On Extensions of supersingular representations of $\mathrm{SL}_2(\mathbb{Q}_p)$.

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January 29, 2017

Abstract

In this note for $p > 5$ we calculate the dimensions of $\mathrm{Ext}^1_{\mathrm{SL}_2(\mathbb{Q}_p)}(\tau, \sigma)$ for any two irreducible supersingular representations τ and σ of $\mathrm{SL}_2(\mathbb{Q}_p)$.

1 Introduction

In this note we calculate the space of extensions of supersingular representations of $\mathrm{SL}_2(\mathbb{Q}_p)$ for $p > 5$. The dimensions of the space of extensions between irreducible supersingular representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ are calculated by Paškūnas in [Paš10]. Understanding extensions between irreducible smooth representations play a crucial role in Paškūnas work on the image of Colmez Montreal functor in (see [Paš13]). We hope that these results have similar application to mod p and p-adic local Langlands correspondence for $\mathrm{SL}_2(\mathbb{Q}_p)$.

Let G be the group $\mathrm{GL}_2(\mathbb{Q}_p)$, K be the maximal compact subgroup $\mathrm{GL}_2(\mathbb{Z}_p)$ and Z be the center of G. We denote by $I(1)$ the pro-p Iwahori subgroup of G. We denote by G_S the special linear group $\mathrm{SL}_2(\mathbb{Q}_p)$. For any subgroup H of $\mathrm{GL}_2(\mathbb{Q}_p)$ we denote by H_S the subgroup $H \cap \mathrm{SL}_2(\mathbb{Q}_p)$. All representations in this note are defined over vector spaces over $\bar{\mathbb{F}}_p$. Let σ be an irreducible smooth representation of K and σ extends uniquely as a representation of KZ such that $p \in Z$ acts trivially. The Hecke algebra $\mathrm{End}_G(\mathrm{ind}_{KZ}^G \sigma)$ is isomorphic to $\bar{\mathbb{F}}_p[T]$. For any constant λ in $\overline{\mathbb{F}}_p^\times$ let μ_λ be the unramified character of Z such that $\mu_\lambda(p) = \lambda$. Let $\pi(\sigma, \mu_\lambda)$ be the representation

$$\frac{\mathrm{ind}_{KZ}^G \sigma}{T(\mathrm{ind}_{KZ}^G \sigma)} \otimes (\mu_\lambda \circ \det).$$

The representations $\pi(\sigma, \mu_\lambda)$ are irreducible (see [Bre03]) and are called **supersingular representations** in the terminology of Barthel–Livné.

Let σ_r be the representation $\mathrm{Sym}^r \bar{\mathbb{F}}_p$ of $\mathrm{GL}_2(\mathbb{F}_p)$. We consider σ_r as a representation of K by inflation. The K-socle of $\pi(\sigma_r, \mu_\lambda)$ is a direct sum of two irreducible smooth representations σ_r and σ_{p-1-r}. Let $\pi_{0,r}$ and $\pi_{1,r}$ be the G_S representations generated by $\sigma_r^{I(1)}$ and $\sigma_{p-1-r}^{I(1)}$. The representations $\pi_{0,r}$ and $\pi_{1,r}$ are irreducible supersingular representations of G_S and

$$\mathrm{res}_{G_S} \pi(\sigma_r, \mu_\lambda) \simeq \pi_{0,r} \oplus \pi_{1,r}.$$

Any irreducible supersingular representation of G_S is isomorphic to $\pi_{i,r}$ for some r such that $0 \leq r \leq p-1$ and $i \in \{0, 1\}$. Moreover the only isomorphisms between $\pi_{i,r}$ are $\pi_{0,r} \simeq \pi_{1,p-1-r}$ and $\pi_{1,r} \simeq \pi_{0,p-1-r}$ (see [Abd14]). Our main theorem on extensions of supersingular representations of G_S is:

Theorem 1.1. *Let $p \geq 5$ and $0 \leq r \leq (p-1)/2$. For any irreducible supersingular representation τ of G_S the space $\mathrm{Ext}^1_{G_S}(\tau, \pi_{i,r})$ is non-zero if and only if $\tau \simeq \pi_{j,r}$ for some $j \in \{0, 1\}$. If $0 \leq r < (p-1)/2$ then $\dim_{\bar{\mathbb{F}}_p} \mathrm{Ext}^1_{G_S}(\pi_{i,r}, \pi_{j,r}) = 2$ for $i \neq j$ and $\dim_{\bar{\mathbb{F}}_p} \mathrm{Ext}^1_{G_S}(\pi_{i,r}, \pi_{i,r}) = 1$. For $r = (p-1)/2$ we have $\dim_{\bar{\mathbb{F}}_p} \mathrm{Ext}^1_{G_S}(\pi_{0,r}, \pi_{0,r}) = 3$.*

