# On the principle of stationary action: An equation to rule them all!

Plan: I) Intro & origins
1) Many eqs
2) Fermat's principle
3) XVIII$^{th}$ cent.: unify laws of ...
4) Calc. of variations
II) Gravity à la Einstein-Hilbert
1) Natural action & EFE
2) Example(s)
III) Degenerate Einstein theory
1) Tensors & curvature
2) Axiomatize Action Principle

## I) Introduction & origins

### 1) (Too) many equations

Newton: $\vec{F} = m\vec{a} = \dot{\vec{p}}$, Maxwell: $\begin{cases} \nabla\cdot\vec{B} = 0 \\ \nabla\cdot\vec{E} = \rho/\varepsilon_0 \\ \nabla\times\vec{B} = \mu_0\left(\varepsilon_0 \frac{\partial\vec{E}}{\partial t} + \vec{j}\right) \\ \nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \end{cases}$

Geodesics: $\nabla_{\dot\gamma}\dot\gamma = 0 \iff \ddot\gamma^i + \left(\sum_{j,k}\right)\Gamma^i_{jk}\dot\gamma^j\dot\gamma^k = 0$

Schrödinger: $i\hbar\frac{\partial|\psi\rangle}{\partial t} = \hat{H}|\psi\rangle$, Einstein: $\underbrace{R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu}}_{=:G_{\mu\nu}} = \underbrace{\frac{8\pi G}{c^4}}_{=:K} T_{\mu\nu}$

Q: See a pattern?

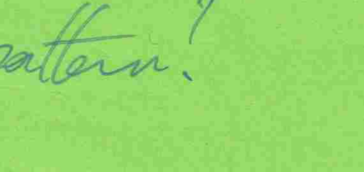
### 2) Fermat's principle (~1662)

"Light rays take the path of least time" ⚠ not distance

In fact: of stationary time (⚠ concave mirror)

**Applications**

Reflection:

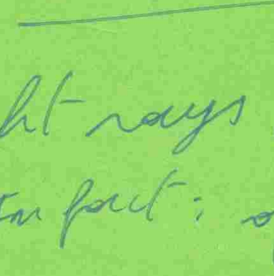



$\begin{cases} L_1 = ct_1 = x/\sin\theta_1 \\ L_2 = ct_2 = (l-x)/\sin\theta_2 \end{cases}$

$\hookrightarrow t = t_1 + t_2 = \frac{1}{c}\left(x/\sin\theta_1 + (l-x)/\sin\theta_2\right)$

$t$ stat. wrt $x \Rightarrow \boxed{\frac{dt}{dx} = 0} \Rightarrow \theta_1 = \theta_2$

Refraction:




$v_1 = \frac{c}{n_1}$

$v_2 = \frac{c}{n_2}$

Ibn Sahl (984)
Ibn Al-Haytam (1021)
Snell-Descartes

$t = \frac{L_1}{v_1} + \frac{L_2}{v_2} = \frac{1}{c}\left(n_1 x/\sin\theta_1 + n_2(l-x)/\sin\theta_2\right)$

$\frac{dt}{dx} = 0 \Rightarrow n_1\sin\theta_1 = n_2\sin\theta_2$

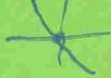
(Define optical path $L_{AB} := \int_{AB} n\, dl$ (⚠ not $\frac{dL}{dx} = 0$), $\delta L_{AB} = 0$)

[Optimization under constraints
fixed extr. in Fermat
~> Lagrange multipliers...

→ Striking use of Fermat's principle:

Ie. Bernoulli's Brachistochrone curve (1696-1718)

challenge: A  B path of least time for falling bead?

idea: many infinit. Fermats  $n_1 \sin\theta_1 = n_2 \sin\theta_2$

$\Rightarrow \frac{\sin\theta}{v} = \text{cst.}$ ⊕ $\sin\theta = \frac{dx}{\sqrt{dx^2+dy^2}}$ & Newton $v = \sqrt{2gy}$
— height

$\Rightarrow \frac{\sin\theta}{\sqrt{y}} = \text{cst} \Rightarrow y(1+y'^2) = \text{cst}$
inverted cycloid

.. powerful!

