GENERALIZED REYNOLDS IDEALS AND DERIVED EQUIVALENCES FOR ALGEBRAS OF DIHEDRAL AND SEMIDIHEDRAL TYPE

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Abstract. Generalized Reynolds ideals are ideals of the center of a symmetric algebra over a field of positive characteristic. They have been shown by the second author to be invariant under derived equivalences. In this paper we determine the generalized Reynolds ideals of algebras of dihedral and semidihedral type (as defined by Erdmann), in characteristic 2. In this way we solve some open problems about scalars occurring in the derived equivalence classification of these algebras.

1. Introduction

Finite-dimensional algebras are distinguished according to their representation type, which is either finite, tame or wild. For blocks of group algebras the representation type is determined by the structure of the defect group. It is finite if and only if the defect groups are cyclic. The structure of such blocks is known; in particular these algebras are Brauer tree algebras. Blocks of tame representation type occur only in characteristic 2, and then precisely if the defect groups are dihedral, semidihedral or generalized quaternion. The structure of such blocks has been determined in a series of seminal papers by K. Erdmann [4]. She introduced the more general classes of algebras of dihedral, semidihedral and quaternion type, and classified them by explicitly describing their basic algebras by quivers and relations. However, some subtle questions remained open in her classification, most of them related to scalars occurring in the relations. Based on Erdmann's Morita equivalence classification, algebras of dihedral, semidihedral and quaternion type have been classified up to derived equivalence by the first author in [5], [6]. Along the way, some of the subtle remaining problems in [4] have been solved, but not all. In particular, for the case of two simple modules still scalars occur in the relations, and it could not be decided whether the algebras for different scalars are derived equivalent, or not. (See the appendix of [7] for tables showing the status of the derived equivalence classifications.)

In this paper, we shall study new invariants for symmetric algebras A over fields of positive characteristic which have been defined in [3]. These are descending sequences of so-called generalized Reynolds ideals, of the center,

$$Z(A) \supseteq T_1(A)^\perp \supseteq T_2(A)^\perp \supseteq \ldots \supseteq T_n(A)^\perp \supseteq \ldots.$$

The precise definition of these ideals is given in Section 2 below.

It has been shown by the second author in [14] that these sequences of ideals are invariant under derived equivalences, i.e. any derived equivalence implies an isomorphism between the centers mapping the generalized Reynolds ideals onto each other.

It turns out that generalized Reynolds ideals can be very useful for distinguishing algebras up to derived equivalence. For instance, in [8], generalized Reynolds ideals have been used successfully to complete the derived equivalence classification of symmetric algebras of domestic representation type.

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In this paper, we are going to compute the generalized Reynolds ideals for algebras of dihedral and semidihedral type. As main application we will settle some of the scalar problems which remained open in the derived equivalence classification [5], [6].

Using the notation of [4], our results can be summarized as follows. The definitions of the algebras under consideration are also recalled below in Sections 4 and 5, respectively.

Theorem 1.1. *Let F be an algebraically closed field of characteristic 2. For any given integers $k, s \geq 1$ consider the algebras of dihedral type $D(2A)^{k,s}(c)$ for the scalars $c = 0$ and $c = 1$. Suppose that if $k = 2$ then $s \geq 3$ is odd, and if $s = 2$ then $k \geq 3$ is odd.*

Then the algebras $D(2A)^{k,s}(0)$ and $D(2A)^{k,s}(1)$ have different sequences of generalized Reynolds ideals. In particular, the algebras $D(2A)^{k,s}(0)$ and $D(2A)^{k,s}(1)$ are not derived equivalent.

The above result has also been obtained earlier by M. Kauer [9], [10], using entirely different methods. However, our new proof seems to be more elementary, just using linear algebra calculations.

For algebras of semidihedral type, we can prove the following result using generalized Reynolds ideals.

Theorem 1.2. *Let F be an algebraically closed field of characteristic 2. For any given integers $k, s \geq 1$, consider the algebras of semidihedral type $SD(2B)_1^{k,s}(c)$ for the scalars $c = 0$ and $c = 1$. Suppose that if $k = 2$ then $s \geq 3$ is odd, and if $s = 2$ then $k \geq 3$ is odd.*

Then the algebras $SD(2B)_1^{k,s}(0)$ and $SD(2B)_1^{k,s}(1)$ have different sequences of generalized Reynolds ideals. In particular, the algebras $SD(2B)_1^{k,s}(0)$ and $SD(2B)_1^{k,s}(1)$ are not derived equivalent.

This settles an important open problem in the derived equivalence classification of algebras of semidihedral type. However, this does not yet complete this classification; there is a second family $SD(2B)_2^{k,s}(c)$ involved for which we can prove the following partial result.

Theorem 1.3. *Let F be an algebraically closed field of characteristic 2. For any given integers $k, s \geq 1$, consider the algebras of semidihedral type $SD(2B)_2^{k,s}(c)$ for the scalars $c = 0$ and $c = 1$. If the parameters k and s are both odd, then the algebras $SD(2B)_2^{k,s}(0)$ and $SD(2B)_2^{k,s}(1)$ have different sequences of generalized Reynolds ideals. In particular, for k and s odd, the algebras $SD(2B)_2^{k,s}(0)$ and $SD(2B)_2^{k,s}(1)$ are not derived equivalent.*

Here in the semidihedral case, in order to distinguish derived equivalence classes in the remaining cases new derived invariants would have to be discovered.

2. Generalized Reynolds ideals

The aim of this section is to briefly give the necessary background on generalized Reynolds ideals, as introduced by B. Külshammer [11]. For more details we refer to the survey [12]. For recent developments we also refer to [2], [3], [14], [15].