We briefly explain the method of proof. We essentially follow [Paš10]. The functor sending a smooth representation to its $I(1)_S$-invariants induces an equivalence of categories of smooth representations of G_S

generated by $I(1)_S$-invariants and the module category of the pro p-Iwahori Hecke algebra (see [Koz16, Theorem 5.2]). We use the Ext spectral sequence thus obtained by this equivalence of categories to calculate $\mathrm{Ext}^1_{G_S}$. Extensions of pro p-Iwahori Hecke algebra modules are calculated from resolutions of Hecke modules due to Schneider and Ollivier. We crucially use results from work of Paškūnas [Paš10]. We first obtain lower bounds on the dimensions of $\mathrm{Ext}^1_{G_S}$ spaces using the spectral sequence and then obtain upper bounds using Paškūnas results on $\mathrm{Ext}^1_K(\sigma, \pi(\sigma, \mu_\lambda))$.

Acknowledgements I thank Eknath Ghate for showing the fundamental paper [Paš10] and for his interest in this work and discussions on the role of extensions in mod p-Langlands. I want to thank Radhika Ganapathy for various discussions on mod p representations and for her mod p seminar at the Tata Institute. After posting this paper on ArXiv the author has learnt that Ramla Abdellatif has independently obtained these results.

2 Pro-p Iwahori Hecke algebra

Let B be the Borel subgroup consisting of invertible upper triangular matrices, U be the unipotent radical of B and T be the maximal torus consisting of diagonal matrices. We denote by $\bar{U}$ the unipotent radical of $\bar{B}$ the Borel subgroup consisting of invertible lower triangular matrices. We denote by I the standard Iwahori-subgroup of G. Let $I(1)$ be the pro-p Iwahori subgroup of G and $I(1)_S$ be the pro-p-Iwahori subgroup of G_S. We note that $I(1)_S(Z \cap I(1))$ is equal to $I(1)$. Let $\mathcal{H}$ be the pro-p Iwahori–Hecke algebra $\mathrm{End}_G(\mathrm{ind}^{G_S}_{I(1)_S}\mathrm{id})$. Let Rep_{G_S} and $\mathrm{Rep}^{I(1)_S}_{G_S}$ be the category of smooth representations of G_S and its full subcategory consisting of those smooth representations generated by $I(1)_S$-invariant vectors respectively. We denote by $\mathrm{Mod}_{\mathcal{H}}$ the category of modules over the ring $\mathcal{H}$. We have two functors

$$\mathcal{I} : \mathrm{Rep}^{I(1)_S}_{G_S} \to \mathrm{Mod}_{\mathcal{H}}$$
$$\mathcal{I}(\pi) = \pi^{I(1)_S}$$

and

$$\mathcal{T} : \mathrm{Mod}_{\mathcal{H}} \to \mathrm{Rep}^{I(1)_S}_{G_S}$$
$$\mathcal{T}(M) = M \otimes_{\mathcal{H}} \mathrm{ind}^{G_S}_{I(1)_S}\mathrm{id}\,.$$

From [Koz16, Theorem 5.2] the functors $\mathcal{T}$ and $\mathcal{I}$ are quasi-inverse to each other. Let σ and τ be any two smooth representations of G_S and σ_1 be the G_S subrepresentation of σ generated by $I(1)_S$-invariants of σ. We have

$$\mathrm{Hom}_G(\tau, \sigma) = \mathrm{Hom}_G(\tau, \sigma_1) = \mathrm{Hom}_H(\mathcal{I}(\tau), \mathcal{I}(\sigma_1)) = \mathrm{Hom}_H(\mathcal{I}(\tau), \mathcal{I}(\sigma)). \tag{1}$$

We get a Grothendieck spectral sequence with E_2^{ij} equal to $\mathrm{Ext}^i(\mathcal{I}(\tau), \mathbb{R}^j\mathcal{I}(\sigma))$ such that

$$\mathrm{Ext}^i(\mathcal{I}(\tau), \mathbb{R}^j\mathcal{I}(\sigma)) \Rightarrow \mathrm{Ext}^{i+j}_G(\tau, \sigma). \tag{2}$$