3) XVIII th century: unify laws of Φ

· Impulse by Leibniz ... (the fewer the principles, the greater the conseq.)

· then E. Du Châtelet [les Institutions] 1740

"Ce principe duquel toutes les vérités contingentes dépendent, et qui n'est ni moins primitif, ni moins universel que celui de contradiction, est le Principe de la raison suffisante: tous les hommes le suivent naturellement; car il n'y a personne qui ne se détermine à une chose plutôt qu'à une autre, sans une raison suffisante qui lui fasse voir que cette chose est préférable à l'autre."

· "then" D'Alembert (1743) $\sum_i (\vec{F}_i - \dot{\vec{p}}_i)\cdot\delta\vec{r}_i = 0$

δ notation of Lagrange — virtual displacement

cf Tinca Prunea-Bretonnet (2023) (PUF/les Études philosophiques)
[Principes Métaphysiques et certitude chez Christian Wolff et Émilie du Châtelet]

· Maupertuis (1744)

"Le chemin qu'elle [la lumière] tient est celui pour lequel la quantité d'action est la moindre"

↳ grows w/ mass, velocity, distance

· Define action $\Big\|$ $S := \int_{path} m \times v \times dl$ $\left( = \int \vec{p}\cdot d\vec{q} \text{ Euler} \right) = S[q]$

ds : inf. length $= v\,dt$ $\qquad p = m\dot{q}$


$\sim [\text{Mass}] \times [L]^2 \times [T]^{-2} \times [T] \sim [\text{Energy}] \times [\text{Time}]$

· State $\boxed{(\delta_q S)_{E_m \text{ fixed}} = 0}$ ⚠ Fixed endpoints & energy (total) in S

Nice but Pbs: $E_m$ fixed + only shape of trajectory (no time parametrization)

⇝ recovers Kepler I but not Kepler II ...

→ Need stronger maths

4) Calculus of variations & Hamilton's principle

Reformulation: $S = \int m v\, ds = \int m v^2 dt = \int 2 E_c\, dt$

$= \int E_c + E_m - E_p\, dt = \int E_c - E_p\, dt + \int E_m\, dt$

↳ vary wrt path $x = x(t)$, since $E_m$ cst

⇝ $\boxed{\delta \int (E_c - E_p)\, dt = 0}$ fixed ext. but dep on time ⇝ parametrization

Let $L(x, \dot{x}(t)) = E_c - E_p = \frac{1}{2} m \dot{x}^2 - V(x)$ Lagrangian

Can be more general $L : TM \to \mathbb{R}$
↳ tangent bundle...

then define $\boxed{S[x] = \int L \circ x(t)\, dt}$

and state $\boxed{\frac{\delta S}{\delta x} = 0}$ Hamilton (1834) "Final form" ...

Q: But what is "δ" really?

→ Lagrange's calculus of variations (1755)

general $S = \int_{t_0}^{t_1} L\, dt$, say $x = x(t)$ = actual path

Perturb: $y = x + \varepsilon\eta$, $0 < \varepsilon << 1$, $\eta$ any path but extremities fixed, so $\eta(t_i) = 0$

Def: $\delta_\eta S[x] := \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} S[x+\varepsilon\eta] = \int_{t_0}^{t_1} \frac{dL}{d\varepsilon}\Big|_{\varepsilon=0} dt$

$= \int \left(\frac{\partial L}{\partial x}\frac{dy}{d\varepsilon} + \frac{\partial L}{\partial \dot{x}}\frac{d\dot{y}}{d\varepsilon}\right) dt = \int \left(\frac{\partial L}{\partial x}\eta + \frac{\partial L}{\partial \dot{x}}\eta'\right) dt$

$\stackrel{IBP}{=} \int \left(\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)\right)\eta\, dt + \underbrace{\left[\frac{\partial L}{\partial \dot{x}}\eta\right]_{t_0}^{t_1}}_{=0}$. So $\delta S = 0 \iff \forall \eta,\ \int \left(\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)\right)\eta\, dt = 0$

↳ Fundamental lemma: Let $f \in L^1_{loc}(\Omega)$ such that $\forall u \in C_0^\infty(\Omega)$, $\int_\Omega f u\, dt = 0$, then $f = 0$.

