ in 0, since $d^2 = 0$. Consider $\ker(d_{-1})$, containing $\mathrm{im}(d_{-2})$. We may choose a basis of $\mathrm{im}(d_{-2})$, extend it to a basis of $\ker(d_{-1})$, and complete the whole to a basis of L_{-1} by the preimages chosen before. By downward induction we chose adapted bases in negative degrees. For positive degrees, we proceed in the same way by upwards induction. The bases elements chosen in the image of the differential, together with their preimages give contractible 2-term complexes. The rest combines to the homology.

Consider the case of

$$L_1 = \ \dots 0 \longrightarrow 0 \longrightarrow M \xrightarrow{id} M \longrightarrow 0 \dots$$

concentrated in degree 0 and 1, and

$$L_2 = \ \dots 0 \longrightarrow N \xrightarrow{id} N \longrightarrow 0 \longrightarrow 0 \dots$$

concentrated in degree -1 and 0. Moreover, suppose that $A = D$ is a skew field. Then $Hom_A(L_1, L_2) = Hom_D(M, N)$ from the spaces in degree 0, using that the differentials do not impose any restriction. However, $Hom_A(L_2, L_1) = 0$ since the compatibility of the differentials impose that the map in degree 0 needs to be 0.

Proposition 4.4. *Let (A, d) be a dg-algebra over a field F, such that A is split simple artinian as an algebra. Then $H(A, d)$ is simple as an algebra.*

Proof. By Proposition 3.3 and Remark 3.4 we see that $(A,d) \simeq (End^\bullet_K(C^\bullet), d_{Hom})$ for some finite dimensional complex $C^\bullet$ of F-vector spaces. By Remark 4.3 any bounded complex of vector spaces is isomorphic to $C^\bullet \simeq D^\bullet \oplus E^\bullet$ for $D^\bullet$ being contractible and $E^\bullet$ a $\mathbb{Z}$-graded vector space with zero differential. Hence, we may replace $C^\bullet$ by this decomposition. In the homotopy category of F-vector spaces we get $C^\bullet \simeq E^\bullet$. Now, denoting by $K^b(F-mod)$ the homotopy category of bounded finite dimensional F-vector spaces,

$$\begin{aligned} H_n((A,d)) &\simeq H_n((End^\bullet_K(C^\bullet), d_{Hom})) \\ &\simeq H_n((End^\bullet_K(D^\bullet \oplus E^\bullet), d_{Hom})) \\ &\simeq Hom_{K^b(F-mod)}(D^\bullet \oplus E^\bullet, D^\bullet \oplus E^\bullet[n]) \\ &\simeq Hom_{K^b(F-mod)}(E^\bullet, E^\bullet[n]) \\ &\simeq Hom_{\text{graded } F\text{-modules}}(E^\bullet, E^\bullet[n]) \end{aligned}$$

Therefore, $H(A,d)$ is a matrix algebra over F, and the grading on the homology gives a grading on the matrix algebra. ■

Remark 4.5. The hypothesis that A is split is not necessary since actually F may well be a skew field in the proof of Proposition 3.3.

Remark 4.6. Consider the situation of Remark 4.3. Then we have a ring homomorphism

$$End^\bullet_A(L) \longrightarrow H_0(End^\bullet_A(L)) = End_{gr-A}(H(L))$$

from the endomorphism complex of L to the ring of graded A-linear homomorphisms of $H(L)$. Since A is semisimple, and since $H(L)$ is a semisimple A-module, $End_{gr-A}(H(L))$ is a semisimple algebra. Moreover, using Lemma 1.1, we get that $(End^\bullet_A(L), d_{Hom})$ is a differential graded algebra, mapping to a semisimple algebra $End_A(H(L))$.

4.2. **Differential graded radicals of differential graded algebras.** We shall study the ring theory of differential graded algebras.

Lemma 4.7. *Let (Λ, d) be a differential graded algebra, let (M,δ) be a differential graded (Λ, d)-module with a differential graded submodule (N,δ). Then there is a differential $\overline{\delta}$ on M/N given by $\overline{\delta}(m+N) := \delta(m)$ for all $m \in M$.*

Proof. M/N is a Λ-module by general ring theory. Since $\delta(N) \subseteq N$, the map $\overline{\delta}$ is well-defined. $\overline{\delta}$ is clearly additive since δ is additive. Since $\delta^2 = 0$, also $\overline{\delta}^2 = 0$. Since δ satisfies Leibniz formula, so is $\overline{\delta}$. ■

Lemma 4.8. *Let (Λ, d) be a differential graded algebra and let (I,d) be a differential graded ideal of (Λ, d). Then there is a differential graded ideal (M,d) in (Λ, d), which is maximal in the partial ordered set of differential graded ideals different from (Λ, d), and containing (I,d).*

Proof. An analogue of the usual proof in the classical situation works. We use Zorn's lemma. Let $\mathcal{X}$ be the set of differential graded ideals of (Λ, d) containing (I,d) but different from (Λ, d). Then $(I,d) \in \mathcal{X}$, and hence $\mathcal{X}$ is not empty. Let $\mathcal{L}$ be a totally ordered subset of $\mathcal{X}$ and let $J := \bigcup_{L \in \mathcal{L}} L$. Then J is an ideal by the classical argument. Moreover, it is differential graded since if $x \in J$, there is $L_x \in \mathcal{L}$ with $x \in L_x$. Then $d(x) \in L_x \subseteq J$. Further the Leibniz formula holds in Λ, whence also in J. If $1 \in J$, then there is $L_1 \in \mathcal{L}$ such that $1 \in L_1$, and hence L_1 is not in $\mathcal{X}$, which provides a contradiction since $\mathcal{L} \subseteq \mathcal{X}$. ■

Lemma 4.9. *Let (Λ, d) be a differential graded algebra and let (I,d) be a differential graded ideal of (Λ, d). Then (I,d) is a differential graded ideal, maximal in the partial ordered set of proper differential graded ideals, if and only if $(\Lambda/I, \overline{d})$ does not contain non trivial differential graded ideals, for $\overline{d}$ the differential induced on Λ/I by d.*

Proof. Let $x \in \Lambda$ and $y \in I$. Then by Lemma 4.7 for $\overline{d}(x+I) := d(x)+I$ defines a differential on Λ/I. Consider the zero ideal in $(\Lambda/I, \overline{d})$. By Lemma 4.8 this is contained in a maximal (in the above sense) differential graded ideal $(\overline{M}, \overline{d})$. Let M be the preimage of $\overline{M}$ in Λ. We claim that (M,d) is a dg-ideal. Since $\overline{M}$ is an ideal, by classical ring theory M is an ideal as well. For $m \in M$ let $d(m)+I = \overline{d}(m+I) \in \overline{M}$ since $(\overline{M}, \overline{d})$ is a dg-ideal. Hence $d(m) \in M$ and by consequence (M,d) is a differential graded ideal. If there is no non trivial differential graded ideal in Λ/I, then $M = I$ and I is a maximal (in the above sense) differential graded ideal. If Λ/I admits non trivial differential graded ideals, then $I \neq M$, and I is not maximal. This shows the statement. ∎

Remark 4.10. Note that the statement of Lemma 4.9 is completely formal and holds for left- or right- or twosided ideals.

Lemma 4.11. *Let (A,d) be a differential graded ring and let (M,δ) be a non zero differential graded (A,d)-module. If (M,δ) is finitely generated as differential graded module, then there is a differential graded submodule (N,δ) of (M,δ), maximal with respect to the lattice of proper differential graded submodules, and such that $(M/N, \overline{\delta})$ is a dg-simple dg-module.*

Proof. Let (M,δ) be finitely generated and denote by $\{g_1,\dots,g_n\}$ a set of generators. We need to show that M contains a proper dg-submodule (N,δ) which is maximal as dg-submodule. Let $\mathcal{X}$ be the set of dg-submodules (L,δ) of (M,δ) such that $L \neq M$. This set is not empty since 0 is trivially a dg-submodule. Let $\mathcal{L}$ be a non empty totally ordered subset of $\mathcal{X}$. Then, analogous to the proof of Lemma 4.8

$$V := \bigcup_{L\in\mathcal{L}} L$$

is a dg-submodule of (M,δ). We need to show that $V \neq M$. Else, $\{g_1,\dots,g_n\} \subseteq M = V$ and hence for any $i \in \{1,\dots,n\}$ there is $L_i \in \mathcal{L}$ with $g_i \in L_i$. Since $\mathcal{L}$ is totally ordered, there is $m \in \{1,\dots,n\}$ such that $L_i \subseteq L_m$ for all $i \in \{1,\dots,n\}$. But then $M = L_m$ since L_m contains all generators $\{g_1,\dots,g_n\}$. This is a contradiction to $L_m \in \mathcal{X}$. By Zorn's lemma there is a maximal element in $\mathcal{X}$, say (N,δ). We claim that N is maximal with respect to the lattice of dg-submodules. Else, let (D,δ) be a strictly bigger proper dg-submodule. Then (D,δ) is superior to (N,δ) in $\mathcal{X}$, a contradiction. Since (N,δ) is maximal as a dg-submodule, $(M/N,\overline{\delta})$ is dg-simple, since else consider a proper quotient of $(M/N,\overline{\delta})$, and the kernel of the composite quotient map from M to the quotient would be strictly larger than (N,δ). We proved the statement. ∎

Lemma 4.12. *Let (Λ,d) be a differential graded algebra and let (M,δ) be a differential graded left (resp. right) module. Then for any homogeneous $a \in ker(\delta)$ we have that $\Lambda\cdot a$ (resp. $a\cdot\Lambda$) is a differential graded (Λ,d) left (resp. right) submodule.*

Proof. Indeed, $\delta(\lambda\cdot a) = d(\lambda)\cdot a \pm \lambda\delta(a) = d(\lambda)\cdot a$ and likewise $\delta(a\cdot\lambda) = \delta(a)\lambda \pm a\cdot d(\lambda) = a\cdot d(\lambda)$. ∎

Remark 4.13. If the homogeneous a is not (necessarily) a cycle, i.e. $a \notin ker(d)$, then $\Lambda\cdot a + \Lambda\cdot d(a)$ is a differential graded ideal. Similarly, if (M,d_M) is a dg-bimodule, and if $m \in \ker(d_M)$, then $\Lambda\cdot m + \Lambda\cdot d_M(m)$ is a left dg-submodule of (M,d_M).

Corollary 4.14. *Let (Λ,d) be a differential graded algebra and let (I,d) be a left (resp. right) differential graded ideal, maximal in the lattice of proper left (resp. right) dg-ideals. If $a \in (ker(d)+I)$ is homogeneous, and if $a \notin I$, then $a+I$ is a left (resp. right) generator of the module Λ/I.*

Proof. If $a \in ker(d)+I$, but $a \notin I$, then $a+I \in ker(\overline{d})$ and then there is $x \in I$ such that $a+x \in \ker(d)$ and hence $\Lambda\cdot(a+x)$ is a differential graded ideal of Λ. Hence, also $I+\Lambda(a+x) = I+\Lambda a$ is a differential graded ideal of Λ. If $a \notin I$, then by the maximality of (I,d) we have that $\Lambda = I+\Lambda a$. Therefore $a+I$ has a left inverse in Λ/I. The statement for the right ideals is analogous. ∎

Definition 4.15. Let (Λ, d) be a differential graded algebra. A non zero differential graded (Λ, d)-module (M, δ) is called *differential graded simple* if there is no differential graded submodule of (M, δ) different from (M, δ) and 0. A differential graded (Λ, d)-module (M, δ) is called *differential graded semisimple* if (M, δ) is the direct sum of simple differential graded (Λ, d)-modules. A *differential graded algebra* (Λ, d) *is called semisimple* if (Λ, d) as left (Λ, d)-module is semisimple.

Remark 4.16. We should remind the reader of different concepts of semisimplicity. In Definition 4.15 we consider a dg-module as being semisimple if it is a direct sum of simple dg-modules, and a dg-module is dg-simple if it does not contain a proper dg-submodule.

In contrast to this is the concept of a dg-module being semisimple if all dg-submodules admit a dg-complement. This concept is the one studied by Aldrich and Garcia-Rozas [1], and they obtain the complete classification as displayed in Theorem 2.1.

For differential graded modules over differential graded algebras these two concepts differ.

Recall that Lam [14] give two notions for these properties. He says (cf [14, I (2.1) Definition]) that a module over an algebra is semisimple if all submodules admit a complement. He calls an algebra Jacobson-semisimple (or J-semisimple) if its Jacobson radical is 0 (cf [14, II (4.7) Definition]).

Example 4.17. Let K be a field and let $A = K[X]/X^2$ for $deg(X) = -1$. Then $d(1) = 0$ and $d(X) = 1$ give a structure of differential graded algebra to A. Indeed, $(a+bX)(c+dX) = ab + (bc + ad)X$ and

$$d((a + bX)(c + dX)) = bc + ad$$

and

$$d(a + bX) \cdot (c + dX) + (a - bX) \cdot d(c + dX) = b(c + dX) + (a - bX)d = bc + ad.$$

Further, $0 = d(0) = d(X^2) = d(X) \cdot X - X \cdot d(X) = X - X = 0$. The algebra A is simple as differential graded algebra in the sense that there is no non trivial twosided differential graded ideal (cf Definition 4.21 below). Indeed, there is only one non trivial algebra ideal, namely $XK[X]$. However, this is not a differential graded ideal, since $d(X) = 1$. Further, the only simple differential graded modules are the regular one and its shifts in degree, which are not concentrated in a single degree. Note that this is the smallest example of algebras mentioned in the second item of Theorem 2.1 studied by Aldrich and Garcia Rozas [1]. Of course, A is not a simple algebra.

