On Equivalences Between Categories Of Representations

Alexander Zimmermann[1]

[1]*LAMFA (UMR 7352 du CNRS)*
Département de Mathématiques
Université de Picardie
33 rue St Leu
80039 Amiens
FRANCE.

URL: http://www.lamfa.u-picardie.fr/alex/azim.html

Abstract. We explain some of the goals of modern representation theory, aiming at categorical methods. We develop one of the most astonishing invariant, Hochschild (co-)homology and we explain on the example of the recent solution of a question due to Rickard how it is possible to reduce fairly abstract questions to explicit methods finally solved by computers.

Keywords: representations of algebras, derived equivalence, stable equivalence, stable equivalence of Morita type, Hochschild cohomology, Hochschild homology, Gerstenhaber structure Brauer group

INTRODUCTION

Representation theory deals with realising abstract structures, such as groups, algebras, Lie algebras, as group of matrices, algebra of matrices, or Lie algebras of matrices over a fixed base ring. Any such realisation is called a module, or synonymously a representation. We will consider here representations of algebras.

It is immediately clear that the concept is most fruitful if not only one single representation is considered, but if all representations at once, possibly subject to certain restrictions such as being finite dimensional, are object of study, and moreover including the relations between them. We are lead automatically to the concept of a module category, denoted $A-\mathrm{mod}$ in case of an algebra A.

Two algebras A and B are called Morita equivalent whenever the categories $A-\mathrm{mod}$ and $B-\mathrm{mod}$ are equivalent as abstract categories. Morita's theorem gives a quite precise description of this situation, and it turns out that it is highly restrictive. A basic example is that the category of vector spaces over a field K is equivalent with the category of modules of an $n\times n$ matrix algebra over K. Morita's theorem states basically that in some sense this is the model for the general case, and in some sense all cases look like this.

In order to be able to include more interesting cases, when representations of two algebras behave similar but are not equivalent, it became desirable to consider weaker equivalences. The general procedure is to form $A-\mathrm{mod}$, the category of modules, and then apply a general procedure to this category, not depending of the underlying algebra, but only on the abstract category. One obtains a new category $C(A)$, and in case $C(A)$ is equivalent to $C(B)$ we can consider properties of A which are encoded in $C(A)$, and hence are shared by B.

This general procedure was applied by Auslander, Reiten and their school in the 1970's to what we call the stable category $A-\underline{\mathrm{mod}}$. Except in special cases properties encoded in $A-\underline{\mathrm{mod}}$ are rare, and the concept was not carried too far, except when A is self-injective and in this case $A-\underline{\mathrm{mod}}$ is a so-called triangulated category, carrying a rich structure. A most prominent example of this situation comes from group representations. Starting from a finite group G and a field K, we may form the so-called group algebra KG, which is then a self-injective algebra and quite a number of properties of G are encoded in $KG-\underline{\mathrm{mod}}$ if K is a field of characteristic p dividing the order of G.

In the general setting it was in the 1980's when Sheila Brenner and Michael Butler discovered what they called tilting of algebras, a special case of what will then later be called derived equivalence. Jeremy Rickard motivated by Dieter Happel's work discovered that a most interesting case appears when one forms $C(A) = D^b(A)$, the bounded derived category. Again, this is a triangulated category, as it was introduced by Grothendieck and Verdier in the context of their revolutionary renewal of algebraic geometry. Algebras A and B with equivalent derived categories $D^b(A)$ and $D^b(B)$ are called derived equivalent. Rickard's first main theorem give an appealing necessary and sufficient criterion when two algebras are derived equivalent.

Derived equivalent algebras share many properties, and, at least philosophically, in particular those of so-called homological nature. Rickard's second main theorem gives the necessary tool for this, showing that if A and B are derived equivalent, then there is an equivalence of a specially nice shape, called of standard type. In case A and B are self-injective, then derived equivalences of standard type induce equivalences between the stable categories of A and of B of a similar particularly nice nature, named stable equivalences of Morita type. This concept was introduced by Michel Broué, mainly motivated by applications to representations of groups, but was then carried further by ChangChang Xi and his school in the general abstract setting.

However, not all stable equivalences of Morita type are induced by derived equivalences. Therefore, one natural question is to ask for those homological properties which are invariant under derived equivalences, and which stay invariant under stable equivalences of Morita type.

We will introduce the reader to this theory and give an overview of some of the relevant questions in more detail. We also explain some invariants, and focus mainly on Hochschild homology and cohomology, together with their rich structure. We explain how degree 0 Hochschild cohomology then allowed us in joint work with Yuming Liu, Guodong Zhou and Serge Bouc to settle a question of Rickard. Finally we answer a question posed in Marco Armenta's 2019 thesis using a number theoretical approach from class field theory.

We made an effort to start most elementary and increase complexity during the text. We do not give any proof, in order to avoid to become technical. Nevertheless, we always give references where to find a complete treatment.

REPRESENTATIONS

Our basic object of study are K-algebras over a field K. In general the field K is left as broad as possible, and could be $\mathbb{Q}$, the rational numbers, $\mathbb{R}$ the real numbers, but most often $\mathbb{C}$ the complex numbers. The latter is appropriate since $\mathbb{C}$ is algebraically closed, meaning that all non constant polynomials with coefficients in $\mathbb{C}$ have roots. The field K may also be a field of finite characteristic, such as $\mathbb{Z}/p\mathbb{Z}$ for some prime p, or any extension. It is a well-known result that a finite field K has $q = p^n$ elements, for some integer $n > 0$ and some prime p. Moreover, for any field K there is an up to isomorphism unique field $\overline{K}$ containing K and which is algebraically closed. When we speak of a ring we always mean a unital, associative ring, but we do not assume commutativity in general.

Definition 1 A K-algebra A is a (not necessarily commutative) ring, equipped with a ring homomorphism

$$\varepsilon : K \to Z(A) := \{a \in A \mid \forall b \in A : ba = ab\}.$$

Morally, a K-algebra is a ring, equipped with an additional and compatible K-vector space structure. In order to shorten notation we write λa instead of $\varepsilon(\lambda)a$ for any $a \in A$ and any $\lambda \in K$. Algebras can be very complicated. However, there is a particularly well understood example.

Example 2 For any integer n the set of $n \times n$ matrices over K form a K-algebra. We denote this algebra by $\mathrm{Mat}_{n\times n}(K)$, or shorter by $\mathrm{Mat}_n(K)$ The ring structure is given by addition of matrices coefficient-wise, and multiplication is given by multiplication of matrices. The homomorphism is given by

$$K \ni \lambda \mapsto \begin{pmatrix} \lambda & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \lambda \end{pmatrix} \in \mathrm{Mat}_{n\times n}(K).$$

Now, comparing two algebras is done by an algebra homomorphism.

Definition 3 Given two algebras A and B an algebra homomorphism $\alpha : A \to B$ is a ring homomorphism α such that in addition $\alpha(\lambda a) = \lambda\alpha(a)$ for any $a \in A$ and any $\lambda \in K$.

We now come to the fundamental

Definition 4 Let K be a field and let A be a K-algebra. A representation of A of dimension n is an algebra homomorphism $\mu : A \to \mathrm{Mat}_n(K)$.

Let $V = K^n$. Then $\mathrm{Mat}_n(K)$ acts on V by simple multiplication of matrices. If $\mu : A \to \mathrm{Mat}_n(K)$ is an A-module, then A acts on V by $a \cdot m := \mu(a)(m)$ for all $a \in A, m \in V$. We say that V is an A-module in this case. The notions of A-module and representation of A are somewhat parallel, and it is not useful to distinguish between these concepts. An A right module is given by a map $\mu : A \to \mathrm{Mat}_n(K)$ such that transpose $\circ\, \mu$ is an algebra homomorphism. Here transpose : $\mathrm{Mat}_n(K) \to \mathrm{Mat}_n(K)$ is the map on matrices given by transposing the matrix. Note that the concept of an A right module M is equivalent with giving an action of A on the right of M: for all $m \in M$ and $a \in A$ let $m \cdot a := \mu(a)(m)$. Using the transpose then ensures that for all $a, b \in A$ and $m \in M$ we get $(ma)b = m(ab)$. Given two K-algebras A and B, an $A - B$-bimodule is a K-vector space M which is an A left module, a B right module and such that the action of K on the left and on the right coincides.