Thm: $\delta S = 0 \iff \boxed{\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0}$ Euler–Lagrange equation

Allows to deal with general situations, by making an energy/work balance instead of forces (often harder) then find $L = T - V$ and write Euler-Lagrange using Hamilton's principle. ("Lets Mathematicians do Physics" Steven Strogatz)

→ Led to modern formulation of Mechanics (Hamiltonian, Poisson structures, symplectic manifolds...)

Examples •) Newton: $\vec{F}_{cons} = -\nabla V$ & $\Delta E_m = \sum W_{nc} = 0$

(↳ without energy loss, otherwise $\partial_x L - \dot{\overline{\partial_{\dot{x}} L}} = -\partial_{\dot{x}} F$ Rayleigh function)

↳ $L = \frac{1}{2} m\dot{x}^2 - V(x)$, EL: $\vec{F} = -\nabla V = \frac{\partial L}{\partial x} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{d}{dt}(m\dot{x}) = m\vec{a}$ ✓

•) Maxwell: $L = -\frac{\varepsilon_0}{4} F_{\mu\nu}F^{\mu\nu} + j^\mu A_\mu$ (Einstein summation), $F_{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix}$ (⚠ 4 coords in space time)

•) Geodesics: $(M,g)$ Riemannian ⇝ $\nabla_X Y$ aff. connection (L.C.) ($\mathbb{R}$-bilin. ⊕ $\nabla_{fX}Y = f\nabla_X Y$, $\nabla_X(fY) = X(f)Y + f\nabla_X Y$)

$\gamma: I \to M$ geodesic if $\boxed{\nabla_{\dot{\gamma}}\dot{\gamma} = 0}$ but if $L[\gamma] := \int_I \|\dot{\gamma}\|\, dt = \int \sqrt{g(\dot{\gamma},\dot{\gamma})}\, dt$ Length then $\boxed{\delta L = 0 \iff \text{geod}}$

# II) Gravity à la Einstein-Hilbert

## 1) Deriving the field equation (EFE) from an action

•) <u>Equivalence principle</u>

gravity is (intrinsic) geometry: <u>matter curves spacetime</u>

→ nice equation for this idea?

<u>1st</u>: <u>Manifold</u> $M$ ($\partial M = \emptyset$) with <u>metric</u> $g = g_{\mu\nu}$
(4 dim) (→ distances)

•) <u>Nice action</u>?

$g_{\mu\nu}$ ⇝ Levi-Civita connection, Riemann + Ricci curvature $R_{\mu\nu}$ But $\boxed{R_{\mu\nu} \propto T_{\mu\nu} \text{ has not } \nabla^\mu T_{\mu\nu} = 0}$

⇝ <u>Ricci scalar</u> $R = tr_g(R_{\mu\nu}) = g^{\mu\nu} R_{\mu\nu}$

<u>Vermeil thm</u>: $R$ is the only function $R = R(g, \partial g, \partial^2 g)$, lin. in $\partial^2 g$. → got a function, need a top-form

⇝ <u>volume form</u> $\omega = dvol_g = \sqrt{|\det(g_{\mu\nu})|}\, d^n x$ in chart.
"canonical"

⇝ let $\boxed{S = S[g] := \int_M R\,\omega}$

•) <u>Vary $S = S[g]$</u> $\quad \delta S = \int \delta R\,\omega + R\,\delta\omega$

* <u>$\delta R = \delta g^{\mu\nu} R_{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}$</u> $\qquad$ <u>Palatini</u>: $g^{\mu\nu}\delta R_{\mu\nu} = \nabla_\mu X^\mu$ for some field $X$

$\Rightarrow \int g^{\mu\nu}\delta R_{\mu\nu}\,\omega = \int \partial_\mu(\sqrt{|g|}X^\mu) = 0$ (<u>Stokes</u>)

* $\delta\omega = \delta\left(\sqrt{\det(g_{\mu\nu})}\right) d^n x = \dfrac{\delta(\det(g_{\mu\nu}))}{2\sqrt{|g|}} d^n x = \dfrac{|g|\, g^{\mu\nu}\delta g_{\mu\nu}}{2\sqrt{|g|}} d^n x = -\dfrac{1}{2} g_{\mu\nu}\delta g^{\mu\nu}\,\omega$