Let F an algebraically closed field of characteristic $p > 0$. (For the theory of generalized Reynolds ideals a perfect ground field would be sufficient.) Generalized Reynolds ideals have originally been defined for symmetric algebras (see [1] for an extension to arbitrary finite-dimensional algebras). Any finite-dimensional symmetric F-algebra A has an associative, symmetric, non-degenerate F-bilinear form $\langle -, - \rangle : A \times A \to F$. With respect to this form we have for any subspace M of A the orthogonal space $M^\perp$. Moreover, let $K(A)$ be the commutator subspace, i.e. the F-subspace of A generated by all commutators $[a, b] := ab - ba$, where $a, b \in A$. For any $n \geq 0$ set

$$T_n(A) = \left\{ x \in A \mid x^{p^n} \in K(A) \right\}.$$

Then, by [11], for any $n \geq 0$, the orthogonal space $T_n(A)^\perp$ is an ideal of the center $Z(A)$ of A. These are called *generalized Reynolds ideals*. They form a descending sequence

$$Z(A) = K(A)^\perp = T_0(A)^\perp \supseteq T_1(A)^\perp \supseteq T_2(A)^\perp \supseteq \ldots \supseteq T_n(A)^\perp \supseteq \ldots$$

In [3] it has been shown that the sequence of generalized Reynolds ideals is invariant under Morita equivalences. More generally, the following theorem has been proven recently by the second author.

Proposition 2.1 ([14], Theorem 1)**.** *Let A and B be finite-dimensional symmetric algebras over a perfect field F of positive characteristic p. If A and B are derived equivalent, then there is an isomorphism $\varphi : Z(A) \to Z(B)$ between the centers of A and B such that $\varphi(T_n(A)^\perp) = T_n(B)^\perp$ for all positive integers n.*

We note that in the proof of [14, Theorem 1] the fact that F is algebraically closed is never used. The assumption on the field F to be perfect is sufficient. Hence the sequence of generalized Reynolds ideals gives a new derived invariant for symmetric algebras over perfect fields of positive characteristic.

The aim of the present note is to show how these new derived invariants can be applied to some subtle questions in the derived equivalence classifications of algebras of dihedral and semidihedral type.

3. A symmetric bilinear form

Symmetric algebras are equipped with an associative, non-degenerate symmetric bilinear form. For actual computations with generalized Reynolds ideals one needs to know such a symmetrizing form explicitly. We should stress that the series of generalized Reynolds ideals is independent of the choice of symmetrizing form (cf. [14, end of the proof of the theorem]). The algebras in our paper are all basic symmetric algebras, defined by a quiver with relations $A = FQ/I$. There is the following standard construction, which provides a bilinear form very suitable for actual calculations. As usual, $\mathrm{soc}(A)$ denotes the socle of the algebra A. Recall that an algebra is called weakly symmetric if for each projective indecomposable module the top and the socle are isomorphic.

Proposition 3.1. *Let $A = FQ/I$ be a weakly symmetric algebra given by the quiver Q and ideal of relations I, and fix an F-basis $\mathcal{B}$ of A consisting of pairwise distinct non-zero paths of the quiver Q. Assume that $\mathcal{B}$ contains a basis of* $\mathrm{soc}(A)$*. Then the following statements hold:*

(1) *Define an F-linear mapping ψ on the basis elements by*

$$\psi(b) = \begin{cases} 1 & \text{if } b \in \mathrm{soc}(A) \\ 0 & \text{otherwise} \end{cases}$$

for $b \in \mathcal{B}$. Then an associative non-degenerate F-bilinear form $\langle -, - \rangle$ for A is given by $\langle x, y \rangle := \psi(xy)$.

(2) *If A is symmetric, then for any $n \geq 0$, the socle* $\mathrm{soc}(A)$ *is contained in the generalized Reynolds ideal $T_n(A)^\perp$.*

Proof. (1) By definition, since A is an associative algebra, ψ is associative on basis elements, hence is associative on all of A.

We observe now that $\psi(xe) = \psi(ex)$ for all $x \in A$ and all primitive idempotents $e \in A$. Indeed, since ψ is linear, we need to show this only on the elements in $\mathcal{B}$. Let $b \in \mathcal{B}$. If b is a path not in the socle of A, then be and eb are either zero or not contained in the socle either, and hence $0 = \psi(b) = \psi(be) = \psi(eb)$. Moreover, by assumption A is weakly symmetric. If $b \in \mathcal{B}$ is in the socle of A, then $b = e_b b = b e_b$ for exactly one primitive idempotent e_b and $e'b = be' = 0$ for each primitive idempotent $e' \neq e_b$. Therefore, $\psi(e'b) = \psi(be') = 0$ and $\psi(e_b b) = \psi(b) = \psi(b e_b)$.

It remains to show that the map $(x, y) \mapsto \psi(xy)$ is non-degenerate. Suppose we had $x \in A\backslash\{0\}$ so that $\psi(xy) = 0$ for all $y \in A$. In particular for each primitive idempotent e_i of A we get $\psi(e_i xy) = \psi(xye_i) = 0$ for all $y \in A$. Hence we may suppose that $x \in e_i A$ for some primitive idempotent $e_i \in A$.

Now, xA is a right A-module. Choose a simple submodule S of xA and $s \in S \setminus \{0\}$. Then, since $s \in S \leq xA$ there is a $y \in A$ so that $s = xy$. Since $S \leq xA \leq A$, and since S is simple, $s \in \mathrm{soc}(A) \setminus \{0\}$. Moreover, since $x \in e_i A$, also $s = e_i s$, i.e. s is in the (1-dimensional) socle of the projective indecomposable module e_iA. So, up to a scalar factor, s is a path contained in the basis $\mathcal{B}$ (recall that by assumption $\mathcal{B}$ contains a basis of the socle). This implies that

$$\psi(xy) = \psi(s) = \psi(e_i s) \neq 0,$$

contradicting the choice of x, and hence proving non-degeneracy.

