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Classification of Iwahori-Hecke modules and p-modular representations of $SL_2(F)$

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Abstract. Let F be a non-archimedean local field complete for a discrete valuation and with finite residue class field of characteristic $p > 0$, and let $\overline{\mathbb{F}}_p$ be an algebraic closure of the finite field with p elements. The study of the pro-p-Iwahori-Hecke algebra of $GL_2(F)$ over $\overline{\mathbb{F}}_p$ and of its finite-dimensional simple right modules, due to Vignéras and Ollivier, provides interesting results when compared to the theory of smooth representations of $GL_2(F)$ over $\overline{\mathbb{F}}_p$. This paper makes the first steps towards an analogue study for the special linear group $SL_2(F)$. After proving a general relation between the standard Iwahori-Hecke algebras of $GL_n(F)$ and $SL_n(F)$ for arbitrary $n \geq 2$, we give an explicit description of the pro-p-Iwahori-Hecke algebra of $SL_2(F)$ and compute all its simple right modules. In particular, we connect these results to those we proved in a previous work on smooth representations of $SL_2(F)$ over $\overline{\mathbb{F}}_p$, and to the corresponding statements obtained by Vignéras for $GL_2(F)$.

1. Introduction

Let p be a prime number and let F be a non-archimedean local field which is complete for a discrete valuation and has finite residue class field k_F of characteristic p and cardinality q. Let C be an algebraically closed field of same characteristic p and let G be the group of F-rational points of a connected reductive group defined over F. Understanding the irreducible smooth representations of G over C is still a very hard problem, even for basic groups as the general linear group $GL_n(F)$. The first results in this domain were proved in the mid-nineties in a remarkable work of Barthel and Livné for $GL_2(F)$ [4, 5]. In these papers, they pointed out a mysterious family of representations that they called *supersingular* and proved that these objects correspond to the supercuspidal representations of $GL_2(F)$; unfortunately, they were not able to describe them explicitely. Several years later, Breuil managed in [6] to compute all these supersingular objects when $F = \mathbb{Q}_p$ is the field of p-adic numbers, what gave an exhaustive classification of irreducible smooth representations of $GL_2(\mathbb{Q}_p)$, but he could not deal with a general F. So far, the only groups other than $GL_2(\mathbb{Q}_p)$ for which an exhaustive classification of irreducible smooth representations over $\overline{\mathbb{F}}_p$ is known are $SL_2(\mathbb{Q}_p)$ [2] and $U(1,1)(\mathbb{Q}_{p^2}/\mathbb{Q}_p)$ with p odd [9], where $\mathbb{Q}_{p^2}$ is a quadratic unramified extension of $\mathbb{Q}_p$. Note that these representations play a key role in the context of (conjectural) modular Langlands correspondences : as they coincide with supercuspidal representations, they should be the counterpart of the irreducible Galois representations appearing in such correspondences.

Various strategies aiming to collect as much information as possible on supersingular representations of $GL_2(F)$ for arbitrary F have been developed by various authors, as in [7, 8, 14, 19, 21] for instance. In this paper, we are interested in the approach initiated

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by Vignéras [21, 22, 23, 25] and fruitfully developed for $GL_n(F)$ by Ollivier [13, 14, 15] and her collaborators [16, 17]. Its starting point is based on the following observation : if π is any smooth representation of G over C and if H is any open compact subgroup of G, compact Frobenius reciprocity implies that the space π^H of H-invariant vectors in π is isomorphic to $\mathrm{Hom}_{C[G]}(\mathrm{ind}_H^G(\mathbf{1}),\pi)$, and is hence naturally endowed with a structure of right module over the C-algebra $\mathcal{H}_C(G,H) := \mathrm{End}_{C[G]}(\mathrm{ind}_H^G(\mathbf{1}))$. If H is chosen such that π^H is non-zero, a hope is that understanding the $\mathcal{H}_C(G,H)$-module π^H could bring some interesting information about π, and maybe even characterize it. For instance, when H is a pro-p-group of G, we know that π^H is non-zero [4, Lemma 3 (1)] and we have a good irreducibility criterion for π [21, Criterium 4.5]. Another powerful application, due to Ollivier [14, Théorème 1.3.a)], is as follows : for H being the standard pro-p-Iwahori subgroup of $G = GL_2(\mathbb{Q}_p)$, any smooth representation π of $GL_2(\mathbb{Q}_p)$ generated by its H-invariant vectors is uniquely determined by the structure of $\mathcal{H}_C(GL_2(\mathbb{Q}_p),H)$-module carried by π^H.

To obtain this kind of results, one must understand the structure of the C-algebra $\mathcal{H}_C(G,H)$ and of its simple right modules, then identify among them those corresponding to H-invariant spaces of irreducible smooth representations of G over C. As we are only interested in admissible representations of G over C, it is enough to classify simple right $\mathcal{H}_C(G,H)$-modules of finite dimension[1] over C. In the present paper, we realize this programme when $G = SL_2(F)$ is the special linear group of rank 1 and $H = I_S(1)$ is its standard pro-p-Iwahori subgroup. We moreover deduce from these results some interesting relationships with the theory of irreducible smooth representations of G over C, as developed in [2], and with the results obtained by Vignéras for $GL_2(F)$ [21, 25]. Before we state our main results, let us mention that they have already been of crucial use in recent works of several authors [10, 11] and that they will certainly be useful in further developments of p-adic and mod p Langlands programmes for $SL_2(F)$. Also note that in a recent work [3], Abe gives a full classification of simple right modules over the standard pro-p-Iwahori-Hecke algebra for an arbitrary p-adic group G in terms of parabolic triples, and our final classification result can be recovered (using more sophisticated tools) from this work. Nevertheless, the results contained in this paper were not only proved before those in [3], but they are also giving finer structure statements together with relations to modular representation theory of G that are not addressed at all in [3].

Presentation of the main results

The first main result of this paper builds a bridge between Iwahori-Hecke algebras of $SL_n(F)$ and $GL_n(F)$, for any $n \geq 2$, that will be useful to state and prove results about compatibility with restriction from $GL_n(F)$ to $SL_n(F)$ (see Corollary 4.27). Recall that I and I_S respectively denote the standard Iwahori subgroups of $GL_n(F)$ and $SL_n(F)$, while W and W_S respectively denote the (infinite) Weyl groups of $GL_n(F)$ and $SL_n(F)$ (see §§2.1 and 2.2 for precise definitions).

Theorem 1.1. *Let A be a commutative ring with unit. Assume that it contains a primitive $(q-1)^{th}$ root of unity and that $q-1$ belongs to $A^\times$.*

1. *The map $[I_S w I_S \to IwI]$ defines an injective group homomorphism $\iota : W_S \hookrightarrow W$ whose image is equal to the affine Weyl group W_{aff} of $GL_n(F)$.*

[1] This assumption will actually be automatically valid in our setting, see Remark 2.7 below.

2. *The isomorphism $\iota : W_S \simeq W_{aff}$ in* (1) *induces an isomorphism of A-algebras from the standard Iwahori-Hecke algebra $\mathcal{H}_A(G_S, I_S)$ of $SL_n(F)$ to the affine Iwahori-Hecke algebra $\mathcal{H}_A^{aff}(G, I)$ of $GL_n(F)$.*

Now assume that $n = 2$, that $A = \overline{\mathbb{F}}_p$ is an algebraic closure of k_F, and let us set $G_S := SL_2(F)$. The next step is to understand the structure of the pro-p-Iwahori-Hecke algebra $\mathbb{H}_S^1 := \mathrm{End}_{\overline{\mathbb{F}}_p[G_S]}(\mathrm{ind}_{I_S(1)}^{G_S}(\mathbf{1}))$. We prove that $\mathbb{H}_S^1$ is a direct sum of Iwahori-Hecke algebras (Theorem 3.1), what reduces the study of simple right modules over $\mathbb{H}_S^1$ to the study of simple right modules over several smaller algebras. Inspired by what is done for $GL_2(F)$ in [21], we dispatch these Iwahori-Hecke algebras into three cases named the *Iwahori case*, the *regular case* and the *exceptional case*. In each case, we give an explicit description by generators and relations of the corresponding algebras (Theorems 3.3, 4.2 and 4.3 for the Iwahori case, Section 5.1 for the regular case, Theorems 6.1 and 6.2 for the exceptional case) and we classify their simple right modules (see Sections 4.2, 5.2 and 6.2), that will all be of finite dimension over $\overline{\mathbb{F}}_p$. As a by-product, we prove the following result, that gives a strong bound on the dimension of simple right $\mathbb{H}_S^1$-modules and comes by combination of Theorems 4.7, 5.6 and 6.7.

THEOREM 1.2. *Any simple right $\mathbb{H}_S^1$-module is of finite dimension at most 2 as vector space over $\overline{\mathbb{F}}_p$.*

Note that understanding the Iwahori case already allows us to build connections with a part of the classification of irreducible smooth representations of $SL_2(F)$ over $\overline{\mathbb{F}}_p$ given in [2] and with some results proved in [21] for $GL_2(F)$. We have for instance the following statement, obtained by gathering Corollaries 4.23 and 4.27.

THEOREM 1.3. *Let I and I_S be the standard Iwahori subgroups of $G = GL_2(F)$ and $G_S = SL_2(F)$ respectively. Let $\mathbb{H} := \mathrm{End}_{\overline{\mathbb{F}}_p[G]}(\mathrm{ind}_I^G(\mathbf{1}))$ and $\mathbb{H}_S := \mathrm{End}_{\overline{\mathbb{F}}_p[G_S]}(\mathrm{ind}_{I_S}^{G_S}(\mathbf{1}))$ be the standard Iwahori-Hecke algebra of G and G_S respectively.*

1. *The functor of I_S-invariants defines a bijection*

$$\left\{\begin{array}{c}\textit{isomorphism classes of}\\ \text{non-supercuspidal } \textit{irreducible smooth}\\ \textit{representations of } SL_2(F) \textit{ over } \overline{\mathbb{F}}_p\\ \textit{generated by their } I_S\textit{-invariant vectors}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\textit{isomorphism classes of}\\ \text{non-supersingular}\\ \textit{simple right } \mathbb{H}_S\textit{-modules}\end{array}\right\}.$$

Furthermore, for any non-supercuspidal *irreducible smooth representation π of G over $\overline{\mathbb{F}}_p$ with central character and non-zero I-invariant vectors, the restriction to $\mathbb{H}_S$ of the $\mathbb{H}$-module π^I is isomorphic to the $\mathbb{H}_S$-module $\left(\pi|_{SL_2(F)}\right)^{I_S}$.*

2. *Assume that $F = \mathbb{Q}_p$. The previous bijection extends to a bijection*

$$\left\{\begin{array}{c}\textit{isomorphism classes of irreducible smooth}\\ \textit{representations of } SL_2(\mathbb{Q}_p) \textit{ over } \overline{\mathbb{F}}_p\\ \textit{generated by their } I_S\textit{-invariant vectors}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\textit{isomorphism classes of}\\ \textit{simple right } \mathbb{H}_S\textit{-modules}\end{array}\right\}.$$

Moreover, if π is an irreducible smooth representation of G over $\overline{\mathbb{F}}_p$ with central character and non-zero I-invariant vectors, the restriction to $\mathbb{H}_S$ of the $\mathbb{H}$-module

π^I *is then isomorphic to the* $\mathbb{H}_S$*-module* $\left(\pi|_{SL_2(F)}\right)^{I_S}$.

Note that the bijections appearing in the statement of Theorem 1.3 are far from taking into account all irreducible smooth representation of G_S over $\overline{\mathbb{F}}_p$, as they forget for instance most of the supercuspidal representations of $SL_2(\mathbb{Q}_p)$. We solve this omission through the following result (Corollary 7.10), which heavily relies on the complete classification of simple right $\mathbb{H}^1_S$-modules we establish in this paper as on the good understanding of irreducible smooth representations of $SL_2(F)$ over $\overline{\mathbb{F}}_p$ provided by [2].

Theorem 1.4. 1. *The functor of* $I_S(1)$*-invariants defines a bijection*

$$\left\{\begin{array}{c}\textit{isomorphism classes of}\\ \text{non-supercuspidal } \textit{irreducible}\\ \textit{smooth representations}\\ \textit{of } SL_2(F) \textit{ over } \overline{\mathbb{F}}_p\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\textit{isomorphism classes of}\\ \text{non-supersingular}\\ \textit{simple right } \mathbb{H}^1_S\textit{-modules}\end{array}\right\}.$$

2. *When* $F=\mathbb{Q}_p$*, the previous bijection extends to a bijection*

$$\left\{\begin{array}{c}\textit{isomorphism classes of}\\ \textit{irreducible smooth representations}\\ \textit{of } SL_2(\mathbb{Q}_p) \textit{ over } \overline{\mathbb{F}}_p\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\textit{isomorphism classes of}\\ \textit{simple right } \mathbb{H}^1_S\textit{-modules}\end{array}\right\}.$$

Structure of the paper

Section 2 contains a short remainder on Weyl groups for $GL_n(F)$ and $SL_n(F)$ gathering what we need to prove Theorem 1.1. We obtain the aforementioned decomposition of $\mathbb{H}^1_S$ in Section 3, where we also relate the Iwahori case to the standard Iwahori-Hecke algebra of $SL_2(F)$ via Theorem 3.3. The study of the Iwahori case and its applications to the structure of I_S-invariant spaces of irreducible smooth representations of G_S over $\overline{\mathbb{F}}_p$ is done in Section 4, and leads in particular to a proof of Theorem 1.3. Sections 5 and 6 respectively deal with the regular and exceptional cases, while Section 7 contains the missing comparisons between the classification of simple right $\mathbb{H}^1_S$-modules coming from the previous sections and the classification of irreducible smooth representations of $SL_2(F)$ over $\overline{\mathbb{F}}_p$ coming from [2] needed to prove Theorem 1.4.

General notations

Fix a prime number $p \geq 2$. Let F be a non-archimedean local field complete for a discrete valuation and with finite residue class field k_F of characteristic p, with ring of integers $\mathcal{O}_F$ and with fixed uniformizer ϖ_F. We let $q=p^f$ be the cardinality of k_F, we fix an algebraic closure $\overline{\mathbb{F}}_p$ of k_F together with an embedding ι of k_F into $\overline{\mathbb{F}}_p$ and we let $[.] : k_F \to \mathcal{O}_F^\times$ be the Teichmüller lift. We also let v be the discrete valution of F normalized by $v(\varpi_F)=1$.

For any integer $n \geq 2$, let $G := GL_n(F)$ be the general linear group with coefficients in F and let $K := GL_n(\mathcal{O}_F)$ be its standard maximal open compact subgroup. Denote by B the Borel subgroup of upper-triangular matrices in G and by T the maximal split torus of diagonal matrices of G. Let I be the standard Iwahori subgroup of K, defined as the set of elements in K whose reduction modulo ϖ_F is an upper-triangular matrix of $GL_n(k_F)$. The pro-p-radical of I, called the *standard pro-p-Iwahori subgroup* of G

and denoted by $I(1)$, is then made of the elements of I whose reduction modulo ϖ_F is moreover unipotent.

Let $G_S := SL_n(F)$ be the special linear group with coefficients in F and let $K_S := SL_n(\mathcal{O}_F)$ be its standard maximal open compact subgroup. Let B_S be the Borel subgroup of upper-triangular matrices in G_S and let T_S be the torus of diagonal matrices in G_S. Finally, let I_S (respectively $I_S(1)$) be the standard Iwahori subgroup (resp. the standard pro-p-Iwahori subgroup) of G_S, defined the same way as I (resp. $I(1)$) previously was in G. We clearly have $B_S = B \cap G_S$, $T_S = T \cap G_S$, $I_S = I \cap G_S$, $I_S(1) = I(1) \cap G_S$, and the reduction modulo ϖ_F defines a group isomorphism from the quotient class group $I_S/I_S(1)$ to the abelian group Γ_S of diagonal matrices in $SL_n(k_F)$.

When $n = 2$, we introduce the following specific elements of G_S :

$$\alpha_0 := \begin{pmatrix} \varpi_F^{-1} & 0 \\ 0 & \varpi_F \end{pmatrix} \ , \ s_0 := \begin{pmatrix} 0 & \varpi_F \\ -\varpi_F^{-1} & 0 \end{pmatrix} \text{ and } s_1 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ .$$

We moreover set, for any $x \in F$,

$$u(x) := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \text{ and } \bar{u}(x) := \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \ .$$

All these elements satisfy the following relations, valid for any integer $k \in \mathbb{Z}$ and any element $x \in F$:

$$\text{(1.1)} \qquad \begin{cases} \alpha_0 = s_1 s_0, \ \alpha_0^k s_1 = s_1 \alpha_0^{-k}, \ s_1 u(x) = \bar{u}(-x) s_1, \\ \alpha_0^k u(x) \alpha_0^{-k} = u(\varpi_F^{-2k} x) \text{ and } \alpha_0^k \bar{u}(x) \alpha_0^{-k} = \bar{u}(\varpi_F^{2k} x) \ . \end{cases}$$

For any $\lambda \in \overline{\mathbb{F}}_p^\times$, let $\mu_\lambda : B_S \to \overline{\mathbb{F}}_p^\times$ be the smooth character obtained by inflation of the smooth unramified (i.e. trivial on $\mathcal{O}_F^\times$) character of $F^\times$ that maps ϖ_F to λ. This means that we set :

$$\forall \ (a,b) \in F^\times \times F, \ \mu_\lambda \left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \right) := \lambda^{v(a)} \ .$$

We will also let $\mu_\lambda \otimes \mathbf{1}$ be the smooth character of B obtained by inflation of the smooth unramified character of $(F^\times)^2$ that maps (a, b) to $\mu_\lambda(a)$. For any integer $r \in \{0, \dots, q-2\}$, we let $\omega^r : k_F^\times \to \overline{\mathbb{F}}_p^\times$ be the character defined by $\omega^r(x) := \iota(x^r)$. We furthermore consider the following objects, as in [2, 4, 6] : let H be an open subgroup of $\Gamma \in \{G_S, G\}$ and (σ, V_σ) be an irreducible smooth representation of H over $\overline{\mathbb{F}}_p$. For any element $g \in \Gamma$ and any vector $v \in V_\sigma$, we let $[g, v] : \Gamma \to V_\sigma$ be the function defined as follows :

$$\forall \ x \in \Gamma, \ [g,v](x) := \begin{cases} \sigma(xg)(v) & \text{if } x \in Hg^{-1} \ ; \\ 0 & \text{otherwise.} \end{cases}$$

This means that $[g, v]$ is the unique element of $\mathrm{ind}_H^\Gamma(\sigma)$ having support equal to Hg^{-1} and value v at g^{-1}. For the reader interested in more properties of these functions, we recommand to have a look at [1, Section 2.2.2]

Finally, we will use the same notation as in [21] : if $\mathcal{M}$ is a right module over an $\overline{\mathbb{F}}_p$-algebra $\mathbb{H}$, we let $m|T$ be the vector given by the action of an element $T \in \mathbb{H}$ on a vector $m \in \mathcal{M}$.

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2. A relationship between Iwahori-Hecke algebras of $GL_n(F)$ and $SL_n(F)$

2.1. A remainder about Weyl groups of $GL_n(F)$

This subsection gathers some results about Weyl groups for $GL_n(F)$ we will use in the sequel. The reader can refer to [12] or [24, Chapitre 3] for more details and proofs.