Corollary 4.18. *Let* (Λ, d) *be a differential graded algebra and let* (I, d) *be differential graded left ideal of* (Λ, d) *which is maximal in the lattice of proper dg-ideals. Then* $(\Lambda/I, \overline{d})$ *is a dg-simple differential graded* (Λ, d)*-module.*

Proof. This is a direct consequence of Lemma 4.9. ∎

Lemma 4.19. *Let* (Λ, d) *be a differential graded algebra and let* (S, δ) *be a dg-simple differential graded* (Λ, d)*-module. Then for any non zero homogeneous* $u \in \ker(\delta)$ *there is a surjective homomorphism* $(\Lambda, d)[-|u|] \xrightarrow{\pi} (S, \delta)$ *given by* $\pi(\lambda) = \lambda u$*. Further, there is always a non zero homogeneous* $u \in \ker(\delta)$*, and* $\ker(\pi)$ *is a differential graded ideal which is maximal in the set of proper dg-ideals.*

Proof. Due to the shift, π is a homomorphism of graded modules. We need to show that π is compatible with the differential. Let $\lambda \in \Lambda$ be homogeneous. Then

$$\pi(d(\lambda)) = d(\lambda)u = d(\lambda)u + (-1)^{|\lambda|}\lambda\delta(u) = \delta(\lambda u) = \delta(\pi(\lambda)),$$

where the third equation is Leibniz formula, and where the second equation holds since $u \in \ker \delta$. Using Lemma 4.12 we see that $\text{im}(\pi)$ is a non trivial dg submodule of S, and by simplicity of S, we see that π is surjective. Note that $\delta^2 = 0$ shows that $\ker(\delta)$ is not zero. Using Lemma 4.9 we see that $\ker(\pi)$ is a maximal dg-ideal. ∎

Lemma 4.20. *Let (Λ, d) be a differential graded algebra, let (M, δ) be a differential graded (Λ, d)-module. Suppose that there are dg-simple differential graded (Λ, d)-left modules (S_1, δ_1), ..., (S_n, δ_n) such that*

$$(M, \delta) \simeq (S_1, \delta_1) \oplus \cdots \oplus (S_n, \delta_n)$$

as differential graded left modules. Then for any differential graded submodule (N, δ) of (M, δ) there is a differential graded submodule (L, δ) of (M, δ) such that $M = N \oplus L$.

Proof. Consider the family of sets $\{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$ such that $\left(\bigoplus_{j=1}^k S_{i_j}\right) \cap N = \{0\}$. Then taking a maximal one $P = \{i_1, \dots, i_k\}$ of these subsets, and put $L := \bigoplus_{j=1}^k S_{i_j}$. Then $L+N = L\oplus N$ is a differential graded (Λ, d)-module. Hence we get that $(S_\ell+L)\cap N \neq \{0\}$ for any $\ell \notin P$. Therefore $(L \oplus N) \cap S_\ell \neq \{0\}$ for any $\ell \notin P$. Since S_ℓ is a differential graded simple (Λ, d)-module, and since $L \oplus N$ is a differential graded module, $S_\ell \subseteq L \oplus N$ for any $\ell \notin P$. But this shows that $L \oplus N \supseteq \bigoplus_{i=1}^n S_i = M$. We proved the statement. ■

Definition 4.21. Let K be a field and let (A, d) be a differential graded K-algebra. The dg-algebra A is called *differential graded simple* if the only differential graded twosided (A, d)-ideals are 0 and (A, d).

Remark 4.22. Recall from the second item of Theorem 2.1 that every differential graded (A, d)-module is semisimple if and only if (A, d) is acyclic and $\ker(d)$ is a semisimple algebra. Moreover, then $A = \ker(d) \oplus z \cdot \ker(d)$ for $z \in d^{-1}(1)$.

Consider for example the case of an algebra concentrated in degree 0, such as the field K. Then there is no non trivial twosided ideal. However, this dg-algebra does not satisfy the condition of Theorem 2.1. The classical result that simple artinian algebras are semisimple cannot be transposed to the dg-situation.

Proposition 4.23. *Let K be a field and let (A, d) be a dg-simple differential graded algebra. Suppose that the regular differential graded left (A, d) module and the regular differential graded right (A, d) module are dg-artinian.*

Denote by $\mathrm{cone}(id_K)$ *the cone of the identity map of the stalk complex K in $D^b(K-mod)$.*

Then, as differential graded left (A, d)-module, the regular module (A, d) is the direct sum of shifts of copies of one specific simple differential graded modules (S, δ) and shifted copies of modules of the form total complex of the tensor product of (S, δ) by $\mathrm{cone}(id_K)$*, i.e.* $tot\left((S, \delta) \otimes_K (0 \longrightarrow K \xrightarrow{id} K \longrightarrow 0)\right)$.

Proof. As A is dg-artinian there is a minimal non zero differential graded left ideal (S, d) and a minimal non zero differential graded right ideal (T, d). Then consider the $(A, d) - (A, d)$-bimodule $X := (S, d) \otimes_K (T, d)$. The multiplication map

$$A \otimes_K A \longrightarrow A$$

is a chain map (replacing $A \otimes_K A$ by $tot(A \otimes_K A)$) by the Leibniz rule and maps X to a bimodule, whence a twosided ideal of A. Hence the image is a differential graded twosided ideal of (A, d). Since there is no non trivial twosided ideal, this image is A. Now, consider X as differential graded left module. Since only the K-structure of (T, d) is of importance then, and since complexes of K-modules are a direct sum of shifted copies of stalk complexes K and shifted copies of complexes $\mathrm{cone}(id_K)$. This shows the statement. ■

Remark 4.24. We should note that the short exact sequence

$$\begin{array}{ccccccc} & & 0 & \longrightarrow & K & \longrightarrow & 0 \\ & & & & \downarrow & & \\ 0 & \longrightarrow & K & \longrightarrow & K & \longrightarrow & 0 \\ & & \downarrow & & & & \\ & & 0 \longrightarrow K \longrightarrow 0 & & & & \end{array}$$

of differential graded K-modules yields a short exact sequence

$$0 \longrightarrow (S,\delta) \longrightarrow tot\left((S,\delta)\otimes_K (0\longrightarrow K \stackrel{id}{\longrightarrow} K \longrightarrow 0)\right) \longrightarrow (S,\delta)[1] \longrightarrow 0$$

of (A,d)-left modules. Hence, this tensor product is not a dg-simple dg-module.

For any differential graded left (Λ,d)-module (M,δ) denote the *annihilator of* (M,δ) by

$$\mathrm{ann}(M,\delta) := \{\lambda\in\Lambda \mid \lambda m = 0\ \ \forall m\in M\}.$$

Lemma 4.25. *We get that* $\mathrm{ann}(M,\delta)$ *is a differential graded twosided ideal of* (Λ,d).

Proof. Indeed, $\mathrm{ann}(M,\delta)$ is a twosided graded ideal by general ring theory. Further, if $\lambda\in\mathrm{ann}(M,\delta)$ is homogeneous, then $\lambda m = 0$ for any $m\in M$, and hence $0=\delta(\lambda m) = d(\lambda)m \pm \lambda\delta(m) = d(\lambda)m$ since $\delta(m)\in M$ and $\lambda\in\mathrm{ann}(M,\delta)$. This show that $d(\lambda)m=0$, and hence $d(\lambda)\in\mathrm{ann}(M,\delta)$. ■

Definition 4.26. Let (Λ,d) be a differential graded algebra.

- Then the *differential graded left radical* $\mathrm{dgrad}_\ell(\Lambda,d)$ is the intersection of all dg-maximal differential graded left ideals of (Λ,d).
- Likewise the *differential graded right radical* $\mathrm{dgrad}_r(\Lambda,d)$ is the intersection of all dg-maximal differential graded right ideals of (Λ,d).
- The *differential graded twosided radical* $\mathrm{dgrad}_2(\Lambda,d)$ is the intersection of all of the annihilators of dg-simple differential graded left modules.

Lemma 4.27. *(differential graded Nakayama's Lemma) Let* (Λ,d) *be a differential graded algebra. Let* M *and* N *be differential graded* Λ*-modules such that* $N\leq M$. *Assume that* (M,δ) *is finitely generated as a dg-module. Then* $\mathrm{dgrad}_2(\Lambda)M=M$ *implies* $M=0$, *and* $N+\mathrm{dgrad}_2(\Lambda)M=M$ *implies* $M=N$.

Proof. We suppose that (M,δ) is finitely generated as a dg-module. Again, in case $M=0$ the statement is trivial. By Lemma 4.11 the dg-module (M,δ) contains a proper maximal dg-submodule (D,δ) and $(M/D,\overline{\delta})$ is a non trivial dg-simple dg-module. By definition of $\mathrm{dgrad}_2(\Lambda,d)$ we get $\mathrm{dgrad}_2(\Lambda,d)\cdot M/D=0$, whence $\mathrm{dgrad}_2(\Lambda,d)\cdot M\subseteq D$. This proves that only $M=0$ satisfies $\mathrm{dgrad}_2(\Lambda,d)\cdot M=M$. As for the second part, $N+\mathrm{dgrad}_2(\Lambda)M=M$ implies $\mathrm{dgrad}_2(\Lambda)(M/N)=M/N$. The first statement then implies $M/N=0$, and therefore $M=N$. Hence the result follows. ■

Remark 4.28. The statement of the dg-Nakayama Lemma for finitely generated dg-modules M was suggested by the referee. I am very grateful for this suggestion.

Lemma 4.29. *A dg-module is dg-Noetherian if and only if all its dg-submodules are finitely generated as dg-modules.*

Proof. The classical proof of this fact transposes to the dg-case, after slight modifications. Let us verify the details.

If (M,δ) is dg-Noetherian, and the dg-submodule (N,δ) is not finitely generated, then take a non zero $g_1\in\ker(\delta)\cap N$. Such an element exists since if $g_1\in N\setminus\ker(\delta)$, then $\delta(g_1)\in\ker(\delta)\cap N$. Using Lemma 4.12 we get that $A\cdot g_1$ is a dg-submodule of N. Suppose we constructed $g_1,\cdots,g_{n-1}$ such that $\sum_{i=1}^j Ag_i$ is a dg-submodule for all j and $\sum_{i=1}^j Ag_i \subsetneq \sum_{i=1}^{j+1} Ag_i$ for all j. Since (N,δ) is not finitely generated, there is $g_n\in N\setminus\sum_{i=1}^{n-1}Ag_i$. Consider the dg-module $(N_1,\overline{\delta}) := (N/\sum_{i=1}^{n-1}Ag_i,\overline{\delta})$. If $\overline{\delta}(g_n+\sum_{i=1}^{n-1}Ag_i)\neq 0$, then replace g_n by $\delta(g_n)$, which still is not in $\sum_{i=1}^{n-1}Ag_i$, and the image in $N/\sum_{i=1}^{n-1}Ag_i$ of this replaced element is in $\ker(\overline{\delta})$. If $\overline{\delta}(g_n+\sum_{i=1}^{n-1}Ag_i)=0$ there is $h_n\in\sum_{i=1}^{n-1}Ag_i$ with $\delta(g_n)=h_n$. But then $\sum_{i=1}^n Ag_i$ is a dg-submodule of N. By induction we obtain a strictly increasing sequence of dg-submodules

$$Ag_1 \subsetneq Ag_1+Ag_2 \subsetneq Ag_1+Ag_2+Ag_3 \subsetneq \cdots$$

of $(N,\delta) \subseteq (M,\delta)$. This contradicts the hypothesis that (M,δ) is dg-Noetherian.

If all dg-submodules of (M,δ) are finitely generated, then let

$$N_1 \subseteq N_2 \subseteq N_3 \subseteq \cdots \subseteq M$$

be a deceasing sequence of dg-submodules. Put $L := \bigcup_{i=1}^{\infty} N_i$. This is a dg-submodule of M, and hence is finitely generated, by $\{g_1,\dots,g_n\}$ say, by hypothesis. For each $i \in \{1,\dots,n\}$ there is $j(i)$ such that $g_i \in N_{j(i)}$. But then, for $m = \max(j(1),\dots,j(n))$ we have $\{g_1,\dots,g_n\} \subseteq N_m$, and therefore $N_m = N_{m+k}$ for all $k \geq 0$. Therefore M is dg-Noetherian. ■

Corollary 4.30. *Let M and N be differential graded Λ-modules such that $N \leq M$. If M is dg-Noetherian, then* $\mathrm{dgrad}_2(\Lambda)M = M$ *implies $M = 0$, and* $N + \mathrm{dgrad}_2(\Lambda)M = M$ *implies $M = N$.*

Proof. By Lemma 4.29 the hypothesis of Lemma 4.27 are satisfied. ■

Remark 4.31. Note that we need Zorn's lemma in the proof of Lemma 4.27. If M is dg-Noetherian and dg-artinian, then a proof without Zorn's lemma is possible by a short induction on the composition length.

If $M = 0$, the (first) statement is trivial. Else, the hypothesis on M implies that there is a dg-simple dg-module, namely a dg-simple dg-submodule of M. If $\mathrm{dgrad}_2(\Lambda)M = M$, then comparing the composition lengths, one needs to have $M = 0$.

Remark 4.32. There is a parallel work on Nakayama's Lemma for differential graded algebras by Goodbody [9], based on the work of Orlov [17]. These results are very different from ours, and in particular Goodbody uses that the algebra is finite dimensional over some field. I discovered both articles after having submitted this manuscript.

Proposition 4.33. *Let (Λ,d) be a differential graded algebra, and suppose that (Λ,d) is artinian as differential graded left and as a differential graded right module. Then*

$$\mathrm{dgrad}_\ell(\Lambda,d) \cap \mathrm{dgrad}_r(\Lambda,d) \supseteq \mathrm{dgrad}_2(\Lambda,d).$$

Proof. We shall prove that $\mathrm{dgrad}_2(\Lambda,d) \subseteq \mathrm{dgrad}_\ell(\Lambda,d)$. The analogous statement for $\mathrm{dgrad}_r(\Lambda,d)$ is symmetric. By Corollary 4.18 every simple differential graded (Λ,d)-module is isomorphic to one of the form $(\Lambda/M,\overline{d})$ for a maximal differential graded ideal (M,d) of (Λ,d). By definition of $\mathrm{dgrad}_2(\Lambda,d)$ the dg-simple dg-module $(\Lambda/M,\overline{d})$ is annihilated by $\mathrm{dgrad}_2(\Lambda,d)$. Hence $\mathrm{dgrad}_2(\Lambda,d) \subseteq M$. This shows $\mathrm{dgrad}_2(\Lambda,d) \subseteq \mathrm{dgrad}_\ell(\Lambda,d)$. This shows the proposition. ■

Remark 4.34. The hypothesis that (Λ,d) is dg-artinian is only used to show that there are dg-simple dg-left modules. Just like in the classical situation, an easy use of Zorn's lemma shows that it is sufficient to ask that (Λ,d) is finitely generated as dg-left module.