Fixing a K-algebra A a homomorphism of A-modules V to W is given by a K-linear map $\varphi : V \longrightarrow W$ such that $\varphi(a \cdot m) = a \cdot \varphi(m)$ for all $a \in A, m \in V$. Isomorphisms are bijective homomorphisms, as usual, and for two A-modules M and N we write $M \simeq N$ if we want to express that there exists an isomorphism $M \to N$. In this case we say that M and N are isomorphic. Finally, let $\mathrm{Hom}_A(M, N)$ be the set of all A-module homomorphisms $M \to N$.

If A is a K-algebra, then A itself is an A-module with $\mu(a)(b) = ab$. This module is called the regular module. Similarly for any $n \in \mathbb{N}$ we have A^n is an A-module, where the above μ is now taken componentwise. We call A^n the rank n free A-module. Given two A-modules M and N we can form by the same procedure $M \times N$ as K-vector space. This becomes an A-module as above and is denoted $M \oplus N$.

Definition 5 An A-module P is called projective if there is another A-module Q such that $P \oplus Q$ is isomorphic to a free module.

Example 6 The setting of a module over an algebra applies also to representations of finite groups. Let G be a finite group and let K be a field. The group ring KG is the K-vector space with basis G. The structural law of a group can be extended K-bilinearly to a multiplicative structure on all of KG. Altogether this produces a K-algebra. A KG-module is, by definition, the same as what is classically known as a representation of G over K.

As the algebra $\mathrm{Mat}_n(K)$ is very easy to understand, compared to an abstract algebra A, it is unlikely to be able to obtain a lot of information by just looking at one module only. We consider $A - \mathrm{mod}$, the class of all A-modules of finite dimension together with the data of all A-module homomorphisms $V \to W$ for any two fixed finite dimensional A-modules V, W. This data now has a richer structure, it is a so-called category.

CATEGORIES

We will give in this section some elements of the concepts in category theory needed for the rest of the paper. All what is sketched in this section can be found in many textbooks. We suggest [26, Chapter 1, Chapter 3] for more details on the concepts in this section.

Some general theory

A category is a rather abstract and very general gadget. However, it can be equipped with a lot of additional structure, and then it is very much part of even classical mathematics. In particular, categories are the appropriate setting to work with as it concerns our question.

Definition 7 A category C consists in a class of objects $obj(C)$, and for all two objects V, W a set $C(V, W)$ called morphisms, and a for all three objects U, V, W a map

$$\circ : C(V, W) \times C(U, V) \longrightarrow C(U, W),$$

called composition, satisfying the following axioms:

- composition is associative

- For each object V there is an object $1_V \in C(V, V)$ such that for all objects U, W and all $\alpha \in C(V, W)$ and all $\beta \in C(U, V)$ we have $1_V \circ \beta = \beta$ and $\alpha \circ 1_V = \alpha$.

For each field K a category C is called K-linear if for each V, W the set $C(V, W)$ is a K-vector space and composition is K-bilinear.

Occasionally we denote by $\mathrm{Hom}_C(V, W)$ the set $C(V, W)$ of a category C and objects V, W of C.

It is easy to see that $A - \mathrm{mod}$ is a K-linear category. It is actually more, an abelian category. We will not go into detail to this technical statement since it would lead us much too far.

Nevertheless, now we can describe our first modification of $A - \mathrm{mod}$. Recall that for a vector space V and a sub vector space W we denote by V/W the quotient space.

Definition 8 Let K be a field and let A be a K-algebra. The stable category $A - \underline{\mathrm{mod}}$ is the category with objects being finite dimensional A-modules (just as those of $A - mod$). For any two A-modules M and N let

$$\mathrm{PHom}_A(M, N) := \{\alpha \in \mathrm{Hom}_A(M, N) \mid \exists P \text{ projective and } \beta \in \mathrm{Hom}_A(M, P), \gamma \in \mathrm{Hom}_A(P, N) : \alpha = \gamma \circ \beta\}$$

and morphisms in $A - \underline{mod}$ are then

$$(A - \underline{\mathrm{mod}})(M, N) := \mathrm{Hom}_A(M, N)/\mathrm{PHom}_A(M, N).$$

Composition is given by composition of A-module homomorphisms.

Note that we need to show that the composition is actually well-defined, i.e. does not depend on the representative taken. But this is clear, since if $\alpha \in \mathrm{PHom}_A(M, N)$, then for all $\sigma \in \mathrm{Hom}_A(N, L)$ and $\tau \in \mathrm{Hom}_A(S, M)$, also $\sigma \circ \alpha \in \mathrm{PHom}_A(M, L)$ and $\alpha \circ \tau \in \mathrm{PHom}_A(S, N)$.

We still need a concept to compare categories. This concept is called a functor.

Definition 9 Let C and $\mathcal{D}$ be two categories. A functor $F : C \to \mathcal{D}$ is given by associating for each object X of C an object $F(X)$ of $\mathcal{D}$ and for any two objects X, Y of C a map $C(X, Y) \to \mathcal{D}(F(X), F(Y))$ such that $F(1_X) = 1_{F(X)}$ for any object X of C and for any objects X, Y, Z of C and any $\alpha \in C(Y, Z)$ and any $\beta \in C(X, Y)$ we have $F(\alpha \circ \beta) = F(\alpha) \circ F(\beta)$.

The identity functor id_C on a category C is the functor being the identity on the objects and the identity on the morphisms. If $F : C \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$ are functors, then the composition $G \circ F$ in the obvious way is again a functor.

Examples of functors are legendary. For any K-algebra A and any fixed A-module M we define $F = \mathrm{Hom}_A(M, -) : A - \mathrm{mod} \to K - \mathrm{mod}$ the functor defined by mapping any A-module N to the K-vector space $\mathrm{Hom}_A(M, N)$ and any morphism $\alpha \in \mathrm{Hom}_A(N, L)$ to the map $\mathrm{Hom}_A(M, N) \ni \beta \mapsto \alpha \circ \beta \in \mathrm{Hom}_A(M, L)$. We denote the above map by $\mathrm{Hom}_A(M, \alpha)$. This type of functor is important for structural reason, and is called representable. Similarly, one has what we call a contravariant functor associating to any N the vector space $\mathrm{Hom}_A(N, M)$ and and any morphism $\alpha \in \mathrm{Hom}_A(N, L)$ the map $\mathrm{Hom}_A(L, M) \ni \beta \mapsto \beta \circ \alpha \in \mathrm{Hom}_A(N, M)$, denoted $\mathrm{Hom}_A(\alpha, M)$.

We sometimes need to compare functors.

Definition 10 Let C and $\mathcal{D}$ be two categories and let $F : C \to \mathcal{D}$ and $G : C \to \mathcal{D}$ be two functors. A natural transformation $\eta : F \to G$ consists in the following data: For every object X of C an element $\eta_X \in \mathcal{D}(FX, GX)$ such that

$$\begin{array}{ccc} C(X, Y) & \xrightarrow{F} & \mathcal{D}(FX, FY) \\ \downarrow G & & \downarrow \mathcal{D}(FX, \eta_Y) \\ \mathcal{D}(GX, GY) & \xrightarrow[\mathcal{D}(\eta_X, GY)]{} & \mathcal{D}(FX, GY) \end{array}$$

is a commutative diagram.

A natural isomorphism is a natural transformation $\eta : F \to G$ such that η_X is an isomorphism for each object X. We denote in this case $\eta : F \xrightarrow{\simeq} G$, or for short $F \simeq G$.

Definition 11 Let C and $\mathcal{D}$ be two categories. Then C is equivalent to $\mathcal{D}$ if there is a functor $F : C \to \mathcal{D}$ and a functor $G : \mathcal{D} \to C$ such that $F \circ G \simeq id_{\mathcal{D}}$ and $G \circ F \simeq id_C$. We call G a quasi-inverse to F.

Example 12 Let A be a K-algebra. Then we get a natural functor $\Pi_A : A - \text{mod} \to A - \underline{\text{mod}}$ given by associating an A-module M to the same A-module M, and mapping an A-module homomorphism $\alpha : M \to N$ to its class $\alpha + \text{PHom}_A(M, N)$ in $(A - \underline{\text{mod}})(M, N)$.