Let $(X, X^\vee, R, R^\vee, \Delta)$ be the root data attached to the triple (G, B, T). In particular, $X \simeq \mathbb{Z}^n$ can be identified with the group $X^*(T)$ of F-characters of the split maximal torus T of G while $X^\vee$ can be identified with the group $X_*(T)$ of F-cocharacters of T. The positive simple roots of this root data are $\{\alpha_1, \dots, \alpha_{n-1}\}$, where α_i is defined for any $i \in \{1, \dots, n-1\}$ by the following formula :

$$\alpha_i : \begin{pmatrix} \varpi_F^{x_1} & & \\ & \ddots & \\ & & \varpi_F^{x_n} \end{pmatrix} \mapsto x_{i+1} - x_i \ .$$

The coroot corresponding to α_i can hence be represented by the diagonal matrix

$$A_i := \mathrm{diag}(1, \dots, 1, \varpi_F^{-1}, \varpi_F, 1 \dots, 1) \ ,$$

where ϖ_F^{-1} is in i^{th} position while ϖ_F is in $(i+1)^{th}$-position. If we let $\sigma_i \in \Delta$ be the reflexion associated to α_i, the group generated by $\{\sigma_1, \dots, \sigma_{n-1}\}$ is called the *finite Weyl group* W_0 *of* G. It is a Coxeter group canonically isomorphic to the quotient class group $N_G(T)/T$, where $N_G(T)$ is the normalizer of T in G, and it parametrizes the double coset space $I\backslash K/I$. Also note that W_0 is naturally isomorphic to the symmetric group $\mathfrak{S}_n$ via the group homomorphism mapping σ_i to the transposition $(i, i+1)$.

The *Weyl group* W *of* G is defined as the quotient class group $N_G(T)/(T \cap K)$. It parametrizes the double coset space $I\backslash G/I$ and can be written as a semi-direct product of W_0 and X, provided X is identified with the multiplicative group of translations it defines. This endows W with a natural length function ℓ [12, Section 1.4]. The Weyl group W contains an interesting subgroup, called the *affine Weyl group* W_{aff} *attached to (the data root of)* G, which is defined as the semi-direct product in W of W_0 and the subgroup of X generated by R. One can prove that W_{aff} is a Coxeter group with $\Sigma_{\text{aff}} := \{\sigma_0 := t^{-1}\sigma_1 t, \sigma_1, \dots, \sigma_{n-1}\}$ as set of simple reflexions [12, Section 1.5] and that its length function coincides with the restriction of ℓ to W_{aff}. Note that we set

$$t := \begin{pmatrix} 0 & 1 & 0 & 0 \dots 0 \\ 0 & 0 & 1 & 0 \dots 0 \\ \vdots & \multicolumn{3}{c}{\dots\dots\dots\dots} & \vdots \\ 0 & 0 & 0 & 0 \dots 1 \\ \varpi_F & 0 & 0 & 0 \dots 0 \end{pmatrix} \in G.$$

Let moreover $\Omega = t^{\mathbb{Z}}$ be the subgroup of G generated by t : it is equal to the subgroup of W made of all the elements of length 0, and one can check that W is a semi-direct product of W_{aff} and Ω [24, Proposition 3.1].

We finally define the *extended Weyl group* $W^{(1)}$ *of* G as the quotient class group $N_G(T)/(T \cap K(1))$, where $T \cap K(1)$ is the kernel of the reduction map $T \cap K \twoheadrightarrow \Gamma$, with Γ being the torus of diagonal matrices in $GL_n(k_F)$. The extended Weyl group parametrizes the double coset space $I(1)\backslash G/I(1)$ and it fits into the following canonical non-split[2] short exact sequence :

$$1 \longrightarrow \Gamma \longrightarrow W^{(1)} \longrightarrow W \longrightarrow 1 \ .$$

In particular, $W^{(1)}$ is endowed with a length function extending the length function ℓ already defined on W and such that any element of Γ is of length 0.

2.2. From $GL_n(F)$ to $SL_n(F)$

Let $(X_S, X_S^\vee, R_S, R_S^\vee, \Delta)$ be the root data attached to (G_S, B_S, T_S). The finite Weyl group attached to this data parametrizes the double coset space $I_S\backslash K_S/I_S$ and is actually equal to W_0 as it only depends on the simple roots of the root data, which are the same for G and for G_S. This allows us to lift any element of W_0 in G_S, and from now we will always consider such lifts. We then define the Weyl group W_S of G_S, that parametrizes the double coset space $I_S\backslash G_S/I_S$, as the quotient class group $N_{G_S}(T_S)/(T_S \cap K_S)$. It is isomorphic to a semi-direct product of W_0 and $X_S \simeq \mathbb{Z}^{n-1}$. Note that we will use the same symbol to denote an element of W (resp. : of W_S, of W_0) and any fixed lift of this element in G (resp. : in G_S, in K_S).

As W and W_S respectively parametrize the double coset spaces $I\backslash G/I$ and $I_S\backslash G_S/I_S$, the equality $I_S = I \cap G_S$ assures that the canonical inclusion map $G_S \hookrightarrow G$ induces an injective group homomorphism $W_S \hookrightarrow W$. The following lemma, which points out an important difference between W and W_S, will be useful to prove Theorem 1.1.

Lemma 2.1. *For any integer $n \geq 2$, the affine Weyl group attached to $SL_n(F)$ is equal to the Weyl group W_S.*

Proof. First recall that we can identify X_S with the hyperplane of $X \simeq \mathbb{Z}^n$ made of all uplets $(x_1, \ldots, x_n)$ satisfying $\sum_{i=1}^{n} x_i = 0$. Moreover note that any $\alpha = \mathrm{diag}(\varpi_F^{x_1}, \ldots, \varpi_F^{x_n})$ in X_S can be written as follows :

$$\alpha = \left(\prod_{i=1}^{n-2} \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & \varpi^{x_1+x_2+\ldots+x_i} & & \\ & & & \varpi^{-(x_1+x_2+\ldots+x_i)} & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} \right) \begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & & \varpi^{\sum_{i=1}^{n-1} x_i} & \\ & & & & & \varpi^{x_n} \end{pmatrix} .$$

[2]Note that this splitting property is specific to the GL_n case, as underlined in [23].

If we set $\lambda_i(\alpha) := -\sum_{j=1}^{i} x_j = \sum_{j=i+1}^{n} x_j$, we hence get the following equality for roots :

$$\alpha = \sum_{i=1}^{n-1} \lambda_i(\alpha)\alpha_i \ .$$

This proves that X_S is generated by R_S, and that W_S is consequently equal to the affine Weyl group attached to G_S. $\square$

As the root systems R and R_S are equal, the affine Weyl groups attached to G and G_S are canonically isomorphic. Together with Lemma 2.1, this gives the following statement.

Corollary 2.2. *For any integer $n \geq 2$, the map sending the double coset $I_S g I_S$ to the double coset IgI for any $g \in G_S$ induces a group isomorphism from the Weyl group W_S of $SL_n(F)$ to the affine Weyl group W_{aff} attached to $GL_n(F)$.*

Remark 2.3. Corollary 2.2 can also be seen as a consequence of [18, Section 2.a.2]. More precisely, let G be a (residually) split connected reductive group over a local field with perfect residue class field and let T be a maximal split torus in G. Pappas and Rapoport attach to the pair (G, T) an Iwahori-Weyl group that can be canonically identified with the Weyl group W appearing in our setting [18, Proposition 2.1], and they prove in [18, Section 2.a.2] that the Iwahori-Weyl group of the simply connected covering of the derived group of G, which canonically corresponds to W_S in our setting, can be naturally identified with the affine Weyl group defined by the affine root system attached to T, i.e. to the affine Weyl group W_{aff} of G.

Remark 2.4. In view of [24, Proposition 3.1], Corollary 2.2 proves in particular that W is isomorphic to a semi-direct product of W_S and $\Omega \simeq \mathbb{Z}$.

2.3. Application to Iwahori-Hecke algebras

Let A be a commutative ring with unit 1_A. Assume that it contains a primitive $(q-1)^{th}$ root of unity and that $q-1$ belongs to $A^\times$: then A can be endowed with a structure of $\mathbb{Z}[q]$-module via the unitary ring homomorphism mapping q on $q1_A$. Now recall that the *standard Iwahori-Hecke algebra of G over A* is the A-algebra $\mathcal{H}_A(G)$ generated by the family $(T_w)_{w \in W}$ satisfying the following braid and quadratic relations.

- Braid relations : if $w, w' \in W$ satisfy $\ell(ww') = \ell(w) + \ell(w')$, then $T_{ww'} = T_w T_{w'}$.
- Quadratic relations : for any $s \in \Sigma_{\text{aff}}$, $(T_s + 1)(T_s - q) = 0$.

Following [22, Example 1], one can check that the family $(T_w)_{w \in W}$ is actually a basis of the A-module $\mathcal{H}_A(G)$. The same construction holds to define the *standard Iwahori-Hecke algebra $\mathcal{H}_A(G_S)$ of G_S over A* as the A-algebra generated by the family $(\mathcal{T}_w)_{w \in W_S}$ satisfying the same braid and quadratic relations as above with W replaced by W_S. One similarly proves that $(\mathcal{T}_w)_{w \in W_S}$ is a basis of the A-module $\mathcal{H}_A(G_S)$.

Let $\mathcal{H}_A^{\text{aff}}(G)$ be the A-subalgebra of $\mathcal{H}_A(G)$ generated by the family $(T_w)_{w \in W_{\text{aff}}}$, called the *affine Iwahori-Hecke algebra of G over A*. By Lemma 2.1, the restriction to W_{aff} of the Bruhat length function of W coincides with the Bruhat length function of W_S under the group isomorphism given by Corollary 2.2. This implies that $\mathcal{H}_A(G_S)$ and $\mathcal{H}_A^{\text{aff}}(G)$

have the same braid relations. As the quadratic relations only depend on the affine part, they are also the same for $\mathcal{H}_A(G_S)$ and $\mathcal{H}_A^{\text{aff}}(G)$, what finally proves the following result.

Corollary 2.5. *With the notations introduced above, the map sending $\mathcal{T}_w$ on T_w for any $w \in W_S$ is a well-defined injective homomorphism of A-algebras from $\mathcal{H}_A(G_S)$ into $\mathcal{H}_A(G)$, with image equal to $\mathcal{H}_A^{\text{aff}}(G)$.*

A convenient reformulation of this statement is the following one : the standard Iwahori-Hecke algebra of G_S over A is canonically isomorphic to the affine Iwahori-Hecke algebra of G over A, and can consequently naturally be seen as an A-subalgebra of the standard Iwahori-Hecke algebra of G.

Remark 2.6. In a recent work [10, Section 3], Kozioł proved further relations of that kind for the standard pro-p-Iwahori-Hecke algebras of $GL_n(F)$ and $SL_n(F)$.

Remark 2.7. Before going further, let us mention here that [22, Theorem 4] implies in particular that any simple right $\mathcal{H}^1_{\overline{\mathbb{F}}_p}(G_S)$-module will be of finite dimension as vector space over $\overline{\mathbb{F}}_p$. Consequently, the classification of simple right $\mathcal{H}^1_{\overline{\mathbb{F}}_p}(G_S)$-modules amounts to classifying finite-dimensional simple right $\mathcal{H}^1_{\overline{\mathbb{F}}_p}(G_S)$-modules, what explains why this apparent restriction shows up in the sequel of this paper.

3. A decomposition of the pro-p-Iwahori-Hecke algebra of $SL_2(F)$

From now on, we assume that $n = 2$ and $A = \overline{\mathbb{F}}_p$, what allows us to use the results we proved in the previous section. In particular, the affine Weyl group of $G = GL_2(F)$ admits $\Sigma_{\text{aff}} = \{s_0, s_1\}$ as Coxeter system. To ease notations, we let $\mathcal{H}$ be the standard Iwahori-Hecke algebra of G over $\overline{\mathbb{F}}_p$ and $\mathcal{H}_S$ be the standard Iwahori-Hecke algebra of $G_S = SL_2(F)$ over $\overline{\mathbb{F}}_p$. We also set $T := T_t \in \mathcal{H}$, $S := T_{s_1} \in \mathcal{H}$ and $\mathcal{T}_i := \mathcal{T}_{s_i} \in \mathcal{H}_S$ for any $i \in \{0, 1\}$.

In this section, we will decompose the $\overline{\mathbb{F}}_p$-algebra $\mathbb{H}^1_S := \text{End}_{\overline{\mathbb{F}}_p[G_S]}(\text{ind}_{I_S(1)}^{G_S}(\mathbf{1}))$ as a direct sum of finitely many smaller algebras easier to compute, in order to describe the simple right $\mathbb{H}^1_S$-modules. As $I_S(1)$ is a normal pro-p-subgroup of I_S, [4, Lemma 3 (1)] implies that any irreducible smooth representation of I_S over $\overline{\mathbb{F}}_p$ comes by inflation from an irreducible representation of the quotient group $I_S/I_S(1) \simeq \Gamma_S$. As $n = 2$, the torus Γ_S is canonically isomorphic to the finite cyclic (hence abelian) group $k_F^\times$. Consequently, any irreducible smooth representation of I_S over $\overline{\mathbb{F}}_p$ comes by inflation from a character of $k_F^\times$ over $\overline{\mathbb{F}}_p$, and is then of the form[3] ω^r for a unique integer $r \in \{0, \dots, q-2\}$. Now recall that the normalizer of Γ_S in $SL_2(k_F)$ acts by conjugation on this set of characters and a direct computation proves that the orbit of ω^r for this action is equal to $\{\omega^r, \omega^{q-1-r}\}$. In particular, it is reduced to one element when r is equal to 0 (what is the *Iwahori case*[4]) or to $\frac{q-1}{2}$ (what is the *exceptional case*, that does not appear when $p = 2$), and consists in two elements otherwise (what is the *regular case*). Considering how $\mathbb{H}^1_S$ is defined, we

[3] The same symbol denotes a character of $k_F^\times$ and the smooth character of I_S it defines by inflation.
[4] The terminology of *Iwahori case* and *regular case* is inspired by the one used in [21] for $GL_2(F)$.

are led to introduce the following $\overline{\mathbb{F}}_p$-algebras : for any $r \in \{0, \ldots, q-2\}$, we set

$$\mathcal{H}_S(r) := \begin{cases} \mathrm{End}_{\overline{\mathbb{F}}_p[G_S]}(\mathrm{ind}_{I_S}^{G_S}(\omega^r)) & \text{if } r \text{ is equal to } 0 \text{ or to } \dfrac{q-1}{2}; \\ \mathrm{End}_{\overline{\mathbb{F}}_p[G_S]}(\mathrm{ind}_{I_S}^{G_S}(\omega^r \oplus \omega^{q-1-r})) & \text{otherwise.} \end{cases} \tag{3.1}$$

THEOREM 3.1. *With the notations above, the $\overline{\mathbb{F}}_p$-algebra $\mathbb{H}_S^1$ admits the following decomposition, where $[x]$ denotes here the floor part of $x \in \mathbb{Q}$:*

$$\mathbb{H}_S^1 \simeq \bigoplus_{r=0}^{\left[\frac{q-1}{2}\right]} \mathcal{H}_S(r) \ , \tag{3.2}$$

Proof. The proof of this statement is organized as the proof of [21, Proposition 3.1]. First, we use the transitivity of compact induction and the knowledge of all irreducible smooth representations of I_S over $\overline{\mathbb{F}}_p$ to get that

$$\mathrm{ind}_{I_S(1)}^{G_S}(\mathbf{1}) = \mathrm{ind}_{I_S}^{G_S}(\mathrm{ind}_{I_S(1)}^{I_S}(\mathbf{1})) = \bigoplus_{r=0}^{q-1} \mathrm{ind}_{I_S}^{G_S}(\omega^r) \ .$$

Hence we have to determine all pairs $(r, k) \in \{0, \ldots, q-2\}^2$ such that the $\overline{\mathbb{F}}_p[G_S]$-modules $\mathrm{ind}_{I_S}^{G_S}(\omega^r)$ and $\mathrm{ind}_{I_S}^{G_S}(\omega^k)$ are intertwinned. To do this, we recall that the adjunction property of $\mathrm{ind}_{I_S}^{G_S}$ gives an isomorphism of vector spaces over $\overline{\mathbb{F}}_p$ of the following form :

$$\mathrm{Hom}_{\overline{\mathbb{F}}_p[G_S]}\left(\mathrm{ind}_{I_S}^{G_S}(\omega^r),\ \mathrm{ind}_{I_S}^{G_S}(\omega^k)\right) \simeq \mathrm{Hom}_{\overline{\mathbb{F}}_p[I_S]}\left(\omega^r,\ \mathrm{ind}_{I_S}^{G_S}(\omega^k)|_{I_S}\right) \ .$$

Moreover, Mackey decomposition [1, Proposition 2.2.7] proves that the restriction to I_S defines an isomorphism of $\overline{\mathbb{F}}_p[I_S]$-modules of the following form :

$$\mathrm{ind}_{I_S}^{G_S}(\omega^k)|_{I_S} \simeq \bigoplus_{w \in W_S} \mathrm{ind}_{I_{S,w}}^{I_S}(\omega^{k,w}),$$

where we set $I_{w,S} := w^{-1} I_S w \cap I_S$ for any $w \in W_S \simeq \{I_2, s_1\} \rtimes \mathbb{Z}$ and

$$\omega^{k,w} := \begin{cases} \omega^k & \text{if } w \in \{I_2\} \rtimes \mathbb{Z} \ ; \\ s_1 \cdot \omega^k = \omega^{q-1-k} & \text{otherwise.} \end{cases}$$

Now remark that for any element $w \in W_S$, the reduction modulo ϖ_F defines a short exact sequence

$$0 \longrightarrow I_S(1) \cap I_{w,S} \longrightarrow I_{w,S} \longrightarrow \Gamma_S \longrightarrow 0 \ .$$

This implies in particular that $I_S = I_S(1) I_{w,S}$, hence that $\omega^k = \omega^{k,w}$ if, and only if, these two smooth characters coincide on $I_{w,S}$. As $I_{w,S}$ is of finite index in I_S, compact induction (from $I_{w,S}$ to I_S) coincides with smooth induction [20, I.5.2.a)]. If we respectively denote these two functors by ind and Ind, their adjunction properties (respectively called *compact Frobenius reciprocity* [1, Proposition 2.2.3] and *(smooth) Frobenius reciprocity* [1, Proposition 2.2.1]) then lead to the following chain of isomorphisms :

$$\begin{aligned}\mathrm{Hom}_{\overline{\mathbb{F}}_p[I_S]}(\omega^r, \mathrm{ind}_{I_S}^{G_S}(\omega^k)) &\simeq \bigoplus_{w\in W_S} \mathrm{Hom}_{\overline{\mathbb{F}}_p[I_S]}(\omega^r, \mathrm{ind}_{I_{w,S}}^{I_S}(\omega^{k,w})) \\ &\simeq \bigoplus_{w\in W_S} \mathrm{Hom}_{\overline{\mathbb{F}}_p[I_S]}(\omega^r, \mathrm{Ind}_{I_{w,S}}^{I_S}(\omega^{k,w})) \\ &\simeq \bigoplus_{w\in W_S} \mathrm{Hom}_{\overline{\mathbb{F}}_p[I_{w,S}]}(\omega^r, \omega^{k,w}) \\ &\simeq \bigoplus_{w\in W_S} \mathrm{Hom}_{\overline{\mathbb{F}}_p[I_S]}(\omega^r, \omega^{k,w}) \ .\end{aligned}$$

Considering how the characters $\omega^{k,w}$ are defined, this finally proves that the space $\mathrm{Hom}_{\overline{\mathbb{F}}_p[G_S]}(\mathrm{ind}_{I_S}^{G_S}(\omega^r), \mathrm{ind}_{I_S}^{G_S}(\omega^k))$ is non-zero if, and only if, r is equal to k or to $q-1-k$. We therefore fall into one of the two following cases :

- either p (and then q) is odd, in which case there are two orbits reduced to one character, corresponding to $r=0$ and $r=\dfrac{q-1}{2}$, and any other orbit is of size 2;
- or q is even, in which case the unique orbit of size one corresponds to $r=0$, while any other orbit is of size 2.