(2) By [3] we have for any symmetric algebra A that

$$\bigcap_{n=0}^{\infty} T_n(A)^{\perp} = \mathrm{soc}(A) \cap Z(A).$$

Moreover, using the proof given in [12], for a basic algebra for which the endomorphism rings of all simple modules are commutative, we always have $\mathrm{rad}(A) \supseteq K(A)$ and hence, taking orthogonal spaces, $\mathrm{soc}(A) \subseteq Z(A)$. □

Remark 3.2. We should mention that the hypothesis on the algebra A in the above proposition is satisfied for the algebras of dihedral and semidihedral type we deal with in this paper. Moreover, these algebras are symmetric algebras, and for all of them the above-described form $\langle -, - \rangle$ is actually symmetric which can be checked directly from the definitions of the algebras given below. Hence, we shall use the form given in Proposition 3.1 throughout as symmetrizing form for our computations of generalized Reynolds ideals.

With a more subtle analysis one might be able to show that if $A = FQ/I$ as in the proposition is assumed to be symmetric then the form $\langle -, - \rangle$ is always symmetric. We do not embark on this aspect here.

4. Algebras of dihedral type

Following K. Erdmann [4, sec. VI.2], an algebra A (over an algebraically closed field) is said to be *of dihedral type* if it satisfies the following conditions:

(i) A is symmetric and indecomposable.
(ii) The Cartan matrix of A is non-singular.
(iii) The stable Auslander-Reiten quiver of A consists of the following components: 1-tubes, at most two 3-tubes, and non-periodic components of tree class $\mathbb{A}_{\infty}^{\infty}$ or $\tilde{\mathbb{A}}_{1,2}$.

K. Erdmann classified these algebras up to Morita equivalence. A derived equivalence classification of algebras of dihedral type has been given in [6]. Any algebra of dihedral type with two simple modules is derived equivalent to a basic algebra $A_c^{k,s} := D(2\mathcal{B})^{k,s}(c)$ where $k, s \geq 1$ are integers and the scalar is $c = 0$ or $c = 1$. These algebras are defined by the following quiver

subject to the relations

$$\beta\eta = 0,\ \eta\gamma = 0,\ \gamma\beta = 0,\ \alpha^2 = c(\alpha\beta\gamma)^k,\ (\alpha\beta\gamma)^k = (\beta\gamma\alpha)^k,\ \eta^s = (\gamma\alpha\beta)^k.$$

Note that the case $s = 1$ has to be interpreted so that the loop η doesn't exist in the quiver.

The algebras $A_0^{k,s}$ and $A_1^{k,s}$ are known to be isomorphic if the underlying field has characteristic different from 2 [4, proof of VI.8.1]. So we assume throughout this section that the underlying field has characteristic 2.

For any $k, s \geq 1$ (and fixed c) the algebras $A_c^{k,s}$ and $A_c^{s,k}$ are derived equivalent [6, lemma 3.2]. So the derived equivalence classes are represented by the algebras $A_c^{k,s}$ where $k \geq s \geq 1$ and $c \in \{0, 1\}$. Moreover, for different parameters $k', s' \geq 1$, i.e. if $\{k, s\} \neq \{k', s'\}$, the algebra $A_d^{k',s'}$ (where $d \in \{0, 1\}$) is not derived equivalent to $A_c^{k,s}$ [6, lemma 3.3].

Blocks of finite group algebras having dihedral defect group of order 2^n and two simple modules are Morita equivalent to algebras $D(2\mathcal{B})^{1,2^{n-2}}(c)$. It is not known whether blocks of finite groups can occur for both scalars $c = 0$ and $c = 1$.

In this section we are going to study the sequence of generalized Reynolds ideals

$$Z(A_c^{k,s}) \supseteq T_1(A_c^{k,s})^\perp \supseteq T_2(A_c^{k,s})^\perp \supseteq \ldots \supseteq T_r(A_c^{k,s})^\perp \supseteq \mathrm{soc}(A_c^{k,s})$$

of the center. It is known by [14] that this sequence is invariant under derived equivalences.

Our main result in this section is the following, partly restating Theorem 1.1 of the Introduction.

Theorem 4.1. *Let $k, s \geq 1$, and suppose that if $k = 2$ then $s \geq 3$ is odd, and if $s = 2$ then $k \geq 3$ is odd.*

Then the factor rings $Z(A_0^{k,s})/T_1(A_0^{k,s})^\perp$ and $Z(A_1^{k,s})/T_1(A_1^{k,s})^\perp$ are not isomorphic.

In particular, the algebras $A_0^{k,s}$ and $A_1^{k,s}$ are not derived equivalent.

Remark 4.2. This result has already been obtained earlier by M. Kauer [9], [10], using entirely different methods, as byproduct of a rather sophisticated study of the class of so-called graph algebras. (With this method, the cases of small parameters excluded above can also be dealt with.) However, our new 'linear algebra' proof seems to be more elementary. Moreover, our methods can successfully be extended to algebras of semidihedral type, as we shall see in the next section, in contrast to the methods in [9], [10].

Before embarking on the proof of Theorem 4.1 we need to collect some prerequisites, and thereby we also set some notation.

4.1. **Bases for the algebras.** We fix the integers $k, s \geq 1$. We have to compute in detail with elements of the algebras $A_c^{k,s}$ where $c = 0$ or $c = 1$. Both algebras are of dimension $9k + s$ (cf. [4]), the Cartan matrix is of the form

$$\begin{pmatrix} 4k & 2k \\ 2k & k+s \end{pmatrix}.$$

A basis of $A_c^{k,s}$ is given by the union of the following bases of the subalgebras $e_i A_c^{k,s} e_j$, where e_1 and e_2 are the idempotents corresponding to the trivial paths at the vertices of the quiver:

$$\mathcal{B}_{1,1} := \{e_1, (\alpha\beta\gamma)^i, \alpha, (\alpha\beta\gamma)^i\alpha, (\beta\gamma\alpha)^i, \beta\gamma, (\beta\gamma\alpha)^i\beta\gamma, (\alpha\beta\gamma)^k = (\beta\gamma\alpha)^k \,:\, 1 \leq i \leq k-1\}$$
$$\mathcal{B}_{1,2} := \{\beta, (\beta\gamma\alpha)^i\beta, \alpha\beta, (\alpha\beta\gamma)^i\alpha\beta \,:\, 1 \leq i \leq k-1\}$$
$$\mathcal{B}_{2,1} := \{\gamma, (\gamma\alpha\beta)^i\gamma, \gamma\alpha, (\gamma\alpha\beta)^i\gamma\alpha \,:\, 1 \leq i \leq k-1\}$$
$$\mathcal{B}_{2,2} := \{e_2, (\gamma\alpha\beta)^i, \eta, \eta^2, \ldots, \eta^{s-1}, \eta^s = (\gamma\alpha\beta)^k \,:\, 1 \leq i \leq k-1\}.$$