If B is another K-algebra, then we get again a functor $\Pi_B : B - \text{mod} \to B - \underline{\text{mod}}$. Suppose now $F : A - \text{mod} \to B - \text{mod}$ is a functor, and suppose that for any projective A-module P again $F(P)$ is a projective B-module. Then $\underline{F} : A - \underline{\text{mod}} \to B - \underline{\text{mod}}$ which is defined by $\underline{F}(X) := F(X)$ for any A-module, and for any $\alpha \in \text{Hom}_A(M, N)$ let $\underline{F}(\alpha + \text{PHom}(M, N)) := F(\alpha) + \text{PHom}_B(F(M), F(N))$. This is well-defined and yields a commutative diagram of functors between categories as follows:

$$\begin{array}{ccc} A - \text{mod} & \xrightarrow{F} & B - \text{mod} \\ \downarrow{\scriptstyle \Pi_A} & & \downarrow{\scriptstyle \Pi_B} \\ A - \underline{\text{mod}} & \xrightarrow[\underline{F}]{} & B - \underline{\text{mod}} \end{array}$$

However, given a functor, or even an equivalence $G : A - \underline{\text{mod}} \to B - \underline{\text{mod}}$, then it is in general not true that there is a functor $F : A - \text{mod} \to B - \text{mod}$ such that $\underline{F} \simeq G$. We will see examples in Theorem 46 below.

Finally we will need the concept of a full subcategory. Given a category C. A subcategory $\mathcal{S}$ of C consists in a subclass of objects, and for any two objects X, Y of $\mathcal{S}$ a subset $\mathcal{S}(X, Y) \subseteq C(X, Y)$ such that $\mathcal{S}$ is again a category with composition of morphisms being the restriction to $\mathcal{S}$ of the composition of morphisms in C.

A subcategory is full if $\mathcal{S}(X, Y) = C(X, Y)$ for any two objects of X, Y of $\mathcal{S}$.

(Co-)homology

Let A be a K-algebra for a field K, and let M be an A-module. A submodule of M is just a subvector space N of M such that for all $a \in A$ and $n \in N$ we have $a \cdot n \in N$. In other words, the action of A on the subspace N gives a module structure on N. The quotient vector space M/N is again an A-module with action of $a \in A$ on $m + N \in M/N$ defined to be $a \cdot (m + N) := (a \cdot m) + N$, if N is an A submodule of M. A non zero A-module M is called simple if the only submodules of M are $\{0\}$ and M. If M is a finite dimensional A-module, then there is a maximal submodule N_1, in the sense that $M/N_1 = S_1$ is simple. Since N_1 is again finite dimensional, there is again a maximal submodule N_2 of N_1 with $S_2 := N_1/N_2$ simple, and continuing this way we obtain for every finite dimensional A-module M a set of simple modules S which are obtained as $U/V = S$ for submodules $V \subseteq V \subseteq M$ of M. We call the set (keeping multiplicities of isomorphism classes of modules!) $\{S_1, S_2, \ldots S_k\}$ of these simple submodules composition factors of M and the above constructed sequence of submodules $M \supset N_1 \supset N_2 \supset \cdots \supset N_k = S_k \supset \{0\}$ with $N_i/N_{i+1} =: S_i$ simple is called a composition series. Of course, a priori the composition factors depend on the composition series.

Theorem 13 (Jordan-Hölder) *Let K be a field and let A be a K-algebra. Let M be a finite dimensional A-module. Then the composition factors, as set of isomorphism classes of simple modules, keeping track of the possible multiplicity of the isomorphism class of a simple module occurring several times in a composition series, is independent of the chosen composition series.*

The situation is even better. If A is a finite dimensional K-algebra, then there are only finitely many isomorphism classes of simple A-modules. So, imagine we know these, and consider them as 'bricks' to build our 'house', i.e. our module. What we need is the 'mortar'. This is what will be considered now.

So, try the converse procedure. Given two finite dimensional A-modules U and V. What are the possible A-modules M such that U is isomorphic to a submodule U' of M, and such that M/U' is isomorphic to V? There is always at least one such module, namely $M = U \oplus V$, but in general this is not the only one. Consider the set of schemes

$$0 \to U \xrightarrow{\iota} M \xrightarrow{\pi} V \to 0$$

such that M is an A-module, such that ι and π are A-module homomorphisms, such that $\ker(\pi) := \pi^{-1}(0) = \iota(U)$, such that ι is injective and such that π is surjective. Two such schemes

$$0 \to U \xrightarrow{\iota} M \xrightarrow{\pi} V \to 0$$

and

$$0 \to U \xrightarrow{\iota'} L \xrightarrow{\pi'} V \to 0$$

are called equivalent if there is an A-module homomorphism $\alpha : M \to L$ such that $\alpha \circ \iota = \iota'$ and $\pi' \circ \alpha = \pi$.

The important observation is the following statement.

Theorem 14 *The equivalence classes* $\mathrm{Ext}^1_A(V, U)$ *of such schemes form a group, actually a* K*-vector space.*

Given a K-algebra A over a field K, and a (finite dimensional, to simplify) A-module V, we can fix a K-basis B of V, and obtain a surjective homomorphism $\pi : A^{|B|} \to V$ by mapping a sequence $(a_b)_{b\in B}$ to $\sum_{b\in B} a_b b \in V$. Let $\Omega(V) := \ker(\pi)$. We may define inductively $\Omega^i(V) := \Omega(\Omega^{i-1}(V))$ for all $i \geq 1$, with $\Omega^0(V) = V$.

Definition 15 Given a field K and a finite dimensional K-algebra A. Then, for all finite dimensional A-modules V and all A-modules U let $\mathrm{Ext}^i_A(V, U) := \mathrm{Ext}^1_A(\Omega^{i-1}(V), U)$ for all $i \geq 1$.

It can be shown that this definition does not depend on the basis B, and, being slightly more careful, can also be used without the hypothesis on the dimension of A and V.

Can we understand the vector spaces $\mathrm{Ext}^i_A(V, U)$ in a systematic fashion? This is done in the next subsection.

Derived categories

We want to construct a category such that $\mathrm{Ext}^1_A(V, U)$ are just homomorphisms in this category. Consider the schemes

$$0 \to U \stackrel{\iota}{\to} M \stackrel{\pi}{\to} V \to 0$$

as above. Observe $\pi \circ \iota = 0$. We consider more generally schemes

$$\cdots \to C_n \stackrel{\delta_{n-1}}{\to} C_{n-1} \stackrel{\delta_{n-2}}{\to} C_{n-2} \stackrel{\delta_{n-3}}{\to} \cdots \stackrel{\delta_m}{\to} C_m \stackrel{\delta_{m-1}}{\to} C_{m-1} \to \cdots$$

such that all C_i are A-modules, all δ_i are A-module homomorphisms, and such that $\delta_{i-1} \circ \delta_i = 0$ for all i. Such schemes are called complexes, the maps δ_i are called differentials, and mostly we will assume that there is $n_0 \in \mathbb{Z}$ such that $C_m = 0$ for all $m < n_0$. The homology of this complex $(C_\bullet, \delta_\bullet)$ is the sequence of A-modules $H_i(C_\bullet) := \ker(\delta_{i-1})/\delta_i(C_{i+1})$. An easy procedure can be applied to such complexes. Let $(C_\bullet, \delta_\bullet)$ be a complex. Then denote by $(C_\bullet, \delta_\bullet)[1]$ the complex with $(C_\bullet[1])_i := C_{i+1}$ and $(\delta_\bullet[1])_i := \delta_{i+1}$. This is again a complex, 'shifted by 1 to the left.'

Let

$$\cdots \to C_n \stackrel{\delta_{n-1}}{\to} C_{n-1} \stackrel{\delta_{n-2}}{\to} C_{n-2} \stackrel{\delta_{n-3}}{\to} \cdots \stackrel{\delta_m}{\to} C_m \stackrel{\delta_{m-1}}{\to} C_{m-1} \to \cdots$$

et

$$\cdots \to D_n \stackrel{d_{n-1}}{\to} D_{n-1} \stackrel{d_{n-2}}{\to} D_{n-2} \stackrel{d_{n-3}}{\to} \cdots \stackrel{d_m}{\to} D_m \stackrel{d_{m-1}}{\to} D_{m-1} \to \cdots$$

be two complexes of A-modules. A homomorphism of complexes $C_\bullet \to D_\bullet$ is a sequence $\alpha_i : C_i \to D_i$ of A-module homomorphisms such that $\alpha_i \circ \delta_i = d_i \circ \alpha_{i+1}$ for all $i \in \mathbb{Z}$. The notion of isomorphism is the obvious one.

It is not difficult to show that a homomorphism of complexes induces a homomorphism of the homology of the complexes. The converse is false in general, and even worse, the homomorphism of complexes may not be an isomorphism even though the homomorphism on the homology of the complexes may be an isomorphism.

There is a rather sophisticated procedure to modify the homomorphism sets of complexes to get that any homomorphism of complexes, inducing an isomorphism on the homology of the complexes is actually an isomorphism.