The 5-term exact sequence associated to the above spectral sequence gives the following exact sequence:

$$\begin{aligned} 0 \to \mathrm{Ext}^1_{\mathcal{H}}(\mathcal{I}(\tau), \mathcal{I}(\sigma)) \xrightarrow{i} \mathrm{Ext}^1_G(\tau, \sigma) \xrightarrow{\delta} \\ \mathrm{Hom}_{\mathcal{H}}(\mathcal{I}(\tau), \mathbb{R}^1\mathcal{I}(\sigma)) \to \mathrm{Ext}^2_{\mathcal{H}}(\mathcal{I}(\tau), \mathcal{I}(\sigma)) \to \mathrm{Ext}^2_G(\tau, \sigma) \end{aligned} \tag{3}$$

for all τ such that $\tau =< G_S\tau^{I(1)} >$. In particular we apply these results when τ and σ are irreducible supersingular representations of G_S. We first recall the structure of the ring $\mathcal{H}$, its modules $M(i, r) = \pi^{I(1)}_{i,r}$ for i in $\{0, 1\}$ and $0 \leq r \leq p-1$. The $\mathcal{H}$ module $M(i, r)$ is a character and we first calculate the dimensions of the spaces $\mathrm{Ext}^1_{\mathcal{H}}(M(i, r), M(j, s))$.

Let T^0_S and T^1_S be the maximal compact subgroup of T_S and its maximal pro-p-subgroup. We denote by s_0, s_1 and θ the matrices $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & -p^{-1} \\ p & 0 \end{pmatrix}$ and $\begin{pmatrix} p & 0 \\ 0 & p^{-1} \end{pmatrix}$ respectively. Let $N(T_S)$ be the normaliser of the torus T_S and W_0 be the Weyl group $N(T_S)/T_S$. The extended Weyl group $W = \theta^{\mathbb{Z}} \coprod s_0\theta^{\mathbb{Z}}$ sits into an exact sequence of the form

$$0 \to \Omega := \frac{T_S^0}{T_S^1} \to \tilde{W} := \frac{N(T_S)}{T_S^1} \to W = \frac{N(T_S)}{T_S^0} \to 0.$$

The length function l on W, given by $l(\theta^i) = |2i|$ and $l(s_0\theta^i) = |1-2i|$, extends to a function on $\tilde{W}$ such that $l(\Omega) = 0$. Let T_w be the element $\mathrm{Char}_{I(1)wI(1)}$ for all $w \in \tilde{W}$. We denote by e_1 the element $\sum_{w\in\Omega} T_w$. The functions T_w span $\mathcal{H}$ and the relations in $\mathcal{H}$ are given by

$$\begin{aligned} T_wT_v &= T_{wv} \text{ whenever } l(v) + l(w) = l(vw), \\ T_{s_i}^2 &= -e_1T_{s_i}. \end{aligned}$$

The pro-p-Iwahori Hecke algebra is generated by $T_wT_{s_i}$ for w in Ω. For any character χ of Ω let e_χ be the element $\sum_{w\in\Omega} \chi^{-1}(w)T_w$. Let γ be a W_0 orbit of the characters χ and e_γ be the element $\sum_{\chi\in\gamma} e_\chi$. The elements $\{e_\gamma; \gamma \in \hat{\Omega}/W_0\}$ are central idempotents in the ring $\mathcal{H}$ and we have

$$\mathcal{H} = \bigoplus_{\hat{\Omega}/W_0} \mathcal{H}e_\gamma. \tag{4}$$

For the group G_S, we know that $\mathcal{H}$ is the affine Hecke algebra. The characters of affine Hecke algebra are described in a simple manner we recall this for G_S. Let I be a subset of $\{s_0, s_1\}$ and W_I be the subgroup of W generated by elements of I and $W_\emptyset$ is trivial group. The characters of $\mathcal{H}$ are parametrised by pairs (λ, I) where λ is a character of Ω and $I \subset S_\lambda$. For such a pair (λ, I) the character $\chi_{\lambda,I}$ associated to it is given by

$$\chi_{\lambda,I}(T_{wt}) = 0 \text{ for all } w \in W\backslash W_I \text{ and for all } \ t \in \Omega, \tag{5}$$

$$\chi_{\lambda,I}(T_{wt}) = \lambda(t)(-1)^{l(w)} \text{ for all } w \in W_I \text{ and for all } \ t \in \Omega. \tag{6}$$

If λ is nontrivial then we have $\chi_{\lambda,\emptyset}(T_t) = \lambda(t)$, for all $t \in \Omega$ and $\chi_{\lambda,\emptyset}(T_{wt}) = 0$ for all $w \neq \mathrm{id}$ and $t \in \Omega$.