Putting all these results together finishes the proof as a direct computation shows that in the first case (p odd), we have

$$\begin{aligned}\mathrm{End}_{\overline{\mathbb{F}}_p[G_S]}(\mathrm{ind}_{I_S(1)}^{G_S}(\mathbf{1})) \ \simeq \ \mathrm{End}_{\overline{\mathbb{F}}_p[G_S]}(\mathrm{ind}_{I_S}^{G_S}(\mathbf{1})) \quad &\oplus \quad \mathrm{End}_{\overline{\mathbb{F}}_p[G_S]}(\mathrm{ind}_{I_S}^{G_S}(\omega^{\frac{q-1}{2}})) \\ &\bigoplus_{0<r<\frac{q-1}{2}} \mathrm{End}_{\overline{\mathbb{F}}_p[G_S]}(\mathrm{ind}_{I_S}^{G_S}(\omega^r \oplus \omega^{q-1-r}))\ ,\end{aligned}$$

while in the second case ($p=2$), we have

$$\mathrm{End}_{\overline{\mathbb{F}}_p[G_S]}(\mathrm{ind}_{I_S(1)}^{G_S}(\mathbf{1})) \ \simeq \ \mathrm{End}_{\overline{\mathbb{F}}_p[G_S]}(\mathrm{ind}_{I_S}^{G_S}(\mathbf{1})) \bigoplus_{1\leq r\leq\left[\frac{q-1}{2}\right]} \mathrm{End}_{\overline{\mathbb{F}}_p[G_S]}(\mathrm{ind}_{I_S}^{G_S}(\omega^r \oplus \omega^{q-1-r}))\ .$$

□

REMARK 3.2. Note that the computation using Mackey decomposition we did in the previous proof also shows that any non-zero element of the (I_S, ω^k)-isotypical component of $\mathrm{ind}_{I_S}^{G_S}(\omega^{q-1-k})$ has support in the double coset $I_S s_1 \alpha_0 I_S$.

A fundamental consequence of Theorem 3.1 is that any simple right $\mathbb{H}_S^1$-module comes from a simple right $\mathcal{H}_S(r)$-module for some well-chosen $r \in \{0, \dots \left[\frac{q-1}{2}\right]\}$, what motivates the study of simple right $\mathcal{H}_S(r)$-modules for any parameter r. Remark 2.7 furthermore implies that all these simple modules will be of finite dimension over $\overline{\mathbb{F}}_p$. We start by the case $r=0$: not only it is interesting by itself, as it is closely related to spaces of I-invariant vectors (see Section 4.4), but it is also connected with the standard Iwahori-Hecke algebra $\mathcal{H}_S$ introduced in the Section 2.3, as can be seen in the following theorem.

THEOREM 3.3. *Let f_0 and f_1 be the elements of $\mathrm{End}_{\overline{\mathbb{F}}_p[G_S]}(\mathrm{ind}_{I_S}^{G_S}(\mathbf{1}))$ that respectively correspond by compact Frobenius reciprocity to the functions φ_0, φ_1 in $\mathrm{ind}_{I_S}^{G_S}(\mathbf{1})^{I_S}$ defined as follows : for any $i \in \{0,1\}$, φ_i has support equal to $I_S s_i I_S$ and value 1 at s_i.*

Then there exists a unique homomorphism of $\overline{\mathbb{F}}_p$-algebras $\mathrm{End}_{\overline{\mathbb{F}}_p[G_S]}(\mathrm{ind}_{I_S}^{G_S}(\mathbf{1})) \to \mathcal{H}_S$ *that maps f_i to $\mathcal{T}_i$ for any $i \in \{0,1\}$, and it is actually an isomorphism of $\overline{\mathbb{F}}_p$-algebras.*

Proof. By compact Frobenius reciprocity, the $\overline{\mathbb{F}}_p$-algebra $\mathrm{End}_{\overline{\mathbb{F}}_p[G_S]}(\mathrm{ind}_{I_S}^{G_S}(\mathbf{1}))$ is isomorphic to the $(I_S, \mathbf{1})$-isotypical component of $\mathrm{ind}_{I_S}^{G_S}(\mathbf{1})$. This component is isomorphic to the convolution algebra $\overline{\mathbb{F}}_p[I_S \backslash G_S / I_S]$ via the morphism of $\overline{\mathbb{F}}_p$-algebras sending, for any $g \in G_S$, the double coset $I_S g I_S$ to the element of $\mathrm{ind}_{I_S}^{G_S}(\mathbf{1})^{I_S}$ with support $I_S g I_S$ and value 1 at g. As W_S parametrizes the double cosets $I_S \backslash G_S / I_S$, we can follow step by step the computations of [21, Appendix 1.3 - Iwahori case] to prove the theorem. □

4. The standard Iwahori-Hecke algebra and its simple modules

Theorems 3.1 and 3.3 explain why the study of the structure of the $\overline{\mathbb{F}}_p$-algebra $\mathcal{H}_S$ and of its simple right modules is a first step towards the classification of simple right $\mathbb{H}_S^1$-modules. We realize this study in this section, which also contains a first relationship with mod p representations of $SL_2(F)$ (see Section 4.4).

4.1. On the structure of the standard Iwahori-Hecke algebra $\mathcal{H}_S$

We start by proving that the standard Iwahori-Hecke algebra $\mathcal{H}_S$ is the $\overline{\mathbb{F}}_p$-algebra generated by the operators $\mathcal{T}_0$ and $\mathcal{T}_1$, then we describe its center. To do this, we need the following result, which is a direct consequence of the braid relations as we have $\ell(s_0 s_1) = \ell(s_0) + \ell(s_1) = \ell(s_1 s_0)$.

Lemma 4.1. *The following identities hold in $\mathcal{H}_S$: $\mathcal{T}_{s_0 s_1} = \mathcal{T}_0 \mathcal{T}_1$ and $\mathcal{T}_{s_1 s_0} = \mathcal{T}_1 \mathcal{T}_0$.*

Theorem 4.2. *The $\overline{\mathbb{F}}_p$-algebra $\mathcal{H}_S$ is generated by $\mathcal{T}_0$ and $\mathcal{T}_1$.*

Proof. We want to prove that for any $w \in W_S$, the operator $\mathcal{T}_w$ can be written as a polynomial in $\mathcal{T}_0$ and $\mathcal{T}_1$ with coefficients in $\overline{\mathbb{F}}_p$. We do this by induction on the Bruhat length of w. As $\ell(w) = 1$ if, and only if, w belongs to $\{s_0, s_1\}$, the statement we want to prove is clearly true for elements of length 1. If $\ell(w) = 2$, then $\mathcal{T}_w$ is of one of the forms considered in Lemma 4.1 and is hence a polynomial in $\mathcal{T}_0$ and $\mathcal{T}_1$ with coefficients in $\overline{\mathbb{F}}_p$. Now assume that $\ell(w) = n \geq 3$ and that $w_1 \dots w_n$ is a reduced decomposition of w. As s_0 and s_1 are both of order 2 in W_S, saying that the decomposition $w_1 \dots w_n$ is reduced implies that for any index $i \in \{1, \dots, n-1\}$, we have $\{w_i, w_{i+1}\} = \{s_0, s_1\}$. Depending on the parity of n, we necessarily fall into one of the two following cases.

- Either $n = 2m$ is even and we hence have $w = (w_1 w_2)^m$. As we started from a reduced decomposition of w, the equality $\{w_1, w_2\} = \{s_0, s_1\}$ implies that $\ell(w) = \ell(w_1 w_2) + \ell((w_1 w_2)^{m-1})$. The braid relations together with the induction hypothesis and Lemma 4.1 then prove that $\mathcal{T}_w = \mathcal{T}_{(w_1 w_2)} \mathcal{T}_{(w_1 w_2)^{m-1}}$ is a polynomial in $\mathcal{T}_0$ and $\mathcal{T}_1$ with coefficients in $\overline{\mathbb{F}}_p$.

- Or $n = 2m+1$ is odd, and we hence have $w = (w_1 w_2)^m w_1$. As we started from a reduced decomposition of w, we now have $\ell(w) = \ell((w_1 w_2)^m) + \ell(w_1)$ with $\ell((w_1 w_2)^m) = 2m = n - \ell(w_1)$. The braid relations together with the even case hence imply that $\mathcal{T}_w = \mathcal{T}_{(w_1 w_2)^m} \mathcal{T}_1$ is again a polynomial in $\mathcal{T}_0$ and $\mathcal{T}_1$ with coefficients in $\overline{\mathbb{F}}_p$. □

To describe the center of $\mathcal{H}_S$, we use the quadratic relations to construct an operator that commutes to any element of $\mathcal{H}_S$, then we prove that any central element in $\mathcal{H}_S$ is a polynomial with coefficients in $\overline{\mathbb{F}}_p$ in this particular operator.

THEOREM 4.3. *The center of $\mathcal{H}_S$ is equal to the $\overline{\mathbb{F}}_p$-algebra $\overline{\mathbb{F}}_p[(\mathcal{T}_0 - \mathcal{T}_1)^2]$.*

Proof. The quadratic relations involved in the definition of $\mathcal{H}_S$ imply that we have

$$-(\mathcal{T}_0 - \mathcal{T}_1)^2 = \mathcal{T}_0 + \mathcal{T}_1 + \mathcal{T}_0\mathcal{T}_1 + \mathcal{T}_1\mathcal{T}_0 \ . \tag{4.1}$$

This implies that $-(\mathcal{T}_0 - \mathcal{T}_1)^2$ commutes with $\mathcal{T}_0$ and $\mathcal{T}_1$, and is thus central in $\mathcal{H}_S$ by Theorem 4.2, as we have

$$\begin{cases} -\mathcal{T}_0(\mathcal{T}_0 - \mathcal{T}_1)^2 = \mathcal{T}_0^2 + \mathcal{T}_0\mathcal{T}_1 + \mathcal{T}_0^2\mathcal{T}_1 + \mathcal{T}_0\mathcal{T}_1\mathcal{T}_0 = -\mathcal{T}_0 + \mathcal{T}_0\mathcal{T}_1\mathcal{T}_0 \ , \\ -(\mathcal{T}_0 - \mathcal{T}_1)^2\mathcal{T}_0 = \mathcal{T}_0^2 + \mathcal{T}_1\mathcal{T}_0 + \mathcal{T}_0\mathcal{T}_1\mathcal{T}_0 + \mathcal{T}_1\mathcal{T}_0^2 = -\mathcal{T}_0 + \mathcal{T}_0\mathcal{T}_1\mathcal{T}_0 \ , \end{cases}$$

and

$$\begin{cases} -\mathcal{T}_1(\mathcal{T}_0 - \mathcal{T}_1)^2 = \mathcal{T}_1\mathcal{T}_0 + \mathcal{T}_1^2 + \mathcal{T}_1\mathcal{T}_0\mathcal{T}_1 + \mathcal{T}_1^2\mathcal{T}_0 = -\mathcal{T}_1 + \mathcal{T}_1\mathcal{T}_0\mathcal{T}_1 \ , \\ -(\mathcal{T}_0 - \mathcal{T}_1)^2\mathcal{T}_1 = \mathcal{T}_0\mathcal{T}_1 + \mathcal{T}_1^2 + \mathcal{T}_0\mathcal{T}_1^2 + \mathcal{T}_1\mathcal{T}_0\mathcal{T}_1 = -\mathcal{T}_1 + \mathcal{T}_1\mathcal{T}_0\mathcal{T}_1 \ . \end{cases}$$

We now prove that any central element in $\mathcal{H}_S$ is a polynomial in $(\mathcal{T}_0 - \mathcal{T}_1)^2$. Thanks to Theorem 4.2, we know that any element of $\mathcal{H}_S$ can be written as a polynomial in $\mathcal{T}_0$ and $\mathcal{T}_1$ with coefficients in $\overline{\mathbb{F}}_p$. Moreover, the quadratic relations satisfied by $\mathcal{T}_0$ and $\mathcal{T}_1$ imply that this polynomial is a finite linear combination of monomials of the following form : $(\mathcal{T}_1\mathcal{T}_0)^n, (\mathcal{T}_0\mathcal{T}_1)^n, \mathcal{T}_0(\mathcal{T}_1\mathcal{T}_0)^n$ or $\mathcal{T}_1(\mathcal{T}_0\mathcal{T}_1)^n$, with $n \in \mathbb{N}$. As the braid relations in $\mathcal{H}_S$ assure that these monomials are respectively equal to $\mathcal{T}_{(s_1 s_0)^n}, \mathcal{T}_{(s_0 s_1)^n}, \mathcal{T}_{s_0(s_1 s_0)^n}$ and $\mathcal{T}_{s_1(s_0 s_1)^n}$, they are in particular linearly independant over $\overline{\mathbb{F}}_p$. To conclude, we proceed by induction on the maximal homogeneous degree of the polynomial in $\mathcal{T}_0$ and $\mathcal{T}_1$ defining the central element $\mathcal{Z} \in \mathcal{H}_S$ we consider.

- First assume that $\mathcal{Z}$ is given by a polynomial of maximal homogeneous degree $d \leq 1$. This means that we have $\mathcal{Z} = \alpha\mathcal{T}_0 + \beta\mathcal{T}_1 + \gamma$ for some $\alpha, \beta, \gamma \in \overline{\mathbb{F}}_p$. The relation $\mathcal{Z}\mathcal{T}_0 = \mathcal{T}_0\mathcal{Z}$ then reduces to $\beta\mathcal{T}_1\mathcal{T}_0 = \beta\mathcal{T}_0\mathcal{T}_1$ and implies that β is null. Since $\mathcal{Z} - \gamma$ is a central element in $\mathcal{H}_S$ while $\mathcal{T}_0$ is not, we necessarily have $\alpha = 0$, what proves that $\mathcal{Z} = \gamma$ is given by a constant polynomial, hence belongs to $\overline{\mathbb{F}}_p[(\mathcal{T}_0 - \mathcal{T}_1)^2]$.

- Assume now that $\mathcal{Z}$ is given by a polynomial of maximal homogeneous degree equal to 2 : it can be written as $\alpha\mathcal{T}_0\mathcal{T}_1 + \beta\mathcal{T}_1\mathcal{T}_0 + \gamma\mathcal{T}_0 + \delta\mathcal{T}_1 + \psi$ with $\alpha, \beta, \gamma, \delta, \psi \in \overline{\mathbb{F}}_p$. The relation $\mathcal{Z}\mathcal{T}_0 = \mathcal{T}_0\mathcal{Z}$ then reduces to

$$(\delta - \alpha)\mathcal{T}_0\mathcal{T}_1 + \beta\mathcal{T}_0\mathcal{T}_1\mathcal{T}_0 = (\delta - \beta)\mathcal{T}_1\mathcal{T}_0 + \alpha\mathcal{T}_0\mathcal{T}_1\mathcal{T}_0 \ ,$$

  what shows that $\alpha = \delta = \beta$. By (4.1), we have $\mathcal{T}_1 + \mathcal{T}_1\mathcal{T}_0 + \mathcal{T}_0\mathcal{T}_1 = -(\mathcal{T}_0 - \mathcal{T}_1)^2 - \mathcal{T}_0$, what implies that $\mathcal{Z} - (\gamma - \alpha)\mathcal{T}_0$ belongs to $\overline{\mathbb{F}}_p[(\mathcal{T}_0 - \mathcal{T}_1)^2]$ and is consequently a central element in $\mathcal{H}_S$. Hence $(\gamma - \alpha)\mathcal{T}_0$ must also be central in $\mathcal{H}_S$, what allows us to deduce from the previous case that we have $\gamma = \alpha$. This finally proves that $\mathcal{Z}$ is equal to $\alpha(\mathcal{T}_0\mathcal{T}_1 + \mathcal{T}_1\mathcal{T}_0 + \mathcal{T}_0 + \mathcal{T}_1) + \psi = -\alpha(\mathcal{T}_0 - \mathcal{T}_1)^2 + \psi$, and therefore belongs to $\overline{\mathbb{F}}_p[(\mathcal{T}_0 - \mathcal{T}_1)^2]$.

- Finally assume that $\mathcal{Z}$ is given by a polynomial of maximal homogeneous degree $d \geq 2$ and make the following induction hypothesis : *any central element in $\mathcal{H}_S$ given by a polynomial of maximal homogeneous degree strictly less than d belongs to $\overline{\mathbb{F}}_p[(\mathcal{T}_0 - \mathcal{T}_1)^2]$.*

  ⋆ If $d = 2n$ is even, we have $\mathcal{Z} = \alpha(\mathcal{T}_0\mathcal{T}_1)^n + \beta(\mathcal{T}_1\mathcal{T}_0)^n + \mathcal{Z}_1$ with $\alpha, \beta \in \overline{\mathbb{F}}_p$ and $\mathcal{Z}_1 \in \mathcal{H}_S$ given by a polynomial of maximal homogeneous degree strictly less than d. Since $\mathcal{Z}$ commutes to $\mathcal{T}_0$, we have

  $$-\alpha(\mathcal{T}_0\mathcal{T}_1)^n + \beta\mathcal{T}_0(\mathcal{T}_1\mathcal{T}_0)^n + \mathcal{T}_0\mathcal{Z}_1 = \alpha(\mathcal{T}_0\mathcal{T}_1)^n\mathcal{T}_0 - \beta(\mathcal{T}_1\mathcal{T}_0)^n + \mathcal{Z}_1\mathcal{T}_0 \ . \tag{4.2}$$

  As $-\alpha(\mathcal{T}_0\mathcal{T}_1)^n, \mathcal{T}_0\mathcal{Z}_1, -\beta(\mathcal{T}_1\mathcal{T}_0)^n$ and $\mathcal{Z}_1\mathcal{T}_0$ are all of maximal homogenous degree strictly less than $2n+1$, the computation of the homogeneous term of degree $2n+1$ in (4.2) shows that $\alpha = \beta$. We hence deduce from (4.1) that

  $$\mathcal{Z} = \alpha((\mathcal{T}_0\mathcal{T}_1)^n + (\mathcal{T}_1\mathcal{T}_0)^n) + \mathcal{Z}_1 = -\alpha(\mathcal{T}_0 - \mathcal{T}_1)^{2n} + \tilde{\mathcal{Z}}_1 \ , \tag{4.3}$$

  where $\tilde{\mathcal{Z}}_1 \in \mathcal{H}_S$ is given by a polynomial of maximal homogeneous degree strictly less than $2n = d$. As $\tilde{\mathcal{Z}}_1 = \mathcal{Z} + \alpha(\mathcal{T}_0 - \mathcal{T}_1)^{2n}$ is also a central element in $\mathcal{H}_S$, it belongs to $\overline{\mathbb{F}}_p[(\mathcal{T}_0 - \mathcal{T}_1)^2]$ by induction hypothesis, and (4.3) hence proves that $\mathcal{Z}$ is contained in $\overline{\mathbb{F}}_p[(\mathcal{T}_0 - \mathcal{T}_1)^2]$.