(since $\delta\det g) = (\det g)\, tr(g^{-1}\delta g)$)

$\Sigma$: $0 = \delta S = \int \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}\right)\delta g^{\mu\nu}\,\omega$, $\forall\, \underline{\delta g^{\mu\nu}}$ (can be anything since $GL_n \cap Sym_n \subset Sym_n$ open)

$\xrightarrow{\text{fund. Lemma}}$ <u>$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} =: G_{\mu\nu} = 0$</u> $\quad$ has <u>$\nabla^\mu G_{\mu\nu} = 0$</u>

If matter, Lagrangian density $\mathcal{L}: M \to \mathbb{R}$, $S := \frac{1}{c}\int_M \left(\frac{1}{2\kappa}(R - 2\Lambda) + \mathcal{L}\right)\omega$

Vary $S$ and let $T_{\mu\nu} := \mathcal{L} g_{\mu\nu} - 2\dfrac{\delta\mathcal{L}}{\delta g^{\mu\nu}}$ then $\boxed{G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}}$ & $\boxed{\nabla^\mu T_{\mu\nu} = 0}$ ($\kappa = \frac{8\pi G}{c^4}$ to match Newton)

## 2) Example: Schwarzschild & beyond...

Simple $\underline{\text{vacuum}}$ sol of EFE! $\left.\begin{matrix} \\ T_{\mu\nu}=0, \Lambda=0 \end{matrix}\right) \Rightarrow \underline{R_{\mu\nu}=0}$

static, sph. sym $\rightsquigarrow ds^2 = -c^2 f(r)\, dt^2 + \frac{dr^2}{f(r)} + r^2 \underbrace{(d\theta^2 + \sin^2\theta\, d\varphi^2)}_{=: d\Omega^2}$

$\rightsquigarrow R_{\mu\nu}$ diagonal, $=0 \Leftrightarrow f + rf' = 1 \Leftrightarrow (rf)' = 1 \rightsquigarrow f(r) = 1 - \frac{2GM}{c^2 r}$

$\rightsquigarrow f(r) = 1 - \frac{r_s}{r}$, $r_s := \frac{2GM}{c^2}$ Schwarzschild radius "Central Mass" (Horizon)

$ds^2 = ds^2_{Mink.} + \underbrace{\frac{R_s}{r}\left(d(ct)^2 + dr^2\right)}_{\text{Key correction...}} + O\left(\frac{1}{r^2}\right)$ many verifications & applications...

{ More involved: FLRW: $ds^2 = -c^2dt^2 + a(t)^2\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right)$ expansion of the Universe..

$\rightsquigarrow$ Friedmann equations, Hubble's law

$\Rightarrow \nabla^\mu T_{\mu\nu} = 0$ is $1^{st}$ principle of thermodynamics here.. }

Quantum gravity (Hartle–Hawking 1983 → Quantum theory of BigBang) $\rightsquigarrow$ Euclidean version

$$ds^2 = +c^2\left(1 - \frac{r_s}{r}\right)dt^2 + \frac{dr^2}{1 - \frac{r_s}{r}} + r^2 d\Omega^2$$

Q: $\rightsquigarrow$ 5-dim manifold over both? idea: $ds^2 = -c^2\left(1 - \frac{r_s}{r}\right)(dt^2 + i\, dT^2) + \dots$ $d\tau^2$

Pb: It has cst rank $4 < 5$, But Battista observed it works the same!

$\rightsquigarrow$ need a theory for degenerate metrics (over $\mathbb{C}$)..

# III) Degenerate Einstein theory

## 1) Tensors & curvature [B.G. 2024]

• $1^{st}$: "Inverse"? Idea: Moore–Penrose $\left\{\begin{matrix} g\tilde{g}g = g \\ \tilde{g}g\tilde{g} = \tilde{g} \\ (g\tilde{g})^\dagger = g\tilde{g} \\ (\tilde{g}g)^\dagger = \tilde{g}g \end{matrix}\right)$ not covariant

Make dagger covariant by def. it from background spectator (flat) metric $\xi_{\mu\nu}$ (Riemann) $\rightsquigarrow \exists!$ MP inverse $\tilde{g}^{\mu\nu}$ of $g_{\mu\nu}$

• Next, need canonical connection. Start w/ $\hat{\nabla}$, Levi-Civita of $\xi_{\mu\nu}$. Thm: $\exists! \nabla \left| \begin{matrix} \nabla g = 0 \\ \tilde{g}(\nabla^* - g\nabla)\tilde{g} = 0 \\ \tilde{g}g(\nabla - \hat{\nabla}) = (\nabla - \hat{\nabla}) \end{matrix}\right.$. $\nabla = \nabla_{LC}$ if $g_{\mu\nu}$ invertible.