We denote by $\chi_{r,\emptyset}$ the character $\chi_{x\mapsto x^r,\emptyset}$. From the above description we get that $M(0,r) = \chi_{r,\emptyset}$ and $M(1,r) = \chi_{p-1-r,\emptyset}$ for $r \notin \{0, p-1\}$. If $r \in \{0, p-1\}$ then [OS16, Proposition 3.9] says that $\chi_{\mathrm{id},\emptyset}$ and $\chi_{\mathrm{id},S}$ are not supersingular characters. This shows that $M(i,r)$ is either $\chi_{\mathrm{id},I}$ or $\chi_{\mathrm{id},J}$, for $r \in \{0, p-1\}$, where $I = \{s_0\}$ and $J = \{s_1\}$. Since the element T_{s_0} belongs to pro-p Iwahori–Hecke algebra of G and using the presentation in [BP12, Corollary 6.4] we obtain that $M(1,0) = \chi_{\mathrm{id},I}$ and $M(0,0) = \chi_{\mathrm{id},J}$. Similarly $M(1,p-1)$ is given by the character $\chi_{\mathrm{id},J}$ and $M(0,p-1)$ is given by the character $\chi_{\mathrm{id},I}$. Let $0 \le r, s \le (p-1)/2$ then (4) shows that

$$\mathrm{Ext}^1_{\mathcal{H}}(M(i,r), M(j,s)) = 0 \tag{7}$$

for $r \neq s$.

2.1 Resolutions of Hecke modules

In order to calculate extensions between the characters $M(i,r)$, we use resolutions constructed by Schneider and Ollivier for $\mathcal{H}$. Let $\mathfrak{X}$ be the Bruhat–Tits tree of G_S and let $A(T_S)$ be the standard apartment associated to T_S. We fix an edge E and vertices v_0 and v_1 contained in E such that the G_S-stabiliser of v_0 is K_S. For any facet F of $\mathfrak{X}$ we denote by $\mathbf{G}_F$ the $\mathbb{Z}_p$-group scheme with generic fibre $\mathbf{SL}_2$ and $\mathbf{G}_F(\mathbb{Z}_p)$ is the G-stabiliser of F. We denote by I_F the subgroup of $\mathbf{G}_F(\mathbb{Z}_p)$ whose elements under mod- $\mathfrak{p}$ reduction of $\mathbf{G}_F(\mathbb{Z}_p)$ belong to the $\mathbb{F}_p$-points of the unipotent radical of $\mathbf{G}_F \times \mathbb{F}_p$. We denote by $\mathcal{H}_F$ the finite subalgebra of $\mathcal{H}$ defined as

$$\mathcal{H}_F := \mathrm{End}_{\mathbf{G}_F(\mathbb{Z}_p)}(\mathrm{ind}_{I_F}^{\mathbf{G}_F(\mathbb{Z}_p)}(\mathrm{id})).$$

In particular $\mathcal{H}_E$ is a semi-simple algebra.

For any $\mathcal{H}$-module $\mathfrak{m}$ the construction of Schneider and Ollivier [OS14, Theorem 3.12, (6.4)] gives us a $(\mathcal{H},\mathcal{H})$-exact resolution

$$0 \to \mathcal{H} \otimes_{\mathcal{H}_E} \mathfrak{m} \xrightarrow{\delta_1} (\mathcal{H} \otimes_{\mathcal{H}_{v_0}} \mathfrak{m}) \oplus (\mathcal{H} \otimes_{\mathcal{H}_{v_1}} \mathfrak{m}) \xrightarrow{\delta_0} \mathfrak{m} \to 0. \tag{8}$$

Using the resolution (8) and the observation that $\mathcal{H}_E$ is semi-simple for $p \neq 2$ we get that

$$0 \to \mathrm{Hom}_{\mathcal{H}}(\mathfrak{m}, \mathfrak{n}) \to \bigoplus_{v_0, v_1} \mathrm{Hom}_{\mathcal{H}_{v_i}}(\mathfrak{m}, \mathfrak{n}) \to \mathrm{Hom}_{\mathcal{H}_E}(\mathfrak{m}, \mathfrak{n}) \xrightarrow{\delta} \mathrm{Ext}^1_{\mathcal{H}}(\mathfrak{m}, \mathfrak{n}) \to \bigoplus_{v_0, v_1} \mathrm{Ext}^1_{\mathcal{H}_{v_i}}(\mathfrak{m}, \mathfrak{n}) \to 0 \tag{9}$$