Example 4.35. Let K be a field. Recall from Example 3.6 that the graded endomorphism algebra $\begin{pmatrix} K & K \\ K & K \end{pmatrix}$ of the complex $\mathrm{cone}(\mathrm{id}_K)$ is a dg-algebra. It is not difficult to verify that $\begin{pmatrix} 0 & K \\ 0 & K \end{pmatrix} =: I$ is the only non trivial differential graded left ideal of A. Hence $\mathrm{dgrad}_\ell(A) = I$ and likewise the first row is the only non trivial differential graded right ideal. Hence $\mathrm{dgrad}_\ell(A) \cap \mathrm{dgrad}_r(A) = \begin{pmatrix} 0 & K \\ 0 & 0 \end{pmatrix}$ is not an ideal at all. Further it strictly contains $\mathrm{dgrad}_2(A) = 0$.

Proposition 4.36. *Let (Λ,d) be a differential graded algebra. Suppose that (Λ,d) is dg-Noetherian and dg-artinian, i.e. Noetherian and artinian as differential graded left module over itself. Then $\Lambda/\mathrm{dgrad}_\ell(\Lambda)$ is a direct sum of dg-simple dg-modules over (Λ,d). In particular,* $\mathrm{dgrad}_\ell(\Lambda) = 0$ *if and only if any finitely generated differential graded (Λ,d)-module is isomorphic to a direct sum of dg-simple differential graded (Λ,d)-modules.*

Proof. In general there are infinitely many dg-maximal dg-left ideals. We shall show that there are only finitely many dg-maximal differential graded left ideals $I_1,\dots,I_n$ of Λ such that $I_j \not\supseteq \bigcap_{i=1}^{j-1} I_i$ for all j. Take a maximal differential graded left ideal (I_1,d). If there is no other maximal differential graded left ideal (I_2,d), we are done. Else we consider $(I_1\cap I_2,d)$. If there is no maximal differential graded left ideal I_3 with $(I_3,d)\not\supseteq I_1\cap I_2$, we are done. Else we consider $(I_1\cap I_2\cap I_3,d)$. Inductively we obtain a chain of strictly decreasing differential graded left ideals. Since Λ is assumed to be dg-artinian, this chain is finite. Hence for any set of maximal differential graded ideals there is a finite subset of dg-maximal differential graded ideals $I_1,\dots,I_n$ such that $\bigcap_{i=1}^n I_i$ is contained in any dg-maximal differential graded left ideal of (Λ,d). Hence

$$I_1\cap\dots\cap I_n = \mathrm{dgrad}_\ell(\Lambda) = \bigcap_{I \text{ dg-maximal dg-ideal}} I.$$

Then there is a homomorphism

$$\Lambda \xrightarrow{\rho} \Lambda/I_1\times\dots\times\Lambda/I_n$$

with kernel $\mathrm{dgrad}_\ell(\Lambda)$. Hence ρ induces a monomorphism

$$\Lambda/\mathrm{dgrad}_\ell(\Lambda) \overset{\overline{\rho}}{\hookrightarrow} \Lambda/I_1\times\dots\times\Lambda/I_n.$$

Lemma 4.20 then shows that there is a differential graded module M such that

$$\Lambda/\mathrm{dgrad}_\ell(\Lambda)\oplus M \simeq \Lambda/I_1\times\dots\times\Lambda/I_n.$$

Since Λ is dg-Noetherian and dg-artinian, the Krull-Schmidt theorem applies and hence, renumbering if necessary, there is k such that

$$\Lambda/\mathrm{dgrad}_\ell(\Lambda) \simeq \Lambda/I_1\times\dots\times\Lambda/I_k \text{ and } M\simeq \Lambda/I_{k+1}\times\dots\times\Lambda/I_n.$$

However, by construction of the sequence of ideals $I_1,\dots,I_n$ we have $k=n$. Hence $\overline{\rho}$ is an isomorphism.

If Λ is a direct sum of dg-simple differential graded left modules, $\Lambda = S_1\times\dots\times S_n$, and hence a maximal dg-ideal is $S_1\times\dots\times S_{i-1}\times S_{i+1}\times\dots S_n$. Therefore the intersection of all dg-maximal dg-ideals is 0.

Let (X,δ) be a finitely generated differential graded (Λ,d)-module. Then there is an epimorphism $\bigoplus_{j=1}^n \Lambda[n_j]\longrightarrow X$. By Lemma 4.20, the kernel is a direct factor of $\bigoplus_{j=1}^n \Lambda[n_j]$ and this shows the statement.

This shows the proposition. ∎

We shall now characterise $\mathrm{dgrad}_2(A,d)$ by universal properties. For this purpose we need to recall some classical concepts.

Recall that an algebra A is called left primitive if A allows a faithful simple left A-module. Simple rings allowing a simple module are trivially primitive, and the converse is true for artinian algebras. In general primitive algebras are not simple. The standard example is the endomorphism algebra (as a vector space) of a vector space of countable dimension. The set of endomorphisms with finite dimensional image is a twosided ideal. However, the endomorphism algebra is primitive. We further mention that in general a simple (hence primitive) algebra can have many non isomorphic simple modules. An example is the Weyl algebra over an algebraically closed field of characteristic 0 (see Block [2, 3]). There are algebras which are left primitive, but not right primitive. Since these properties hold for ordinary algebras, and since an algebra is a dg-algebra with trivial grading and differential 0, we cannot expect better properties than these in the dg-version.

Recall further that for algebras A_i indexed by elements of a set $i\in I$, a subdirect product is defined to be a subalgebra S of $\prod_{i\in I} A_i$ such that the composition of the injection of S into the product, followed by the projection onto A_i is onto for any $i\in I$. In particular, the intersection of the kernels of these composition of injection and projections is 0 in S.

We come to the straight forward corresponding dg-concept.

Definition 4.37. A dg-algebra (A,d) is called *dg-primitive* if there is a faithful dg-simple differential graded left module (S,δ) over (A,d). A twosided dg-ideal (I,d) of (A,d) is called *left dg-primitive* if $(A/I,\overline{d})$ is a left dg-primitive algebra

Lemma 4.38. *Let (Λ,d) be a differential graded algebra and let (S,δ) be a dg-simple differential graded module. Then,* $\mathrm{ann}(S,\delta)$ *is a dg-primitive differential graded twosided ideal.*

Proof. The fact that $\mathrm{ann}(S,\delta)$ is a twosided dg-ideal was already established. Since (S,δ) is a simple (Λ,d)-module, it is still a simple $(\Lambda/\mathrm{ann}(S,\delta))$-module. By definition, it is a faithful $(\Lambda/\mathrm{ann}(S,\delta))$-module. ■

Remark 4.39. Note that we always have $\mathrm{ann}(M,\delta)=\mathrm{ann}((M,\delta)[1])$, and if $(M,\delta_M)\simeq(N,\delta_N)$, then $\mathrm{ann}(M,\delta_M)=\mathrm{ann}(N,\delta_N)$.

Proposition 4.40. *Let (Λ,d) be a differential graded algebra and suppose that (Λ,d) allows a dg-simple dg-module. Then* $\mathrm{dgrad}_2(\Lambda,d)$ *is a twosided differential graded ideal. Then* $\mathrm{dgrad}_2(\Lambda,d)$ *is the smallest twosided differential graded ideal I such that $(\Lambda/I,\overline{d})$ is a subdirect product of dg-primitive differential graded algebras.*

Proof. The fact that $\mathrm{dgrad}_2(\Lambda,d)$ is a twosided dg-ideal follows from Lemma 4.38. By definition there is a dg-simple dg-module, and

$$\mathrm{dgrad}_2(\Lambda,d)=\bigcap_{(S,\delta)\text{ dg-simple dg module}}\mathrm{ann}(S,\delta)$$

is an intersection over a non empty index set. We have a canonical dg-ring homomorphism

$$\Lambda\longrightarrow\prod_{\substack{(S,\delta)\text{ representative of}\\ \text{an isoclass of dg-simple}\\ \text{dg-module up to shift}}}\Lambda/\mathrm{ann}(S,\delta).$$

Composing this map with the projection onto $\Lambda/\mathrm{ann}(S,\delta)$ is the natural projection $\Lambda\longrightarrow\Lambda/\mathrm{ann}(S,\delta)$, which clearly is surjective. Further, the kernel is precisely $\mathrm{dgrad}_2(\Lambda,d)$. Hence, we obtain that $(\Lambda/\mathrm{dgrad}_2(\Lambda,d),\overline{d})$ is a subdirect product of dg-primitive dg-algebras.

Let (I,d) be a twosided dg-ideal such that $(\Lambda/I,\overline{d})$ is a subdirect product of dg-primitive dg-algebras. Then the projection onto each of the direct factors is a dg-primitive dg-algebra quotient, yields hence a dg-simple faithful dg-module. Hence, its annihilator contains I. As I is a subdirect product of all these dg-primitive algebras, it is contained in the intersection of all the annihilators of the corresponding dg-simples. Hence, $\mathrm{dgrad}_2(\Lambda,d)\subseteq I$. ■

Remark 4.41. Recall that an artinian primitive algebra is simple. One might ask if similar properties hold in the differential graded case. One major difficulty is that we do not have a Wedderburn Artin theorem in the dg-case. Our favorite example $(Mat_{2\times 2}(K,d_1)$ provides a striking observation. The dg-algebra has only one dg-simple ideal, namely the right hand column. This is a faithful dg-simple dg-module, and hence the algebra is dg-primitive. Since it is simple, it is also dg-simple (and therefore also dg-primitive, giving a second argument). However, the left regular dg-module is not a direct sum of dg-simples. Actually, the dg-left module structure is one of the type studied in Remark 4.24. Nevertheless, the algebra is acyclic.

5. Differential Graded Orders

Let R be a Dedekind domain with field of fractions K. Recall that K is a flat R-module.

Definition 5.1. A *differential graded R-order (dg-R-order)* is a differential graded R-algebra (Λ,d) such that Λ is finitely generated R-projective and such that $K\otimes_R A$ is semisimple artinian as an algebra. A differential graded R-order is a *proper differential graded R-order* if in addition $H(\Lambda)$ is finitely generated R-projective,

Definition 5.2. Let (Λ, d) be a dg-R-order. A *differential graded Λ-lattice (dg-R-lattice, or dg-lattice)* is a differential graded Λ-module (L, d_L) such that L is R-projective. A dg-lattice (L, d_L) is a *proper dg-lattice* if in addition $H(L)$ is R-projective.

Remark 5.3. Note that if (Λ, d) is a dg-order, then a dg-lattice (L, d_L) can be proper or not. If (Λ, d) is a proper dg-order, then a dg-lattice (L, d_L) can be proper or not.

Remark 5.4. Let R be a Dedekind domain with field of fractions K. Following Theorem 2.1 we may as well consider differential graded K-algebras (A, d) which are acyclic and $\ker(d)$ semisimple. Then we may define an R-order as a differential graded R-subalgebra (Λ, d) projective as an R-module with $K \otimes_R \Lambda = A$. This will give a different definition, implying that the homology is necessarily R-torsion. Further, the fact that the algebra A is very general does not seem to allow a rich theory. Moreover, as a classical order is not a dg-order with differential 0 in this case, this alternative theory will not generalize the classical theory of lattices over orders, and in particular it is unclear how one might recover in a profitable way complexes of lattices over a classical order in any way as differential graded module over a dg-order in this alternative definition.

Example 5.5. We recall some examples of classical orders and explain differential graded structures on them.

(1) For R a complete discrete valuation ring containing $\widehat{\mathbb{Z}}_3$ the 3-adic integers, following [26, Example 1.2.43.4] the group ring $R\mathfrak{S}_3$ of the symmetric group of order 6 is isomorphic to

$$\{(d_1, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}, a_3) \in R \times Mat_{2\times 2}(R) \times R \mid a_2 - d_1 \in 3R, c_2 \in 3R, a_3 - d_2 \in 3R\}.$$

This algebra is differential graded with the grading given by $deg(a_i) = deg(d_i) = 0$, $deg(c_2) = -1$ and $deg(b_2) = 1$ for all i, using the designation of the variables as in the above definition. The differential is given as in Example 3.6 by d_x for any fixed $x \in R$. More precisely

$$d_x((d_1, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}, a_3)) = (0, \begin{pmatrix} xc_2 & x(d_2 - a_2) \\ 0 & xc_2 \end{pmatrix}, 0).$$

We can even consider suborders, such as those given by asking in addition that $a_2 - d_2 \in 3R$.

(2) Fix a prime $p > 2$. By a result due to Roggenkamp [21] we know that the principal block of $R\mathfrak{S}_p$, for R being a complete discrete valuation ring containing $\widehat{\mathbb{Z}}_p$, the p-adic integers and $\mathfrak{S}_p$ the symmetric group of degree p, is Morita equivalent to

$$\{(d_1, \prod_{i=2}^{p-1} \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}, a_p) \in R \times \prod_{i=2}^{p-1} Mat_{2\times 2}(R) \times R \mid a_{j+1} - d_j \in pR, c_j \in pR, \forall j\}.$$

Again, following Example 3.6, for every of the 2×2 matrix algebras in the product we may impose a differential graded algebra structure with differential d_{x_i} by choosing $x_2, \ldots, x_{p-1} \in R$.