Definition 16 (sketch) The derived category of A-modules $D(A)$ is the category with objects being complexes, and morphisms being some classes of morphisms of complexes, modified in such a way that an isomorphism on the homology of the complexes is actually an isomorphism.

The bounded derived category of A-modules $D^b(A)$ is the subcategory of $D(A)$ with objects being complexes $(D_\bullet, \delta_\bullet)$ such that there is n_0 with $C_i = 0$ for all i with $|i| > n_0$.

For a more precise definition see e.g. [26, Chapter 3]. A precise definition would go much too far here. However, the above "Definition 16" is precise enough for our purpose. We note however that any A-module M can be considered as an object in $D^b(A)$. Indeed, M can be identified with the complex $M_\bullet$ where $M_0 = M$, $M_i = 0$ for all $i \neq 0$, and of course all differentials 0. This actually gives a functor $L : A - \mathrm{mod} \to D^b(A)$. It has nice properties such as that it induces an isomorphism

$$\mathrm{Hom}_A(M, N) \simeq \mathrm{Hom}_{D^b(A)}(LM, LN).$$

Theorem 17 *Consider two A-modules U and V. Then, for all $i \geq 1$ we get*

$$\mathrm{Ext}^i_A(V,U) \simeq \mathrm{Hom}_{D^b(A)}(LV, LU[i]).$$

Already this property convinces us that the derived category is an interesting object. It encodes our 'mortar' in a structural fashion. Moreover, it is precisely the long term project we were aiming at. Starting from the module category we form another one, the derived category by some universal procedure, having different, somehow less rigid properties.

We should emphasize that the derived category has a rich structure. It is a triangulated category. This highly sophisticated property goes far beyond our introduction. The interested reader may like to look up these details from e.g. [26, Chapter 3].

Some astonishing homological invariant: Hochschild homology

The derived category is well-suited to define some of the most important and most used invariants which are of relevance for us.

First, for a K-algebra A we denote by A^{op} the K-vector space A equipped with a multiplicative structure

$$a \cdot_{op} b := b \cdot a.$$

Here we denote by $\cdot_{op}$ the multiplicative structure of A^{op} and by $\cdot$ the multiplicative structure of A.

Then for any two K-algebras A and B we get that $B \otimes_K A^{op}$ is a K-algebra again, and the $B \otimes_K A^{op}$-modules are precisely the $B - A$-bimodules, in the sense that such a bimodule M is an A right module and a B left module, and the field K acts the same way on the left and on the right. If $B = A$, then A is clearly an $A \otimes_K A^{op}$-module. But also $\mathrm{Hom}_K(A, K)$ is an $A \otimes_K A^{op}$-module by putting for any $f \in \mathrm{Hom}_K(A, K)$ and any $a, b, c \in A$

$$(a \cdot f \cdot b)(c) := f(bca).$$

Definition 18

- The degree n Hochschild cohomology $HH^n(A)$ of A is $\mathrm{Ext}^n_{A\otimes_K A^{op}}(A, A)$.
- The K-dual of the degree n Hochschild homology $HH_n(A)$ of A is $\mathrm{Ext}^n_{A\otimes_K A^{op}}(A, \mathrm{Hom}_K(A, K))$.

The slightly cumbersome definition of the Hochschild homology is valid for finite dimensional algebras over a field, and follows by some expression due to Cartan-Eilenberg. In general Hochschild homology is defined by some torsion group and would need some more preparation. I learned the definition of Hochschild homology as above from the thesis of Marco Antonio Armenta.

Remark 19 Note the degree 0 cases. First, $HH^0(A) = Z(A) = \{b \in A \mid \forall a \in A : ba = ab\}$ is the centre of the algebra A. Second, $HH_0(A) = A/[A, A]$ where $[A, A]$ is the K-subvector space of A generated by the expressions $ab - ba$ for all $a, b \in A$.

Hochschild cohomology and homology have stunning properties. It is a very rich and quite rigid structure in itself.

First, let

$$HH^\bullet(A) := \bigoplus_{n\in\mathbb{N}} HH^n(A).$$

This is clearly a $\mathbb{Z}$-graded vector space. Recall that a commutative K-algebra is a K-algebra which is a commutative ring. A graded K-algebra is a K-algebra G such that we have a decomposition as vector spaces

$$G = \bigoplus_{n\in\mathbb{Z}} G_n$$

such that for any $a \in G_n$ and $b \in G_m$ we have $a \cdot b \in G_{n+m}$. A graded K-algebra is graded commutative if for any $a \in G_n$ and $b \in G_m$ we have $a \cdot b = (-1)^{nm} b \cdot a$. Hence graded commutative K-algebras are almost commutative in the sense that it is commutative up to a sign. The sum of all even degree subspaces give a commutative algebra.

Theorem 20 (Gerstenhaber [7]) *The vector space $HH^\bullet(A)$ is a graded commutative K-algebra. The multiplicative structure is called cup product, and denoted as, the $\cup$ product.*

Actually, the cup product is not complicated. It is basically the composition of maps in the derived category.

But this is not all. There is an additional graded Lie algebra structure. Recall that a Lie algebra is a K-vector space $\mathfrak{g}$ and K-bilinear law

$$[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$$

such that

$$[x, y] + [y, x] = 0 \text{ and } [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

for all $x, y, z \in \mathfrak{g}$. A graded Lie algebra $\mathfrak{g}$ is a $\mathbb{Z}$-graded vector space $\mathfrak{g} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_n$ such that for all $x \in \mathfrak{g}_n$, $y \in \mathfrak{g}_m$, $z \in \mathfrak{g}_p$ we get

- $[x, y] \in \mathfrak{g}_{n+m}$,
- $[x, y] + (-1)^{mn}[y, x] = 0$
- $(-1)^{np}[x, [y, z]] + (-1)^{mn}[y, [z, x]] + (-1)^{pm}[z, [x, y]] = 0$

Theorem 21 (Gerstenhaber [7]) *Let K be a field and let A be a finite dimensional K-algebra. Then the vector spaces $\mathfrak{g}_n := HH^{n+1}(A)$ allow a graded Lie algebra structure $[\ ,\ ]$ such that in addition*

$$[x \cup y, z] = [x, z] \cup y + (-1)^{(m+1)p} x \cup [y, z]$$

for all $x \in \mathfrak{g}_n$, $y \in \mathfrak{g}_m$, $z \in \mathfrak{g}_p$.

Definition 22 A Gerstenhaber algebra is a graded vector space $G^\bullet := \bigoplus_{n\in\mathbb{Z}} G^n$ together with a multiplicative structure $\cup$ such that $(G, \cup)$ is a graded commutative K-algebra, a bracket $[\ ,\ ] : G^\bullet \times G^\bullet \longrightarrow G^\bullet$ such that with $\mathfrak{g}_n := G^{n+1}$ and $\mathfrak{g} := \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_n$ we get $(\mathfrak{g}, [\ ,\ ])$ is a graded Lie algebra satisfying in addition

$$[x \cup y, z] = [x, z] \cup y + (-1)^{(m+1)p} x \cup [y, z]$$

for all $x \in \mathfrak{g}_n$, $y \in \mathfrak{g}_m$, $z \in \mathfrak{g}_p$.

And it is still not finished. $HH^\bullet(A)$ acts on $HH_\bullet(A)$ in the sense that $HH_\bullet(A)$ is an $HH^\bullet(A)$-module. Not only this, the action respects the associative and the Lie structure.

More precisely, there is a K-bilinear map

$$\cap : HH^\bullet(A) \times HH_\bullet(A) \longrightarrow HH_\bullet(A)$$

such that for all $x \in HH^n(A)$ and for all $z \in HH_m(A)$ we get $x \cap z \in HH_{m-n}(A)$ whenever $m \geq n$ and $x \cap z = 0$ if $m < n$.

EQUIVALENCES OF DERIVED, STABLE, MODULE CATEGORIES

In the previous section we had three levels.

- We considered the category of modules over an algebra,
- we considered the stable category,
- and we considered the derived category.

What can we say when for two algebras A and B we have equivalences of the categories on one of these levels. We shall give a brief introduction in the present section.

Morita equivalences

The most rigid case is the case of equivalences between the module categories of A and of B. We already have seen the concept of a projective module. We need the concept of a generator.

Definition 23 An object G of a category C is a generator if for every two objects X and Y of C and all morphisms $f, g \in C(X, Y)$ there is a morphism $h \in C(G, X)$ such that $h \circ f \neq h \circ g$.

