Completions And Isomorphism Type: Eighty Years of Algebra

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February 25, 2013 at the ITB Indonesia

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Theorem (Normal basis theorem)

Let K be a Galois extension of the field k and let G be the Galois group of K over k. Then K is as kG-module isomorphic to the rank one free kG-module:

$$K \simeq kG$$

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$$K \otimes_k M \simeq K \otimes_k N \text{ as } K \otimes_k A\text{-modules},$$

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And this is what we are going to consider in the sequel.

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and $Gal(K:k) =: G$ acts on each component of the direct product by permuting factors.

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Galois Module Problem

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Is $S \simeq RG$ as RG-modules?

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In other words, Λ contains a K-basis of A.

Local versus Complete

If R is a complete discrete valuation domain, it has a residue field k and there are many methods for passing informations from $k \otimes_R \Lambda$ to Λ.

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And so,

$$k \otimes_R M = \overline{M_1} \oplus \overline{M_2}$$

as $k \otimes_R \Lambda$ module implies

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as Λ-module for $\overline{M_i} = k \otimes_R M_i$.

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Can we have a Noether-Deuring theorem for the local-complete case?

Roggenkamp's Noether-Deuring Theorem

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Theorem (Roggenkamp 1972)

If $End_\Lambda(M)$ is finitely presented as R-module. Then $\hat{R} \otimes_R M \simeq \hat{R} \otimes_R N$ as $\hat{R} \otimes_R \Lambda$-modules implies $M \simeq N$ as Λ-modules.

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Roggenkamp's Noether-Deuring Theorem

The proof of Roggenkamp's theorem

- uses some methods of homological algebra
- applies to classical orders
- is valid even for the property of being direct factor instead of isomorphism
- applies also when replacing $\hat{R}$ by S, a commutative R-algebra, which is faithfully projective of finite type as R-module.

Derived Categories; Introduction

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"when you take homological algebra seriously".

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The derived category admits an endo-functor [1] given by shift in the degree of the grading, and forms a so-called triangulated category.

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Theorem (Keller 1990; Rickard 1989)

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Tilting complexes have many interesting properties.

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Let R be a complete local ring with residue field k and let Λ be an R-free R-algebra of finite rank over R. Let $\overline{\Lambda} := k \otimes_R \Lambda$.

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$$End_{D^b(\overline{\Lambda})}(\overline{T}) \simeq k \otimes_R End_{D^b(\Lambda)}(T).$$

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As for modules, the lifting theorem is not true for local coefficient rings R only.

Equivalences between derived categories

As for modules, the lifting theorem is not true for local coefficient rings R only.
Hence, a Noether-Deuring like theorem passing from the complete local situation to the local situation is desirable.

The Noether-Deuring theorem for derived categories

We obtain

Theorem (A.Z. (2012))

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$$S \otimes_R X \simeq S \otimes_R Y \text{ in } D^b(S \otimes_R \Lambda)$$

implies

$$X \simeq Y \text{ in } D^b(\Lambda).$$

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- It is necessary to work over the bonded derived category. For technical reasons concerning tensor products in the unbounded case the proof will not work.
- Λ can be a classical order.
- The proof really uses derived category techniques and the triangulated structure.

The Noether-Deuring theorem for derived categories

Corollary

The same statement as in the theorem holds if R is commutative, semilocal, Noetherian, S a commutative R-algebra so that $\hat{R} \otimes_R S$ is a faithful projective $\hat{R}$-module of finite type.

Some remarks on the proof

We first show that

Lemma

$$Hom_{D^b(S\otimes_R\Lambda)}(S\otimes_R X, S\otimes_R Y)\simeq S\otimes_R Hom_{D^b(\Lambda)}(X,Y)$$

if S is faithfully flat as R-module.

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Lemma (Change of Rings Theorem)

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Again, this is a very classical change of rings theorem.

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and use $R/rad(R) = S/rad(S)$ to modify s_i by r_i so that the difference is in the radical.
$\varphi := \sum_i \varphi_i r_i$ is then an isomorphism since invertible module radical is invertible. (Nakayama's lemma)

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Some remarks on the proof

Then the corollary

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The same statement as in the theorem holds if R is commutative, semilocal, Noetherian, S a commutative R-algebra so that $\hat{R} \otimes_R S$ is a faithful projective $\hat{R}$-module of finite type.

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Some remarks on the proof

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to the theorem is shown the following way:

$$\hat{R} \otimes_R S \simeq \prod_{\wp_i \in Spec(R)} (\widehat{R}_{\wp_i})^{n_i}$$

(think of ramifications)

Some remarks on the proof

and hence

$$S \otimes_R \hat{R} \otimes_R X \simeq S \otimes_R \hat{R} \otimes_R Y$$

$$\Rightarrow$$

$$\prod_{\wp_i \in Spec(R)} (\widehat{R}_{\wp_i})^{n_i} \otimes X \simeq \prod_{\wp_i \in Spec(R)} (\widehat{R}_{\wp_i})^{n_i} \otimes_R Y$$

Some remarks on the proof

Comparing coefficient domains gives

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