(3) Fix a prime $p > 2$ and let R be a complete discrete valuation ring containing $\widehat{Z}_p$, the p-adic integers. By a result due to König [13] the principal block of the Schur algebra $S_R(p, p) = End_{R\mathfrak{S}_p}((R^p)^{\otimes p})$ with parameters (p, p) is known to be Morita equivalent to an algebra of similar shape.

$$\{(\prod_{i=1}^{p-1} \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}, a_p) \in \prod_{i=1}^{p-1} Mat_{2\times 2}(R) \times R \mid a_{j+1} - d_j \in pR, c_j \in pR, \forall j\}.$$

The same statement on differential graded structures holds.

(4) Fix a prime p. By [24] ([7, 8] for the case $p=2$ and $p=3$) the category $\mathcal{F}^p_{\mathbb{Z}}$ of polynomial functors of degree at most p from finitely generated free abelian groups to finitely generated $\widehat{\mathbb{Z}}_p$-modules is equivalent to a module category, having a 'principal block' and a number of trivial direct factors. The principal block is Morita equivalent to

$$\{(d_0,\prod_{i=1}^{p-1}\left(\begin{array}{cc} a_i & b_i\\ c_i & d_i\end{array}\right),a_p)\in\widehat{\mathbb{Z}}_p\times\prod_{i=1}^{p-1}Mat_{2\times 2}(\widehat{\mathbb{Z}}_p)\times\widehat{\mathbb{Z}}_p\mid a_{j+1}-d_j\in p\widehat{\mathbb{Z}}_p, c_j\in p\widehat{\mathbb{Z}}_p,\forall j\}.$$

Again, following Example 3.6, for every of the 2×2 matrix algebras in the product we may impose a differential graded algebra structure with differential d_{x_i} by choosing $x_1,\dots,x_{p-1}\in p\widehat{\mathbb{Z}}_p$.

Remark 5.6. Le Bruyn et al studied in [15, Section II.4] graded orders over Krull domains in a very general setting. They define for a Krull domain R with field of fractions K and a central simple K-algebra A a subring Λ with $R\subset\Lambda\subset A$ an R-order if each element a of Λ is integral over R, and if $K\Lambda=A$. If a is integral over R, then for the reduced trace $Tr:A\to K$ one has $Tr(a)\in R$ and Tr induces an isomorphism $A\to Hom_K(A,K)$ by $A\ni a\mapsto(A\ni b\mapsto Tr(ab)\in K)\in Hom_K(A,K)$.

Recall in this context the classical result

Theorem 5.7. *(cf. e.g.* [19], [26, Theorem 8.3.7]*) Let R be a Dedekind domain of characteristic* 0 *with field of fractions K and let Λ be an R-subalgebra of the semisimple K-algebra A, containing a K-basis of A. Then Λ is an R-order if and only if every element of Λ is integral over R.*

Hence the definition of [15, Section II.4] coincides with the classical definition of an R-order in case of a Dedekind domain R.

Recall that a dg-module (V,δ) with $\delta(V_i)\subseteq V_{i+1}$ for all i is *right bounded* if there is $n_0\in\mathbb{N}$ such that $V_n=0$ for all $n>n_0$. Analogously we define *left bounded* dg-modules. A dg-module is *bounded* if it is at once left and right bounded. Note that for a differential graded R-order (Λ,d) in a finite dimensional semisimple differential graded K-algebra (A,d) the lattice (Λ,d) is always bounded. Indeed, since (A,d) is finite dimensional, (A,d) is bounded, and hence so is (Λ,d).

Lemma 5.8. *Let R be a Dedekind domain with field of fractions K. Let (Λ,d) be a differential graded R-order in a finite dimensional differential graded K-algebra (A,d) and let (V,δ) be finite dimensional differential graded (hence bounded) (A,d)-module. Then there is a differential graded (Λ,d)-lattice (L,δ) in (V,d) such that $K\otimes_R L=V$.*

Proof. Recall that classically for an R-order Λ and a $K\Lambda$-module V there is always a full Λ-lattice L in V. Consider $\Lambda^+:=\bigoplus_{k\geq 1}\Lambda_k$ and $\Lambda_0^+:=\Lambda_0\oplus\Lambda^+$. Now, Λ_0^+ is a (differential graded) subalgebra of Λ and Λ^+ is a dg-ideal of Λ_0^+.

Let us perform the differential graded construction. We construct (L,δ) by downward induction on the degree. Let $V_k=0\neq V_{n-1}$ for $k\geq n$ and take any Λ_0^+-lattice in V_{n-1}. Its existence is a well-known classical property for lattices. Note that Λ^+ acts as 0. Hence this is automatically a (Λ_0^+,d)-lattice since $\delta(V_{n-1})=0$ and hence the Leibniz formula automatically holds.

Let $\widetilde{L}_{n-2}:=\delta^{-1}(L_{n-1})\subseteq V_{n-2}$ and choose a full Λ_0-lattice $\widehat{L}_{n-2}$ in $\widetilde{L}_{n-2}$. Then choose an $r\in R\setminus\{0\}$ such that $r\cdot\Lambda^+\cdot\widehat{L}_{n-2}\subseteq L_{n-1}$. This is possible since $\Lambda^+\cdot\widehat{L}_{n-2}$ is a Λ_0^+-lattice. Put $L_{n-2}:=r\cdot\widehat{L}_{n-2}$. Then by construction $\delta(L_{n-2})\subseteq L_{n-1}$ and $L_{n-2}\oplus L_{n-1}$ is a Λ_0^+-lattice, and moreover the Leibniz formula holds since it holds for the (A,d)-action on (V,δ). We suppose having constructed $L_k,L_{k+1},\dots,L_n$. Then let $\delta^{-1}(L_k)=:\widetilde{L}_{k-1}$ and choose a Λ_0-lattice $\widehat{L}_{k-1}$ in $\widetilde{L}_{k-1}$. Then there is again an $r \in R \setminus \{0\}$ such that

$$r \cdot \Lambda^+ \cdot \widehat{L}_{k-1} \subseteq \bigoplus_{\ell=k}^{n} L_\ell.$$

Put $L_{k-1} := r \cdot \widehat{L}_{k-1}$. Again, $\bigoplus_{\ell=k-1}^{n} L_\ell$ this is a full differential graded Λ_0^+-lattice. Since (V, δ) is bounded, after a finite number of steps we constructed a Noetherian differential graded Λ_0^+-lattice (L, δ) as $\bigoplus_{k\in\mathbb{Z}} L_k$.

Now, since Λ is Noetherian as well, $\Lambda \cdot L$ is Noetherian again, and since it is a submodule of V, it is R-torsion free. Hence $\Lambda \cdot L$ is a lattice in (V, δ). Moreover, the Leibniz formula holds. Since

$$\delta(\lambda \cdot x) = d(\lambda) \cdot x \pm \lambda \cdot \delta(x) \in \Lambda \cdot L$$

for homogeneous elements $\lambda \in \Lambda$ and $x \in L$. This shows that

$$\delta(\Lambda \cdot L) \subseteq \Lambda \cdot L$$

is a full differential graded (Λ, d)-lattice in (V, d). ■

Remark 5.9. Note that we do not really need Λ_0^+. It is possible to first find a full and finitely generated R-submodule $\check{L}$ of V, stable under δ, and then consider $\Lambda \cdot \check{L}$. Noetherianity and the last argument of the above proof then provides the result. However, the above proof of Lemma 5.8 gives a more direct construction for coconnective dg-algebras. A similar construction can be given for connective dg-algebras.

In the classical theory of orders the following result is a main tool. We shall need to transpose it to the differential graded situation.

Proposition 5.10. *Let R be a Dedekind domain with field of fractions K. For each $\wp \in \mathrm{Spec}(R)$ and any R-module M denote $M_\wp := R_\wp \otimes_R M$ and $\mathrm{id}_{R_\wp} \otimes d := d_\wp$ for any map d, in particular the differential.*

1. *If (Λ, d_Λ) is a (resp. proper) dg-R-order in the semisimple dg-K-algebra (A, d_A), then $(\Lambda_\wp, 1_{R_\wp} \otimes d_\Lambda)$ is a (resp. proper) dg-$R_\wp$-order in the semisimple dg-K-algebra (A, d_A).*
2. *If (L, d_L) is a (resp. proper) dg-R-lattice, then $(L_\wp, (d_L)_\wp)$ is a (resp. proper) dg-$(\Lambda_\wp, (d_\Lambda)_\wp)$-lattice.*
3. *For any (resp. proper) dg-(Λ, d_Λ)-lattice (L, d_L) we have*
$$L = \bigcap_{\wp\in\mathrm{Spec}(R)} (L_\wp, (d_L)_\wp)$$
where the intersection is taken inside $K \otimes_R L$.
4. *Fix a dg-(A, d_A)-module (V, d_V). For each $\wp \in \mathrm{Spec}(R)$ fix (resp. proper) $(\Lambda_\wp, (d_\Lambda)_\wp)$-lattices $(M(\wp), (d_M(\wp)))$ such that $(KM(\wp), (Kd_M(\wp))) = (V, d_V)$ for all $\wp \in \mathrm{Spec}(R)$. Suppose moreover that there is a (resp. proper) dg-(Λ, d_Λ)-lattice (N, d_N) such that $(N_\wp, (d_N)_\wp) = (M(\wp), d_M(\wp))$ for all but a finite number of $\wp \in \mathrm{Spec}(R)$. Then there is a (resp. proper) dg-(Λ, d_Λ)-lattice (L, d_L) with $(L_\wp, (d_L)_\wp) = (M(\wp), d_M(\wp))$ for all $\wp \in \mathrm{Spec}(R)$.*
5. *The analogous statements hold replacing the localisation by the completion.*

Proof. Item (1) follows from the classical non dg-statement (cf e.g. [26, Proposition 8.1.14]).

For item (2) we need to see that $H(L_\wp, (d_L)_\wp)$ is $R_\wp$-torsion free if $H(L, d_L)$ is R-torsion free. But this follows from the fact that localisation is flat (cf e.g. [26, Lemma 6.5.7]).

Item (3) is again a direct consequence of the classical non dg-statement (cf e.g. [26, Proposition 8.1.14]).

As for item (4) we first get a lattice L again from the classical situation (cf e.g. [26, Proposition 8.1.14]). The differential is fixed as the restriction of the differential d_V on L. We need to verify that $d_V(L) \subseteq L$. But this is true at every prime, i.e.

$$d_V(L_\wp) = d_M(\wp)(M(\wp)) \subseteq M(\wp) = L_\wp.$$

Hence by the non dg-version of item (3) we have $d_V(L) \subseteq L$.

As for item (5) we first see that the non dg-version is again classical (cf e.g. [26, Proposition 8.1.14]). The analogous of the second statement first uses the localisation, then the uniqueness and existence of the differential follows by continuity. The third item is clear by the localisation case. As for the fourth item we first restrict from the completion to A. Then, in a second step we use continuity again. ■

An important tool in case of lattices for orders is the conductor. Recall that given a Dedekind domain R with field of fractions K and a finite dimensional semisimple K-algebra A, then for an R-lattice L in A with $KL = A$ we define

$$\mathcal{O}_\ell(L) := \{\lambda \in A \mid \lambda L \subseteq L\} \text{ and } \mathcal{O}_r(L) := \{\lambda \in A \mid L\lambda \subseteq L\}.$$

Then by [26, Lemma 8.3.14] we get that $\mathcal{O}_\ell(L)$ and $\mathcal{O}_r(L)$ are R-orders in A.

We can show a dg-version of this lemma.

Lemma 5.11. *Let R be a Dedekind domain with field of fractions K and let (A, d) be a differential graded K-algebra (A, d), semisimple as an algebra. Let (L, d_L) be a differential graded R-lattice in (A, d) such that $K\cdot(L, d_L) = (A, d)$. Then $(\mathcal{O}_\ell(L), d|_{\mathcal{O}_\ell(L)})$ is a differential graded R-order in (A, d). Similar statements hold for $\mathcal{O}_r(L)$.*

Proof. By symmetry it is enough to consider the case $(\mathcal{O}_\ell(L), d|_{\mathcal{O}_\ell(L)})$. By [26, Lemma 8.3.14] we see that $\mathcal{O}_\ell(L)$ is an R-order in A. If L is graded, then also $\mathcal{O}_\ell(L)$ is graded. We need to see that it is differential graded. Let $\lambda \in \mathcal{O}_\ell(L)$ be homogeneous. Then $\lambda L \subseteq L$. We need to see that $d(\lambda)L \subseteq L$. Since $L \subseteq A$, and since (A, d) is a differential graded algebra, we have for any $x \in L$ and homogeneous $\lambda \in \mathcal{O}_\ell(L)$

$$d(\lambda x) = d(\lambda)x + (-1)^{|\lambda|}\lambda d(x)$$

Hence

$$d(\lambda)x = d(\lambda x) - (-1)^{|\lambda|}\lambda d(x)$$

Since $\lambda \in \mathcal{O}_\ell(L)$ and since $d(L) \subseteq L$, using that (L, d) is a dg-lattice, we have $\lambda d(x) \in L$. Since $\lambda \in \mathcal{O}_\ell(L)$, and since $x \in L$, we get $\lambda x \in L$, and hence $d(\lambda x) \in L$ again. The differential on $\mathcal{O}_\ell(L)$ is the restriction of the differential d of A to $\mathcal{O}_\ell(L)$, and hence the defining equation on products holds. Hence $(\mathcal{O}_\ell(L), d|_{\mathcal{O}_\ell(L)})$ is a dg-order in (A, d). ■

Remark 5.12. If the category of dg-modules over (A, d) is semisimple, then by [1] we have that $H(A, d) = 0$ and therefore any order in (A, d) is either acyclic or has R-torsion homology. In particular, in this case, if the conductor is not acyclic, then $(\mathcal{O}_\ell(L), d|_{\mathcal{O}_\ell(L)})$ cannot be a proper differential graded R-order in (A, d), whatever the choice of a dg-lattice L may be.

6. On maximal differential graded orders

Let R be a Dedekind domain with field of fractions K. It is a classical fact that for any semisimple K-algebra A and an R-order Λ in A there is a maximal R-order Γ containing Λ. Maximal orders have many striking properties, and behave very much as the base ring R (cf [19]). We shall consider maximal dg-orders.