Note that this basis $\mathcal{B} := \mathcal{B}_{1,1} \cup \mathcal{B}_{1,2} \cup \mathcal{B}_{2,1} \cup \mathcal{B}_{2,2}$ is independent of the scalar c.

4.2. **The centers.** The center of $A_c^{k,s}$ has dimension $k + s + 2$ (cf. [4]), a basis of the center $Z(A_c^{k,s})$ is given by

$$\mathcal{Z} := \{1, (\alpha\beta\gamma)^i + (\beta\gamma\alpha)^i + (\gamma\alpha\beta)^i, (\beta\gamma\alpha)^{k-1}\beta\gamma, (\alpha\beta\gamma)^k, \eta^j \,:\, 1 \leq i \leq k-1,\, 1 \leq j \leq s\}.$$

Note that this basis is also independent of the scalar c.

4.3. **The commutator spaces.** The algebras $A_c^{k,s}$ are symmetric, so the commutator space $K(A_c^{k,s})$ has dimension

$$\dim K(A_c^{k,s}) = \dim A_c^{k,s} - \dim Z(A_c^{k,s}) = 9k + s - (k + s + 2) = 8k - 2.$$

Indeed, the center of an algebra is the degree 0 Hochschild cohomology of the algebra, the quotient space of the algebra modulo the commutators is the degree 0 Hochschild homology of the algebra, and the k-linear dual of the Hochschild homology of an algebra is isomorphic to the Hochschild cohomology of the algebra with values in the space of linear forms of the the algebra (cf. [13, Chapter 1, Exercice 1.5.3, Corollary 1.1.8 and Section 1.5.2]). A basis of $K(A_c^{k,s})$ is given by the union

$$\mathcal{K} := \mathcal{B}_{1,2} \cup \mathcal{B}_{2,1} \cup \mathcal{K}_{1,1} \cup \mathcal{K}_{2,2}$$

where $\mathcal{B}_{1,2}$ and $\mathcal{B}_{2,1}$ have been defined above and where

$$\mathcal{K}_{1,1} := \{\beta\gamma, (\alpha\beta\gamma)^i + (\beta\gamma\alpha)^i, (\alpha\beta\gamma)^i\alpha, (\beta\gamma\alpha)^i\beta\gamma \,:\, 1 \le i \le k-1\}$$

and

$$\mathcal{K}_{2,2} := \{(\alpha\beta\gamma)^i + (\gamma\alpha\beta)^i, (\alpha\beta\gamma)^k + (\gamma\alpha\beta)^k \,:\, 1 \le i \le k-1\}.$$

4.4. **The spaces T_1.** We now consider the spaces

$$T_1(A_c^{k,s}) := \{x \in A_c^{k,s} \,:\, x^2 \in K(A_c^{k,s})\}.$$

Note that the commutator space is always contained in T_1 [12, eq. (16)]. Recall that a basis for $K(A_c^{k,s})$ was given in Section 4.3. The codimension of the commutator space inside the entire algebra is

$$\dim A_c^{k,s}/K(A_c^{k,s}) = \dim Z(A^{k,s}) = 9k + s - (8k-2) = k + s + 2.$$

A basis of $A_c^{k,s}/K(A_c^{k,s})$ is given by the cosets of the following paths

$$\{e_1, e_2, \alpha, \alpha\beta\gamma, \ldots, (\alpha\beta\gamma)^{k-1}, \eta, \ldots, \eta^{s-1}, \eta^s\}. \tag{1}$$

¿From this, we determine bases of the spaces $T_1(A_c^{k,s})$. It turns out that they depend on the parity of k and s (and on the scalar c). Recall that we denoted the above basis of the commutator space by $\mathcal{K}$. By $\lfloor . \rfloor$ and $\lceil . \rceil$ we denote the usual floor and ceiling functions, respectively.

Lemma 4.3. *A basis of $T_1(A_c^{k,s})$ is given by the union*

$$\mathcal{T} := \mathcal{K} \cup \{(\alpha\beta\gamma)^{\lceil \frac{k+1}{2} \rceil}, \ldots, (\alpha\beta\gamma)^{k-1}, \eta^{\lceil \frac{s+1}{2} \rceil}, \ldots, \eta^s\} \cup \mathcal{N}$$

where the set $\mathcal{N}$ is equal to

$$\begin{cases} \{\alpha\} & \text{if } k, s \text{ odd and } c = 0 \\ \emptyset & \text{if } k, s \text{ odd and } c = 1 \\ \{\alpha\} & \text{if } k \text{ odd, } s \text{ even and } c = 0 \\ \{\alpha + \eta^{\lfloor s/2 \rfloor}\} & \text{if } k \text{ odd, } s \text{ even and } c = 1 \\ \{\alpha\} & \text{if } k \text{ even, } s \text{ odd and } c = 0 \\ \{\alpha + (\alpha\beta\gamma)^{\lfloor k/2 \rfloor}\} & \text{if } k \text{ even, } s \text{ odd and } c = 1 \\ \{\alpha, (\alpha\beta\gamma)^{\lfloor k/2 \rfloor} + \eta^{\lfloor s/2 \rfloor}\} & \text{if } k, s \text{ even and } c = 0 \\ \{\alpha + (\alpha\beta\gamma)^{\lfloor k/2 \rfloor}, (\alpha\beta\gamma)^{\lfloor k/2 \rfloor} + \eta^{\lfloor s/2 \rfloor}\} & \text{if } k, s \text{ even and } c = 1 \end{cases}$$