  ⋆ If $d = 2n+1$ is odd, we have $\mathcal{Z} = \alpha(\mathcal{T}_0\mathcal{T}_1)^n\mathcal{T}_0 + \beta(\mathcal{T}_1\mathcal{T}_0)^n\mathcal{T}_1 + \mathcal{Z}_1$ with $\alpha, \beta \in \overline{\mathbb{F}}_p$ and $\mathcal{Z}_1 \in \mathcal{H}_S$ given by a polynomial of maximal homogeneous degree strictly less than d. The computation of the homogeneous term of degree $2n+2$ in the equality $\mathcal{T}_0\mathcal{Z} = \mathcal{Z}\mathcal{T}_0$ (respectively $\mathcal{T}_1\mathcal{Z} = \mathcal{Z}\mathcal{T}_1$) shows that $\beta = 0$ (resp. $\alpha = 0$). This implies that $\mathcal{Z} = \mathcal{Z}_1$ should be of maximal homogeneous degree strictly less than d, what contradicts our assumption saying that $\mathcal{Z}$ is of maximal homogeneous degree equal to d.

This shows that any central element in $\mathcal{H}_S$ belongs to $\overline{\mathbb{F}}_p[(\mathcal{T}_0 - \mathcal{T}_1)^2]$, what ends the proof of Theorem 4.3. □

4.2. Classification of simple $\mathcal{H}_S$-modules

For any pair $(\varepsilon_0, \varepsilon_1) \in \{0, -1\}^2$ of parameters, let $M_1^S(\varepsilon_0, \varepsilon_1)$ be the $\mathcal{H}_S$-character over $\overline{\mathbb{F}}_p$ that maps $\mathcal{T}_i$ to ε_i for any $i \in \{0, 1\}$. Note that the quadratic relations involved in the definition of $\mathcal{H}_S$ imply that any $\mathcal{H}_S$-character over $\overline{\mathbb{F}}_p$ is necessarily of that form. To ease notations, we denote by $M_1^S(\varepsilon)$ the $\mathcal{H}_S$-character $M_1^S(\varepsilon, \varepsilon)$ for any $\varepsilon \in \{0, -1\}$. By analogy with the terminology used in [21], the $\mathcal{H}_S$-characters $M_1^S(0)$ and $M_1^S(-1)$ are respectively called the *trivial character* and the *sign character*.

We now introduce the analogue of the standard $\mathcal{H}$-modules defined in [21] : for any scalar $\lambda \in \overline{\mathbb{F}}_p$, let $M_2^S(\lambda)$ be the two-dimensional $\mathcal{H}_S$-module $\overline{\mathbb{F}}_p x \oplus \overline{\mathbb{F}}_p y$ endowed with the actions of $\mathcal{T}_0$ and $\mathcal{T}_1$ that are respectively defined by the following matrices in the $\overline{\mathbb{F}}_p$-basis $\{x, y\}$:

$$\begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 & \lambda \\ 0 & 0 \end{pmatrix} \ . \tag{4.4}$$

We call $M_2^S(\lambda)$ the *standard $\mathcal{H}_S$-module with parameter* λ. Moreover, any basis $\{x, y\}$ of the $\overline{\mathbb{F}}_p$-vector space $M_2^S(\lambda)$ in which the matrices describing the actions of $\mathcal{T}_0$ and $\mathcal{T}_1$ are as in (4.4) is called an *adapted basis for* $M_2^S(\lambda)$.

REMARK 4.4. The central element $-(\mathcal{T}_0 - \mathcal{T}_1)^2$ acts on $M_2^S(\lambda)$ via the scalar $\lambda - 1$. Consequently, two standard $\mathcal{H}_S$-modules with distinct parameters cannot be isomorphic.

We now establish the irreducibility properties of the standard $\mathcal{H}_S$-modules.

THEOREM 4.5. *Let* $\lambda \in \overline{\mathbb{F}}_p$.

1. *The* $\mathcal{H}_S$*-module* $M_2^S(\lambda)$ *is irreducible if, and only if,* λ *does not belong to* $\{0, 1\}$.
2. *The standard* $\mathcal{H}_S$*-module with parameter* 1 *is indecomposable of length* 2 *and fits into the following non-split short exact sequence of* $\mathcal{H}_S$*-modules :*

$$0 \longrightarrow M_1^S(0) \longrightarrow M_2^S(1) \longrightarrow M_1^S(-1) \longrightarrow 0 \ . \tag{4.5}$$

3. *The standard* $\mathcal{H}_S$*-module with parameter* 0 *is indecomposable of length* 2 *and fits into the following non-split short exact sequence of* $\mathcal{H}_S$*-modules :*

$$0 \longrightarrow M_1^S(-1,0) \longrightarrow M_2^S(0) \longrightarrow M_1^S(0,-1) \longrightarrow 0 \ . \tag{4.6}$$

Proof. First note that $M_2^S(1)$ contains the trivial character $M_1^S(0)$, generated by the vector $x + y$ if $\{x, y\}$ is an adapted basis of $M_2^S(1)$, while $M_2^S(0)$ contains the character $M_1^S(-1,0)$, generated by y if $\{x, y\}$ is an adapted basis of $M_2^S(0)$. This proves a sufficient condition for $M_2^S(\lambda)$ to be reducible is $\lambda \in \{0, 1\}$.

Now fix an adapted basis $\{x, y\}$ of $M_2^S(\lambda)$ and assume that $M_2^S(\lambda)$ is a reducible $\mathcal{H}_S$-module, what means that it contains a one-dimensional $\mathcal{H}_S$-submodule M generated by a non-zero vector $v := ax + by$ with $a, b \in \overline{\mathbb{F}}_p$. Note that x generates the $\mathcal{H}_S$-module $M_2^S(\lambda)$, what implies that b is non-zero as M is strictly contained in $M_2^S(\lambda)$. Up to scaling, we can assume that $b = 1$, i.e. that $v = \alpha x + y$ for some $\alpha \in \overline{\mathbb{F}}_p$. If $\alpha = 0$, then $v = y$ generates $M_2^S(\lambda)$ whenever λ is non-zero, what contradicts the fact that v generates a one-dimensional $\mathcal{H}_S$-module, so λ must be null in this case. If α is non-zero, note that $v|\mathcal{T}_0 = (\alpha - 1)y$ and that $v|\mathcal{T}_1 = (\lambda - \alpha)x$. As x generates the $\mathcal{H}_S$-module $M_2^S(\lambda)$, the second equality implies that we must have $\lambda = \alpha$. The first equality $v|\mathcal{T}_0 = (\alpha - 1)y$ now implies that whenever α is different from 1, the vector y belongs to M, and so does $x = \alpha^{-1}(v - y)$, what contradicts the fact that M is one-dimensional over $\overline{\mathbb{F}}_p$. We thus have $\lambda = \alpha = 1$, and we deduce from this dichotomy that a necessary condition for $M_2^S(\lambda)$ to be reducible is $\lambda \in \{0, 1\}$. This ends the proof of statement (1) by contraposition.

Now assume that $\lambda = 0$, in which case $M_1^S(-1,0)$ is an $\mathcal{H}_S$-submodule of $M_2^S(0)$ as we already mentionned it. One checks immediately that the quotient of $M_2^S(0)$ by this $\mathcal{H}_S$-submodule is generated by the image of x and is isomorphic to the $\mathcal{H}_S$-character $M_1(0,-1)$. This leads to the short exact sequence (4.6), which is necessarily non-split as x generates the whole $\mathcal{H}_S$-module $M_2^S(0)$. This ends the proof of (3).

Finally assume that $\lambda = 1$, in which case we already noticed that $x + y$ generates the trivial character $M_1^S(0)$. One immediately checks that the quotient of $M_2^S(1)$ by this $\mathcal{H}_S$-submodule is generated by the image of y and is isomorphic to the sign character $M_1^S(-1)$. This leads to the short exact sequence (4.5), which is necessarily non-split as y generates the $\mathcal{H}_S$-module $M_2^S(0)$, and completes the proof of Theorem 4.5. $\square$

Theorem 4.6. *Any two-dimensional simple right $\mathcal{H}_S$-module is isomorphic to a standard $\mathcal{H}_S$-module $M_2^S(\lambda)$ for a unique parameter $\lambda \in \overline{\mathbb{F}}_p \backslash \{0,1\}$.*

Proof. Assume that $\mathcal{M}$ is a simple right $\mathcal{H}_S$-module of dimension 2 over $\overline{\mathbb{F}}_p$. As $\mathcal{M}$ is a finite-dimensional vector space over the algebraically closed field $\overline{\mathbb{F}}_p$, the central element $-(\mathcal{T}_0 - \mathcal{T}_1)^2$ of $\mathcal{H}_S$ acts on $\mathcal{M}$ by a scalar $\mu \in \overline{\mathbb{F}}_p$. Let φ be the endomorphism of $\mathcal{M}$ defined by the action of $\mathcal{T}_1$ on $\mathcal{M}$: then the quadratic relation $\mathcal{T}_1(\mathcal{T}_1+1) = 0$ implies that the minimal polynomial of φ divides $X(X+1)$. However, φ cannot be a homothety : otherwise, Theorem 4.2 would imply that any non-zero eigenvector for the action of $\mathcal{T}_0$ on $\mathcal{M}$ generates a one-dimensional $\mathcal{H}_S$-submodule of $\mathcal{M}$, what contradicts the simplicity of the two-dimensional $\mathcal{H}_S$-module $\mathcal{M}$. This shows that $X(X+1)$ is the minimal polynomial of φ and that $\mathcal{M}$ is the direct sum - as vector space over $\overline{\mathbb{F}}_p$ - of $\ker\varphi$ and $\ker(\varphi+1)$, these spaces being both non-zero, hence of dimension 1 over $\overline{\mathbb{F}}_p$. Pick a non-zero vector $x \in \ker(\varphi+1)$: as $\mathcal{M}$ is a simple $\mathcal{H}_S$-module of dimension 2 over $\overline{\mathbb{F}}_p$, the line $\ker(\varphi+1)$ cannot be stable under the action of $\mathcal{T}_0$ on $\mathcal{M}$, and the family $\{x, x|\mathcal{T}_0\}$ is hence linearly independant over $\overline{\mathbb{F}}_p$, what means that it defines a basis of the vector space $\mathcal{M}$. A direct computation gives the following relations in $\mathcal{M}$:

$$x|\mathcal{T}_1 = -x \ , \ (x|\mathcal{T}_0)|\mathcal{T}_0 = -x|\mathcal{T}_0 \ , \ (x|\mathcal{T}_0)|\mathcal{T}_1 = (\mu+1)x \ ,$$

where the last equality comes from the relation (4.1). This way, we obtain that in the basis $\{x, x|\mathcal{T}_0\}$, the actions of $\mathcal{T}_0$ and $\mathcal{T}_1$ are respectively given by the matrices

$$\begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \text{ and } \begin{pmatrix} -1 & 1+\mu \\ 0 & 0 \end{pmatrix} .$$

This shows that $\mathcal{M}$ is isomorphic to the standard $\mathcal{H}_S$-module $M_2^S(1+\mu)$, what finishes the proof as the uniqueness of the parameter comes from Remark 4.4 while the restriction on its possible values comes from Theorem 4.5. □

Our next result states that any simple right $\mathcal{H}_S$-module necessarily appears among the $\mathcal{H}_S$-modules we built so far. Recall that Remark 2.7 and Theorem 3.1 ensure that any simple $\mathcal{H}_S$-module is of finite dimension over $\overline{\mathbb{F}}_p$.

Theorem 4.7. *Any simple right $\mathcal{H}_S$-module is either a character or an irreducible standard $\mathcal{H}_S$-module. In particular, its dimension over $\overline{\mathbb{F}}_p$ is at most 2.*

Proof. This proof essentially reduces to the proof of Theorem 4.6. Assume that $\mathcal{M}$ is a (finite-dimensional) simple right $\mathcal{H}_S$-module which is not one-dimensional. As in the proof of Theorem 4.6, the central element $-(\mathcal{T}_0 - \mathcal{T}_1)^2$ acts on $\mathcal{M}$ by a scalar $\lambda \in \overline{\mathbb{F}}_p$ and $\ker(\mathcal{T}_1 + 1)$ contains a non-zero vector x. The argument ending the proof of Theorem 4.6 implies here that x and $y := x|\mathcal{T}_0$ generate a two-dimensional $\mathcal{H}_S$-submodule of $\mathcal{M}$ which is isomorphic to $M_2^S(\lambda+1)$. As $\mathcal{M}$ is simple, we must have $\mathcal{M} = M_2^S(\lambda+1)$. □

Corollary 4.8. *Any simple right $\mathcal{H}_S$-module is isomorphic to one, and only one, of the following $\mathcal{H}_S$-modules :*

- *a character $M_1^S(\varepsilon_0, \varepsilon_1)$ for a unique pair of parameters $(\varepsilon_0, \varepsilon_1) \in \{0, -1\}^2$;*
- *a standard $\mathcal{H}_S$-module $M_2^S(\lambda)$ for a unique parameter $\lambda \in \overline{\mathbb{F}}_p \backslash \{0,1\}$.*

4.3. Relationships with the corresponding objects for $GL_2(F)$ - Part 1

We explained in Section 2.3 how $\mathcal{H}_S$ can naturally be identified to an $\overline{\mathbb{F}}_p$-subalgebra of the standard Iwahori-Hecke algebra $\mathcal{H}$ of $GL_2(F)$. In particular, any element of $\mathcal{H}_S$ can be seen as an element of $\mathcal{H}$, hence can be written as a polynomial in T, T^{-1} and S with coefficients in $\overline{\mathbb{F}}_p$ [21, Section 1.1]. Moreover, the restriction to $\mathcal{H}_S$ endows any right $\mathcal{H}$-module with a structure of $\mathcal{H}_S$-module. The two following questions then naturally arise and will be solved in this subsection.

- To which elements of $\overline{\mathbb{F}}_p[T, T^{-1}, S]$ do correspond the operators $\mathcal{T}_0, \mathcal{T}_1 \in \mathcal{H}_S$ when considered as elements of $\mathcal{H}$?
- What is the structure of $\mathcal{H}_S$-module carried by the finite-dimensional simple right $\mathcal{H}$-modules that Vignéras classified in [21, Theorem 1.2]?

The next statement answers to the first question.

LEMMA 4.9. *The following relations hold in the standard Iwahori-Hecke algebra $\mathcal{H}$ of $GL_2(F)$: $\mathcal{T}_0 = T^{-1}ST$ and $\mathcal{T}_1 = S$.*

Proof. The second equality directly comes from the definitions of $\mathcal{T}_1$ and S. Now recall that $s_0 = t^{-1}st$, where st is an element of length 1 in the extended affine Weyl group of $GL_2(F)$ [21, Annexe A.2]. As t and t^{-1} are both of length 0, the braid relations in $\mathcal{H}$ imply that $T_{s_0} = T_{t^{-1}}T_{st} = (T_t)^{-1}T_sT_t = T^{-1}ST$, what proves the first equality. □

As T^2 and T^{-2} are central elements in $\mathcal{H}$, we also have $\mathcal{T}_0 = TST^{-1}$ in $\mathcal{H}$. Using Lemma 4.9, we obtain the following reformulation of Corollary 2.5.

COROLLARY 4.10. *The standard Iwahori-Hecke algebra $\mathcal{H}_S$ of $SL_2(F)$ is isomorphic to the $\overline{\mathbb{F}}_p$-subalgebra of $\mathcal{H}$ generated by the operators S and $TST^{-1} = T^{-1}ST$.*

We now address the second question and start by a recall of the classification of finite-dimensional simple right $\mathcal{H}$-modules as stated in [21]. For any pair $(\tau, \varepsilon) \in \overline{\mathbb{F}}_p^\times \times \{0, -1\}$, let $M_1(\tau, \varepsilon)$ be the one-dimensional $\mathcal{H}$-module over $\overline{\mathbb{F}}_p$ that maps T to τ and S to ε. For any pair $(a, z) \in \overline{\mathbb{F}}_p \times \overline{\mathbb{F}}_p^\times$, the standard $\mathcal{H}$-module $M_2(a, z)$ is the two-dimensional vector space $\overline{\mathbb{F}}_p x \oplus \overline{\mathbb{F}}_y$ endowed with the following action of $\mathcal{H}$: in the adapted basis $\{x, y\}$, the actions of T and S are respectively given by the matrices $\begin{pmatrix} 0 & z \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} -1 & a \\ 0 & 0 \end{pmatrix}$. The classification of finite-dimensional simple right $\mathcal{H}$-modules given by [21] is then as follows.

THEOREM 4.11. 1. *Any finite-dimensional simple right $\mathcal{H}$-module is isomorphic to one of the following modules :*

- *an $\mathcal{H}$-character $M_1(\tau, \varepsilon)$ with $(\tau, \varepsilon) \in \overline{\mathbb{F}}_p^\times \times \{0, -1\}$;*
- *a standard $\mathcal{H}$-module $M_2(a, z)$ with $(a, z) \in \overline{\mathbb{F}}_p \times \overline{\mathbb{F}}_p^\times$ satisfying $a^2 \neq z$.*

2. *For any non-zero parameter $a \in \overline{\mathbb{F}}_p^\times$, $M_2(a, a^2)$ is an indecomposable $\mathcal{H}$-module of length 2 that fits into the following non-split short exact sequence of $\mathcal{H}$-modules :*

$$0 \longrightarrow M_1(a, 0) \longrightarrow M_2(a, a^2) \longrightarrow M_1(-a, -1) \longrightarrow 0 \ .$$

Corollary 4.10 directly leads to the following statement for one-dimensional modules.

LEMMA 4.12. *For any pair of parameters* $(\tau, \varepsilon) \in \overline{\mathbb{F}}_p^\times \times \{0, -1\}$, *the* $\mathcal{H}_S$*-module carried by* $M_1(\tau, \varepsilon)$ *is equal to the* $\mathcal{H}_S$*-character* $M_1^S(\varepsilon)$.

Proof. By definition of $M_1(\tau, \varepsilon)$ and Corollary 4.10, one easily checks that $M_1(\tau, \varepsilon)$ maps both operators $\mathcal{T}_0$ and $\mathcal{T}_1$ to ε. □

Now fix a pair $(a, z) \in \overline{\mathbb{F}}_p \times \overline{\mathbb{F}}_p^\times$ and consider the two-dimensional $\mathcal{H}$-module $M_2(a, z)$ together with an adapted basis $\{x, y\}$. A direct computation based on Corollary 4.10 shows that in this adapted basis, the actions of $\mathcal{T}_0$ and $\mathcal{T}_1$ on $M_2(a, z)$ are respectively given by the matrices $\begin{pmatrix} 0 & 0 \\ az^{-1} & -1 \end{pmatrix}$ and $\begin{pmatrix} -1 & a \\ 0 & 0 \end{pmatrix}$. Doing a suitable base change when a is non-zero hence leads to the following result.

LEMMA 4.13. *Let* $(a, z) \in \overline{\mathbb{F}}_p \times \overline{\mathbb{F}}_p^\times$ *be a pair of parameters.*

1. *If* a *is non-zero, the restriction to* $\mathcal{H}_S$ *of the standard* $\mathcal{H}$*-module* $M_2(a, z)$ *is isomorphic to the standard* $\mathcal{H}_S$*-module* $M_2^S(a^2 z^{-1})$.
2. *The restriction to* $\mathcal{H}_S$ *of the standard* $\mathcal{H}$*-module* $M_2(0, z)$ *is a split* $\mathcal{H}_S$*-module isomorphic to* $M_1^S(0, -1) \oplus M_1^S(-1, 0)$.

As z is always assumed to be non-zero, the following statement is a direct consequence of Lemma 4.13 and Corollary 4.8.