Theorem 6.1. *Let R be a Dedekind domain with field of fractions K of characteristic 0 and let (Λ, d) be a differential graded R-order in the semisimple finite dimensional differential graded K-algebra (A, d). Then there is a differential graded R-order (Γ, d), which is maximal with respect to being a dg-order and containing (Λ, d). If (Λ, d) is a proper dg-order, then* *there is a proper differential graded R-order* (Γ_p, d) *which is maximal with respect to being a proper dg-order and containing* (Λ, d).

Proof. The set of dg-orders in (A, d) is partially ordered. We need to make precise the partial ordering we consider. Let (Λ_1, d_1) and (Λ_2, d_2) be dg-R-orders in a semisimple dg-K-algebra (A, d). Then $(\Lambda_1, d_1) \leq (\Lambda_2, d_2)$ if $\Lambda_1 \subseteq \Lambda_2$. Since $d_1 = d|_{\Lambda_1}$ and $d_2 = d|_{\Lambda_2}$, we get that then automatically $d_1 = d_2|_{\Lambda_1}$.

Consider

$$\mathcal{X} := \{(\Sigma, d) \mid (\Lambda, d) \leq (\Sigma, d) \text{ and } (\Sigma, d) \text{ is a dg-}R\text{-order in } (A, d)\}.$$

Since $(\Lambda, d) \in \mathcal{X}$, this set $\mathcal{X}$ is not empty. Let $\mathcal{Y}$ be a totally ordered subset in $\mathcal{X}$ and put $\Gamma := \bigcup_{(\Delta,d)\in\mathcal{Y}} \Delta$. It is graded since the embedding of one order in the other in $\mathcal{Y}$ preserves the grading, and hence so is the union. It allows a differential by $d|_\Gamma$ and if $\gamma \in \Gamma$, then $\gamma \in \Delta$ for some $\Delta \in \mathcal{Y}$, and therefore $d(\gamma) \in \Delta \subseteq \Gamma$. Moreover, since $\gamma \in \Delta$, and since Δ is an order, Theorem 5.7 shows that γ is integral, and, using Theorem 5.7 again, Γ is an order in A. Hence $(\Gamma, d) \in \mathcal{X}$ and (Γ, d) dominates any element in $\mathcal{Y}$. By Zorn's lemma there are maximal elements in $\mathcal{X}$. Any such element is a maximal differential graded order containing (Λ, d).

Now consider

$$\widehat{\mathcal{X}} := \{(\Sigma, d) \mid (\Lambda, d) \leq (\Sigma, d) \text{ and } (\Sigma, d) \text{ is a proper dg-}R\text{-order in } (A, d) \text{ and } \Sigma\}$$

and if (Λ, d) is a proper dg-R-order in (A, d), then $(\Lambda, d) \in \widehat{\mathcal{X}}$. Hence this set is not empty neither. The first steps are as above. Let $\widehat{\mathcal{Y}}$ be a totally ordered subset in $\widehat{\mathcal{X}}$ and put $\Gamma := \bigcup_{(\Delta,d)\in\widehat{\mathcal{Y}}} \Delta$. It is graded since the embedding of one order in the other in $\mathcal{Y}$ preserves the grading, and hence so is the union. It allows a differential by $d|_\Gamma$ and if $\gamma \in \Gamma$, then $\gamma \in \Delta$ for some $\Delta \in \widehat{\mathcal{Y}}$. Hence γ is integral, and therefore Γ is an order in A. Since R is Noetherian (cf [26, Lemma 7.5.3]), Γ is Noetherian as well, and therefore Γ actually in $\widehat{\mathcal{Y}}$. Therefore (Γ, d) is a proper dg-order. By Zorn's lemma, $\widehat{\mathcal{X}}$ contains maximal elements. This proves the statement. ∎

We call an R-order, which is maximal with respect to being a dg-order in a fixed dg-algebra, a dg-maximal dg-order.

Corollary 6.2. *Let R be a Dedekind domain with field of fractions K, let* (A, d) *be a finite dimensional semisimple differential graded K-algebra, and let* Λ *be a maximal R-order in A and suppose that* $d(\Lambda) \subseteq \Lambda$. *Then* $(\Lambda, d|_\Lambda)$ *is a dg-maximal differential graded R-order in* (A, d).

Proof. The hypotheses imply that $(\Lambda, d|_\Lambda)$ is a differential R-graded order. If $(\Gamma, d|_\Gamma)$ is a differential graded order containing $(\Lambda, d|_\Lambda)$, then Γ is an R-order containing Λ. Hence, since Λ is a dg-maximal order, $\Lambda = \Gamma$ and therefore $(\Lambda, d|_\Lambda)$ is a dg-maximal differential R-graded order. This shows the statement. ∎

Remark 6.3. Note that in the situation of Corollary 6.2 there is no reason why $H(\Lambda, d_\Lambda)$ should be R-projective, and indeed $H(A, d) = 0$, as in Example 3.6, is possible and implies this case. Recall from Remark 2.1 that orders in algebras with semisimple category of dg-module are actually necessarily of this form. Let (Λ, d) be an order in (A, d) with $H(\Lambda, d) = 0$. Then Aldrich and Garcia-Rocas show in [1, Theorem 4.7] that in this case the category of differential graded (Λ, d)-modules is equivalent with the category of graded $\ker(d)$-modules.

Example 6.4. Let R be an integral domain with field of fractions K. We consider the differential graded semisimple K-algebra $(Mat_{2\times 2}(K), d_x)$ from Example 3.6. Recall that for each $x \in K$ there is a differential d_x on $Mat_{2\times 2}(K)$ with the grading chosen in Example 3.6.

Choose $x = 1$ for the moment. Then $Mat_{2\times 2}(R)$ is a maximal differential graded R-order, and actually a proper differential graded R-order since the homology of $(Mat_{2\times 2}(R), d_1)$ is 0.

However, choosing an element $x \in R \setminus R^\times$, then again $Mat_{2\times 2}(R)$ is a maximal differential graded R-order. The homology in degree 1 is R/xR, as is easily seen. The kernel of the differential in degree 0 is $C_0 := \{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a \in R\}$ and the image if the differential from degree -1 is xC_0. Clearly, the differential in degree -1 is injective. Hence $H_*(Mat_{2\times 2}(R), d_x) \simeq R/xR \oplus R/xR$, where the first copy is in degree 0 and the second copy is in degree 1. Therefore $H_*(Mat_{2\times 2}(R), d_x) \simeq (R/xR)[\epsilon]/\langle \epsilon^2 \rangle$ where ϵ is an element in degree 1.

Note that $Mat_{2\times 2}(R)$ is hereditary, as an order, whereas, as soon as $x \notin R^\times$, the homology algebra $H_*(Mat_{2\times 2}(R), d_x)$ is not, even of infinite global dimension.

Example 6.5. Let R be an integral domain with field of fractions K. We consider again the differential graded semisimple K-algebra $(Mat_{2\times 2}(K), d_x)$ from Example 3.6. If $R = \mathbb{Z}$ and $x = \frac{1}{2}$, then for $\Lambda = Mat_{2\times 2}(R)$ we obtain $d_x(\Lambda)$ is not a subset of Λ. Actually,

$$d_{\frac{1}{2}}(Mat_{2\times 2}(\mathbb{Z})) = \frac{1}{2}\mathbb{Z}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2}\mathbb{Z}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Since

$$\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{4} \\ 0 & 0 \end{pmatrix}$$

and iterating, we have that for each integer n the element $\begin{pmatrix} 0 & \frac{1}{2^n} \\ 0 & 0 \end{pmatrix}$ is in the ring generated by $d_{\frac{1}{2}}(Mat_{2\times 2}(\mathbb{Z}))$. Hence, there is no differential graded order $(\Gamma, d|_\Gamma)$ such that $Mat_{2\times 2}(K) \subseteq \Gamma$. However, for any $x \in K \setminus \{0\}$ the set

$$\Lambda := \begin{pmatrix} R & xR \\ \frac{1}{x}R & R \end{pmatrix}$$

is a subring of $Mat_{2\times 2}(K)$ and is stable under d_x. Further, if R is a Dedekind domain, then xR is projective, and hence Λ is an R-order if R is a Dedekind domain. We observe that

$$\begin{pmatrix} \frac{1}{x} & 0 \\ 0 & 1 \end{pmatrix} \cdot \Lambda \cdot \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} = Mat_{2\times 2}(R)$$

and hence Λ is conjugate to the maximal order $Mat_{2\times 2}(R)$. Therefore, using Corollary 6.2, (Λ, d_x) is a maximal differential graded order. We get $H(\Lambda, d_x) = 0$, and hence this is actually a proper differential graded R-order.

Question 6.6. If (Λ, d) is a dg-maximal differential graded order, can one show that Λ is a maximal order.

Question 6.7. If (Λ, d) is a dg-maximal proper differential graded order, can we show that $H(\Lambda, d)$ is hereditary?

7. Class groups of differential graded orders

Recall that a differential graded R-order (Λ, d) in a semisimple differential graded K-algebra (A, d) is at first an R-order Λ in a semisimple K-algebra A. The locally free class group of an order proved to be a useful invariant in integral representation theory. We want to provide a definition and first properties of a locally free class group of differential graded orders.

We first need to elaborate on what we mean by a free differential graded module. Let (Λ, d) be a differential graded algebra. A differential graded (Λ, d)-module (L, δ) is free of rank $n \in \mathbb{N}$ if $(L,\delta) \simeq \bigoplus_{i=1}^n (\Lambda,d)[k_i]$ for some integers k_i, as differential graded (Λ,d)-modules. Later we shall mainly consider the case $k_i = 0$ for all i, and say that such a dg-module is degree 0-free of rank n.

Since $d(1) = 0$ in Λ, we get that for an homomorphism $\varphi : \bigoplus_{i=1}^n (\Lambda,d)[k_i] \longrightarrow (L,\delta)$ we have $\varphi(0,\dots,0,1,0,\dots,0) \in \ker(\delta)$, where 1 is in position i, for each position i. However, for any homogeneous $z \in \ker(\delta)$ we get that $\varphi(\lambda) := \lambda z$ defines a homomorphism $(\Lambda,d)[-|z|] \longrightarrow (L,\delta)$.

Lemma 7.1. *Let R be a Dedekind domain with field of fractions K. Let (Λ,d) be a differential graded algebra, and denote $(A,d) := (K \otimes_R \Lambda, \mathrm{id}_K \otimes d)$. Then the set of free differential graded (Λ,d)-ideals is in bijection with the group of left regular homoegenous elements in* $\ker(d) \subseteq A$ *modulo the subgroup* $(\ker(d) \cap \Lambda^\times)$.

Proof. We now consider dg-ideals (I,d) of (Λ,d). Such an ideal is free if there is an isomorphism $\varphi : (\Lambda,d)[k] \longrightarrow (I,d)$ and this is equivalent with the choice of a $z \in (I \cap \ker(d))$ homogeneous of degree k such that $I = \Lambda z$ and such that $\lambda z = 0 \Rightarrow \lambda = 0$. We hence obtain that any left regular homogeneous generator of I in the cycles gives an isomorphism, and hence there is a surjective map from left regular homogeneous elements in the cycles to the set of rank one free dg (Λ,d) ideals. Two such left regular elements homogeneous z_1, z_2 of I in the cycles give the same ideal if and only if $\Lambda z_1 = \Lambda z_2$. This is equivalent with $z_1 = \lambda_2 z_2$ and $z_2 = \lambda_1 z_1$ for homogeneous units $\lambda_1, \lambda_2 \in \Lambda$. However, $z_1, z_2 \in \ker(d)$ shows that

$$0 = d(\lambda_2 z_2) = d(\lambda_2) z_2 \pm \lambda_2 d(z_2) = d(\lambda_2) z_2$$

and the fact that z_2 is regular gives that $\lambda_2 \in \ker(d)$. Likewise $\lambda_1 \in \ker(d)$.

Conversely, if $\lambda \in \ker(d)$ is a homogeneous unit, then $\Lambda z = \Lambda\lambda z$ as differential graded ideal. Indeed, if $\lambda \in \ker(d)$ and $z \in \ker(d)$, then

$$d(\mu\lambda z) = d(\mu)\lambda z + (-1)^{|\mu|}\mu d(\lambda) z + (-1)^{|\mu|+|\lambda|}\mu\lambda d(z) = d(\mu)\lambda z$$

and hence mapping μz to $\mu\lambda z$ for any $\mu \in \Lambda$ is an isomorphism of differential graded ideals. ■

We shall need to study invertible elements in the subring of cycles.

Lemma 7.2. *Let (Λ,d) be a differential graded algebra. Then* $\ker(d) \cap \Lambda^\times = \ker(d)^\times$.

Proof. We have seen in Corollary 1.3 that $\ker(d)$ is a graded subalgebra. Trivially $\ker(d) \cap \Lambda^\times \supseteq \ker(d)^\times$. Further, if u is homogeneous and invertible in Λ, and if $u \in \ker(d)$, then also $u^{-1} \in \ker(d)$. Indeed, $0 = d(1) = d(uu^{-1}) = d(u)u^{-1} \pm ud(u^{-1})$ and hence $d(u^{-1}) = 0$ since u is invertible. Hence $\ker(d) \cap \Lambda^\times \subseteq \ker(d)^\times$. ■

Let R be a Dedekind domain with field of fractions K. Suppose now that (A,d) is a differential graded K-algebra, which is assumed to be left and right artinian as an algebra. By Wedderburn's theorem, in an artinian semisimple K-algebra an element u is left regular if and only if it is represented by a tuple of non singular matrices, whence an invertible element. In any left and right artinian ring a left or right regular element is invertible. Indeed, by the above, this is clear modulo the Jacobson radical. Then, for an artinian ring the Jacobson radical is nilpotent, and hence any lift of a unit in the semisimple quotient to an element in the artinian ring is again a unit.