Proof. As mentioned above, the commutator space is always contained in $T_1(A_c^{k,s})$ [12, eq. (16)]. So it remains to deal with the elements outside the commutator, and we use the basis of $A_c^{k,s}/K(A_c^{k,s})$ given above in (1). So we consider a linear combination

$$\lambda := a_0\alpha + a_1(\alpha\beta\gamma) + \ldots + a_{k-1}(\alpha\beta\gamma)^{k-1} + b_1\eta + \ldots + b_{s-1}\eta^{s-1} + b_s\eta^s,$$

where $a_i, b_j \in F$, and the question is, when is $\lambda^2 \in K(A_c^{k,s})$? (Note that for this question the idempotents occurring in the basis (1) can be disregarded.) Since we are working in characteristic 2, we get

$$\lambda^2 = a_0^2\alpha^2 + \ldots + a_{\lfloor k/2 \rfloor}^2 \left((\alpha\beta\gamma)^{\lfloor k/2 \rfloor}\right)^2 + b_1^2\eta^2 + \ldots + b_{\lfloor s/2 \rfloor}^2 \left(\eta^{\lfloor s/2 \rfloor}\right)^2.$$

Thus we can deduce that $\lambda^2 \in K(A_c^{k,s})$ if and only if the following conditions are satisfied:

(i) $a_1 = \ldots = a_{\lfloor k/2 \rfloor - 1} = 0$ and $b_1 = \ldots = b_{\lfloor s/2 \rfloor - 1} = 0$

(ii)

$$\begin{cases} a_{\lfloor k/2 \rfloor} = 0 = b_{\lfloor s/2 \rfloor} & \text{if } k, s \text{ odd and } c = 0 \\ a_0 = 0, \ a_{\lfloor k/2 \rfloor} = 0 = b_{\lfloor s/2 \rfloor} & \text{if } k, s \text{ odd and } c = 1 \\ a_{\lfloor k/2 \rfloor} = 0 = b_{\lfloor s/2 \rfloor} & \text{if } k \text{ odd, } s \text{ even and } c = 0 \\ a_0 = b_{\lfloor s/2 \rfloor}, \ a_{\lfloor k/2 \rfloor} = 0 & \text{if } k \text{ odd, } s \text{ even and } c = 1 \\ a_{\lfloor k/2 \rfloor} = 0 = b_{\lfloor s/2 \rfloor} & \text{if } k \text{ even, } s \text{ odd and } c = 0 \\ a_0 = a_{\lfloor k/2 \rfloor}, \ b_{\lfloor s/2 \rfloor} = 0 & \text{if } k \text{ even, } s \text{ odd and } c = 1 \\ a_{\lfloor k/2 \rfloor} = b_{\lfloor s/2 \rfloor} & \text{if } k, s \text{ even and } c = 0 \\ a_0 + a_{\lfloor k/2 \rfloor} + b_{\lfloor s/2 \rfloor} = 0 & \text{if } k, s \text{ even and } c = 1 \end{cases}$$

These conditions directly translate into the statement on the basis elements in the set $\mathcal{N}$, thus proving the lemma. □

4.5. **Proof of Theorem 4.1.** We are now in the position to prove Theorem 4.1, the main result of this section. To this end, we have to distinguish cases according to the parity of k and s. In each case we have to show that the algebras $A_0^{k,s}$ and $A_1^{k,s}$ are not derived equivalent.

4.5.1. *Case k, s odd.* By Lemma 4.3, the spaces $T_1(A_0^{k,s})$ and $T_1(A_1^{k,s})$ have different dimensions. But these dimensions are invariant under derived equivalences [14, Theorem 1], the dimension of the center being invariant and the bilinear form being non-degenerate. Hence, the algebras $A_0^{k,s}$ and $A_1^{k,s}$ are not derived equivalent.

4.5.2. *Case k odd, s even.* We first determine bases of the ideals $T_1(A_c^{k,s})^\perp$. Recall that these are ideals of the center $Z(A_c^{k,s})$. We are going to work with the bases $\mathcal{Z}$ of the center given in 4.2. A straightforward computation yields that a basis for the orthogonal space $T_1(A_c^{k,s})^\perp$ is given by

$$\mathcal{T}^\perp := \mathcal{N}^\perp \cup \{(\alpha\beta\gamma)^i + (\beta\gamma\alpha)^i + (\gamma\alpha\beta)^i, (\alpha\beta\gamma)^k, \eta^j \ : \ \lceil k/2 \rceil \le i \le k-1, \ s/2 \le j \le s\}$$

where

$$\mathcal{N}^\perp := \begin{cases} \{\eta^{s/2}\} & \text{if } c = 0 \\ \{\eta^{s/2} + (\beta\gamma\alpha)^{k-1}\beta\gamma\} & \text{if } c = 1 \end{cases}$$

We set $Z_c := Z(A_c^{k,s})$ for abbreviation and consider the factor rings $\overline{Z}_c := Z_c/T_1(A_c^{k,s})^\perp$.

A basis of these factor rings can be given (independently of c) by the cosets of the following central elements

$$\{1, (\alpha\beta\gamma)^i + (\beta\gamma\alpha)^i + (\gamma\alpha\beta)^i, (\beta\gamma\alpha)^{k-1}\beta\gamma, \eta^j \ : \ 1 \le i \le \lceil k/2 \rceil - 1, \ 1 \le j \le s/2 - 1\}.$$

In order to show that these factor rings are not isomorphic, we consider their Jacobson radicals $\overline{J}_c := \mathrm{rad}(\overline{Z}_c)$. Clearly, a basis for $\overline{J}_c$ is obtained from the above basis of $\overline{Z}_c$ by removing the unit element 1.