In case of K-linear categories, replacing f by $f-g$ the condition is equivalent to finding for any $f \in C(X,Y)$ an $h \in C(G,X)$ with $h \circ f \neq 0$. For module categories a slightly more complicated argument shows that a generator is an object G such that for each object X there is an index set I_X and a surjective homomorphism $G^{I_X} \to X$.

The most famous Morita theorem characterises equivalences between module categories completely. We formulate the result in a way suitable for our text, though the theorem is much more general.

Theorem 24 (Morita; cf e.g. [26] Chapter 4) *Let K be a field and let A and B be finite dimensional K-algebras. Then $A-\mathrm{mod} \simeq B-\mathrm{mod}$ if and only if there is a projective A-module M, which is a generator for $A-\mathrm{mod}$, such that $End_A(M) \simeq B^{op}$. In this case M is naturally a projective B-right module, a generator for $B-\mathrm{mod}$, and $End_B(M) \simeq A^{op}$. Moreover, $M \otimes_B - : B-\mathrm{mod} \to A-\mathrm{mod}$ is an equivalence.*

Definition 25 We say that A and B are Morita equivalent if $A-\mathrm{mod} \simeq B-\mathrm{mod}$. and a bimodule M as in Theorem 24 is called a Morita bimodule.

This is our model. We cannot expect a better situation.

Example 26 Consider the case of a skew field D with centre containing K, (and being finite dimensional over K to stay in our setting). Then D is a K-algebra. As for fields, there is a basis theorem also for skew fields and hence a finitely generated D-module is isomorphic to some D^n for some $n \in \mathbb{N}$. Hence, D is Morita equivalent to $\mathrm{Mat}_n(D)$ for any integer $n > 0$. We shall come back to this observation in Proposition 48 below.

In general however, there are more sophisticated possibilities. For example for indecomposable ring direct factors of the group ring of symmetric groups over algebraically closed fields of characteristic $p > 0$ Morita equivalence classes are given by involved combinatorial data by work of Scopes [22].

Derived equivalences

Rickard showed a classification result similar to the Morita Theorem 24. Again we will stay in our restricted framework of finitely dimensional algebras over some field K. The results hold in a larger generality, but need more preparation then.

Definition 27 Let A be a finite dimensional K-algebra for some field K. A complex T in $D^b(A)$ is called a tilting complex if

- T is isomorphic to some bounded complex of finitely generated projective modules.
- $\mathrm{Hom}_{D^b(A)}(T, T[n]) = 0$ for every $n \neq 0$.
- the smallest triangulated category containing T and all its direct sums and direct summands also contains $L(A)$.

Recall that the derived category is a triangulated category. With this notation we get

Theorem 28 (Rickard [18]) *Let K be a field and let A and B be two finite dimensional K-algebras. Then $D^b(A) \simeq D^b(B)$ as triangulated categories if and only if there is a tilting complex T in $D^b(A)$ such that $End_{D^b(A)}(T) \simeq B^{op}$.*

Observe that we do not get a functor yet. This is the purpose of the following result in a version due to Keller, generalising a result of Rickard. Again the result holds much more generally, but in order to stay in our setting, and in order to simplify technical difficulties we restrict the presentation to some special case.

Theorem 29 (Rickard [20], Keller [11]) *Let K be a field and let A and B be two finite dimensional K-algebras. Let T be a bounded complex of finitely generated projective B-modules and let $\alpha : A \to End_{D^b(B)}(T)$ be an algebra homomorphism. Suppose that $\mathrm{Hom}_{D^b(B)}(T, T[n]) = 0$ for all $n > 0$. Then there is a complex X in $D^b(B \otimes_K A^{op})$ and an isomorphism $\varphi : T \longrightarrow \mathrm{res}_B^{B \otimes_K A^{op}} X$ in $D^b(B)$, where res denotes the restriction of the complex of $B-A$-bimodules to a complex with action of B only,*

$$\begin{array}{ccc} T & \xrightarrow{\varphi} & X \\ {\scriptstyle \alpha(a)}\downarrow & & \downarrow{\scriptstyle \cdot a} \\ T & \xrightarrow[\varphi]{} & X \end{array}$$

is commutative for all $a \in A$. In this case (left derived) tensor product with X over A gives an equivalence $D^b(A) \to D^b(B)$. We call such an equivalence an equivalence of standard type.

The result above just implies that given a tilting complex T in $D^b(B)$ with endomorphism algebra A, then we may replace T by an isomorphic copy X, admitting an action of A on each component, such that each endomorphism of T can be realised as a multiplication on each homogeneous component of the complex. Hence, if there is an equivalence between derived categories of algebras, then there is a derived equivalence of standard type. Nevertheless, it is not known if in general every equivalence $D^b(A) \simeq D^b(B)$ is an equivalence of standard type. Recent work of Xiao-Wu Chen shows that this is true for some small class of algebras.

With respect to many aspects, except the last mentioned ambiguity, this situation is a quite satisfactory replacement of Morita's theorem, and has similar implications.

Stable equivalences

The stable category is a very loose invariant. Only few properties are encoded in the stable category. A striking example was given by Auslander and Reiten.

Example 30 (Auslander, Reiten [3]) Since projective modules are isomorphic to 0 in the stable category, and since any module over $\mathrm{Mat}_n(K)$ is projective for any $n \in \mathbb{N}$, we get for any algebra A

$$A - \underline{\mathrm{mod}} \simeq (A \times \mathrm{Mat}_n(K)) - \underline{\mathrm{mod}}$$

for any $n \in \mathbb{N}$.

The situation is even worse. Let K be a field, and let

$$A = \begin{pmatrix} K & K & K \\ 0 & K & K \\ 0 & 0 & K \end{pmatrix}$$

be the K-algebra of upper triangular 3×3 matrices with coefficients in K. This algebra has a two-sided ideal

$$I = \begin{pmatrix} 0 & 0 & K \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

as is readily verified. Let $B = A/I$. Let

$$C := \begin{pmatrix} K & K \\ 0 & K \end{pmatrix}$$

be the algebra of 2×2 upper triangular matrices with coefficients in K. Then

$$B - \underline{\mathrm{mod}} \simeq (C \times C) - \underline{\mathrm{mod}}.$$

In other words, B and $C \times C$ are stably equivalent. In particular, B is indecomposable, $C \times C$ is not, and hence the stable category does not even preserve indecomposability of an algebra. An indecomposable algebra may be stably equivalent to a product of two algebras, each of which is stably non trivial.

Nevertheless, there is a very nice result due to Reiten. The result deals with the notion of self-injectivity. In order to introduce this notation, let K be a field and let A be a finite-dimensional K-algebra. Then the K-space of K-linear forms $\mathrm{Hom}_K(A, K)$ on A is an A-module again. Indeed, let $f : A \to K$ be a linear form, and let $a \in A$. Then $a \cdot f$ is the linear form given by $(a \cdot f)(b) := f(ba)$ for all $b \in A$.

Definition 31 Let K be a field, and let A be a finite dimensional K-algebra. Then A is called self-injective if $\mathrm{Hom}_K(A, K)$ is a projective A-module. The algebra A is symmetric if $A \simeq \mathrm{Hom}_K(A, K)$ as $A - A$-bimodules.

Note that symmetric algebras are self-injective. The converse is false, and it is easy to find examples.

Let A be a finite dimensional algebra over some field. Let $rad(A)$ be the intersection of all maximal left ideals of A. Recall that for two ideals I and J of A we denote by $I \cdot J$ the smallest ideal of A containing all elements of the form xy where $x \in I$ and $y \in J$. Denote, $I^2 := I \cdot I$ and

$$I^n := \underbrace{I \cdot \ldots \cdot I}_{n \text{ terms}}$$

for each integer $n \geq 2$. Then $rad(A)$ is nilpotent in the sense that there is an integer n such that $rad^n(A) = 0$, and we get the following result.

Theorem 32 (Reiten [17]) *Let K be a field, let A be a finite dimensional K-algebra, and let B be an algebra such that $A-\underline{\mathrm{mod}} \simeq B-\underline{\mathrm{mod}}$. Suppose that for each indecomposable direct factor S of B we have $rad^2(S) \neq 0$. Then, if A is self-injective, also B is self-injective.*

However, success like this result is rare, and disappointment about the properties of stable equivalences prevails. Fortunately there is a concept due to Broué which is more promising, and actually there are many properties which were shown to be invariants.