Let (Λ,d) be a differential graded subalgebra of (A,d) such that Λ is R-projective and $K \otimes_R \Lambda = A$. As in the classical case we need to work with two versions, the localised version and the completed version. Let $X_\wp$ be the localisation at a prime $\wp$, and denote by $\widehat{X}_\wp$ the completion at the prime $\wp$. Hence we denote

$$J(A,d) := \{(u(\wp))_{\wp \in Spec(R)} \in \prod_{\wp \in Spec(R)} \left(A_\wp^\times \cap \ker(d)\right) \mid$$
$$u(\wp) \in \Lambda_\wp^\times \text{ for almost all } \wp \in Spec(R) \text{ and } u_\wp \text{ homogeneous for all } \wp\}.$$

Note that this definition does not depend on the choice of Λ. Define its subgroup

$$U(\Lambda,d) := J(A,d) \cap \prod_{\wp\in Spec(R)} (\Lambda_\wp^\times \cap \ker(d)),$$

which does depend on Λ. Likewise

$$\widehat{J}(A,d) := \{(u(\wp))_{\wp\in Spec(R)} \in \prod_{\wp\in Spec(R)} \left(\widehat{A}_\wp^\times \cap \ker(d)\right) \mid$$
$$u(\wp) \in \widehat{\Lambda}_\wp^\times \text{ for almost all } \wp \in Spec(R) \text{ and } u_\wp \text{ homogeneous for all } \wp\}.$$

and its subgroup

$$\widehat{U}(\Lambda,d) := \widehat{J}(A,d) \cap \prod_{\wp\in Spec(R)} (\widehat{\Lambda}_\wp^\times \cap \ker(d)).$$

Note that if A is finite dimensional, then any homogeneous $u_\wp$ being a factor in an element in $\widehat{J}(A,d)$ (resp. in $J(A,d)$) has to be in degree 0. Then the set of left classes

$$U(\Lambda,d)\backslash J(A,d)$$

is in bijection with the set of locally free differential graded fractional ideals of (Λ,d) and

$$\widehat{U}(\Lambda,d)\backslash \widehat{J}(A,d)$$

is in bijection with the set of completion locally free differential graded fractional ideals of (Λ,d).

We associate to a representative $\alpha = (\alpha(\wp))_{\wp\in Spec(R)}$ of a class $U(\Lambda,d)\alpha \in U(\Lambda,d)\backslash J(A,d)$ the fractional ideal

$$\Lambda\alpha := A \cap \bigcap_{\wp\in Spec(R)} \Lambda_\wp \cdot \alpha(\wp).$$

Likewise we get the completed version. Then, $\Lambda\alpha \simeq \Lambda\beta$ as fractional differential graded ideal if and only if there is a homogeneous $x \in A^\times \cap \ker(d)$ of degree 0 with $\alpha = \beta x$. If A is semisimple, then we may apply Proposition 5.10 we have that

$$(\Lambda\alpha)_\wp = \Lambda_\wp \cdot \alpha(\wp)$$

for all $\wp \in Spec(R)$.

We now define a category $(\Lambda,d)-LF_0-dgmod$. This is defined to be the full subcategory of $(\Lambda,d)-dgmod$ containing all locally degree 0 rank one free differential graded (Λ,d)-modules, and all direct sums of these objects.

In the definition below we shall need to consider $K_0((\Lambda,d)-LF_0-dgmod)$. This is defined as the quotient of the free abelian group on isomorphism classes of objects in $(\Lambda,d)-LF_0-dgmod$ modulo the relation $[(X,\delta_X)]-[(Y,\delta_Y)]-[(Z,\delta_Z)]$ whenever $(X,\delta_X) \simeq (Y,\delta_Y) \oplus (Z,\delta_Z)$. We further define $G_0((\Lambda,d)-LF_0-dgmod)$, which is the quotient of the free abelian group on isomorphism classes of objects in $(\Lambda,d)-LF_0-dgmod$ modulo the relation $[(X,\delta_X)]-[(Y,\delta_Y)]-[(Z,\delta_Z)]$ whenever there is a short exact sequence

$$0 \longrightarrow (Y,\delta_Y) \longrightarrow (X,\delta_X) \longrightarrow (Z,\delta_Z) \longrightarrow 0$$

in $(\Lambda,d)-LF_0-dgmod$, considered as a subcategory of dg-modules. Here, denote by $[(M,\delta)]$ the image of a locally degree 0-free dg-module in $K_0((\Lambda,d)-LF-dgmod)$, respectively $G_0((\Lambda,d)-LF-dgmod)$. Note that this setting makes sense for any differential graded R-algebra (Λ,d) whenever R is a Dedekind domain with field of fractions K.

Definition 7.3. Let R be a Dedekind domain with field of fractions K.

- Let (Λ,d) be a differential graded order in the semisimple differential graded K-algebra (A,d).
  - The group $\widehat{I}(\Lambda,d) := \widehat{U}(\Lambda,d)\backslash \widehat{J}(A,d)/(A^\times \cap \ker(d))$ is the group of *completion differential graded idèles* of the dg-order (Λ,d).

  – The group $I(\Lambda,d):=U(\Lambda,d)\backslash J(A,d)/(A^\times\cap\ker(d))$ is the group of *differential graded idèles* of the dg-order (Λ,d).
- If (Λ,d) is a differential graded R-algebra, then the *class group* $Cl((\Lambda,d))$ *of* (Λ,d) is the subgroup of $G_0((\Lambda,d)-LF_0-dgmod)$ generated by elements $[L]-[\Lambda]$ for degree 0-locally free differential graded (Λ,d)-lattices (L,δ) of rank 1.
- If (Λ,d) is a differential graded R-algebra, then the *idèle class group* $Cl^{(I)}((\Lambda,d))$ *of* (Λ,d) is the subgroup of $K_0((\Lambda,d)-LF_0-dgmod)$ generated by elements $[L]-[\Lambda]$ for degree 0-locally free differential graded (Λ,d)-lattices (L,δ) of rank 1.

As in the classical case we get

Theorem 7.4. *Let R be a Dedekind domain with field of fractions K. Let (Λ,d) be a differential graded order in the semisimple differential graded K-algebra (A,d). Then in $G_0((\Lambda,d)-LF_0-dgmod)$ we have*

$$[\Lambda\alpha]+[\Lambda\beta]-2[\Lambda]=[\Lambda\alpha\beta]-[\Lambda]$$

for any two completion differential graded idèles α and β. In particular, there is a surjective group homomorphism Φ from the group of completion differential graded idèles to the differential graded class group given by $\Phi(\alpha)=[\Lambda\alpha]-[\Lambda]$.

Proof. We need to verify that the constructions of idèles in [26, Theorem 8.5.11] do not lead out of $\ker(d)$. Besides this the proof of [26, Theorem 8.5.11] holds in the dg-concept verbatim as in the non dg-concept, until the very last argument. Let us go through the arguments of [26, Theorem 8.5.11]. As in the original proof we may replace α by a version after having multiplied by an $r\in R$ such that α is an integral idèle. Again there is an integer k such that $\alpha_\wp^{-1}\in\wp^{-k}\widehat{\Lambda}_\wp$ and an integer t and $x\in A^\times\cap\ker(d)$ such that $\beta_\wp x-1\in\wp^t\widehat{\Lambda}_\wp$, and again that βx is an integral idèle. We observe that $\beta x\in\ker(d)$. Replacing β by βx is possible since we do not quit $\ker(d)$ by this operation. Again, by the same computation $\alpha_\wp\beta_\wp\alpha_\wp^{-1}\beta_\wp^{-1}$ is invertible and in $\ker(d)$ as α and β are. By Lemma 7.2 units which are cycles are precisely the units in the ring of cycles. The morphism f in [26, page 310] is easily seen to be a morphism of dg-modules. The fact that $\ker(f)=\Lambda\alpha\beta$ still holds by the same proof. The fact that $\mathrm{im}(f)=\Lambda$ still holds also in the dg-version by the very same proof. Hence there is a short exact sequence of locally free dg-modules

$$0\longrightarrow\Lambda\alpha\beta\longrightarrow\Lambda\alpha\oplus\Lambda\beta\stackrel{f}{\longrightarrow}\Lambda\longrightarrow 0$$

and $\Lambda\alpha\beta\simeq\Lambda\beta\alpha$. All these terms are in the category $(\Lambda,d)-LF_0-dgmod$ and we hence get a short exact sequence n the category giving a relation in the relevant Grothendieck group. Further, this short exact sequence shows

$$[\Lambda\alpha\beta]+[\Lambda]=[\Lambda\alpha]+[\Lambda\beta]$$

in the class group. Recall that the group law in $Cl(\Lambda,d)$ is the group law of the Grothendieck group, where we consider the Grothendieck group modulo direct sums. Hence

$$\begin{array}{rcl}\widehat{U}(\Lambda,d)\backslash\widehat{J}(\Lambda,d)/(A^\times\cap\ker(d)) & \stackrel{\Phi}{\longrightarrow} & Cl(\Lambda,d)\\ \alpha & \mapsto & [\Lambda\alpha]-[\Lambda]\end{array}$$

is a group homomorphism. Indeed,

$$\begin{array}{rcl}\Phi(\alpha\beta) & = & [\Lambda\alpha\beta]-[\Lambda]\\ & = & ([\Lambda\alpha\beta]+[\Lambda])-2[\Lambda]\\ & = & ([\Lambda\alpha]+[\Lambda\beta])-2[\Lambda]\\ & = & ([\Lambda\alpha]-[\Lambda])+([\Lambda\beta]-[\Lambda])\\ & = & \Phi(\alpha)+\Phi(\beta)\end{array}$$

. By construction of $Cl(\Lambda,d)$ we see that Φ is surjective. This proves the theorem. ∎

Remark 7.5. Note that Λ is projective, and therefore

$$0 \longrightarrow \Lambda\alpha\beta \longrightarrow \Lambda\alpha \oplus \Lambda\beta \longrightarrow \Lambda \longrightarrow 0$$

splits as Λ-modules. However, a differential graded module (M, δ) is projective in the category of dg-modules if and only if (M, δ) is acyclic (cf [1]). Consider the short exact sequence

$$0 \longrightarrow \Lambda\gamma \stackrel{\iota}{\longrightarrow} \Lambda\alpha + \Lambda\beta \stackrel{f}{\longrightarrow} \Lambda \longrightarrow 0$$

of dg-modules over (Λ, d). Here, f is just the map sending the two components to a sum, and $\gamma = \alpha\beta$. Consider now the completion at primes $\wp$. We only need to prove that the sequence splits at all completions to show that the class in the Ext-group is 0.

If we could have Theorem 7.4 for the Grothendieck group $K_0((\Lambda, d) - LF_0 - dgmod)$ instead, then we had that the group of completion differential graded idèles is isomorphic to the group $Cl^{(I)}(\Lambda, d)$ and this group actually parameterizes stable isomorphisms of these lattices, such as in the classical situation.

Remark 7.6. If $d = 0$ and the grading is trivial, the concepts of class groups coincide and $Cl(\Lambda, 0) = Cl(\Lambda)$.

Corollary 7.7. *Let R be a Dedekind domain with field of fractions K and let (A, d) be a differential graded algebra, semisimple as an algebra. Let (Λ, d) be a differential graded order in (A, d). Then*

$$\widehat{U}(\Lambda, d)\backslash\widehat{J}(A, d)/(A^\times \cap \ker(d)^\times) \simeq \widehat{U}(\ker(d))\backslash\widehat{J}(K \cdot \ker(d))/(K \cdot \ker(d)^\times).$$

Proof. By Lemma 7.2 we have $\widehat{U}(\Lambda, d) = \widehat{U}(\ker(d))$ and $A^\times \cap \ker(d) = \ker(d)^\times$. ■

More generally, we get the following

Proposition 7.8. *Let R be a Dedekind domain with field of fractions K. Let (A, d) be a differential graded semisimple finite dimensional K-algebra and let (Λ, d) be a differential graded R-order in (A, d). Then there is a group homomorphism*

$$\widehat{U}(\Lambda, d)\backslash\widehat{J}(A, d)/(A^\times \cap \ker(d)^\times) \longrightarrow Cl(\Lambda).$$

Proof. Indeed, since $A^\times \cap \ker(d) \subseteq A^\times$ and since $\Lambda^\times \cap \ker(d) \subseteq \Lambda^\times$, we get a group homomorphism $J(A, d) \longrightarrow J(A)$. Further, by the same argument the same embedding shows that $U(\Lambda, d)$ maps to $U(\Lambda)$ and hence we get a group homomorphism $U(\Lambda, d)\backslash J(A, d) \longrightarrow U(\Lambda)\backslash J(A)$. Again, since $(A^\times \cap \ker(d)) \subseteq A^\times$ the same map yields a group homomorphism

$$U(\Lambda, d)\backslash J(A, d)/(A^\times \cap \ker(d)) \longrightarrow U(\Lambda)\backslash J(A)/A^\times$$

and since $Cl(\Lambda) \simeq U(\Lambda)\backslash J(A)/A^\times$, we get the statement. ■

Example 7.9. Consider the $\mathbb{Z}$-order

$$\Lambda = \mathbb{Z} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + p \cdot Mat_{2\times 2}(\mathbb{Z})$$

in $Mat_{2\times 2}(\mathbb{Q})$. Then, following Example 3.6 this is a differential graded order with the differential d_1. Indeed, for $a, x, y, z, u \in \mathbb{Z}$ we compute

$$d_1(\begin{pmatrix} a + px & pu \\ pz & a + py \end{pmatrix}) = \begin{pmatrix} pz & p(y - x) \\ 0 & pz \end{pmatrix} \in \Lambda$$

Further, by [5, §34D] we have that $Cl(\Lambda)$ is of order 2 whenever $p - 1$ is divisible by 4 (and trivial otherwise). However, $\ker(d_1) = \mathbb{Z}[\epsilon]/\epsilon^2$, and hence $Cl(\Lambda, d_1)$, as well as the dg-idèle group are trivial as is easily seen using the idèle description. Therefore, the map in Proposition 7.8 is not surjective in general.