COROLLARY 4.14. *Let* $(a, z) \in \overline{\mathbb{F}}_p \times \overline{\mathbb{F}}_p^\times$ *be a pair of parameters.*

1. *If* a *is non-zero and satisfies* $a^2 \neq z$, *the* $\mathcal{H}_S$*-module carried by* $M_2(a, z)$ *is simple.*
2. *For* $a \neq 0$, *the* $\mathcal{H}_S$*-module carried by* $M_2(a, a^2)$ *is indecomposible of length* 2 *and fits into the following non-split short exact sequence of* $\mathcal{H}_S$*-modules :*
$$0 \longrightarrow M_1^S(0) \longrightarrow M_2(a, a^2)|_{\mathcal{H}_S} \longrightarrow M_1^S(-1) \longrightarrow 0 \ .$$
3. *The* $\mathcal{H}_S$*-module carried by* $M_2(0, z)$ *is isomorphic to* $M_1^S(0, -1) \oplus M_1^S(-1, 0)$.

4.4. A first correspondence with mod p representations of $SL_2(F)$

Before we start the study of the Iwahori-Hecke algebras attached to non-zero values of r, let us use the results we proved so far to establish some connections between simple right $\mathcal{H}_S$-modules and certain irreducible smooth representations of $SL_2(F)$ over $\overline{\mathbb{F}}_p$. We explained in the introduction that for any smooth representation π of G_S over $\overline{\mathbb{F}}_p$, the space π^{I_S} is naturally endowed with a structure of right $\mathcal{H}_S(0)$-module, hence of right $\mathcal{H}_S$-module by Theorem 3.3. This subsection aims to use the classification results given in [2] to describe the $\mathcal{H}_S$-module π^{I_S} whenever π is an irreducible smooth representation of G_S over $\overline{\mathbb{F}}_p$ admitting non-zero I_S-invariant vectors.

REMARK 4.15. As we will use some results of [2], we assume from now on that any choice made in this paper is done in a compatible way with those made in [2].

4.4.1. The non-supercuspidal case

We first recall two important results. The first one is due to Vignéras [25, Section 6.5] and gives the structure of $\mathcal{H}$-module carried by the space of I-invariant vectors of any non-supercuspidal irreducible smooth representation of $GL_2(F)$ with central character and with non-zero I-invariant vectors.

PROPOSITION 4.16. *Let St denote the Steinberg representation of $GL_2(F)$.*

1. *The $\mathcal{H}$-module carried by $\mathbf{1}^I$ is equal to the $\mathcal{H}$-character $M_1(1,0)$.*
2. *The $\mathcal{H}$-module carried by St^I is equal to the $\mathcal{H}$-character $M_1(1,-1)$.*
3. *For any $\lambda \in \overline{\mathbb{F}}_p^\times$, the $\mathcal{H}$-module carried by $(\mathrm{Ind}_B^G(\mu_\lambda \otimes \mathbf{1}))^I$ is isomorphic to the standard $\mathcal{H}$-module $M_2(\lambda^{-1}, \lambda^{-1})$.*

The second result was proved by the author [2, Corollaire 2.11, Théorème 2.16 and Remarque 2.17] and gathers all the results we need about the relationships between non-supercuspidal irreducible smooth representations of $SL_2(F)$ and $GL_2(F)$ over $\overline{\mathbb{F}}_p$.

PROPOSITION 4.17. 1. *Any non-supercuspidal irreducible smooth representation of $SL_2(F)$ over $\overline{\mathbb{F}}_p$ that has non-zero I_S-invariant vectors is isomorphic to one, and only one, of the following representations :*

- *the trivial character $\mathbf{1}$;*
- *the Steinberg representation St_S;*
- *the parabolically induced representation $\mathrm{Ind}_{B_S}^{G_S}(\mu_\lambda)$ for a unique $\lambda \in \overline{\mathbb{F}}_p \backslash \{0,1\}$.*

2. *We have $\mathbf{1}^{I_S} = \mathbf{1}^I$ and the standard injection[5] of St^I into $St_S^{I_S}$ is an isomorphism of $\overline{\mathbb{F}}_p$-vector spaces.*
3. *For any $\lambda \in \overline{\mathbb{F}}_p^\times$, the standard injection of $(\mathrm{Ind}_B^G(\mu_\lambda \otimes \mathbf{1}))^I$ into $(\mathrm{Ind}_{B_S}^{G_S}(\mu_\lambda))^{I_S}$ is an isomorphism of $\overline{\mathbb{F}}_p$-vector spaces.*

Combined with Corollary 2.5 and Lemmas 4.12 and 4.13, these two propositions lead to the following statement.

THEOREM 4.18. 1. *The $\mathcal{H}_S$-module carried by $\mathbf{1}^{I_S}$ is the trivial character $M_1^S(0)$.*

2. *The $\mathcal{H}_S$-module carried by $St_S^{I_S}$ is the sign character $M_1^S(-1)$.*
3. *For any parameter $\lambda \in \overline{\mathbb{F}}_p \backslash \{0,1\}$, the $\mathcal{H}_S$-module carried by $(\mathrm{Ind}_{B_S}^{G_S}(\mu_\lambda))^{I_S}$ is isomorphic to the standard $\mathcal{H}_S$-module $M_2^S(\lambda^{-1})$.*

REMARK 4.19. The same comparison process proves that the $\mathcal{H}_S$-module carried by $(\mathrm{Ind}_{B_S}^{G_S}(\mathbf{1}))^{I_S}$ is isomorphic to the (reducible) standard $\mathcal{H}_S$-module $M_2^S(1)$.

[5] Via the restriction of St to G_S, as explained in [2, Section 2.6].

4.4.2. The supercuspidal case

We assume in this paragraph that $F = \mathbb{Q}_p$, as it is the only case so far where supercuspidal representations of $SL_2(F)$ and $GL_2(F)$ are fully understood (see respectively [2] and [4, 6]). In particular, we know from [2, Théorème 4.12] that there are p isomorphism classes of supercuspidal representations for $SL_2(\mathbb{Q}_p)$, for which an explicit system of representatives $\{\pi_r,\ 0 \leq r \leq p-1\}$ is built in [2, Section 4.1]. Using the same notations as in [2] and [6], we can moreover connect as follows supercuspidal representations of $GL_2(\mathbb{Q}_p)$ of the form $\pi(r, 0, \mathbf{1})$ with those of $SL_2(\mathbb{Q}_p)$ [2, Théorème 4.12] :

$$\forall\, r \in \{0, \ldots, p-1\},\ \pi(r, 0, \mathbf{1})|_{G_S} \simeq \pi_r \oplus \pi_{p-1-r}\ . \tag{4.7}$$

As in [2, Section 4], we let $v_r := \overline{[I_2, x^r]}$ be the element of $\pi_r^{I_S(1)}$ that naturally generates the representation π_r of G_S over $\overline{\mathbb{F}}_p$. We have the following result [25, Section 6.5], which is an analogue of Proposition 4.16 in the supercuspidal case.

PROPOSITION 4.20. *The $\mathcal{H}$-module carried by $\pi(0, 0, 1)^I$ is isomorphic to $M_2(0, 1)$. Moreover, it has an adapted basis of the form $\{v_0, y\}$ with y being an element of $\pi(0, 0, 1)^I$ that generates the $\overline{\mathbb{F}}_p[G_S]$-submodule π_{p-1} of $\pi(0, 0, \mathbf{1})|_{G_S}$ appearing in* (4.7).

We also know from [2, Propositions 4.7 et 4.11] that up to isomorphism, π_0 and π_{p-1} are the only supercuspidal representations of G_S having non-zero I_S-invariant vectors, that $\dim_{\overline{\mathbb{F}}_p} \pi_0^{I_S} = \dim_{\overline{\mathbb{F}}_p} \pi_{p-1}^{I_S} = 1$, and that v_0 belongs to $\pi_0^{I_S}$. Using [2, Théorème 4.3] together with Proposition 4.20 and Lemma 4.13 (3), we get the following result.

THEOREM 4.21. *We assume $F = \mathbb{Q}_p$ and we use the notations introduced above.*

1. *The $\mathcal{H}_S$-module $\pi_0^{I_S}$ is equal to the $\mathcal{H}_S$-character $M_1^S(-1, 0)$.*
2. *The $\mathcal{H}_S$-module $\pi_{p-1}^{I_S}$ is equal to the $\mathcal{H}_S$-character $M_1^S(0, -1)$.*

Proof. Proposition 4.20 implies that $\pi_0^{I_S}$ and $\pi_{p-1}^{I_S}$ are the lines generated by the vectors of an adapted basis of the $\mathcal{H}$-module $\pi(0, 0, \mathbf{1})^I \simeq M_2(0, 1)$, what ends up the proof via the computations leading to Lemma 4.13 (2). □

4.4.3. Application to the functor of I_S-invariants

As in the dictionnary established by Vignéras [21] for $GL_2(F)$, we have a notion of supersingular $\mathcal{H}_S$-module.

DEFINITION 4.22. A simple right $\mathcal{H}_S$-module is *supersingular* if it is not isomorphic to a subquotient of some $(\mathrm{Ind}_{B_S}^{G_S}(\eta))^{I_S}$ for a smooth character $\eta : B_S \to \overline{\mathbb{F}}_p^\times$.

Recall here that the map sending a smooth representation of G_S over $\overline{\mathbb{F}}_p$ to its space of I_S-invariant vectors comes from a functor (called the *functor of I_S-invariants*) going from the category of smooth representations of G_S over $\overline{\mathbb{F}}_p$ generated by their I_S-invariant vectors to the category of right $\mathcal{H}_S$-modules. The comparison of Theorems 4.18 and 4.21 to the classification established in Corollary 4.8 leads to the following result.

Corollary 4.23. 1. *The functor of I_S-invariants defines a bijection :*

$$\left\{\begin{array}{c}\textit{isomorphism classes of}\\ \text{non-supercuspidal } \textit{irreducible smooth}\\ \textit{representations of } SL_2(F) \textit{ over } \overline{\mathbb{F}}_p\\ \textit{generated by their } I_S\textit{-invariant vectors}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\textit{isomorphism classes of}\\ \text{non-supersingular}\\ \textit{simple right } \mathcal{H}_S\textit{-modules}\end{array}\right\}.$$

2. *When $F = \mathbb{Q}_p$, the previous bijection extends to a bijection :*

$$\left\{\begin{array}{c}\textit{isomorphism classes of irreducible smooth}\\ \textit{representations of } SL_2(\mathbb{Q}_p) \textit{ over } \overline{\mathbb{F}}_p\\ \textit{generated by their } I_S\textit{-invariant vectors}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\textit{isomorphism classes of}\\ \textit{simple right } \mathcal{H}_S\textit{-modules}\end{array}\right\}.$$

Remark 4.24. As all $\mathcal{H}_S$-modules appearing in Corollary 4.23 are simple modules, Theorem 3.1 implies that they also define simple right $\mathbb{H}^1_S$-modules. In particular, they define $\mathbb{H}^1_S$-submodules of the corresponding spaces of $I_S(1)$-invariants vectors. We will see later (Propositions 7.3 and 7.4) that they actually completely define the $\mathbb{H}^1_S$-modules carried by these spaces of $I_S(1)$-invariant vectors.

Remark 4.25. As it only deals with irreducible smooth representations having non-zero I_S-invariant vectors, the statement of Corollary 4.23 is highly partial : for instance, it puts aside most of the supercuspidal representations of $SL_2(\mathbb{Q}_p)$ when p is odd.

4.5. Relationships with the corresponding objects for $GL_2(F)$ - Part 2

We close this section with a result underlining the interplay between the two kinds of restriction maps considered above. On the one hand, the restriction to G_S of any smooth representation of G over $\overline{\mathbb{F}}_p$ defines a smooth representation of G_S over $\overline{\mathbb{F}}_p$. On the other hand, what we did in Section 2.3 allows us to restrict to $\mathcal{H}_S$ any right $\mathcal{H}$-module to obtain a right $\mathcal{H}_S$-module. The next result is a direct consequence of [2, Remarque 2.17 et Théorème 4.3] together with [5, Lemma 27 and Theorem 28] and [6, Théorème 3.2.4].

Proposition 4.26. *Let π be an irreducible smooth representation of G over $\overline{\mathbb{F}}_p$ having a central character and non-zero I-invariant vectors. If one of the following conditions holds, then I acts trivially on π^{I_S} :*

- *either π is non-supercuspidal;*
- *or $F = \mathbb{Q}_p$.*

Together with Lemmas 4.12 and 4.13, Propositions 4.16 and 4.20, Theorems 4.18 and 4.21, and [2, Théorème 4.12], Proposition 4.26 leads to the following statement, which roughly says that taking invariant vectors under the standard Iwahori subgroups commutes with restriction functors.

Corollary 4.27. 1. *For any* non-supercuspidal *irreducible smooth representation π of $GL_2(F)$ over $\overline{\mathbb{F}}_p$ having a central character and non-zero I-invariant vectors, the restriction to $\mathcal{H}_S$ of the $\mathcal{H}$-module π^I is isomorphic to the $\mathcal{H}_S$-module $(\pi|_{SL_2(F)})^{I_S}$.*

2. *For any irreducible smooth representation π of $GL_2(\mathbb{Q}_p)$ over $\overline{\mathbb{F}}_p$ having a central character and non-zero I-invariant vectors, the restriction to $\mathcal{H}_S$ of the $\mathcal{H}$-module π^I is isomorphic to the $\mathcal{H}_S$-module $(\pi|_{SL_2(\mathbb{Q}_p)})^{I_S}$.*

5. The regular Iwahori-Hecke algebras and their simple modules

In this section, we describe the structure of the Iwahori-Hecke algebra $\mathcal{H}_S(r)$ and of its simple right modules for any parameter $r \in \{1, \dots, \left[\frac{q-1}{2}\right]\}$ distinct from $\frac{q-1}{2}$ when p is odd. This is the counterpart of Vignéras did for $GL_2(F)$ in [21, Section 2].

5.1. Structure of the Iwahori-Hecke algebra $\mathcal{H}_S(r)$

As we assume $r \neq q-1-r$, the $\overline{\mathbb{F}}_p$-algebra $\mathcal{H}_S(r) := \mathrm{End}_{\overline{\mathbb{F}}_p[G_S]}(\mathrm{ind}_{I_S}^{G_S}(\omega^r \oplus \omega^{q-1-r}))$ can be decomposed as follows :

$$\mathcal{H}_S(r) = \begin{pmatrix} A_r & B_r \\ B_{q-1-r} & A_{q-1-r} \end{pmatrix} ,$$

with $A_k := \mathrm{End}_{\overline{\mathbb{F}}_p[G_S]}(\mathrm{ind}_{I_S}^{G_S}(\omega^k))$ and $B_k := \mathrm{Hom}_{\overline{\mathbb{F}}_p[G_S]}(\mathrm{ind}_{I_S}^{G_S}(\omega^k), \mathrm{ind}_{I_S}^{G_S}(\omega^{q-1-k}))$ for $k \in \{r, q-1-r\}$. The understanding of $\mathcal{H}_S(r)$ hence reduces to the study of each of its four components and of the relations existing between them. Before going further, note that A_k is naturally an algebra over $\overline{\mathbb{F}}_p$ while B_k is certainly not. Nevertheless, the composition of functions endows B_k with a natural structure of (A_{q-1-k}, A_k)-bimodule as we have $A_{q-1-k} \circ B_k \circ A_k \subset B_k$.

5.1.1. Structure of the $\overline{\mathbb{F}}_p$-algebra A_k

Let k be equal to r or to $q-1-r$. Compact Frobenius reciprocity implies that the $\overline{\mathbb{F}}_p$-algebra A_k is isomorphic to the convolution algebra $\mathbb{H}_k$ of functions $f : G_S \to \overline{\mathbb{F}}_p$ with compact support modulo I_S that satisfy $f(igj) = \omega^k(ij)f(g)$ for any $i, j \in I_S$ and any $g \in G_S$. Any element of A_k can consequently be seen as a function over G_S, what allows us to consider its *support in G_S* or its *value at some element of G_S*. We can now state the following structure result for A_k, where $\mathcal{A}$ denotes the commutative $\overline{\mathbb{F}}_p$-algebra $\overline{\mathbb{F}}_p[X,Y]/(XY, YX)$.

Theorem 5.1. *Let T_k and S_k be the elements of A_k defined as follows : T_k has support equal to $I_S\alpha_0^{-1}I_S$ and value 1 at α_0^{-1}, while S_k has support equal to $I_S\alpha_0 I_S$ and value 1 at α_0. The $\overline{\mathbb{F}}_p$-linear map sending T_k to X and S_k to Y then defines an isomorphism of $\overline{\mathbb{F}}_p$-algebras from A_k to $\mathcal{A}$.*

Proof. Things work the same way as in the proof of [21, Appendix 1.3]. For any integer n, let $T_n \in A_k$ be the element with support equal to $I_S\alpha_0^{-n}I_S$ and value 1 at α_0^{-n} : then we know from [2, Proposition 3.30] that $\{T_n, n \in \mathbb{Z}\}$ is a basis of the $\overline{\mathbb{F}}_p$-vector space A_k. Now recall that we have the following decompositions into disjoint left cosets, where $\mathcal{C}_m := \{[x_0] + \varpi_F[x_1] + \ldots + \varpi_F^{m-1}[x_{m-1}], x_i \in k_F\}$ is a set of representatives for the elements of $\mathcal{O}_F/\varpi_F^m\mathcal{O}_F$ (for any $m \in \mathbb{N}$) :

$$\begin{cases} \forall\ n \in \mathbb{N}^*,\ I_S\alpha_0^{-n}I_S = \displaystyle\bigsqcup_{x \in \mathcal{C}_{2n}} u(-x)\alpha_0^{-n}I_S\ ; \\ \forall\ n \in \mathbb{N},\ I_S\alpha_0^{n}I_S = \displaystyle\bigsqcup_{x \in \mathcal{C}_{2n}} \bar{u}(-\varpi_F x)\alpha_0^{n}I_S\ . \end{cases}$$

As [4, Lemma 3 (1)] implies that $\omega^k(\bar{u}(\varpi_F x)) = \omega^k(u(x)) = 1$ for any $x \in \mathcal{O}_F$, we deduce from [4, Equation (9)] that for any integer n, the operator T_n acts as follows on the standard function $[I_2, 1] \in \mathrm{ind}_{I_S}^{G_S}(\omega^k)$:

$$\begin{cases} \forall\ n \in \mathbb{N},\ T_n([I_2,1]) = \sum\limits_{x\in\mathcal{C}_{2n}} [\bar{u}(-\varpi_F x)\alpha_0^n, 1]\ ; \\ \forall\ n \in \mathbb{N}^*,\ T_{-n}([I_2,1]) = \sum\limits_{x\in\mathcal{C}_{2n}} [u(-x)\alpha_0^{-n}, 1]\ . \end{cases}$$

These equalities completely determine T_n as it is a G_S-equivariant and $\overline{\mathbb{F}}_p$-linear operator. A direct computation based on [4, Equation (9)] and (1.1) now proves that we have :

$$\begin{cases} \forall\ n \in \mathbb{N},\ T_1 \circ T_n = T_{n+1} = T_n \circ T_1\ ; \\ \forall\ n \in \mathbb{N}^*,\ T_{-1} \circ T_{-n} = T_{-(n+1)} = T_{-n} \circ T_{-1}\ . \end{cases} \tag{5.1}$$