Definition 33 (Broué [5]) Let K be a field and let A and B be finite dimensional K-algebras. Let M be an $A-B$-bimodule, and let N be a $B-A$-bimodule. The pair (M,N) induces a stable equivalence of Morita type if the following conditions are satisfied:

1.
   - M considered as A left module is finitely generated projective,
   - N considered as A right module is finitely generated projective,
   - M considered as B right module is finitely generated projective,
   - N considered as B left module is finitely generated projective,
2.
   - there is a projective $A-A$-bimodule P such that $M \otimes_B N \simeq A \oplus P$ as $A-A$-bimodules,
   - there is a projective $B-B$-bimodule Q such that $N \otimes_A M \simeq B \oplus Q$ as $B-B$-bimodules.

We say that A and B are stably equivalent of Morita type if there is a pair of bimodules (M,N) inducing a stable equivalence of Morita type.

It is easy to see that if P is a projective $A-A$-bimodule, then $P \otimes_A X$ is a projective A-module for all A-modules X, and likewise if Q is a projective $B-B$-bimodule, then $Q \otimes_B Y$ is a projective B-module for all B-modules Y.

Hence if (M,N) induces a stable equivalence of Morita type between A and B, then $M \otimes_B - : B-\underline{\mathrm{mod}} \longrightarrow A-\underline{\mathrm{mod}}$ is an equivalence with quasi-inverse $N \otimes_A - : A-\underline{\mathrm{mod}} \longrightarrow B-\underline{\mathrm{mod}}$. In other words, a stable equivalence of Morita type is an equivalence between stable categories of a particularly nice form.

Links between the three types of equivalences

First, it is somewhat clear that Morita equivalent algebras are derived equivalent, stably equivalent and stably equivalent of Morita type.

Proposition 34 (folklore; cf e.g. [26] Chapter 3) *Let K be a field, and let A and B be finite dimensional K-algebras. Then*

$$A-\mathrm{mod} \simeq B-\mathrm{mod} \Rightarrow D^b(A) \simeq D^b(B)$$

and

$$A-\mathrm{mod} \simeq B-\mathrm{mod} \Rightarrow A-\underline{\mathrm{mod}} \simeq B-\underline{\mathrm{mod}}.$$

Moreover, for a Morita $A-B$-bimodule M as in Definition 25 there is a $B-A$-bimodule N such that (M,N) induce a stable equivalence of Morita type. The equivalence given by tensoring with M is a derived equivalence of standard type.

For triangulated categories in general, and derived categories in particular there is an important construction, called Verdier quotient construction. This can be applied to triangulated categories $\mathcal{T}$ and so-called thick subcategories $\mathcal{S}$, and produces a triangulated category denoted $\mathcal{T}/\mathcal{S}$ which has the property that all objects in $\mathcal{S}$ become 0, and which is universal in some sense. Moreover, there is a canonical functor $\mathcal{T} \to \mathcal{T}/\mathcal{S}$. We refer to [26, Chapter 3] for some introduction and to Verdier [24] for a detailed exposition.

The following result was proved by Keller and Vossieck, and independently by Rickard. For a finite dimensional K-algebra A denote by $K^b(A-proj)$ the full subcategory of $D^b(A)$ formed by the bounded complexes of finitely generated projective A-modules.

Theorem 35 (Keller-Vossieck [9], Rickard [19]) *Let K be a field, and let A be a self-injective finite dimensional K-algebra. Then*

$$D^b(A)/K^b(A-proj) \simeq A-\underline{\mathrm{mod}}.$$

The main motivation for Broué to have given Definition 33 is the following result.

Theorem 36 (Broué [5]) *Let K be a field, and let A and B be self-injective finite dimensional K-algebras. Then any derived equivalence of standard type induces a stable equivalence of Morita type.*

More precisely, if X is a complex of $A-B$-bimodules such that $X \otimes_B^{\mathbb{L}} - : D^b(B) \to D^b(A)$ is an equivalence, then the canonical functor $D^b(A \otimes_K B^{op}) \to D^b(A \otimes_K B^{op})/K^b(A \otimes_K B^{op} - proj)$ maps X to an $A-B$-bimodule M to which there is a $B-A$-bimodule N such that (M, N) induce a stable equivalence of Morita type.

We therefore get the following scheme.

$$\begin{array}{ccc} \text{Morita equivalence} & \Longrightarrow & \text{derived equivalence of standard type} \\ \Downarrow & & \Uparrow \exists \\ \text{stable equivalence of Morita type} & & \text{derived equivalence} \end{array}$$

For self-injective algebras A and B we get a little more, as indicated below.

$$\begin{array}{ccc} \text{Morita equivalence} & \Longrightarrow & \text{derived equivalence of standard type} \\ \Downarrow & \swarrow\!\!\!\swarrow & \Uparrow \exists \\ \text{stable equivalence of Morita type} & & \text{derived equivalence} \end{array}$$

SOME INVARIANTS FROM HOCHSCHILD THEORY

A rather trivial fact is the following. Hochschild cohomology and Hochschild homology are invariant under Morita equivalence. But even the Gerstenhaber structure is invariant, and also the Hochschild cohomology algebra module structure of Hochschild homology is invariant. But even more is true.

Theorem 37 *Let K be a field and let A and B be a finite dimensional K-algebras. If $D^b(A) \simeq D^b(B)$, then*

- *(Rickard [20]) the Hochschild cohomology algebras $(HH^\bullet(A), \cup)$ and $(HH^\bullet(B), \cup)$ are isomorphic as graded algebras via an isomorphism κ.*
- *(Keller [12]) the Hochschild cohomology Lie algebras $(HH^{\bullet+1}(A), [\ ,\ ])$ and $(HH^{\bullet+1}(B), [\ ,\ ])$ are isomorphic as graded Lie algebras.*
- *the Gerstenhaber algebra structures $(HH^\bullet(A), \cup, [\ ,\ ])$ and $(HH^\bullet(B), \cup, [\ ,\ ])$ are isomorphic,*
- *(Keller [10]; see also [25]) the Hochschild homology structures $HH_\bullet(A) \simeq HH_\bullet(B)$ are isomorphic via an isomorphism λ,*
- *(Armenta and Keller [1]) the $(HH^\bullet(A), \cup)$ module structure on $HH_\bullet(A)$ is isomorphic to the $(HH^\bullet(B), \cup)$ module structure on $HH_\bullet(B)$ in the sense that the diagram*

$$\begin{array}{ccc} HH^\bullet(A) \times HH_\bullet(A) & \xrightarrow{\cap} & HH_\bullet(A) \\ \downarrow{\scriptstyle \kappa\times\lambda} & & \downarrow{\scriptstyle \lambda} \\ HH^\bullet(B) \times HH_\bullet(B) & \xrightarrow{\cap} & HH_\bullet(B) \end{array}$$

is commutative.

Remark 38 It is important to note that the invariance of Hochschild (co-)homology follows from a derived equivalence of standard type. An abstract equivalence is not enough. However, for algebras over fields if there is a derived equivalence, then there is also a derived equivalence of standard type.

The situation is much less satisfactory in the case of stable equivalences. Since we have seen in Example 30 that stable equivalences do not even preserve indecomposability of algebras, the centre is not an invariant. Indeed, in Example 30 we get $Z(B) = K$ and $Z(C \times C) = K \times K$. Note that Remark 19 shows that

$$HH^0(B) = Z(B) = K \neq K \times K = Z(C \times C) = HH^0(C \times C).$$

Nevertheless, for degrees different from 0 we get at least for self-injective algebras the following

Proposition 39 *Let A and B be a finite dimensional self-injective K-algebras over a field K. Then, for any two finite dimensional A-modules U and V we have* $\mathrm{Ext}^1_A(U,V) \simeq \underline{\mathrm{Hom}}_A(\Omega U, V)$. *In particular, if there is an equivalence* $F : A - \underline{\mathrm{mod}} \longrightarrow B - \underline{\mathrm{mod}}$, *then* $\mathrm{Ext}^1_A(U,V) \simeq \mathrm{Ext}^1_B(F(U), F(V))$.

The situation is even better for stable equivalences of Morita type and symmetric algebras.

Theorem 40 (Linckelmann [14], König-Liu-Zhou [13]) *Let A and B be a finite dimensional symmetric K-algebras over a field K. If (M,N) induces a stable equivalence of Morita type between A and B, then the self-injective algebras $A \otimes_K A^{op}$ and $B \otimes_K B^{op}$ are stably equivalent of Morita type with a functor*

$$F : (A \otimes_K A^{op}) - \underline{\mathrm{mod}} \longrightarrow (B \otimes_K B^{op}) - \underline{\mathrm{mod}}$$

satisfying $F(A) \simeq B$. In particular, $HH^n(A) \simeq HH^n(B)$ for all $n \geq 1$.