Since localisation is flat, we get $(H(\Lambda,d))_\wp \simeq H(\Lambda_\wp, d_\wp)$. For any dg-module (M,δ) we get $(H(M,\delta))_\wp \simeq H(M_\wp,\delta_\wp)$. Moreover, if (M,δ) is a differential graded (Λ,d)-module, then by Lemma 1.6 we get that $H(M,\delta)$ is a graded $H(\Lambda,d)$-module. Further, if (M,δ) is a locally free dg (Λ,d)-module, then $(M_\wp,\delta_\wp)$ is a free $(\Lambda_\wp,d_\wp)$-module. Hence $H(M,\delta)$ is a graded locally free $H(\Lambda,d)$-module.

If we identify

$$Cl(H(\Lambda,d)) = \ker\left(G_0(H(\Lambda,d) - LF - mod) \longrightarrow G_0(H(A,d) - proj)\right)$$

taking homology is therefore a group homomorphism

$$Cl(\Lambda,d) \stackrel{CH_{(\Lambda,d)}}{\longrightarrow} Cl(H(\Lambda,d)).$$

Note however that $H(\Lambda,d)$ is not an order in general.

Definition 7.10. Let R be the ring of integers in a number field K. Let (Λ,d) be a differential graded order in the semisimple differential graded K-algebra (A,d). Then define the *homology-isomorphism class group kernel* $\ker(CH_{(\Lambda,d)}) =: Cl(\Lambda,d)_{hi}$.

Remark 7.11. Note that for each locally free differential graded (Λ,d)-module (L,δ) we have that

$$Cl(\Lambda,d)_{hi} + [(L,\delta)] = CH^{-1}_{(\Lambda,d)}(CH_{(\Lambda,d)}([(L,\delta)])$$

parameterizes those locally free dg-(Λ,d)-modules which have the same homology as (L,δ). Those which are quasi-isomorphic to (L,δ) do share the homology with (L,δ). However, being quasi-isomorphic is stronger in general than just having isomorphic homology.

We now consider locally free $H(\Lambda,d)$-modules.

Lemma 7.12. *Let R be a Dedekind domain with field of fractions K and let B be a finitely generated R-algebra. For any B-module M let*

$$t(M) := \{x \in M \mid \exists r \in R \setminus \{0\} : rx = 0\}$$

be the torsion submodule of M. If L is a locally free B-module of rank n, then $t(L) \simeq t(B)^n$.

Proof. By hypothesis $L_\wp \simeq B_\wp^n$ for all primes $\wp \in Spec(R)$. Since R is Dedekind, $R_\wp$ is a principal ideal domain (cf [26, Lemma 7.5.9]). Hence $t(L_\wp) = t(L)_\wp$. Since R is Dedekind, any non zero prime ideal is maximal (cf [26, Lemma 7.5.15]) and hence the primary decomposition (cf [26, Theorem 7.2.5]) of $t(L)$ gives a direct product decomposition $t(L) = \prod_{\wp \in Spec(R)} t(L)_\wp$. We hence get $t(L) \simeq t(B)^n$. ∎

Corollary 7.13. *Let R be a Dedekind domain with field of fractions K, let (A,d) be a finite dimensional and semisimple and differential graded K-algebra. Let (Λ,d) be a differential graded R-order in (A,d). Then*

$$Cl(H(\Lambda,d)) \simeq Cl(H(\Lambda,d)/t(H(\Lambda,d))).$$

In particular, if $H(A,d) = 0$, then $Cl(\Lambda,d)_{hi} = 0$.

Proof. By Lemma 7.12 for any rank 1 locally free $H(\Lambda,d)$-module L we have $t(L) = t(H(\Lambda,d))$ and therefore

$$H(\Lambda,d)/t(H(\Lambda,d)) \otimes_{H(\Lambda,d)} - : Cl(H(\Lambda,d)) \longrightarrow Cl(H(\Lambda,d)/t(H(\Lambda,d)))$$

is an isomorphism. Note that $H(\Lambda,d)/t(H(\Lambda,d)) \otimes_{H(\Lambda,d)} - \simeq (R/tR) \otimes_R -$.

If now (A,d) is acyclic, then $H(\Lambda,d)$ is torsion, and therefore locally free $H(\Lambda,d)$-modules are actually free. This proves the lemma. ∎

We shall need a result due to Wehlen. Recall (cf e.g. [10, Chapter 3]) that the Baer lower radical $L(A)$ of an algebra A is the intersection of all prime ideals of A. Every element of $L(A)$ is nilpotent and if A is Noetherian, then $L(A)$ is a nilpotent ideal [10, Theorem 3.11].

Theorem 7.14. [23, Theorem 2.4] *Let R be a Prüfer domain and let A be a finitely generated algebra over R. Let $L(A)$ be the (Baer) lower radical of A. If $B := A/L(A)$ is separable over R, i.e. B is projective as an $B \otimes_R B^{op}$-module, then there are idempotents e_p and e_t of A such that*

$$A = \begin{pmatrix} e_pAe_p & e_tAe_p \\ e_pAe_t & e_tAe_t \end{pmatrix}$$

and $e_pAe_p/e_pL(A)e_p$ is R-projective, such that $\begin{pmatrix} 0 & e_tAe_p \\ e_pAe_t & e_tAe_t \end{pmatrix} \subseteq t(A)$, the R-torsion ideal of A.

Note that obviously e_p and e_t are orthogonal and $e_p+e_t = 1$. Further, note that $A/t(A) = e_pAe_p/t(e_pAe_p)$.

Corollary 7.15. *Suppose the hypotheses of Theorem 7.14. Then any unit u in $A/t(A)$ can be lifted to a unit u_2 of A.*

Proof. If u is a unit of $A/t(A)$, then we get that u is actually a unit of $e_pAe_p/t(e_pAe_p)$. Further, since $e_pAe_p/e_pL(A)e_p$ is R-projective, we have that $t(e_pAe_p) \subseteq e_pL(A)e_p$ and since the right hand side of the inclusion is a nil ideal, so is the left hand side. Hence there is u_1 and v_1 in e_pAe_p such that u_1 maps to u in $A/t(A)$ and such that $u_1v_1 - 1$ is in $e_pL(A)e_p$, whence nilpotent. But since $1 + n$ for a nilpotent element n is a unit, we have that u_1 is a unit. Therefore we may find a unit $u_1 \in e_pAe_p$ such that u_1 maps to u in $A/t(A)$. But then there is a unit $u_2 := \begin{pmatrix} u_1 & 0 \\ 0 & 1 \end{pmatrix}$ in A (actually in the Pierce decomposition above) such that u_2 maps to u in $A/t(A)$. ■

Theorem 7.16. *Let R be a Dedekind domain with field of fractions K and let (A, d) be a differential graded algebra such that A is finite dimensional separable over K. Suppose that* $\ker(d)$ *is separable as a graded algebra. Let (Λ, d) be a differential graded R-order in (A, d). Denote the canonical map $\pi : \ker(d) \longrightarrow H(\Lambda, d)$ and denote $\overline{H(\Lambda, d)} := H(\Lambda, d)/t(H(\Lambda, d))$. Then $\overline{H(\Lambda, d)}$ is a classical R-order in $H(A, d)$, and we have an exact sequence*

$$0 \longrightarrow Cl(\Lambda, d)_{hi} \longrightarrow Cl(\Lambda, d) \longrightarrow Cl(\overline{H(\Lambda, d)}) \longrightarrow 0.$$

Proof. It is a classical fact that separable algebras are semisimple. Since $\ker(d)$ is a semisimple algebra, Lemma 4.2 shows that we see that $H(A, d)$ is a semisimple algebra. Hence, $H(\Lambda, d)/t(H(\Lambda, d))$ is an R-order in the semisimple K-algebra $H(A, d)$.

We need to see that the right hand map is surjective. For this we shall use the interpretation of class groups as idèles.

We hence need to see that

$$\widehat{U}(\Lambda, d)\backslash\widehat{J}(A, d)/(A^\times \cap \ker(d)) \longrightarrow \widehat{U}(\overline{H(\Lambda, d)})\backslash\widehat{J}(H(A, d))/H(A, d)^\times$$

is surjective, since the map on the level of idèles factors through the right hand map.

First, by Lemma 4.2 any unit of $H(A, d)$ lifts to a unit in $\ker(d)$. Now, clearly $\ker(d)^\times \subseteq (A^\times \cap \ker(d))$. For a finitely generated R-algebra B satisfying that $B/L(B)$ is separable, if $\overline{u}$ is a unit in B/tB, then by Corollary 7.15 any unit u in B/tB lifts to a unit u in B. Now, by hypothesis $\ker(d)$ is finite dimensional separable. This is equivalent to the fact that in the Wedderburn decomposition the centres of the skew fields are separable extensions of the base field. Hence $H(A, d)$ is separable as well. Therefore $H(\Lambda_\wp, d)/t(H(\Lambda_\wp, d))$ is separable for all $\wp \in Spec(R)$, and so is its quotient modulo its Baer (lower) radical. Following Corollary 7.15 we can hence lift any unit of $H(\Lambda_\wp, d)/t(H(\Lambda_\wp, d))$ to a unit of $H(\Lambda_\wp, d)$

By semisimplicity $A_\wp^\times \cap \ker(d) \longrightarrow H(A_\wp, d)^\times$ is surjective. By the same argument $\widehat{J}(A, d) \longrightarrow \widehat{J}(H(A, d))$ is surjective as well. Since $\pi(\widehat{U}(\Lambda, d)) \subseteq \widehat{U}(H(\Lambda, d))$, we get indeed a map $Cl(\Lambda, d) \longrightarrow Cl(\overline{H(\Lambda, d)})$.

Now, for any locally free $\overline{H(\Lambda,d)}$-ideal $\overline{I}$ we can find an idèle $\overline{\alpha} \in J(\overline{H(A,d)})$ such that $\overline{I} = \bigcap_{\wp\in Spec(R)} \overline{H(\Lambda,d)} \cdot \overline{\alpha}_\wp$. Since $\widehat{J}(A,d) \longrightarrow \widehat{J}(H(A,d))$ is surjective, there is an idèle $\alpha \in J(A,d)$ which maps to $\overline{\alpha}$. But then the locally free ideal

$$I := \bigcap_{\wp\in Spec(R)} \Lambda \cdot \alpha_\wp$$

maps to $\overline{I}$. This shows the surjectivity of $Cl(\Lambda,d) \longrightarrow Cl(\overline{H(\Lambda,d)})$. The statement on the kernel of this map follows by definition. ■

8. Reducing to dg-orders in differential graded simple algebras

A major tool in studying class groups of orders is the following result.

Theorem 8.1. *(Reiner-Ullom [20])(Mayer-Vietoris theorem for class groups of orders) Let R be a Dedekind domain with field of fractions K and let Λ be an R-order in the finite dimensional semisimple K-algebra A satisfying the Eichler condition. Let $e^2 = e \in Z(A)$ be a non trivial central idempotent of A. Put $f := 1 - e$. Then Λe is an R-order in Ae and Λf is an R-order in Af and we have a pullback diagram*

$$\begin{array}{ccc} \Lambda & \xrightarrow{\cdot e} & \Lambda\cdot e \\ {\scriptstyle \cdot f}\downarrow & & \downarrow{\scriptstyle \pi_e} \\ \Lambda f & \xrightarrow[\pi_f]{} & \overline{\Lambda} \end{array}$$

for $\overline{\Lambda} := \Lambda e/\Lambda \cap \Lambda e \simeq \Lambda f/\Lambda\cap\Lambda f$ and π_e, resp. π_f being the canonical morphisms. Further, there is a group homomorphism δ such that δ and the canonical maps induce an exact sequence

$$\Lambda^\times \longrightarrow (\Lambda e)^\times \times (\Lambda f)^\times \longrightarrow \overline{\Lambda}^\times \xrightarrow{\delta} Cl(\Lambda) \longrightarrow Cl(\Lambda e)\times Cl(\Lambda f).$$

Keep the assumptions and notations of Theorem 8.1 for the moment , and suppose that K is of characteristic different from 2. In addition if (A,d) is a finite dimensional semisimple differential graded K-algebra, and if (Λ,d) is a differential graded R-order in (A,d), then by Remark 2.2 we have $d(e) = 0 = d(f)$ and hence $(\Lambda e,d)$ and $(\Lambda f,d)$ are differential graded orders in (Ae,d) respectively (Af,d). Since $\overline{\Lambda} = \Lambda e/(\Lambda\cap\Lambda e)$, and since d induces a differential on both Λ and on Λe, it also induces a differential $\overline{d}$ on $\overline{\Lambda}$. Therefore, the maps in

$$\begin{array}{ccc} (\Lambda,d) & \xrightarrow{\cdot e} & (\Lambda\cdot e,d) \\ {\scriptstyle \cdot f}\downarrow & & \downarrow{\scriptstyle \pi_e} \\ (\Lambda\cdot f,d) & \xrightarrow[\pi_f]{} & (\overline{\Lambda},\overline{d}) \end{array}$$

are still well-defined maps of differential graded algebras. Since

$$\begin{array}{ccc} \Lambda & \xrightarrow{\cdot e} & \Lambda\cdot e \\ {\scriptstyle \cdot f}\downarrow & & \downarrow{\scriptstyle \pi_e} \\ \Lambda\cdot f & \xrightarrow[\pi_f]{} & \overline{\Lambda} \end{array}$$

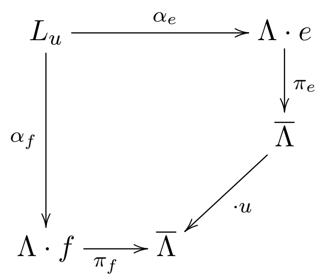
is a pullback diagram of R-algebras, the diagram

$$\begin{array}{ccc} (\Lambda,d) & \xrightarrow{\cdot e} & (\Lambda\cdot e,d) \\ {\scriptstyle \cdot f}\downarrow & & \downarrow{\scriptstyle \pi_e} \\ (\Lambda\cdot f,d) & \xrightarrow[\pi_f]{} & (\overline{\Lambda},\overline{d}) \end{array}$$

is a pullback diagram of differential graded algebras. Recall that $Cl(\Lambda, d)$ is generated by $[L] - [\Lambda]$ for L being rank 1 locally free differential graded modules. Hence the map

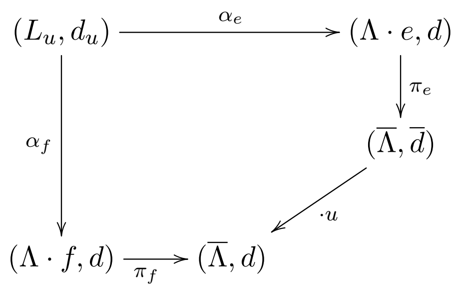
$$Cl(\Lambda, d) \stackrel{Cl(e)\times Cl(f)}{\longrightarrow} Cl(\Lambda e, d) \times Cl(\Lambda f, d)$$

is still well-defined. We recall the definition of δ. Let $u \in \overline{\Lambda}^\times$. Then consider the pullback diagram



and define $\delta(u) := [L_u] - [\Lambda]$.