Note that we have an isomorphism of algebras

$$\mathcal{H}_{v_0} \simeq \mathcal{H}_{v_1} \simeq \mathrm{End}_{\mathrm{SL}_2(\mathbb{F}_p)}(\mathrm{ind}_{N(\mathbb{F}_p)}^{\mathrm{SL}_2(\mathbb{F}_p)} \mathrm{id}).$$

The above isomorphism is not a canonical isomorphism. Let K_0 and K_1 be the compact open subgroups $K \cap G_S$ and $K^{\Pi} \cap G_S$ respectively.

2.2 Extensions of supersingular modules over pro-p Iwahori–Hecke algebra.

The Hecke algebra $\mathcal{H}_{v_i}$ is isomorphic to $\mathrm{End}_{K_i}(\mathrm{ind}_{I(1)}^{K_i} \mathrm{id})$. The Hecke algebra $\mathcal{H}_{v_i}$ is generated by T_t and T_{s_i} for $t \in \Omega$. The relations among them are given by

$$\begin{aligned} &T_{t_1} T_{t_2} = T_{t_1 t_2}, \\ &T_t T_{s_i} = T_{t s_i} = T_{s_i t^{-1}} = T_{s_i} T_{t^{-1}}, \\ &T_{s_i}^2 = -e_1 T_{s_i} \end{aligned}$$

where $e_1 = \sum_{t \in \Omega} T_t$.

Lemma 2.1. *Let $0 < r < (p-1)/2$ the space $\mathrm{Ext}^1_{\mathcal{H}}(M(i,r), M(j,s))$ is non-zero if and only if $i \neq j$ and has dimension 2 when $i \neq j$. If $r = (p-1)/2$ then the space $\mathrm{Ext}^1_{\mathcal{H}}(M(i,r), M(i,r))$ has dimension 2.*

Proof. Since $r \neq 0$ the characters $M(0,r)$ and $M(1,r)$ are isomorphic to $\chi_{r,\emptyset}$ and $\chi_{p-1-r,\emptyset}$ respectively (see (5)). Let E_c be a 2-dimensional $\bar{\mathbb{F}}_p$ module $\bar{\mathbb{F}}_p e_1 \oplus \bar{\mathbb{F}}_p e_2$ and $\bar{\mathbb{F}}_p[\Omega]$ acts on E by $T_t e_0 = t^r e_0$ and $T_t e_1 = t^{p-1-r} e_1$. We set $T_{s_i} e_0 = 0$ and $T_{s_i} e_1 = c e_0$ for some $c \neq 0$. This makes E a $\mathcal{H}_{v_i}$ module and is a non-trivial extension

$$0 \to \chi_{r,\emptyset} \to E \to \chi_{p-1-r,\emptyset} \to 0.$$

Let E be a $\mathcal{H}_{v_i}$-extension of $W := \chi_{s,\emptyset}$ by $V := \chi_{r,\emptyset}$ i.e, we have an exact sequence

$$0 \to V \to E \xrightarrow{f} W \to 0.$$

There exists a $\bar{\mathbb{F}}_p[\Omega]$-equivariant section $s : W \to E$ of f. Let V' be the image of this section. Now $E = V \oplus V'$. The action of T_{s_i} is trivial on V and observe that $f(T_{s_i}(V')) = T_{s_i}(f(V')) = 0$. **If E is nontrivial then** $T_{s_i}(V') = V$. This implies that $r + s = p - 1$ and hence E is isomorphic to E_c for some $c \neq 0$. This shows that the space of $\mathcal{H}_{v_i}$ extensions of W by V is one dimensional if $r + s = p - 1$ and zero otherwise. Now consider the exact sequence (9) when $\mathfrak{m}$ is $M(i,r)$ and $\mathfrak{n}$ is $M(j,r)$. For $i = j$ the map δ in zero (9) hence the space $\mathrm{Ext}^1_{\mathcal{H}}(M(i,r), M(i,r))$ is trivial for $0 < r < (p-1)/2$ and has dimension 2 when $r = (p-1)/2$. When $i \neq j$ the Hom spaces in (9) are all trivial. This shows that the dimension of the space $\mathrm{Ext}^1_{\mathcal{H}}(M(i,r), M(i,r))$ is 2 from our calculations. □