The degree 0 is more subtle. For cohomology, i.e. the centre, this was nevertheless the case first studied by Broué.

Definition 41 Let K be a field and let A be a finite dimensional K-algebra.

- (Broué [5]) $Z^{st}(A) := \underline{End}_{A \otimes A^{op}}(A)$ is the endomorphism algebra of A in the stable category of $A \otimes_K A^{op}$-modules. The projective centre is $Z^{pr}(A) := \ker(Z(A) \longrightarrow Z^{st}(A)$.
- (Liu-Zhou-Zimmermann [15]) The stable degree zero Hochschild homology is $HH_0^{st}(A) := \bigcap_{P \text{ projective } A\text{-module}} \ker(\mathrm{trace}_P : HH_0(A) \to K)$ where trace is the trace map of the K-linear map given by multiplication by $a \in A$ on P.

Clearly, $Z^{st}(A)$ is a K-algebra and $Z^{pr}(A)$ is an ideal of $Z(A)$. An easy consequence, almost by definition is the

Proposition 42 (Broué [5]) *Let K be a field and let A and B be two K-algebras which are stably equivalent of Morita type. Then*

$$Z^{st}(A) \simeq Z^{st}(B)$$

as K-algebras.

Theorem 43 (Liu-Zhou-Zimmermann [15]) *Let K be an algebraically closed field and let A and B be two finite dimensional K-algebras. If A and B are stably equivalent of Morita type, then*

$$HH_0^{st}(A) \simeq HH_0^{st}(B).$$

APPLICATION: A QUESTION DUE TO RICKARD

We shall now give an application of the Hochschild (co-)homology invariance to a question posed by Rickard in [21]. In order to formulate the question we first cite a result due to Rickard.

Theorem 44 (Rickard [20]) *Let K be a field, and let A_1, A_2, B_1, B_2 be finite dimensional K-algebras. If A_1 is derived equivalent to A_2, and if B_1 is derived equivalent to B_2. Then $A_1 \otimes_K B_1$ is derived equivalent to $A_2 \otimes_K B_2$.*

Rickard posed the question if this holds true if we replace the concept 'derived equivalence' by 'stable equivalence of Morita type', and this in particular when all the algebras involved are self-injective.

The non self-injective counterexample

In this direction we obtained

Theorem 45 (Liu-Zhou-Zimmermann [16]) *Let K be a field and let A and B be two finite dimensional self-injective K-algebras. Suppose that neither A nor B have a matrix ring over a skew field as a direct factor. If* $\begin{pmatrix} A & 0 \\ A & A \end{pmatrix}$ *is stably equivalent to* $\begin{pmatrix} B & 0 \\ B & B \end{pmatrix}$, *then A and B are actually Morita equivalent.*

Note that this gives a counterexample to Rickard's question, even for abstract stable equivalence, since

$$\begin{pmatrix} A & 0 \\ A & A \end{pmatrix} \simeq A \otimes_K \begin{pmatrix} K & 0 \\ K & K \end{pmatrix}$$

and likewise for B. Moreover, there are many examples of self-injective algebras A and B which are stably equivalent but not Morita equivalent. We will encounter one example below in Theorem 46, but there are much simpler examples known (cf e.g. [26, Chapter 6]). The proof of this result is not terribly complicate but uses not very complicated parts of some more involved theory, known as Auslander-Reiten theory.

The symmetric counterexample

However, $\begin{pmatrix} K & 0 \\ K & K \end{pmatrix}$ is not self-injective. Hence Rickard's question is not completely solved by Theorem 45. In order to do so, in joint work with Serge Bouc we used the computer algebra program GAP [6].

We first need some preparations on group representations. Let G be a finite group and let K be an algebraically closed field of characteristic $p > 0$. The field K is a simple KG-module when we define $g \cdot x = x$ for all $x \in K$ and $g \in G$. This is called the trivial module. Then the group algebra KG decomposes into indecomposable ring direct factors

$$KG \simeq B_0(G) \times \ldots \times B_s(G)$$

and we call $B_i(G)$ for $i \in \{0, \ldots, s\}$ the blocks of G. For each simple KG-module S there is a unique block $B_{i(S)}$ such that S is a $B_{i(S)}(G)$-module. We say S belongs to $B_{i(S)}(G)$. The trivial module belongs to the principal block $B_0(G)$.

Now, let P be a Sylow p-subgroup of G, and let $H := N_G(P) := \{g \in G \mid g \cdot P \cdot g^{-1} = P\}$ be the normaliser of P in G. Suppose moreover that $P \cap g \cdot P \cdot g^{-1} = \{1\}$ is the neutral element of G for each $g \in G \setminus H$. We say in this case that G has TI Sylow p-subgroups. Then the classical Green correspondence [26, Chapter 2] gives a stable equivalence of Morita type between $B_0(G)$ and $B_0(H)$.

Now we consider a quite particular group. Classical field theory gives that for every integer $s > 1$ there is an up to isomorphism unique field $\mathbb{F}_q$ of cardinal $q = p^s$. The field $\mathbb{F}_{q^2}$ has a field automorphism of order 2 given by

$$\mathbb{F}_{q^2} \ni x \mapsto x^q =: \overline{x} \in \mathbb{F}_{q^2}.$$

Then consider the group $GL_3(q^2)$ of invertible 3×3 matrices with coefficients in $\mathbb{F}_{q^2}$. For a matrix $M = \left(m_{i,j}\right)_{1 \le i,j \le 3} \in GL_3(q^2)$ denote by M^{tr} the transpose of the matrix M, and let $\overline{M}$ be the matrix obtained by applying the automorphism $\overline{}$ to each coefficient, i.e. $\overline{M} := \left(\overline{m_{i,j}}\right)_{1 \le i,j \le 3}$. Consider the special matrix $C := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \in GL_3(q^2)$. Let then

$$U(q) := \{g \in GL_3(q^2) \mid g^{tr} \cdot C \cdot \overline{g} = C\}$$

and

$$SU(q) := \{g \in U(q) \mid \det(g) = 1\}.$$

Finally, for a group Γ let $Z(\Gamma) := \{g \in \Gamma \mid \forall x \in \Gamma : xg = gx\}$ be the centre of Γ. This is a normal subgroup and

$$PSU(q) := SU(q)/Z(SU(q)).$$

It is a fact that $PSU(q)$ has TI Sylow p-subgroups (cf e.g. [8, II, §10, 10.12]).

Theorem 46 (Bouc-Zimmermann [4]; Jürgen Müller for $q \in \{9, 11\}$) *Let K be an algebraically closed field of characteristic $p > 0$ and let $G(q) := PSU(3, q)$, let $P(q)$ be a Sylow p-subgroup of $G(q)$ and let $H(q) = N_{G(q)}(P(q))$. Then for $q \in \{3, 4, 5, 7, 8, 9, 11\}$ the centres of the principal blocks of $KG(q)$ and of $KH(q)$ are not isomorphic, i.e. $Z(B_0(G(q))) \not\simeq Z(B_0(H(q)))$.*

We proved this result by comparing the dimensions over K of the quotients of the centres modulo increasing powers of the radical of the centres. This was possible abstractly for all q for the group $H(q)$, but the group $G(q)$ is quite complicated and there we used GAP. The size of the group is rapidly increasing, as is shown by the following table:

q	size of $PSU(q)$	size of the normalizer of a p-Sylow
3	5616	216
4	20160	960
5	372000	1000
7	1876896	16464
8	16482816	10752
9	42456960	58320
11	212427600	53240

How does this answer Rickard's question? This is a corollary to a result from [15].

Corollary 47 *Let p be a prime, let $q = p^s$ for some integer $s > 0$, and let K be an algebraically closed field of characteristic $p > 0$. Let C_p be the cyclic group of order p. Then for all symmetric finite dimensional K-algebras A we have*

$$Z^{st}(A \otimes_K KC_p) = Z(A \otimes_K KC_p) = Z(A) \otimes_K KC_p.$$

Luckily, principal blocks of group rings are symmetric algebras, and

$$Z(B_0(G(q))) \not\simeq Z(B_0(H(q))) \Rightarrow Z(B_0(G(q))) \otimes_K KC_p \not\simeq Z(B_0(H(q))) \otimes_K KC_p.$$

This then gives a counterexample to Rickard's original question.

THE BRAUER GROUP

We have seen that for a finite dimensional K-algebra over a field K the Hochschild (co-)homology structure is a very rich algebraic gadget. Could it be that the Hochschild structure already determines the algebra up to, say, derived equivalence? The question was posed by Marco Armenta in his thesis [2]. This is very false, as we will see in this section. We note that Marco Armenta produced in collaboration with Claude Cibils another example, namely the tame hereditary algebras of tree class A_4 and D_4.