To finish the proof of Theorem 5.1, it is left to show that $T_1 \circ T_{-1} = T_{-1} \circ T_1 = 0$, as we set $T_k := T_1$ and $S_k := T_{-1}$. Once again, it is enough to check it is true after evaluation at the standard function $[I_2, 1]$. Applying [4, Equation (9)] to $T_1 \circ T_{-1}$, we obtain that

$$\begin{aligned}(T_1 \circ T_{-1})([I_2,1]) &= \sum_{x\in\mathcal{C}_2}\sum_{y\in\mathcal{C}_2} [u(-x)\alpha_0^{-1}\bar{u}(-\varpi_F y)\alpha_0, 1] \\ &= \sum_{x\in\mathcal{C}_2}\sum_{y\in\mathcal{C}_2} [u(-x)\bar{u}(-\varpi_F^{-1} y), 1] \\ &= \sum_{x\in\mathcal{C}_2} [u(-x), 1] + \sum_{y\in\mathcal{C}_2\backslash\{0\}}\sum_{x\in\mathcal{C}_2} [u(-x)\bar{u}(-\varpi_F^{-1} y), 1]\ .\end{aligned}$$

As ω^k is trivial on $I_S(1)$, where $u(x)$ sits for any $x \in \mathcal{O}_F$, the first sum in the last expression above is equal to $\mathrm{Card}(\mathcal{C}_2)[I_2, 1]$, i.e. to 0 as we have $\mathrm{Card}(\mathcal{C}_2) \equiv 0 \mod p$. Now remark that for any $y \in \mathcal{O}_F^\times$ and any $x \in F$, we have :

$$u(-x)\bar{u}(-\varpi_F^{-1} y) = u(-x)u(-\varpi_F y^{-1})s_1\alpha_0 \begin{pmatrix} -y & \varpi_F \\ 0 & -y^{-1} \end{pmatrix}\ .$$

This implies that $u(-x)\bar{u}(-\varpi_F^{-1} y)$ belongs to the double coset $I_S s_1 \alpha_0 I_S$ for any pair $(x, y) \in \mathcal{O}_F^2$ with $y = [y_0] + \varpi_F[y_1]$ and $y_0 \neq 0$. If y is such that $y_0 = 0$ (i.e. of the form $[y_1]\varpi_F$ with $y_1 \in k_F^\times$), then we have

$$u(-x)\bar{u}(-\varpi_F^{-1} y) = u(-x)u(-[y_1]^{-1})s_1 \begin{pmatrix} -[y_1] & 1 \\ 0 & -[y_1]^{-1} \end{pmatrix} \in I_S s_1 I_S\ .$$

Putting this two statements together, we obtain that $\sum\limits_{y\in\mathcal{C}_2\backslash\{0\}}\sum\limits_{x\in\mathcal{C}_2} [u(-x)\bar{u}(-\varpi_F^{-1} y), 1]$ has support in $I_S s_1 I_S \sqcup I_S s_1 \alpha_0 I_S$. As k is not in $\{0\ , \frac{q-1}{2}\}$, [2, Proposition 3.30][6] ensures that there is no non-zero element in $\mathbb{H}_k$ with support contained in $I_S s_1 \alpha_0^{\mathbb{Z}} I_S$. Consequently, we have $\sum\limits_{y\in\mathcal{C}_2\backslash\{0\}}\sum\limits_{x\in\mathcal{C}_2} [u(-x)\bar{u}(-\varpi_F^{-1} y), 1] = 0$, hence $(T_1 \circ T_{-1})([I_2, 1]) = 0$, what ends the proof as a similar argument shows that $(T_{-1} \circ T_1)([I_2, 1]) = 0$. □

[6]With the notations used in [2], we have $I_S s_1 \alpha_0 I_S = I_S w_0 \alpha_0 I_S$.

REMARK 5.2. Recall that $\{\omega^r, \omega^{q-1-r}\}$ is an orbit for the action of t by conjugation on the set of smooth characters of Γ_S. Let ψ_k denote the isomorphism of $\overline{\mathbb{F}}_p$-algebras $A_k \to A_{q-1-k}$ induced by this action and $\varphi_k : A_k \to \mathcal{A}$ be the isomorphism given by Theorem 5.1 : the composite map $\varphi_{q-1-k} \circ \psi_k \circ \varphi_k^{-1}$ defines an automorphism of the $\overline{\mathbb{F}}_p$-algebra $\mathcal{A}$ which is not the identity map but the automorphism swapping X and Y.

5.1.2. Structure of the (A_{q-1-k}, A_k)-bimodule B_k

For any integer $n \in \mathbb{Z}$, let $f_{n,k}$ be the element of the (I_S, ω^k)-isotypical component of $\mathrm{ind}_{I_S}^{G_S}(\omega^{q-1-k})$ with support equal to $I_S s_1 \alpha_0^{-n} I_S = I_S \alpha_0^n s_1 I_S$ and value 1 at $\alpha_0^n s_1$, and let $S_{n,k}$ be the element of B_k corresponding to $f_{n,k}$ by compact Frobenius reciprocity. The computation of $\mathrm{Hom}_{\overline{\mathbb{F}}_p[I_S]}(\omega^i, \mathrm{ind}_{I_S}^{G_S}(\omega^j))$ done in the proof of Theorem 3.1 shows that the family $\{S_{n,k}, n \in \mathbb{Z}\}$ is a basis of the $\overline{\mathbb{F}}_p$-vector space B_k. Now recall that $I_S(\alpha_0^n s_1)^{-1} I_S$ admits the following decomposition into disjoint left cosets :

$$I_S(\alpha_0^n s_1)^{-1} I_S = \begin{cases} \displaystyle\bigsqcup_{x \in \mathcal{C}_{2(-n)+1}} u(x)(\alpha_0^n s_1)^{-1} I_S & \text{if } n \geq 0 \text{ ;} \\ \displaystyle\bigsqcup_{x \in \mathcal{C}_{2n-1}} \bar{u}(\varpi_F x)(\alpha_0^n s_1)^{-1} I_S & \text{if } n \geq 1 \text{ .} \end{cases}$$

Using [4, Equation (9)] together with the relation $(\alpha_0^n s_1)^{-1} = -\alpha_0^n s_1$, we obtain the following expression of $S_{n,k}([I_2, 1])$, that completely determines $S_{n,k}$ by $\overline{\mathbb{F}}_p$-linearity and G_S-equivariance :

$$S_{n,k}([I_2,1]) = \begin{cases} \displaystyle\sum_{x \in \mathcal{C}_{2(-n)+1}} [u(x)(\alpha_0^n s_1)^{-1}, 1] & \text{if } n \leq 0 \text{ ;} \\ \displaystyle\sum_{x \in \mathcal{C}_{2n-1}} [\bar{u}(\varpi_F x)(\alpha_0^n s_1)^{-1}, 1] & \text{if } n \geq 1 \text{ ;} \end{cases}$$

$$= \begin{cases} \displaystyle\sum_{x \in \mathcal{C}_{2(-n)+1}} [u(x)\alpha_0^n s_1, (-1)^{q-1-k}] & \text{if } n \leq 0 \text{ ;} \\ \displaystyle\sum_{x \in \mathcal{C}_{2n-1}} [\bar{u}(\varpi_F x)\alpha_0^n s_1, (-1)^{q-1-k}] & \text{if } n \geq 1 \text{ .} \end{cases}$$

Note that B_k is also endowed with a structure of left A_{q-1-k}-module (by pre-composition) and of right A_k-module (by post-composition). The decompositions above will be useful to prove the next lemma, that describes the structure of (A_{q-1-k}, A_k)-bimodule carried by B_k.

LEMMA 5.3. *Let k be equal to r or to $q-1-r$ with $r \neq q-1-r$.*

1. *For any integer $n \geq 0$, the following equalities hold in B_k :*

$$T_{q-1-k} \circ S_{-n,k} = S_{-n,k} \circ S_k = S_{-(n+1),k} \text{ ;} \tag{5.2}$$

$$S_{q-1-k} \circ S_{-n,k} = S_{-n,k} \circ T_k = 0 \text{ .} \tag{5.3}$$

2. *For any integer $n \geq 1$, the following equalities hold in B_k :*

$$T_{q-1-k} \circ S_{n,k} = S_{n,k} \circ S_k = 0 \ ; \tag{5.4}$$

$$S_{q-1-k} \circ S_{n,k} = S_{n,k} \circ T_k = S_{n+1,k} \ . \tag{5.5}$$

Proof. By $\overline{\mathbb{F}}_p$-linearity and G_S-equivariance of the operators involved in the statement of Lemma 5.3, we only have to check that the equalities above are true after evaluation at $[I_2, 1] \in \mathrm{ind}_{I_S}^{G_S}(\omega^k)$. First assume that $n \geq 0$, in which case we have :

$$\begin{aligned}
(T_{q-1-k} \circ S_{-n,k})([I_2,1]) &= T_{q-1-k}\left(\sum_{x \in \mathcal{C}_{2n+1}} [u(x)\alpha_0^{-n} s_1, (-1)^{q-1-k}]\right) \\
&= \sum_{x \in \mathcal{C}_{2n+1}} \sum_{y \in \mathcal{C}_2} [u(x)\alpha_0^{-n} s_1 \bar{u}(-\varpi_F y)\alpha_0, (-1)^{q-1-k}] \\
&= \sum_{x \in \mathcal{C}_{2n+1}} \sum_{y \in \mathcal{C}_2} [u(x)\alpha_0^{-n} u(\varpi_F y) s_1 \alpha_0, (-1)^{q-1-k}] \\
&= \sum_{x \in \mathcal{C}_{2n+1}} \sum_{y \in \mathcal{C}_2} [u(x) u(\varpi_F^{2n+1} y)\alpha_0^{-n} s_1 \alpha_0, (-1)^{q-1-k}] \\
&= \sum_{z \in \mathcal{C}_{2n+3}} [u(z)\alpha_0^{-(n+1)} s_1, (-1)^{q-1-k}] \\
&= S_{-(n+1),k}([I_2,1]) \ .
\end{aligned}$$

By construction, we have $S_{q-1-r} \circ T_{q-1-r} = 0$, hence this first computation already shows that for any integer $n \geq 1$, we have $S_{q-1-k} \circ S_{-n,k} = S_{q-1-k} \circ (T_{q-1-k})^n \circ S_{0,k} = 0$. When $n = 0$, we have

$$\begin{aligned}
(S_{q-1-k} \circ S_{0,k})([I_2,1]) &= S_{q-1-k}\left(\sum_{x \in \mathcal{C}_1} [u(x) s_1, (-1)^{q-1-k}]\right) \\
&= \sum_{x \in \mathcal{C}_1} \sum_{y \in \mathcal{C}_2} [u(x) s_1 u(-y)\alpha_0^{-1}, (-1)^{q-1-k}] \\
&= \sum_{x \in \mathcal{C}_1} \sum_{y \in \mathcal{C}_2} [u(x) s_1 \alpha_0^{-1} u(-\varpi_F^{-2} y), (-1)^{q-1-k}] \ ,
\end{aligned}$$

and the argument used in the proof of Theorem 5.1 to show that $T_1 \circ T_{-1} = 0$ allows us to conclude here[7] that $S_{q-1-k} \circ S_{0,k} = 0$.

[7] For the sceptical (or lazy) reader, all computations are explicitely written in [1, page 184].

Now assume that $n \geq 1$, in which case we have :

$$\begin{aligned}
(S_{q-1-k} \circ S_{n,k})([I_2, 1]) &= S_{q-1-k}\left(\sum_{x\in\mathcal{C}_{2n-1}} [\bar{u}(\varpi_F x)\alpha_0^n s_1, (-1)^{q-1-k}]\right) \\
&= \sum_{x\in\mathcal{C}_{2n-1}} \sum_{y\in\mathcal{C}_2} [\bar{u}(\varpi_F x)\alpha_0^n s_1 u(-y)\alpha_0^{-1}, (-1)^{q-1-k}] \\
&= \sum_{x\in\mathcal{C}_{2n-1}} \sum_{y\in\mathcal{C}_2} [\bar{u}(\varpi_F x)\alpha_0^n \bar{u}(y) s_1\alpha_0^{-1}, (-1)^{q-1-k}] \\
&= \sum_{x\in\mathcal{C}_{2n-1}} \sum_{y\in\mathcal{C}_2} [\bar{u}(\varpi_F x)\bar{u}(\varpi_F^{2n} y)\alpha_0^n s_1\alpha_0^{-1}, (-1)^{q-1-k}] \\
&= \sum_{x\in\mathcal{C}_{2n-1}} \sum_{y\in\mathcal{C}_2} [\bar{u}(\varpi_F(x + \varpi_F^{2n-1} y))\alpha_0^{n+1} s_1, (-1)^{q-1-k}] \\
&= \sum_{z\in\mathcal{C}_{2n+1}} [\bar{u}(\varpi_F z)\alpha_0^{n+1} s_1, (-1)^{q-1-k}] \\
&= S_{n+1,k}([I_2, 1]) \ .
\end{aligned}$$

For any integer $n \geq 2$, this proves that $T_{q-1-k} \circ S_{n,k} = T_{q-1-k} \circ (S_{q-1-k})^{n-1} \circ S_{1,k} = 0$. When $n = 1$, we have

$$\begin{aligned}
(T_{q-1-k} \circ S_{1,k})([I_2, 1]) &= T_{q-1-k}\left(\sum_{x\in\mathcal{C}_1} [\bar{u}(\varpi_F x)\alpha_0 s_1, (-1)^{q-1-k}]\right) \\
&= \sum_{x\in\mathcal{C}_1} \sum_{y\in\mathcal{C}_2} [\bar{u}(\varpi_F x)\alpha_0 s_1 \bar{u}(-\varpi_F y)\alpha_0, (-1)^{q-1-k}] \\
&= \sum_{x\in\mathcal{C}_1} \sum_{y\in\mathcal{C}_2} [\bar{u}(\varpi_F x) u(\varpi_F^{-1} y) s_1, (-1)^{q-1-k}] \ .
\end{aligned}$$

As in the computation of $S_{q-1-k} \circ S_{0,k}$, we obtain[8] that $T_{q-1-k} \circ S_{1,k} = 0$. The same kind of calculations[9] for T_k and S_k gives the other relations, what finishes the proof. □

5.1.3. Description of the elements of $B_{q-1-k} \circ B_k$ as elements of A_k

To obtain a full understanding of the $\overline{\mathbb{F}}_p$-algebra $\mathcal{H}_S(r)$, we now have to express the element $S_{n,q-1-k} \circ S_{m,k} \in A_k$ as a polynomial in T_k and S_k for any integers $n, m \in \mathbb{Z}$. By Lemma 5.3, it is enough to compute $S_{1,q-1-k} \circ S_{1,k}$, $S_{1,q-1-k} \circ S_{1,k} \circ S_{0,q-1-k}$, $S_{0,q-1-k} \circ S_{1,k}$ and $S_{0,q-1-k} \circ S_{0,k}$: this forms the content of the next statement.

[8] The reader can this time refer to [1, pages 185 and 186] to get all the details.
[9] Left as an exercise to the reader, but also extensively written in [1, pages 186 à 188].

LEMMA 5.4. *The following equalities hold in A_k :*

$$\begin{cases} S_{1,q-1-k} \circ S_{0,k} = (-1)^{q-1-k} S_k \ ; \\ S_{0,q-1-k} \circ S_{1,k} = (-1)^{q-1-k} T_k \ ; \\ S_{1,q-1-k} \circ S_{1,k} = S_{0,q-1-k} \circ S_{0,k} = 0 \ . \end{cases}$$

Proof. As usual, it is enough to check that these equalities are true after evaluation at $[I_2, 1] \in \mathrm{ind}_{I_S}^{G_S}(\omega^k)$. First note that we have

$$\begin{aligned}(S_{1,q-1-k} \circ S_{0,k})([I_2,1]) &= S_{1,q-1-k}\left(\sum_{x\in\mathcal{C}_1}[u(x)s_1,(-1)^{q-1-k}]\right)\\ &= \sum_{x\in\mathcal{C}_1}\sum_{y\in\mathcal{C}_1}[u(x)s_1\bar{u}(\varpi_F y)\alpha_0 s_1,(-1)^{q-1}]\\ &= \sum_{x\in\mathcal{C}_1}\sum_{y\in\mathcal{C}_1}[u(x)u(-\varpi_F y)\alpha_0^{-1},(-1)^{q-1-k}]\\ &= \sum_{z\in\mathcal{C}_2}[u(z)\alpha_0^{-1},(-1)^{q-1-k}]\\ &= (-1)^{q-1-k}S_k([I_2,1]) \ ,\end{aligned}$$

so the first equality of Lemma 5.4 holds. Similarly, the second equality is true as we have

$$\begin{aligned}(S_{0,q-1-r} \circ S_{1,k})([I_2,1]) &= S_{0,q-1-k}\left(\sum_{x\in\mathcal{C}_1}[\bar{u}(\varpi_F x)\alpha_0 s_1,(-1)^{q-1-k}]\right)\\ &= \sum_{x\in\mathcal{C}_1}\sum_{y\in\mathcal{C}_1}[\bar{u}(\varpi_F x)\alpha_0 s_1 u(y)s_1,(-1)^{q-1}]\\ &= \sum_{x\in\mathcal{C}_1}\sum_{y\in\mathcal{C}_1}[\bar{u}(\varpi_F(x-\varpi_F y))\alpha_0,(-1)^{q-1-k}]\\ &= \sum_{z\in\mathcal{C}_2}[\bar{u}(-\varpi_F z)\alpha_0,(-1)^{q-1-k}]\\ &= (-1)^{q-1-k}T_k([I_2,1]) \ .\end{aligned}$$

We now prove that $S_{1,q-1-k} \circ S_{1,k}$ is equal to 0. First note that we have

$$\begin{aligned}(S_{1,q-1-k} \circ S_{1,k})([I_2,1]) &= S_{1,q-1-k}\left(\sum_{x\in\mathcal{C}_1}[\bar{u}(\varpi_F x)\alpha_0 s_1,(-1)^{q-1-k}]\right)\\ &= \sum_{x\in\mathcal{C}_1}\sum_{y\in\mathcal{C}_1}[\bar{u}(\varpi_F x)\alpha_0 s_1\bar{u}(\varpi_F y)\alpha_0 s_1,(-1)^{q-1}] \ ,\end{aligned}$$

hence

$$(S_{1,q-1-k}\circ S_{1,k})([I_2,1])=\sum_{x\in\mathcal{C}_1}\sum_{y\in\mathcal{C}_1}[\bar{u}(\varpi_F x)u(-\varpi_F^{-1}y),(-1)^{q-1-k}]\ .$$

This shows that $(S_{1,q-1-k}\circ S_{1,k})([I_2,1])$ is actually equal to
(5.6)

$$\sum_{x\in\mathcal{C}_1}[\bar{u}(\varpi_F x),(-1)^{q-1-k}]+\sum_{x\in\mathcal{C}_1}\sum_{y\in\mathcal{C}_1\setminus\{0\}}[\bar{u}(\varpi_F x)\bar{u}(-\varpi_F y^{-1})\begin{pmatrix}\varpi_F^{-1}y & 1\\ 0 & \varpi_F y^{-1}\end{pmatrix}s_1,(-1)^{q-1}]\ .$$

As in the proof of Theorem 5.1, the first sum in (5.6) vanishes because it is equal to $\mathrm{Card}(\mathcal{C}_1)[I_2,(-1)^{q-1-k}]$ with $\mathrm{Card}(\mathcal{C}_1)\equiv 0 \bmod p$. The remaining part of (5.6) defines an element of A_k having support in $I_S s_1\alpha_0^{\mathbb{Z}}I_S$, hence is zero by [2, Proposition 3.30]. This proves that $S_{1,q-1-k}\circ S_{1,k}$ is equal to 0 and finishes the proof as the vanishing of $S_{0,q-1-k}\circ S_{0,k}$ can be obtain by similar computations (left as an exercise to the reader and written in full detail in [1, page 191]). ☐

Corollary 5.5. *Let $m\in\mathbb{Z}$ and $n\in\mathbb{N}$. We have the following identities in A_k.*

1. *For any integer $n\geq 0$, we have*

$$S_{-n,q-1-k}\circ S_{m,k}=\begin{cases}(-1)^{q-1-k}T_k^{n+m} & \text{if } m\geq 1\ ;\\ 0 & \text{if } m\leq 0\ .\end{cases}$$

2. *For any integer $n\geq 1$, we have*

$$S_{n,q-1-k}\circ S_{m,k}=\begin{cases}(-1)^{q-1-k}S_k^{n-m} & \text{if } m\leq 0\ ;\\ 0 & \text{if } m\geq 1\ .\end{cases}$$

Proof. This directly comes from the relations proved in Lemmas 5.3 and 5.4. For instance, if n and m are two nonnegative integers with $m\geq 1$, we have

$$\begin{aligned}S_{-n,q-1-k}\circ S_{m,k}=T_k^n\circ S_{0,q-1-k}\circ S_{1,q-1-k}\circ T_k^{m-1}&=T_k^n\circ((-1)^{q-1-k}T_k)\circ T_k^{m-1}\\&=(-1)^{q-1-k}T_k^{m+n}\ ,\end{aligned}$$

what proves the first case of assertion (1) of the corollary. ☐

5.2. Classification of simple $\mathcal{H}_S(r)$-modules

We will classify all simple right $\mathcal{H}_S(r)$-modules as follows : we will first (re)prove that any such module is of finite dimension at most 2 over $\overline{\mathbb{F}}_p$, then we will determine all such modules of fixed dimension over $\overline{\mathbb{F}}_p$. To do this, we introduce the idempotent elements e_1 and e_2 of $\mathcal{H}_S(r)$ respectively given by the identity maps of A_r and of A_{q-1-r} via the diagonal embedding of $A_r\oplus A_{q-1-r}$ into $\mathcal{H}_S(r)$. They satisfy the usual relations, namely $e_1+e_2=1$ and $e_ie_j=\delta_{ij}e_i$ for any $(i,j)\in\{1,2\}^2$, where δ_{ij} is the Kronecker symbol (equal to 1 if $i=j$ and to 0 otherwise).