The crucial observation now is that for $c = 1$, we have that $\eta^{s/2} = (\beta\gamma\alpha)^{k-1}\beta\gamma$.

If $s > 2$ this implies that $(\beta\gamma\alpha)^{k-1}\beta\gamma$ is contained in the square of the radical. So the space $\overline{J}_1/\overline{J}_1^2$ has dimension 2, spanned by the cosets of η and $\alpha\beta\gamma + \beta\gamma\alpha + \gamma\alpha\beta$.

On the other hand, if $c = 0$ then $\overline{J}_0/\overline{J}_0^2$ has dimension 3, spanned by the cosets of η, $\alpha\beta\gamma + \beta\gamma\alpha + \gamma\alpha\beta$ and $(\beta\gamma\alpha)^{k-1}\beta\gamma$.

Hence, if $s > 2$, the factor rings $\overline{Z}_0$ and $\overline{Z}_1$ are not isomorphic. In particular, $A_0^{k,s}$ and $A_1^{k,s}$ are not derived equivalent.

By assumption we have that $s > 2$ or $k > 2$. The case $s = 2$ and $k > 2$ follows from the above argumentation using the fact that the algebras $A_c^{k,s}$ and $A_c^{s,k}$ are derived equivalent [6, lem. 3.2].

4.5.3. *Case k even, s odd.* This case follows from Subsection 4.5.2 once we use that, for given c, the algebra $A_c^{k,s}$ is derived equivalent to $A_c^{s,k}$ [6, lem. 3.2].

4.5.4. *Case k,s even.* We first determine bases of the ideals $T_1(A_c^{k,s})^\perp$. Again, a direct calculation yields that a basis for the orthogonal space $T_1(A_c^{k,s})^\perp$ is given by

$$\mathcal{T}^\perp := \mathcal{N}^\perp \cup \{(\alpha\beta\gamma)^i + (\beta\gamma\alpha)^i + (\gamma\alpha\beta)^i, (\alpha\beta\gamma)^k, \eta^j \ : \ k/2+1 \le i \le k-1,\ s/2+1 \le j \le s\}$$

where

$$\mathcal{N}^\perp := \begin{cases} \{\eta^{s/2} + (\alpha\beta\gamma)^{k/2} + (\beta\gamma\alpha)^{k/2} + (\gamma\alpha\beta)^{k/2}\} & \text{if } c=0 \\ \{\eta^{s/2} + (\alpha\beta\gamma)^{k/2} + (\beta\gamma\alpha)^{k/2} + (\gamma\alpha\beta)^{k/2} + (\beta\gamma\alpha)^{k-1}\beta\gamma\} & \text{if } c=1 \end{cases} \tag{2}$$

As in Subsection 4.5.2, we consider the factor rings $\overline{Z}_c := Z_c/T_1(A_c^{k,s})^\perp$, where $Z_c := Z(A_c^{k,s})$. A basis of $\overline{Z}_c$ is given by the cosets of the following central elements

$$\{1, (\alpha\beta\gamma)^i + (\beta\gamma\alpha)^i + (\gamma\alpha\beta)^i, \eta^j, (\beta\gamma\alpha)^{k-1}\beta\gamma, \eta^{s/2} \ : \ 1 \le i \le k/2-1,\ 1 \le j \le s/2-1\}.$$

Note that this basis is independent of the scalar c.

In order to show that these factor rings are not isomorphic, we consider their Jacobson radicals $\overline{J}_c := \mathrm{rad}(\overline{Z}_c)$. Clearly, a basis for $\overline{J}_c$ is obtained from the above basis of $\overline{Z}_c$ by removing the unit element 1.

The crucial observation now is that for $c=1$, it follows from (2) that in $\overline{Z}_c$ we have

$$(\beta\gamma\alpha)^{k-1}\beta\gamma = \eta^{s/2} + (\alpha\beta\gamma)^{k/2} + (\beta\gamma\alpha)^{k/2} + (\gamma\alpha\beta)^{k/2}. \tag{3}$$

On the other hand, if $c=0$, there is no relation whatsoever in $\overline{Z}_0$ involving $(\beta\gamma\alpha)^{k-1}\beta\gamma$.

By assumption we have that $k>2$ and $s>2$. Then equation (2) implies that $(\beta\gamma\alpha)^{k-1}\beta\gamma \in \overline{J}_1/\overline{J}_1^2$. Hence, for $c=1$ the space $\overline{J}_1/\overline{J}_1^2$ has dimension 2, spanned by the cosets of η and $\alpha\beta\gamma+\beta\gamma\alpha+\gamma\alpha\beta$. On the other hand, for $c=0$ the space $\overline{J}_0/\overline{J}_0^2$ has dimension 3, spanned by the cosets of η, $\alpha\beta\gamma+\beta\gamma\alpha+\gamma\alpha\beta$ and $(\beta\gamma\alpha)^{k-1}\beta\gamma$.

Hence the factor rings $\overline{Z}_0$ and $\overline{Z}_1$ are not isomorphic. In particular, the algebras $A_0^{k,s}$ and $A_1^{k,s}$ are not derived equivalent.

5. Algebras of semidihedral type

Algebras of semidihedral type have been defined by Erdmann. An algebra A (over an algebraically closed field) is said to be *of semidihedral type* if it satisfies the following conditions:

(i) A is symmetric and indecomposable.
(ii) The Cartan matrix of A is non-singular.
(iii) The stable Auslander-Reiten quiver of A has the following components: tubes of rank at most 3, at most one 3-tube, and non-periodic components isomorphic to $\mathbb{Z}\mathbb{A}_\infty^\infty$ and $\mathbb{Z}\mathbb{D}_\infty$.

Note that the original definition in [4, VIII.1] contains the additional requirement that A should be of tame representation type. It has been shown by the first author [6, thm. 6.1] that tameness already follows from the properties given in the above definition.