⚠ Now, $\nabla$ has torsion: $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] \neq 0$. (In charts, $T_{\mu\nu}{}^\rho$ ...) ...

With $\nabla$, can construct Riemann and Ricci tensors, and $\tilde{R} := \tilde{g}^{\mu\nu} R_{\mu\nu}$ as well.

Next, copy-paste EFE! w/ new tensors and $\tilde{R}$

Pb: Energy not conserved locally: $\nabla^\mu G_{\mu\nu} \neq 0$

⊕ What is $T_{\mu\nu}$ ?...

Idea: $G_{\sharp\mu\nu} = K\, T_{\sharp\mu\nu}$

($X_{\sharp\mu\nu} := g_{\mu\alpha} \tilde{g}^{\alpha\beta} X_{\beta\nu}$)

→ but data is $T_{\mu\nu}$, not $T_{\sharp\mu\nu}$

Solution: Copy action, not EFE

2) Ansatzage action principle (B.G. 2016, to appear)

Have Ricci, what is $\omega$! (det unavailable)

Key fact: Writing $\omega = e^f d^n x$, $\nabla\omega = 0 \iff \boxed{\partial_i f = \Gamma^j_{ij}}$ ODE

fix value of $\omega|_{\mathcal{J}}$, $\mathcal{J}$ = non-deg. hypersurface

(⇝ technicalities, tricky...)

⇝ get action $S = \frac{1}{c}\int_M \left(\frac{1}{2K}(\tilde{R} - 2\Lambda) + \mathcal{L}\right)\omega$

$\boxed{\delta S = 0}$ $\overset{\text{Ansatz}}{\Longleftrightarrow}$

$$\boxed{\begin{cases} R_{\sharp\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} + \Lambda g_{\mu\nu} = K\, T_{\sharp\mu\nu} & \text{"Classical EFE"} \\ T_{\alpha(\mu}{}^{[\beta} T_{\nu)\beta}{}^{\alpha]} - \tilde{g}^{\alpha[\beta} T_{\beta(\mu}{}^{\beta]} g_{\nu)\lambda}\left(\Gamma^\lambda{}_{\gamma\alpha} - \tilde{\Gamma}^\lambda{}_{\gamma\alpha}\right) = \frac{K}{2}\left(T_{\mu\nu} - T_{\sharp\mu\nu}\right) \end{cases}}$$

skip fundamental lemma, unusable anyway...

$\underbrace{T_{\cdot(\mu}{}^{[\cdot} T_{\nu)\cdot}{}^{\cdot]} - g^{\cdot[\cdot} T^{\cdot]}{}_{\cdot(\mu} g_{\nu)\cdot}(\Gamma - \tilde{\Gamma})}_{\text{torsion part}} = \frac{K}{2}\left(T - T_{\sharp}\right)$ ↘ error on matter

New Conservation laws: $\boxed{\nabla^\mu T_{\mu\nu} + \nabla(\text{torsion part}) = 0}$

→ yields $R_{\mu\nu} = 0$ for Schwarzschild

⊕ Nice FLRW case: $ds^2 = -d\tau^2 + a(\tau)^2\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right)$, $a: \mathbb{C} \to \mathbb{C}$

$T_{\mu\nu}$ = perfect fluid w/ $\rho, p: \mathbb{C} \to \mathbb{C} \Rightarrow T_{\sharp\mu\nu} = T_{\mu\nu}$

So EFE ⇒ a holomorphic ⊕ holo version of Friedmann Hubble...

⇝ Torsion = 0

⇒ Class. EFE, holomorphic version...

That's all Folks!