The key observation for our general approach is the following.

Proposition 48 (folklore; cf e.g. [26] Chapter 6) *Let K be a field and let D be a skew field with centre containing K. Let $A = \mathrm{Mat}_n(D)$. Then each tilting complex T in $D^b(A)$ is isomorphic to $M[n]$ for some $n \in \mathbb{N}$ and Morita bimodule M. In particular, if D_1 and D_2 are skew fields with centres containing K such that D_1 and D_2 are finite dimensional over K, then $D^b(D_1) \simeq D^b(D_2)$ if and only if D_1 and D_2 are Morita equivalent. The skew fields D_1 and D_2 are Morita equivalent if and only if there are integers n_1 and n_2 such that $\mathrm{Mat}_{n_1}(D_1) \simeq \mathrm{Mat}_{n_2}(D_2)$.*

Proposition 48 indicates that the following definition is appropriate.

Definition 49 Let K be a field. A finite dimensional K-algebra A is called central simple if there is some skew field D and an integer n such that $A \simeq \mathrm{Mat}_n(D)$ and $Z(D) = K$.

We note first that if K is a field of characteristic 0, and if A_1 and A_2 are central simple K-algebras, then $A_1 \otimes_K A_2$ is again a central simple K-algebra.

Definition 50 Two central simple K-algebras A_1 and A_2 are called similar, denoted $A_1 \sim A_2$, if there are integers m_1 and m_2 such that $\mathrm{Mat}_{m_1}(A_1) \simeq \mathrm{Mat}_{m_2}(A_2)$.

Clearly similarity is an equivalence relation. Moreover, by Proposition 48 similarity classes are precisely the derived equivalence classes, and are moreover precisely the Morita equivalence classes. By definition, for any central simple K-algebra A there is a skew field D with centre K such that $A \simeq \mathrm{Mat}_n(D)$. Hence similarity classes of central simple K-algebras are represented by skew fields D with centre K.

Definition 51 The similarity classes $[A]$ of central simple K-algebras form a group, the Brauer group $Br(K)$ with group law being $[A_1] \cdot [A_2] := [A_1 \otimes_K A_2]$.

The neutral element is $[K]$ and the inverse of $[A]$ is $[A^{op}]$. Note that since we are dealing with finite dimensional K-algebras, the Brauer group is actually based on the set of similarity classes, rather than a class. It is not hard to verify that the group law on the Brauer group is well-defined.

Recall that for any prime p we can fix an ultra-metric on $\mathbb{Q}$. This is defined as follows. For any $x \in \mathbb{Q}$ there is a unique integer s_x such that $x = p^{s_x} \cdot \frac{n(x)}{d(x)}$ and such that $n(x)$ and $d(x)$ are integers, both relatively prime to p. We then put $v_p(x) := 2^{-s_x}$. Then, $\delta(x, y) := v_p(x - y)$ defines a metric, and completion of $\mathbb{Q}$ with respect to this metric yields the field $\mathbb{Q}_p$ of p-adic numbers.

The following result is a fairly deep consequence of class field theory.

Theorem 52 ([23], Theorem 1 and Corollary) *$Br(\mathbb{Q}_p) \simeq \mathbb{Q}/\mathbb{Z}$, where $\mathbb{Q}/\mathbb{Z}$ is equipped with the additive group law of $\mathbb{Q}$.*

We infer that there are many non trivial skew-fields with centre $\mathbb{Q}_p$. Any skew field D with centre $\mathbb{Q}_p$ satisfies $HH^\bullet(D) = HH^0(D) = \mathbb{Q}_p$ and $HH_\bullet(D) = HH_0(D) = \mathbb{Q}_p$ since skew fields are clearly symmetric algebras. Hence the Hochschild cohomology and the Hochschild homology does not depend on D.

REFERENCES

[1] M. A. Armenta, B. Keller, *Derived invariance of the cap product in Hochschild theory*, C.R.Math. Acad. Sci. Paris **355** (2017), 1205-1207.

[2] M. A. Armenta, *Derived invariance of the Tamarkin-Tsygan Calculus of an Associative Algebra*, thesis Montpellier-CIMAT Mexico 2019.

[3] M. Auslander, I. Reiten, "Stable equivalences of Artin algebras", in *Proceedings of the Conference on Order, Group Rings and Related Topics*. Springer Lecture Notes in Mathematics (Ohio State University Columbus 1972), pp 8-71, vol. 353 (1973).

[4] S. Bouc and A. Zimmermann, *On a question of Rickard on tensor products of stably equivalent algebras*, Exp. Math. **26** (2017) 31-44.

[5] M. Broué, "Equivalences of blocks of group algebras." Finite dimensional algebras and related topics (Ottawa, 1992), pp. 1-26, NATO Advanced Science Institute Series C Mathematics Physics Science, 424, Kluwer Academic Publisher, Dordrecht (1994)

[6] GAP–Groups, Algorithms, and Programming, Version4.7.6, The GAP Group, `http://www.gap-system.org`.

[7] M. Gerstenhaber, *The cohomology structure of an associative ring*, Annals of Math. **78** (1963) 267-288.

[8] B. Huppert, *Endliche Gruppen I*, Springer Verlag Heidelberg, Berlin, New York 1983.

[9] B. Keller, D. Vossieck, *Sous les catégories dérivées*, C. R. Acad. Sci. Paris Sér. I Math. **305** (1987), 225-228.

[10] B. Keller, "Invariance of cyclic homology under derived equivalence." Representation theory of algebras (Cocoyoc, 1994), 353-361, CMS Conf. Proc., 18, Amer. Math. Soc., Providence, RI, 1996.

[11] B. Keller, *Bimodule complexes via strong homotopy actions*, Algebras and Representation Theory **3** (2000) 357-376.

[12] B. Keller, *Hochschild cohomology and derived Picard groups*, J. Pure Appl. Algebra **190** (2004), 177-196.

[13] S. König, Y.M. Liu, G.D. Zhou, *Transfer maps in Hochschild (co)homology and applications to stable and derived invariants and to the Auslander–Reiten conjecture.* Trans. Amer. Math. Soc. 364 (2012), 195-232

[14] M. Linckelmann, *Transfer in Hochschild cohomology of blocks of finite groups.* Algebras and Representation Theory 2 (1999), 107-135.

[15] Y.M. Liu, G.D. Zhou, A. Zimmermann, *Higman ideal, stable Hochschild homology and Auslander-Reiten conjecture*, Math. Z. **270** (2012) 759-781.

[16] Y.M. Liu, G.D. Zhou, A. Zimmermann, *Stable equivalences of Morita type do not preserve tensor products and trivial extensions of algebras*, Proc. Amer. Math. Soc. **145** (2017) 1881-1890.

[17] I. Reiten, *Stable equivalence of selfinjective algebras*. J. Algebra **40** (1976) 63-74.

[18] J. Rickard, *Morita theory for derived categories*, J. London Math. Soc. (2) **39** (1989) 436-456.

[19] J. Rickard, *Derived categories and stable equivalence*. J. Pure and Appl. Algebra **61** (1989), 303-317.

[20] J. Rickard, *Derived equivalences as derived functors* J. London Math. Soc. (2) **43** (1991), 37-48.

[21] J. Rickard, "Some recent advances in modular representation theory," in *Algebras and Modules, I, Trondheim (1996)*, CMS Conference Proceedings **23**, (American Mathematical Society, Providence, Rhode Island, 1998), pp. 157-178.

[22] J. Scopes, *Cartan matrices and Morita equivalenxe for blocks of the symmetric groups*. J. Algebra **142** (1991), 441-455.

[23] J.-P. Serre, *Local Class Field Theory*, Chapter VI in Algebraic Number Theory, edt. J.W.S. Cassels, A. Fröhlich, Proceedings of an Instructional Conference organized by the London Mathematical Society (A NATO Advanced Study Institute) with the support of the International Mathematical Union, Academic Press London, 1967

[24] J.-L. Verdier, *Des catégories dérivées des catégories abéliennes*. With a preface by Luc Illusie. Edited with a note by by Georges Maltsiniotis. Société Mathématique de France. Astérisque **239** (1996) xii+253 pp.

[25] A. Zimmermann, *Fine Hochschild invariants of derived categories for symmetric algebras.* J. Algebra **308** (2007), 350-367.

[26] A. Zimmermann, *Representation Theory; A homological algebra point of view*, Algebra and Applications **19**, Springer Verlag, Cham 2014.