Theorem 5.6. *Let $r\in\{1,\dots,\left[\frac{q-1}{2}\right]\}$ be such that $r\neq q-1-r$. Any simple right $\mathcal{H}_S(r)$-module is of finite dimension at most 2 over $\overline{\mathbb{F}}_p$.*

Proof. Let M be a simple right $\mathcal{H}_S(r)$-module. For any $i \in \{1,2\}$, denote by $M_i := M|e_i$ the image of M under the action of the idempotent element e_i : the vector space defining M is then decomposed as $M_1 \oplus M_2$ and the action of $\mathcal{H}_S(r)$ on M naturally endows M_1 (respectively M_2) with a structure of A_r-submodule (resp. of A_{q-1-r}-submodule) of M.

Assume that M_1 is non-zero and pick a non-zero A_r-submodule V_1 of M_1. The relations between the elements of $\mathcal{H}_S(r)$ we proved in the previous section imply that $V := V_1 + (V_1|B_r)$ is stable under the action of $\mathcal{H}_S(r)$ on M and is hence equal to M since M is a simple $\mathcal{H}_S(r)$-module. Applying e_1, we obtain that $V_1 = M_1$, what proves that M_1 is a simple A_r-module. As Theorem 5.1 asserts that A_r is a commutative $\overline{\mathbb{F}}_p$-algebra, we conclude that M_1 is either zero or one-dimensional over $\overline{\mathbb{F}}_p$. A similar argument proves that the A_{q-1-r}-module M_2 is either zero or one-dimensional over $\overline{\mathbb{F}}_p$. As M is non-zero by simplicity, this finally shows that M is of finite dimension equal to 1 or 2 over $\overline{\mathbb{F}}_p$. □

The proof of Theorem 5.6 suggests that the classification of all simple right $\mathcal{H}_S(r)$-modules heavily relies on the description of the one-dimensional $\mathcal{A}$-modules over $\overline{\mathbb{F}}_p$. They are naturally dispatched into two families both parametrized by $\lambda \in \overline{\mathbb{F}}_p$: the first one consists in the characters $\mu_1(\lambda)$ mapping X to λ and Y to 0, while the second one is made of characters $\mu_2(\lambda)$ mapping X to 0 and Y to λ. Note that the unique common element to these families is the character $\mu_1(0) = \mu_2(0)$: it will be denoted $\mu(0)$.

We will also use the following notations, where k denotes either r or $q-1-r$: for any index $i \in \{1,2\}$ and any parameter $\lambda \in \overline{\mathbb{F}}_p^{\times}$, we let $\mu_{i,k}(\lambda)$ be the character of A_k defined by $\mu_i(\lambda)$ via the isomorphism $A_k \simeq \mathcal{A}$ given by Theorem 5.1. For instance, $\mu_{1,r}(\lambda)$ is the character of A_r that maps T_r to λ and S_r to 0, while $\mu_k(0)$ is the character of A_k that maps both T_k and S_k to 0.

Let M be a simple right $\mathcal{H}_S(r)$-module. For any $i \in \{1,2\}$, we let $M_i := M|e_i$ be the image of M under the action of the idempotent element e_i. As we already noticed in the proof of Theorem 5.6, the vector space M splits as $M_1 \oplus M_2$ and the action of $\mathcal{H}_S(r)$ on M naturally endows M_1 (respectively M_2) with a structure of A_r-submodule (resp. of A_{q-1-r}-submodule) of M. The end of the proof of Theorem 5.6 moreover shows that each of M_1 and M_2 is either reduced to $\{0\}$ or one-dimensional over $\overline{\mathbb{F}}_p$.

5.2.1. Classification of one-dimensional simple right $\mathcal{H}_S(r)$-modules

Assume that M is one-dimensional over $\overline{\mathbb{F}}_p$. The observations we just made above imply the existence of a parameter λ and of an index $i \in \{1,2\}$ such that we have either $(M_1, M_2) = (\mu_{i,r}(\lambda), 0)$ or $(M_1, M_2) = (0, \mu_{i,q-1-r}(\lambda))$. As M_1 and M_2 play symmetric roles, we can assume for instance that we have $M_2 = \{0\}$ and $M_1 = \mu_{i,r}(\lambda)$, and pick a non-zero vector m in M_1. Seen as elements of $\mathcal{H}_S(r)$, the operators T_r and S_r respectively act on m by the scalars $\delta_{1i}\lambda$ and $\delta_{2i}\lambda$:

$$\begin{cases} m|T_r = \delta_{1i}\lambda m \ , \\ m|S_r = \delta_{2i}\lambda m \ . \end{cases}$$

Now recall that Lemma 5.4 shows in particular that $T_r = (-1)^{q-1-r}(S_{0,q-1-r} \circ S_{1,r})$ and that $S_r = (-1)^{q-1-r}(S_{1,q-1-r} \circ S_{0,r})$. As the right action of B_r on M maps any element of M_1 into $M_2 = \{0\}$, we necessarily have

$$\begin{cases} m|T_r = (-1)^{q-1-r} m|(S_{0,q-1-r} \circ S_{1,r}) = (-1)^{q-1-r}(m|S_{1,r})|S_{0,q-1-r} = 0 \ , \\ m|S_r = (-1)^{q-1-r} m|(S_{1,q-1-r} \circ S_{0,r}) = (-1)^{q-1-r}(m|S_{0,r})|S_{1,q-1-r} = 0 \ . \end{cases}$$

The comparison of the two expressions we have for $m|T_r$ and for $m|S_r$ shows that λ must be equal to 0, as either δ_{1i} or δ_{2i} is non-zero. The relations given by Lemma 5.3 and Corollary 5.5 conversely assure that $M_1^r(0) := \mu_r(0) \oplus \{0\}$ and $M_2^r(0) := \{0\} \oplus \mu_{q-1-r}(0)$ are right $\mathcal{H}_S(r)$-modules of dimension 1 over $\overline{\mathbb{F}}_p$. The following statement summarizes what we just proved.

Theorem 5.7. *Let $r \in \{1, \dots, \left[\frac{q-1}{2}\right]\}$ be such that $r \neq q-1-r$. There are exactly two simple right $\mathcal{H}_S(r)$-modules of dimension 1 over $\overline{\mathbb{F}}_p$, namely*

$$M_1^r(0) := \mu_r(0) \oplus \{0\} \quad \textit{and} \quad M_2^r(0) := \{0\} \oplus \mu_{q-1-r}(0) \ .$$

5.2.2. Classification of two-dimensional simple right $\mathcal{H}_S(r)$-modules

Now assume that M is of dimension 2 over $\overline{\mathbb{F}}_p$, what means that M_1 and M_2 are both non-zero and that there exist two indices $i, j \in \{1, 2\}$ and two parameters $\lambda_i, \lambda_j \in \overline{\mathbb{F}}_p$ satisfying $M_1 = \mu_{i,r}(\lambda_i)$ and $M_2 = \mu_{j,q-1-r}(\lambda_j)$. By simplicity of M, Theorem 5.7 forces λ_i and λ_j to be both non-zero.

Fix a non-zero vector $m_1 \in M_1$, so that $m_1|S_{0,r}$ and $m_1|S_{1,r}$ both belong to M_2. They cannot be simultaneously equal to 0 : otherwise, the computations led in the proof of Theorem 5.7 would imply here that either λ_i or λ_j is zero, what is not. Assume for instance that $m_1|S_{0,r}$ is non-zero : as it lies in M_2, Lemma 5.3 assures that we have

$$\delta_{2j}\lambda_j(m_1|S_{0,r}) = (m_1|S_{0,r})|S_{q-1-r} = m_1|(S_{q-1-r} \circ S_{0,r}) = 0 \ .$$

As λ_j is non-zero, this chain of equalities proves that $\delta_{2j} = 0$, i.e. that $j = 1$. The same argument with T_{q-1-r} replacing S_{q-1-r} proves that

$$\delta_{1j}\lambda_j(m_1|S_{0,r}) = m_1|(T_{q-1-r} \circ S_{0,r}) = m_1|S_{-1,r} = (m_1|S_r)|S_{0,r} = \delta_{i2}\lambda_i(m_1|S_{0,r}) \ ,$$

what implies that $\delta_{1j}\lambda_j = \delta_{i2}\lambda_i$. Knowing that $\delta_{1j}\lambda_j = \lambda_j$ is non-zero, this proves that we necessarily have $i = 2$ and $\lambda_j = \lambda_i$. Consequently, the non-vanishing of $m_1|S_{0,r}$ leads to the existence of a non-zero parameter $\lambda \in \overline{\mathbb{F}}_p^\times$ satisfying $M_1 = \mu_{2,r}(\lambda)$ and $M_2 = \mu_{1,q-1-r}(\lambda)$. One similarly checks that the non-vanishing of $m_1|S_{1,r}$ implies the existence of a non-zero parameter $\lambda \in \overline{\mathbb{F}}_p^\times$ satisfying $M_1 = \mu_{1,r}(\lambda)$ and $M_2 = \mu_{2,q-1-r}(\lambda)$.

Conversely, let $\lambda \in \overline{\mathbb{F}}_p^\times$ be a non-zero parameter and set the following relations on the vector spaces $M_{12}^r(\lambda) := \overline{\mathbb{F}}_p m_1 \oplus \overline{\mathbb{F}}_p m_2$ and $M_{21}^r(\lambda) := \overline{\mathbb{F}}_p n_1 \oplus \overline{\mathbb{F}}_p n_2$:

$$\begin{cases} m_1|S_{0,r} = 0 \ ; \\ m_1|S_{1,r} = m_2 \ ; \\ m_1|T_r = \lambda m_1 \ ; \\ m_1|S_r = 0 \ ; \\ m_1|T_{q-1-r} = 0 \ ; \\ m_1|S_{q-1-r} = 0 \ ; \\ m_1|S_{0,q-1-r} = 0 \ ; \\ m_1|S_{1,q-1-r} = 0 \ ; \end{cases} \quad \text{and} \quad \begin{cases} n_1|S_{0,r} = n_2 \ ; \\ n_1|S_{1,r} = 0 \ ; \\ n_1|T_r = 0 \ ; \\ n_1|S_r = \lambda n_1 \ ; \\ n_1|T_{q-1-r} = 0 \ ; \\ n_1|S_{q-1-r} = 0 \ ; \\ n_1|S_{0,q-1-r} = 0 \ ; \\ n_1|S_{1,q-1-r} = 0 \ . \end{cases}$$

Recall that the right action of $\mathcal{H}_S(r)$ we consider here satisfies the following cancellation relations[10] :

$$\begin{cases} A_r B_{q-1-r} = B_r A_r = B_{q-1-r} A_{q-1-r} = A_{q-1-r} B_r = 0 \ ; \\ A_r A_{q-1-r} = B_r B_r = B_{q-1-r} B_{q-1-r} = A_{q-1-r} A_r = 0 \ . \end{cases}$$

Together with the relations given by Lemma 5.3 and Corollary 5.5, these relations assure that $M_{12}^r(\lambda)$ and $M_{21}^r(\lambda)$ are both stable under the right action of $\mathcal{H}_S(r)$. One can for instance check that the following relations hold in $M_{12}^r(\lambda)$:

$$\begin{cases} m_2|T_{q-1-r} = m_1|(T_{q-1-r} \circ S_{1,r}) = 0 \ ; \\ m_2|S_{q-1-r} = m_1|(S_{q-1-r} \circ S_{1,r}) = m_1|(S_{1,r} \circ T_r) = \lambda m_2 \ ; \\ m_2|S_{1,q-1-r} = m_1|(S_{1,q-1-r} \circ S_{1,r}) = 0 \ ; \\ m_2|S_{0,q-1-r} = m_1|(S_{0,q-1-r} \circ S_{1,r}) = (-1)^{q-1-r} m_1|T_r = (-1)^{q-1-r} \lambda m_1 \ ; \\ m_2|T_r = 0 \ ; \\ m_2|S_r = 0 \ ; \\ m_2|S_{1,r} = 0 \ ; \\ m_2|S_{0,r} = 0 \ . \end{cases}$$

These computations moreover prove that the vector spaces $M_{12}^r(\lambda)$ and $M_{21}^r(\lambda)$ respectively split as $\mu_{1,r}(\lambda) \oplus \mu_{2,q-1-r}(\lambda)$ and $\mu_{2,r}(\lambda) \oplus \mu_{1,q-1-r}(\lambda)$. By Theorem 5.7, this shows that $M_{12}^r(\lambda)$ and $M_{21}^r(\lambda)$ are simple right $\mathcal{H}_S(r)$-modules. Moreover, a direct computation (by contradiction) shows that these modules are pairwise non-isomorphic. The next statement summarizes what we just proved.

THEOREM 5.8. *Let $r \in \{1, \ldots, [\frac{q-1}{2}]\}$ be such that $r \neq q-1-r$ and let M be a simple right $\mathcal{H}_S(r)$-module of dimension 2 over $\overline{\mathbb{F}}_p$. There exists a unique parameter $\lambda \in \overline{\mathbb{F}}_p^{\times}$ such that M is isomorphic to one, and only one, of the two following simple right $\mathcal{H}_S(r)$-modules :*

$$M_{12}^r(\lambda) := \mu_{1,r}(\lambda) \oplus \mu_{2,q-1-r}(\lambda) \quad \text{or} \quad M_{21}^r(\lambda) := \mu_{2,r}(\lambda) \oplus \mu_{1,q-1-r}(\lambda) \ .$$

Gathering Theorems 5.6, 5.7 and 5.8 finally leads to the following classification result for simple right $\mathcal{H}_S(r)$-modules.

COROLLARY 5.9. *Let $r \in \{1, \ldots, [\frac{q-1}{2}]\}$ be such that $r \neq q-1-r$. Any simple right $\mathcal{H}_S(r)$-module is isomorphic to one, and only one, element of the following list :*

- *the character $M_1^r(0)$;*
- *the character $M_2^r(0)$;*
- *the two-dimensional module $M_{12}^r(\lambda)$ for a unique parameter $\lambda \in \overline{\mathbb{F}}_p^{\times}$;*
- *the two-dimensional module $M_{21}^r(\lambda)$ for a unique parameter $\lambda \in \overline{\mathbb{F}}_p^{\times}$.*

[10] If one wants to write these relations using the composition law, one has to reverse them (as we did any time we needed to use the composition law) since we consider a right action, which is the one that naturally appears when one works with modules coming from spaces of invariant vectors. This notation is extremely convenient in the regular case, as it allows us to compute the action of $\mathcal{H}_S(r)$ by multiplying matrices in the usual way.

Remark 5.10. For the sake of completeness, let us mention here that it is pointless to define $M_{12}^r(0)$ or $M_{21}^r(0)$ as these two objects would just be the direct sum of the characters $M_1^r(0)$ and $M_2^r(0)$.

6. The exceptional Iwahori-Hecke algebra and its simple modules

6.1. Introduction of the exceptional Iwahori-Hecke algebra $\mathcal{H}_S^\star$

When p is odd, the decomposition of $\mathbb{H}_S^1$ given by Theorem 3.1 makes appear an algebra that does fit neither into the regular case nor into the Iwahori case, namely $\mathrm{End}_{\overline{\mathbb{F}}_p[G_S]}(\mathrm{ind}_{I_S}^{G_S}(\omega^{\frac{q-1}{2}}))$. This phenomenon has no analogue in the GL_2 case as it reflects the existence of a smooth quadratic character of I_S that cannot be extended to a smooth character of G_S; this cannot happen in the GL_2 case as such a character would factorize through the determinant map, hence extend to $GL_2(F)$ [21, Section 2.1.1].

To understand the structure of this new algebra and of its simple right modules, we mimic what we did in the Iwahori case by defining the *exceptional Iwahori-Hecke algebra* $\mathcal{H}_S^\star$ as the $\overline{\mathbb{F}}_p$-algebra generated by the family $(\mathcal{T}_w^\star)_{w\in W_S}$ satisfying the following braid and quadratic relations.

- Braid relations : if $w, w' \in W_S$ satisfy $\ell(ww') = \ell(w) + \ell(w')$, then $\mathcal{T}_{ww'}^\star = \mathcal{T}_w^\star \mathcal{T}_{w'}^\star$.
- Quadratic relations : for any $i \in \{0,1\}$, $\left(\mathcal{T}_{s_i}^\star\right)^2 = 0$.

As in the Iwahori case, one can check that $\mathcal{H}_S^\star$ is a free $\overline{\mathbb{F}}_p$-module having $(\mathcal{T}_w^\star)_{w\in W_S}$ as a basis. Moreover, the proofs of Theorems 4.2 and 4.3 can be directly transposed to this setting, as it is for instance written in [1, Théorème 6.3.42], to give the following result.