K. Erdmann gave a classification of algebras of semidihedral type up to Morita equivalence. A derived equivalence classification has been given in [6, sec.4]. It turns out that every algebra of semidihedral type is derived equivalent to an algebra in one of the two following families.

For any integers $k \ge 1$, $t \ge 2$ and a scalar $c \in \{0,1\}$ define the algebra $A_c^{k,t} = SD(2\mathcal{B})_1^{k,t}(c)$ by the quiver

subject to the relations

$$\gamma\beta = 0, \eta\gamma = 0, \beta\eta = 0, \alpha^2 = (\beta\gamma\alpha)^{k-1}\beta\gamma + c(\alpha\beta\gamma)^k, \eta^t = (\gamma\alpha\beta)^k, (\alpha\beta\gamma)^k = (\beta\gamma\alpha)^k.$$

Secondly, for any $k \geq 1$, $t \geq 2$ such that $k + t \geq 4$ and $c \in \{0,1\}$ we define the algebras $B_c^{k,t} = SD(2\mathcal{B})_2^{k,t}(c)$ by the same quiver as above, subject to the relations

$$\beta\eta = (\alpha\beta\gamma)^{k-1}\alpha\beta, \gamma\beta = \eta^{t-1}, \eta\gamma = (\gamma\alpha\beta)^{k-1}\gamma\alpha, \beta\eta^2 = 0, \eta^2\gamma = 0, \alpha^2 = c(\alpha\beta\gamma)^k.$$

We remark that every block of a finite group with semidihedral defect group of order 2^n ($n \geq 4$) and two simple modules is derived equivalent to one of the following algebras: $SD(2\mathcal{B})_1^{1,2^{n-2}}(c)$ or to $SD(2\mathcal{B})_2^{2,2^{n-2}}(c)$ where the scalar c is either 0 or 1. It is an open question whether blocks of finite groups can occur for both scalars $c = 0$ and $c = 1$.

5.1. **The algebras $A_c^{k,t}$.** We first consider the algebras $A_c^{k,t}$ defined above. The aim of this section is to prove Theorem 1.2, distinguishing these algebras for different scalars up to derived equivalence.

To this end, we are going to study the sequence of generalized Reynolds ideals

$$\mathcal{Z}_c \,:\, Z(A_c^{k,t}) \supseteq T_1(A_c^{k,t})^\perp \supseteq T_2(A_c^{k,t})^\perp \supseteq \ldots \supseteq T_r(A_c^{k,t})^\perp \supseteq \mathrm{soc}(A_c^{k,t})$$

of the center.

Let us compare the algebras $A_c^{k,t}$ of semidihedral type defined above with the corresponding algebras $A_c^{k,s} = D(2\mathcal{B})^{k,s}(c)$ of dihedral type considered in Section 4. These algebras are defined by the same quiver, and the only difference in the relations is that now in the semidihedral case we have that $\alpha^2 = (\beta\gamma\alpha)^{k-1}\beta\gamma + c(\alpha\beta\gamma)^k$, whereas we had $\alpha^2 = c(\alpha\beta\gamma)^k$ in the dihedral case. Note that the new summand occurring,

$$(\beta\gamma\alpha)^{k-1}\beta\gamma = [(\beta\gamma\alpha)^{k-1}\beta, \gamma]$$

is a commutator in $A_c^{k,t}$ (using that $\gamma\beta = 0$). This actually means that the proof in the dihedral case given in Subsections 4.1 - 4.5 carries over *verbatim* to the algebras $A_c^{k,t}$ of semidihedral type.

We refrain from reproducing the proof again here, and leave the details of the verification to the reader.

5.2. **The algebras $B_c^{k,t}$.** We now consider the second family of algebras $B_c^{k,s} = SD(2\mathcal{B})_2^{k,t}(c)$ where $k \geq 1$, $t \geq 2$ such that $k + t \geq 4$ and $c \in \{0,1\}$. The sequence of generalized Reynolds ideals takes the form

$$\mathcal{Z}_c \,:\, Z(B_c) \supseteq T_1(B_c)^\perp \supseteq T_2(B_c)^\perp \supseteq \ldots \supseteq T_r(B_c)^\perp \supseteq \mathrm{soc}(B_c).$$

We can not distinguish the algebras completely, but we shall prove the following partial result.

Theorem 5.1. *Suppose $k \geq 1$ and $t \geq 3$ are both odd. Then the spaces $T_1(B_0^{k,t})$ and $T_1(B_1^{k,t})$ have different dimensions.*

In particular, the algebras $B_0^{k,s}$ and $B_1^{k,s}$ are not derived equivalent.

5.2.1. *Bases for the algebras.* We fix integers $k \geq 1$ and $t \geq 2$ such that $k+t \geq 4$ (not necessarily both odd). The algebras $B_c^{k,t}$ have dimension $9k + t$, the Cartan matrix has the form (cf. [4])

$$\begin{pmatrix} 4k & 2k \\ 2k & k+t \end{pmatrix}.$$

A basis of the algebras, consisting of non-zero paths in the quiver, is given by the union

$$\mathcal{B} := \mathcal{B}_{1,1} \cup \mathcal{B}_{1,2} \cup \mathcal{B}_{2,1} \cup \mathcal{B}_{2,2},$$

where

$$\mathcal{B}_{1,1} := \{e_1, (\alpha\beta\gamma)^i, \alpha, (\alpha\beta\gamma)^i\alpha, (\beta\gamma\alpha)^i, \beta\gamma, (\beta\gamma\alpha)^i\beta\gamma, (\alpha\beta\gamma)^k = (\beta\gamma\alpha)^k = \beta\eta\gamma \,:\, 1 \leq i \leq k-1\}$$

$$\mathcal{B}_{1,2} := \{\beta, (\beta\gamma\alpha)^i\beta, \alpha\beta, (\alpha\beta\gamma)^i\alpha\beta \,:\, 1 \leq i \leq k-1\}$$

$$\mathcal{B}_{2,1} := \{\gamma, (\gamma\alpha\beta)^i\gamma, \gamma\alpha, (\gamma\alpha\beta)^i\gamma\alpha \,:\, 1 \leq i \leq k-1\}$$

$$\mathcal{B}_{2,2} := \{e_2, (\gamma\alpha\beta)^i, \eta, \ldots, \eta^{t-2}, \eta^{t-1} = \gamma\beta, \eta^t = (\gamma\alpha\beta)^k = \eta\gamma\beta \,:\, 1 \leq i \leq k-1\}.$$