In the dg-case, if $u \in \ker(\overline{d}) \cap \overline{\Lambda}^\times$ is homogeneous, then we get again a pullback diagram



defining a locally free differential graded (Λ, d)-module (L, d_u). We may define a map δ_d by $\delta_d(u) := [(L_u, d_L)] - [(\Lambda, d)]$ in this case. It remains to show that $\ker(Cl(\Lambda e) \times Cl(\Lambda f)) = \mathrm{im}(\delta_d)$.

Suppose that $u \in \ker(\overline{d}) \cap \overline{\Lambda}^\times$. Then

$$Cl(f) \circ \delta_d(u) = [(L_u \cdot f, d_u)] - [(\Lambda f, d)] = [\alpha_f(L_u, d_u)] - [(\Lambda f, d)] = [(\Lambda f, d)] - [(\Lambda f, d)] = 0$$

and likewise $Cl(e) \circ \delta_d(u) = 0$.

Let (L, d_L) be a rank 1 locally free dg (Λ, d)-module in the kernel of $Cl(e) \times Cl(f)$. Let α be the idèle of (L, d_L). Then we know that $\alpha \cdot e$ is the principal idèle and also $\alpha \cdot f$ is the principal idèle in the corresponding algebras. Now

$$A^\times \cap \ker d = (Ae^\times \cap \ker d) \times (Af^\times \cap \ker d).$$

Further, $\ker(d) = \ker(de) \times \ker(df)$. We hence only need to see consider ordinary idèles in order to examine the kernel of $Cl(e) \times Cl(f)$. But then, by the classical Mayer-Vietoris-like theorem for class groups of orders we obtain that

$$\ker(Cl(e) \times Cl(f)) = \mathrm{im}(\delta_d).$$

We hence obtained the following

Theorem 8.2. *Let R be a Dedekind domain with field of fractions K of characteristic different from 2 and let (Λ, d) be a differential graded R-order in the finite dimensional semisimple differential graded K-algebra (A, d), and suppose that A satisfies the Eichler condition. Let $e^2 = e \in Z(A)$ be a non trivial central idempotent of A. Put $f := 1 - e$. Then $(\Lambda e, de)$ is a differential graded R-order in (Ae, de) and $(\Lambda f, df)$ is a differential graded*

R-order in (Af, df) and we have a pullback diagram

$$\begin{array}{ccc} \Lambda & \xrightarrow{\cdot e} & \Lambda\cdot e \\ {\scriptstyle \cdot f}\downarrow & & \downarrow{\scriptstyle \pi_e} \\ \Lambda\cdot f & \xrightarrow[\pi_f]{} & \overline{\Lambda}_e \end{array}$$

for $\overline{\Lambda}_e := \Lambda e/\Lambda\cap\Lambda e \simeq \Lambda f/\Lambda\cap\Lambda f$ and π_e, resp. π_f being the canonical morphisms. Further, there is a group homomorphism δ_d such that δ_d and the canonical maps induce an exact sequence

$$\Lambda^\times\cap\ker(d)\longrightarrow((\Lambda e)^\times\times(\Lambda f)^\times)\cap\ker(d)\longrightarrow\overline{\Lambda}_e^\times\cap\ker(\overline{d})\stackrel{\delta_d}{\longrightarrow}Cl(\Lambda,d)\longrightarrow Cl(\Lambda e,de)\times Cl(\Lambda f,df).$$

Proof. The result follows from the comments leading to the exactness of

$$\overline{\Lambda}_e^\times\cap\ker(\overline{d})\stackrel{\delta_d}{\longrightarrow}Cl(\Lambda,d)\longrightarrow Cl(\Lambda e,de)\times Cl(\Lambda f,df).$$

Since Λ is an order in a semisimple algebra A and $e^2=e\in Z(A)$, the sequence

$$\Lambda^\times\stackrel{\gamma}{\longrightarrow}((\Lambda e)^\times\times(\Lambda f)^\times)\stackrel{\pi}{\longrightarrow}\overline{\Lambda}_e^\times$$

is exact by the classical situation. Since

$$\begin{array}{ccc} \Lambda & \xrightarrow{\cdot e} & \Lambda\cdot e \\ {\scriptstyle \cdot f}\downarrow & & \downarrow{\scriptstyle \pi_e} \\ \Lambda\cdot f & \xrightarrow[\pi_f]{} & \overline{\Lambda}_e \end{array}$$

is a pullback diagram, we get an induced sequence of group homomorphisms

$$\Lambda^\times\cap\ker(d)\longrightarrow((\Lambda e)^\times\times(\Lambda f)^\times)\cap\ker(d)\stackrel{\pi_d}{\longrightarrow}\overline{\Lambda}_e^\times\cap\ker(\overline{d})$$

which composes to 0, and where π_d is the restriction of π to $((\Lambda e)^\times\times(\Lambda f)^\times)\cap\ker(d)$. If $x\in\ker(\pi_d)$, then there is $v\in\Lambda^\times$ with $\gamma(v)=x$ (where $\gamma:\Lambda^\times\longrightarrow(\Lambda e)^\times\times(\Lambda f)^\times$). We need to see that $v\in\ker(d)$. However, $K\Lambda=K\Lambda e\times K\Lambda f$ and $K\overline{\Lambda}=0$. Hence,

$$\mathrm{id}_K\otimes_R\gamma=\mathrm{id}_A$$

which shows that $x\in\ker(d)$ implies $v\in\ker(d)$. This proves the theorem. ∎

Remark 8.3. Taking homology we can compare the Mayer-Vietoris sequence of Theorem 8.2 with the Mayer-Vietoris sequence of Theorem 8.1. Denoting as before

$$\overline{H(\Lambda,d)}=H(\Lambda,d)/tH(\Lambda,d),$$

and likewise for the other orders occurring in the statement, observing that there is a map $\ker(\overline{d})\longrightarrow H(\overline{\Lambda}_e,\overline{d})$, we get a commutative diagram

$$\begin{array}{ccccccc} ((\Lambda e)^\times\times(\Lambda f)^\times)\cap\ker(d) & \longrightarrow & \overline{\Lambda}_e^\times\cap\ker(\overline{d}) & \stackrel{\delta_d}{\longrightarrow} & Cl(\Lambda,d) & \longrightarrow & Cl(\Lambda e,de)\times Cl(\Lambda f,df) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \overline{H(\Lambda e,de)}^\times\times\overline{H(\Lambda f,df)}^\times & \longrightarrow & \overline{H(\overline{\Lambda}_e,\overline{d})}^\times & \stackrel{\overline{\delta}}{\longrightarrow} & Cl(\overline{H(\Lambda,d)}) & \longrightarrow & Cl(\overline{H(\Lambda e,de)})\times Cl(\overline{H(\Lambda f,df)}) \end{array}$$

In the situation of Theorem 8.2 we shall need a particular subgroup of the unit group of $\overline{\Lambda}$.

Definition 8.4. Let R be a Dedekind domain with field of fractions K of characteristic different from 2 and let (Λ,d) be a differential graded R-order in the finite dimensional semisimple differential graded K-algebra (A,d). Let $e^2=e\in Z(A)$ and $f=1-e$. Let D_e be the image of $((\Lambda e)^\times\times(\Lambda f)^\times)\cap\ker(d)\longrightarrow\overline{\Lambda}^\times\cap\ker(\overline{d})$. Put

$$X_e:=\ker((\overline{\Lambda}_e^\times\cap\ker\overline{d})\to\overline{H(\overline{\Lambda}_e,\overline{d})}^\times)$$

we define homology-isomorphism units of $\overline{\Lambda}_e := \Lambda e/(\Lambda\cap\Lambda e)$ as

$$\overline{\Lambda}^\times_{hi}(e) := X_e/\langle [X_e,X_e],D_e\rangle.$$

Proposition 8.5. *Let R be a Dedekind domain with field of fractions K of characteristic different from 2 and let (Λ,d) be a differential graded R-order in the finite dimensional semisimple differential graded K-algebra (A,d). Suppose that $H(A,d)$ is semisimple as an algebra. Suppose that A and $H(A,d)$ both satisfy the Eichler condition. Let $e^2=e\in Z(A)$ be a non trivial central idempotent of A. Put $f:=1-e$ and $\overline{\Lambda}:=\Lambda e/(\Lambda\cap\Lambda e)$. Then the sequence*

$$\overline{\Lambda}^\times_{hi}(e)\longrightarrow Cl_{hi}(\Lambda,d)\longrightarrow Cl_{hi}(\Lambda e,de)\times Cl_{hi}(\Lambda f,df)$$

is exact.

Proof. The proof only uses standard diagram chasing. Here are the details. Recall from Remark 8.3 that there is a commutative diagram with exact rows

$$\begin{array}{ccccccc}
((\Lambda e)^\times\times(\Lambda f)^\times)\cap\ker(d) & \longrightarrow & \overline{\Lambda}_e^\times\cap\ker(\overline{d}) & \xrightarrow{\delta_d} & Cl(\Lambda,d) & \longrightarrow & Cl(\Lambda e,de)\times Cl(\Lambda f,df)\\
\downarrow & & \downarrow & & \downarrow & & \downarrow\\
\overline{H(\Lambda e,de)}^\times\times\overline{H(\Lambda f,df)}^\times & \longrightarrow & \overline{H(\overline{\Lambda},\overline{d})}^\times & \xrightarrow{\overline{\delta}} & Cl(\overline{H(\Lambda,d)}) & \longrightarrow & Cl(\overline{H(\Lambda e,de)})\times Cl(\overline{H(\Lambda f,df)}).
\end{array}$$

Since the class groups are abelian groups, we can consider the abelianisations of the various unit groups, denoted by the index ab, for short. Put F_e the image of

$$\overline{H(\Lambda e,de)}^\times\times\overline{H(\Lambda f,df)}^\times\longrightarrow\overline{H(\overline{\Lambda},\overline{d})}^\times.$$

This then gives a commutative diagram with exact rows (where we replace D_e and F_e by the image in the abelianisation

$$\begin{array}{ccccc}
\left(\overline{\Lambda}_e^\times\cap\ker(\overline{d})\right)_{ab}/D_e & \overset{\delta_d}{\hookrightarrow} & Cl(\Lambda,d) & \longrightarrow & Cl(\Lambda e,de)\times Cl(\Lambda f,df)\\
\downarrow & & \downarrow & & \downarrow\\
\left(\overline{H(\overline{\Lambda}_e,\overline{d})}^\times\right)_{ab}/F_e & \overset{\overline{\delta}}{\hookrightarrow} & Cl(\overline{H(\Lambda,d)}) & \longrightarrow & Cl(\overline{H(\Lambda e,de)})\times Cl(\overline{H(\Lambda f,df)})
\end{array}$$

Taking kernels of the vertical maps can be extended to a commutative diagram with exact columns

$$\begin{array}{ccccc}
\overline{\Lambda}^\times_{hi} & \xrightarrow{\alpha_K} & Cl_{hi}(\Lambda,d) & \xrightarrow{\beta_K} & Cl_{hi}(\Lambda e,de)\times Cl_{hi}(\Lambda f,df)\\
\downarrow{\scriptstyle\iota_1} & & \downarrow{\scriptstyle\iota_2} & & \downarrow{\scriptstyle\iota_3}\\
\left(\overline{\Lambda}_e^\times\cap\ker(\overline{d})\right)_{ab}/D_e & \overset{\delta_d}{\hookrightarrow} & Cl(\Lambda,d) & \xrightarrow{\epsilon} & Cl(\Lambda e,de)\times Cl(\Lambda f,df)\\
\downarrow{\scriptstyle\pi_1} & & \downarrow{\scriptstyle\pi_2} & & \downarrow{\scriptstyle\pi_3}\\
\left(\overline{H(\overline{\Lambda}_e,\overline{d})}^\times\right)_{ab}/F_e & \overset{\overline{\delta}}{\hookrightarrow} & Cl(\overline{H(\Lambda,d)}) & \longrightarrow & Cl(\overline{H(\Lambda e,de)})\times Cl(\overline{H(\Lambda f,df)})
\end{array}$$

Clearly $\beta_K\circ\alpha_K=0$, which shows $\mathrm{im}(\alpha_K)\subseteq\ker(\beta_K)$.

If $x\in\ker(\beta_K)$, then

$$\iota_2(x)=\delta_d(y)\in\ker(\epsilon)\cap\ker(\pi_2)$$

for some $y\in\overline{\Lambda}^\times\cap\ker(\overline{d})$ and

$$(\overline{\delta}\circ\pi_1)(y)=(\pi_2\circ\delta_d)(y)=\pi_2(x)=0.$$

Since $\overline{\delta}$ is injective, $\pi_1(y) = 0$, which shows that there is $y_1 \in \overline{\Lambda}_{hi}^{\times}$ such that $\iota_1(y_1) = y$. Then

$$(\iota_2 \circ \alpha_K)(y_1) = \delta_d(\iota_1(y_1)) = \delta_d(y) = \iota_2(x).$$

Since ι_2 is injective, $\alpha_K(y_1) = x$. This shows the statement. ■

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