Theorem 6.1. *Assume that p is odd.*

1. *The $\overline{\mathbb{F}}_p$-algebra $\mathcal{H}_S^\star$ is generated by the operators $\mathcal{T}_0^\star := \mathcal{T}_{s_0}^\star$ and $\mathcal{T}_1^\star := \mathcal{T}_{s_1}^\star$.*
2. *The center of $\mathcal{H}_S^\star$ is equal to the polynomial $\overline{\mathbb{F}}_p$-subalgebra $\overline{\mathbb{F}}_p[(\mathcal{T}_0^\star - \mathcal{T}_1^\star)^2]$.*

The introduction of $\mathcal{H}_S^\star$ is motivated by the following statement, whose proof is the exact analogue for $r = \frac{q-1}{2}$ of the argument leading to Theorem 3.3. Note here that compact Frobenius reciprocity induces an isomorphism from $\mathrm{End}_{\overline{\mathbb{F}}_p[G_S]}(\mathrm{ind}_{I_S}^{G_S}(\omega^{\frac{q-1}{2}}))$ to the $(I_S, \omega^{\frac{q-1}{2}})$-isotypical component of $\mathrm{ind}_{I_S}^{G_S}(\omega^{\frac{q-1}{2}})$.

Theorem 6.2. *Let f_0, f_1 be the elements of $\mathrm{End}_{\overline{\mathbb{F}}_p[G_S]}(\mathrm{ind}_{I_S}^{G_S}(\omega^{\frac{q-1}{2}}))$ respectively corresponding by compact Frobenius reciprocity to the elements ψ_0, ψ_1 of the $(I_S, \omega^{\frac{q-1}{2}})$-isotypical component of $\mathrm{ind}_{I_S}^{G_S}(\omega^{\frac{q-1}{2}})$ defined as follows : for $i \in \{0,1\}$, ψ_i has support equal to $I_S s_i I_S$ and value 1 at s_i. There exists a unique homomorphism of $\overline{\mathbb{F}}_p$-algebras*

$$\mathrm{End}_{\overline{\mathbb{F}}_p[G_S]}(\mathrm{ind}_{I_S}^{G_S}(\omega^{\frac{q-1}{2}})) \to \mathcal{H}_S^\star$$

that maps ψ_0 to $\mathcal{T}_0^\star$ and ψ_1 to $\mathcal{T}_1^\star$, and this homomorphism is actually an isomorphism of $\overline{\mathbb{F}}_p$-algebras.

This motivates the study of simple right $\mathcal{H}_S^\star$-modules (that are again of finite dimension over $\overline{\mathbb{F}}_p$ via Remark 2.7) to complete the classification of simple right $\mathbb{H}_S^1$-modules.

6.2. Classification of simple $\mathcal{H}_S^\star$-modules

The first assertion of Theorem 6.1 implies that any one-dimensional $\mathcal{H}_S^\star$-module is completely determined by its values on $\mathcal{T}_0^\star$ and $\mathcal{T}_1^\star$, which must be equal to zero because of the quadratic relations satisfied by these operators. This proves the following result.

Lemma 6.3. *There exists a unique one-dimensional right $\mathcal{H}_S^\star$-module, namely the one mapping both $\mathcal{T}_0^\star$ and $\mathcal{T}_1^\star$ to 0. We denote it by $M_1^\star(0)$.*

To describe the simple right $\mathcal{H}_S^\star$-modules of higher dimension over $\overline{\mathbb{F}}_p$, we introduce an analogue of the standard $\mathcal{H}_S$-modules for the exceptional case.

Definition 6.4. For any $\lambda \in \overline{\mathbb{F}}_p$, we define the *standard $\mathcal{H}_S^\star$-module $M_2^\star(\lambda)$ with parameter* λ as the two-dimensional $\mathcal{H}_S^\star$-module $\overline{\mathbb{F}}_p x \oplus \overline{\mathbb{F}}_p y$ endowed with the actions of $\mathcal{T}_0^\star$ and $\mathcal{T}_1^\star$ respectively given by the following matrices in the basis $\{x, y\}$:

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix} .$$

Any basis $\{x, y\}$ as above is called an *adapted basis* for the module $M_2^\star(\lambda)$.

Remark 6.5. As in the Iwahori case (Remark 4.4), a direct computation shows that the central element $-(\mathcal{T}_0^\star - \mathcal{T}_1^\star)^2$ acts on $M_2^\star(\lambda)$ by the scalar λ, hence two standard $\mathcal{H}_S^\star$-modules with distinct parameters cannot be isomorphic.

The next result establishes irreducibility properties of standard $\mathcal{H}_S^\star$-modules.

Theorem 6.6. *Let $\lambda \in \overline{\mathbb{F}}_p$.*

1. *The standard $\mathcal{H}_S^\star$-module $M_2^\star(\lambda)$ is irreducible if, and only if, λ is non-zero.*
2. *The standard $\mathcal{H}_S^\star$-module $M_2^\star(0)$ is indecomposable of length 2, and is hence a non-trivial extension of the $\mathcal{H}_S^\star$-character $M_1^\star(0)$ by itself.*

Proof. If the $\mathcal{H}_S^\star$-module $M_2^\star(\lambda)$ is reducible, Lemma 6.3 implies that $M_2^\star(\lambda)$ contains the $\mathcal{H}_S^\star$-character $M_1^\star(0)$. Let $\{x, y\}$ be an adapted basis for $M_2^\star(\lambda)$ and pick a non-zero vector $v = ax + by$ that generates an $\mathcal{H}_S^\star$-submodule isomorphic to $M_1^\star(0)$. As $\mathcal{T}_0^\star$ and $\mathcal{T}_1^\star$ both act on v by 0, we deduce from Definition 6.4 that $a = 0$ (as $v|\mathcal{T}_0^\star = 0$), hence $\lambda b = 0$ (as $v|\mathcal{T}_1^\star = 0$). As v is non-zero, b cannot be null and λ is hence equal to zero. Conversely, if $\{x, y\}$ is an adapted basis for $M_2^\star(0)$, the $\mathcal{H}_S^\star$-submodule generated by y is isomorphic to $M_1^\star(0)$, what ends the proof of the first statement of Theorem 6.6. The $\mathcal{H}_S^\star$-module $M_2^\star(0)$ is nevertheless indecomposable as it is generated by the first vector of any adapted basis, what ends the proof of Theorem 6.6 thanks to Lemma 6.3. $\square$

We close this section with an analogue of Theorem 4.7 in the exceptional case.

Theorem 6.7. *Up to isomorphism, any (finite-dimensional) simple right $\mathcal{H}_S^\star$-module is either the $\mathcal{H}_S^\star$-character $M_1^\star(0)$, or a standard $\mathcal{H}_S^\star$-module with non-zero parameter. Moreover, there is no non-trivial isomorphism between two such modules.*

Proof. Remark 6.5 assures that two standard $\mathcal{H}_S^\star$-modules with distinct parameters cannot be isomorphic, and a standard $\mathcal{H}_S^\star$-module cannot be isomorphic to $M_1^\star(0)$ as they have different dimensions over $\overline{\mathbb{F}}_p$. By Theorem 6.6 and Lemma 6.3, we are hence reduced to prove that any simple right $\mathcal{H}_S^\star$-module which is not a character contains some standard $\mathcal{H}_S^\star$-module with a non-zero parameter.

Let M be a simple right $\mathcal{H}_S^\star$-module which is not a character and let $\lambda \in \overline{\mathbb{F}}_p$ be the scalar that defines the action of the central element $-(\mathcal{T}_0^\star - \mathcal{T}_1^\star)^2 \in \mathcal{H}_S^\star$ on the finite-dimensional $\overline{\mathbb{F}}_p$-vector space M. The action of $\mathcal{T}_1^\star$ on M is not given by a homothety : otherwise, any eigenvector of $\mathcal{T}_0^\star$ would generate an $\mathcal{H}_S^\star$-submodule of M of dimension 1, what contradicts the simplicity of M as M is not one-dimensional. Fix a non-zero vector $v \in \ker \mathcal{T}_1^\star$, what makes sense as $(\mathcal{T}_1^\star)^2 = 0$: the simplicity of M then implies that the family $\{v, v|\mathcal{T}_0^\star\}$ is linearly independant over $\overline{\mathbb{F}}_p$. As in the proof of Theorem 4.6, a direct computation shows that $\{v, v|\mathcal{T}_0^\star\}$ generates a two-dimensional $\mathcal{H}_S^\star$-module and that the action of $\mathcal{T}_0^\star$ and $\mathcal{T}_1^\star$ in this basis are respectively given by the following matrices :

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & \lambda \\ 0 & 0 \end{pmatrix} .$$

This means that $\{v, v|\mathcal{T}_0^\star\}$ generates an $\mathcal{H}_S^\star$-submodule of M isomorphic to $M_2^\star(\lambda)$, what shows that $M \simeq M_2^\star(\lambda)$ by simplicity of M. Theorem 6.6 then implies that λ is non-zero, what finishes the proof of Theorem 6.7. □

7. Pro-p-Iwahori-Hecke modules and mod p representations of $SL_2(F)$

We start with a reformulation of Theorem 3.1 based on Theorems 3.3 and 6.2.

Corollary 7.1. *The $\overline{\mathbb{F}}_p$-algebra $\mathbb{H}_S^1$ decomposes as follows :*

1. *when p is odd, we have*

$$\mathbb{H}_S^1 \simeq \mathcal{H}_S \oplus \mathcal{H}_S^\star \bigoplus_{0<r<\frac{q-1}{2}} \mathcal{H}_S(r) \ ; \tag{7.1}$$

2. *when $p = 2$, we have*

$$\mathbb{H}_S^1 \simeq \mathcal{H}_S \oplus \bigoplus_{0<r<\frac{q-1}{2}} \mathcal{H}_S(r) \ . \tag{7.2}$$

Any right $\mathbb{H}_S^1$-module then defines by restriction a right $\mathcal{H}_S(r)$-module for any parameter $r \in \{0, \dots \left[\frac{q-1}{2}\right]\}$. Starting from a simple right $\mathbb{H}_S^1$-module, there is exactly one value of the parameter r for which the $\mathcal{H}_S(r)$-module we get is non-zero. This leads to the following definition.

Definition 7.2. Let $r \in \{0, \dots \left[\frac{q-1}{2}\right]\}$ be a parameter.

1. A simple right $\mathbb{H}_S^1$-module *comes from a simple $\mathcal{H}_S(r)$-module* when the unique non-zero module it defines via decompositions (7.1) or (7.2) is an $\mathcal{H}_S(r)$-module.
2. A simple right $\mathbb{H}_S^1$-module is :
   - *put on the component $k = 0$* when it comes from a simple right $\mathcal{H}_S$-module.

- *put on the component* $k = \frac{q-1}{2}$ when it comes from a simple right $\mathcal{H}_S^\star$-module via the decomposition (7.1), assuming that p is odd.
- *put on the component* $k = r$ with $r \notin \{0, \frac{q-1}{2}\}$ when it comes from a simple right $\mathcal{H}_S(r)$-module.

7.1. The Iwahori case

Recall that we still assume the choices made in this paper to be done in a compatible way with those of [2]. Gathering Theorems 4.18 and 4.21, Remarks 4.19 and 4.24 and the results of [2, Sections 2.4 and 4.1], we directly obtain the two following propositions. Note that while Proposition 7.3 deals with non-supercuspidal representations, and is thus valid for arbitrary F, Proposition 7.4 is about supercuspidal representations, and hence only holds in the case $F = \mathbb{Q}_p$.

Proposition 7.3. *Recall that St_S denotes the Steinberg representation of $SL_2(F)$.*

1. *The $\mathbb{H}_S^1$-module carried by $\mathbf{1}^{I_S(1)}$ is put on the component $k = 0$ and isomorphic to the character $M_1^S(0)$.*
2. *The $\mathbb{H}_S^1$-module carried by $St_S^{I_S(1)}$ is put on the component $k = 0$ and isomorphic to the character $M_1^S(-1)$.*
3. *For any non-zero scalar $\lambda \in \overline{\mathbb{F}}_p^\times$, the $\mathbb{H}_S^1$-module carried by $\left(\mathrm{Ind}_{B_S}^{G_S}(\mu_\lambda)\right)^{I_S(1)}$ is put on the component $k = 0$ and isomorphic to the standard module $M_2^S(\lambda^{-1})$.*

Proposition 7.4. *Assume that $F = \mathbb{Q}_p$.*

1. *The $\mathbb{H}_S^1$-module carried by $\pi_0^{I_S(1)}$ is isomorphic to the character $M_1^S(-1,0)$ put on the component $k = 0$.*
2. *The $\mathbb{H}_S^1$-module carried by $\pi_{p-1}^{I_S(1)}$ is isomorphic to the character $M_1^S(0,-1)$ put on the component $k = 0$.*

7.2. The regular case

Fix a parameter $r \in \{1, \dots [\frac{q-1}{2}]\}$ satisfying $r \neq \frac{q-1}{2}$ when p is odd and $\lambda \in \overline{\mathbb{F}}_p^\times$. Thanks to [2, Proposition 2.9 and Lemme 2.10], we know that $\mathrm{Ind}_{B_S}^{G_S}(\mu_\lambda)$ has exactly two non-zero I_S-isotypical components, namely those attached to ω^r and to ω^{q-1-r}. The $\mathbb{H}_S^1$-module carried by $\mathrm{Ind}_{B_S}^{G_S}(\mu_\lambda)$ hence comes from an $\mathcal{H}_S(r)$-module and a direct computation left to the reader[11] proves the following result.

Proposition 7.5. *Let $\lambda \in \overline{\mathbb{F}}_p^\times$ be a non-zero scalar and let $r \in \{1 \dots [\frac{q-1}{2}]\}$ be a parameter satisfying $r \neq \frac{q-1}{2}$ when p is odd.*

1. *The $\mathbb{H}_S^1$-module carried by $\left(\mathrm{Ind}_{B_S}^{G_S}(\mu_\lambda \omega^r)\right)^{I_S(1)}$ is isomorphic to $M_{12}(\lambda^{-1})$ put on the component $k = r$.*
2. *The $\mathbb{H}_S^1$-module carried by $\left(\mathrm{Ind}_{B_S}^{G_S}(\mu_\lambda \omega^{q-1-r})\right)^{I_S(1)}$ is isomorphic to $M_{21}(\lambda^{-1})$ put on the component $k = r$.*

[11] But extensively done in [1, Appendice 7.6].

When $F = \mathbb{Q}_p$, [2, Section 4.1] gives a complete description of supercuspidal representations of G_S over $\overline{\mathbb{F}}_p$. In particular, the first part of [2, Proposition 4.11] implies that the structure of one-dimensional right $\mathbb{H}^1_S$-module carried by $\pi_r^{I_S(1)}$ and $\pi_{p-1-r}^{I_S(1)}$ comes from an $\mathcal{H}_S(r)$-module respectively defined by a non-trivial action of A_r and A_{p-1-r}. We hence have the following result.

Proposition 7.6. *We keep the notations and assumptions of Proposition 7.5 and we furthermore assume that $F = \mathbb{Q}_p$.*

1. *The $\mathbb{H}^1_S$-module carried by $\pi_r^{I_S(1)}$ is isomorphic to the character $M_1^r(0)$ put on the component $k = r$.*
2. *The $\mathbb{H}^1_S$-module carried by $\pi_{p-1-r}^{I_S(1)}$ is isomorphic to the character $M_2^r(0)$ put on the component $k = r$.*

7.3. The exceptional case

Assume that p is odd. The next result shows how the simple right $\mathbb{H}^1_S$-modules that are left can be realized as spaces of $I_S(1)$-invariant vectors of some irreducible smooth representations of G_S over $\overline{\mathbb{F}}_p$.

Proposition 7.7. *Assume that p is odd.*

1. *For any non-zero scalar $\lambda \in \overline{\mathbb{F}}_p^\times$, the $\mathbb{H}^1_S$-module carried by $\left(\mathrm{Ind}_{B_S}^{G_S}(\mu_\lambda \omega^{\frac{q-1}{2}})\right)^{I_S(1)}$ is isomorphic to the standard $\mathcal{H}^\star_S$-module $M_2^\star(\lambda^{-1})$ put on the component $k = \frac{q-1}{2}$.*
2. *Assuming that $F = \mathbb{Q}_p$, the $\mathbb{H}^1_S$-module carried by $\pi_{\frac{p-1}{2}}^{I_S(1)}$ is then isomorphic to the character $M_1^\star(0)$ put on the component $k = \frac{p-1}{2}$.*

Proof. By compact Frobenius reciprocity and [2, Lemme 2.10 and Proposition 4.11], the $\mathbb{H}^1_S$-modules we consider are put on the component $k = \frac{q-1}{2}$. By [2, Proposition 4.7], we know that $\pi_{\frac{p-1}{2}}^{I_S(1)}$ is one-dimensional over $\overline{\mathbb{F}}_p$ so the second part of the proposition directly comes from Lemma 6.3. The first part of the proposition follows from the argument we used in the Iwahori case (Section 4.4.1) and is left as an exercise to the reader. □

7.4. Application to the functor of $I_S(1)$-invariants

As in the Iwahori case (Definition 4.22), the formulation of our correspondence requires the introduction of a suitable notion of supersingular module.

Definition 7.8. A simple right $\mathbb{H}^1_S$-module is called *supersingular* if it is not isomorphic to a subquotient of some $(\mathrm{Ind}_{B_S}^{G_S}(\eta))^{I_S(1)}$ for a smooth character $\eta : B_S \to \overline{\mathbb{F}}_p^\times$.

Remark 7.9. This definition of supersingularity, which fits to the « historical » one (see also [21, Definition 5.1]) and is motivated by the representation theory of $SL_2(F)$, is equivalent to the other definitions that appeared later in the litterature. For instance, Vignéras defined supersingular modules as simple modules whose central character is null, i.e. vanishes on all central elements of positive length [22, Definitions 3 and 4]. It is straightforward to check that for $SL_2(F)$, these modules are exactly those we defined as supersingular.

In a recent work [26, Definition 6.10], Vignéras gave another definition of supersingularity and proved that supersingular modules are all simple modules that contain a supersingular character of the affine Iwahori-Hecke algebra [26, Corollary 6.13]. As noticed at the beginning of this paper, we are in a setting where the (pro-p-)Iwahori-Hecke algebra coincides with its affine subalgebra, hence supersingular modules are necessary one-dimensional. As [26, Theorem 6.15] asserts that supersingular characters are those whose restriction to each « irreducible component » (see [26, bottom of page 26] for the definition) is neither the trivial character nor the sign character, one can immediately check that for $SL_2(F)$, Definition 7.8 is equivalent to [26, Definition 6.10].

The comparison of the statements we proved so far with the classification of irreducible smooth representations of G_S over $\overline{\mathbb{F}}_p$ given in [2] directly leads to the following result.

Corollary 7.10. 1. *The functor of $I_S(1)$-invariants defines a bijection :*

$$\left\{\begin{array}{c}\textit{isomorphism classes of}\\ \text{non-supercuspidal } \textit{irreducible}\\ \textit{smooth representations}\\ \textit{of } SL_2(F) \textit{ over } \overline{\mathbb{F}}_p\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\textit{isomorphism classes of}\\ \text{non-supersingular}\\ \textit{simple right } \mathbb{H}^1_S\textit{-modules}\end{array}\right\}.$$

2. *When $F = \mathbb{Q}_p$, the previous bijection extends to a bijection :*

$$\left\{\begin{array}{c}\textit{isomorphism classes of irreducible}\\ \textit{smooth representations of } SL_2(\mathbb{Q}_p) \textit{ over } \overline{\mathbb{F}}_p\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\textit{isomorphism classes of}\\ \textit{simple right } \mathbb{H}^1_S\textit{-modules}\end{array}\right\}.$$

Remark 7.11. In a recent work [10, Theorem 4.6 and Corollary 4.7], Kozioł proved that for $F = \mathbb{Q}_p$ with p odd, the bijection given in Corollary 7.10 comes from an equivalence of categories. Note that [10, Theorem 4.6] and [14, Theorem 1.3] also prove that such an equivalence fails if we do not assume $F = \mathbb{Q}_p$.

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