5.2.2. *The centers.* The center of $B_c^{k,t}$ has dimension $k+t+2$ (cf. [4]), a basis of the center $Z(B_c^{k,t})$ is given by

$\mathcal{Z} := \{1, (\alpha\beta\gamma)^i+(\beta\gamma\alpha)^i+(\gamma\alpha\beta)^i, (\beta\gamma\alpha)^{k-1}\beta\gamma, (\alpha\beta\gamma)^k, \eta^j, \eta+(\alpha\beta\gamma)^{k-1}\alpha \,:\, 1\le i\le k-1,\, 2\le j\le s\}.$

Note that this basis is also independent of the scalar c.

5.2.3. *The commutator spaces.* The algebras $B_c^{k,t}$ are symmetric, so the commutator space $K(B_c^{k,t})$ has dimension

$$\dim K(B_c^{k,t}) = \dim B_c^{k,t} - \dim Z(B_c^{k,t}) = 9k+t-(k+t+2) = 8k-2.$$

A basis of $K(B_c^{k,t})$ is given by the union

$$\mathcal{K} := \mathcal{B}_{1,2}\cup\mathcal{B}_{2,1}\cup\mathcal{K}_{1,1}\cup\mathcal{K}_{2,2}$$

where $\mathcal{B}_{1,2}$ and $\mathcal{B}_{2,1}$ have been defined above and where

$$\mathcal{K}_{1,1} := \{\beta\gamma+\gamma\beta, (\alpha\beta\gamma)^i+(\beta\gamma\alpha)^i, (\alpha\beta\gamma)^i\alpha, (\beta\gamma\alpha)^i\beta\gamma \,:\, 1\le i\le k-1\}$$

and

$$\mathcal{K}_{2,2} := \{(\alpha\beta\gamma)^i+(\gamma\alpha\beta)^i, \,:\, 1\le i\le k\}.$$

5.2.4. *The spaces T_1.* We now consider the spaces

$$T_1(B_c^{k,t}) := \{x\in B_c^{k,t} \,:\, x^2\in K(B_c^{k,t})\}.$$

The commutator space is always contained in T_1 [12, eq. (16)]. A basis for $K(B_c^{k,t})$ was given in Section 5.2.3. A basis of $B_c^{k,t}/K(B_c^{k,t})$ is given by the cosets of the following paths

$$\{e_1, e_2, \alpha, \alpha\beta\gamma, \ldots, (\alpha\beta\gamma)^{k-1}, \eta, \ldots, \eta^{t-1}, \eta^t\}. \tag{4}$$

We now turn to the the spaces $T_1(B_c^{k,t})$. It turns out that they depend on the parity of k and s (and on the scalar c).

¿From now on, we apply our assumption that k and t are both odd.

Recall that we denoted the above basis of the commutator space by $\mathcal{K}$.

Lemma 5.2. *Let $k\ge 1$ and $t\ge 3$ be both odd. A basis of $T_1(B_c^{k,t})$ is given by the union*

$$\mathcal{T} := \mathcal{K}\cup\{(\alpha\beta\gamma)^{\frac{k+1}{2}}, \ldots, (\alpha\beta\gamma)^{k-1}, \eta^{\frac{t+1}{2}}, \ldots, \eta^t\}\cup\mathcal{N}$$

where the set $\mathcal{N}$ is equal to

$$\begin{cases} \{\alpha\} & \text{if } c=0 \\ \emptyset & \text{if } c=1 \end{cases}$$

Proof. Since the commutator space is contained in $T_1(B_c^{k,t})$ [12, eq. (16)], it remains to consider the basis of $B_c^{k,t}/K(B_c^{k,t})$ given in (4). So we consider a linear combination

$$\lambda := a_0\alpha + a_1(\alpha\beta\gamma)+\ldots+a_{k-1}(\alpha\beta\gamma)^{k-1}+b_1\eta+\ldots+b_{t-1}\eta^{t-1}+b_t\eta^t,$$

where $a_i, b_j\in F$, and we have to determine when $\lambda^2\in K(B_c^{k,t})$. By assumption k and t are odd, so we get

$$\lambda^2 = a_0^2\alpha^2+\ldots+a_{\frac{k-1}{2}}^2(\alpha\beta\gamma)^{k-1}+b_1^2\eta^2+\ldots+b_{\frac{t-1}{2}}^2\eta^{t-1}$$

(recall that we are working in characteristic 2). A basis for the commutator space has been given in 5.2.3. ¿From this we can deduce that $\lambda^2\in K(B_c^{k,t})$ if and only if the following conditions are satisfied:

(i) $a_1=\ldots=a_{\frac{k-1}{2}}=0$ and $b_1=\ldots=b_{\frac{t-1}{2}}=0$,
(ii) if $c=1$, also $a_0=0$.

¿From these conditions, the claim of the lemma follows directly. □

Remark 5.3. In case k is even, in the square of λ above a term $(\alpha\beta\gamma)^k$ appears and analogously to Lemma 4.3 it becomes impossible to distinguish the parameters c just by the dimension of the generalized Reynolds ideals. Similar phenomena appear for t even.

5.2.5. *Proof of Theorem 5.1.* ¿From Lemma 5.2, we deduce that the spaces $T_1(B_0^{k,t})$ and $T_1(B_1^{k,t})$ have different dimensions. But these dimensions are invariant under derived equivalences, thus proving Theorem 5.1. □

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