Lemma 2.2. *The space of extensions $\mathrm{Ext}^1_{\mathcal{H}}(M(i,0), M(j,0))$ is trivial for $i = j$ and has dimension 1 for $i \neq j$.*

Proof. The algebra $e_1 \mathcal{H}_{v_i}$ is semi-simple algebra and hence we get that

$$\mathrm{Ext}^i_{\mathcal{H}_{v_i}}(\chi_{\mathrm{id},S}, \chi_{\mathrm{id},S'}) = 0 \tag{10}$$

for all $i > 0$ and for subsets S and S' of $\{s_0, s_1\}$. Now consider the exact sequence (9) when $\mathfrak{m}$ is $M(i,r)$ and $\mathfrak{n}$ is $M(j,r)$. For $i = j$ the map δ in (9) is zero hence the space $\mathrm{Ext}^1_{\mathcal{H}}(M(i,r), M(i,r))$ is trivial. When $i \neq j$ the first two Hom spaces in (9) are trivial. The space $\mathrm{Hom}_{\mathcal{H}_E}(\mathfrak{m}, \mathfrak{n})$ has dimension one. This shows that the dimension of the space $\mathrm{Ext}^1_{\mathcal{H}}(M(i,r), M(i,r))$ is 1 for $i \neq j$. □

3 The Hecke module $\mathbb{R}^1\mathcal{I}(\pi_{i,r})$.

Paškūnas calculated the cohomology groups $\mathbb{R}^1\mathcal{I}(\pi_{i,r})$ and we now recall his results. Let $\tilde{\pi}_r$ be the supersingular representation $\pi(\sigma_r,\mu_1)$ of G. Recall that the K-socle of $\tilde{\pi}_r$ is isomorphic to $\sigma_r \oplus \sigma_{p-1-r}$ and the space of $I(1)$ invariants has a basis $(\mathbf{v}_0,\mathbf{v}_1)$ where $\mathbf{v}_0$ and $\mathbf{v}_1$ belong to $\sigma_r^{N_p}$ and $\sigma_{p-1-r}^{N_p}$ respectively. Let I^+ and I^- be the groups $I\cap U$ and $I\cap\bar{U}$ respectively. Consider the spaces

$$M_0 :=< I^+\theta^n\,\mathbf{v}_1;\ n\geq 0 > \text{ and } M_1 :=< I^+\theta^n\,\mathbf{v}_2;\ n\geq 0 >$$

and let Π be the matrix $\begin{pmatrix}0 & 1\\ p & 0\end{pmatrix}$ which normalizes I and $I(1)$. We denote by π_0 and π_1 the spaces $M_0+\Pi M_1$ and $M_1+\Pi M_0$. Let G^0 be subgroup of G consisting of elements with integral discriminant. Let G^+ be the group ZG^0. We denote by Z_1 the group $I(1)\cap Z$.

Proposition 3.1 (Paškūnas)**.** *The spaces π_0 and π_1 are G^+ stable. The space $\tilde{\pi}_r$ is the direct sum of the representations π_1 and π_0 as G^+ representations and hence $\pi_{i,r}$ is isomorphic to π_i as G_S representations for $i\in\{0,1\}$. If r be an integer such that $0<r<(p-1)/2$ then the Hecke module $\mathbb{R}^1\mathcal{I}(\pi_{i,r})$ is isomorphic to $\mathcal{I}(\pi_{i,r})\oplus\mathcal{I}(\pi_{i,r})$. In the Iwahori case (i.e, $r=0$) the Hecke module $\mathbb{R}\mathcal{I}(\pi_{0,0})\oplus\mathbb{R}\mathcal{I}(\pi_{1,0})$ is isomorphic to $\mathcal{I}(\pi_{0,0})^{\oplus 2}\oplus\mathcal{I}(\pi_{0,0})^{\oplus 2}$.*

Proof. The first part follows from [Paš10, Corollary 6.5]. The second part follows from [Paš10, Proposition 10.5, Theorem 10.7 and equation (49)]. □

Corollary 3.2. *Let τ be an irreducible supersingular representation of G_S. If the space of extensions $\mathrm{Ext}^1_{G_S}(\tau,\pi_{i,r})$ is non-trivial then $\tau\simeq\pi_{j,r}$ for some $j\in\{0,1\}$.*

Proof. This follows from (3), (7) and Proposition 3.1. □

Corollary 3.3. *Let $0<r<(p-1)/2$ and $i\neq j$ then the dimensions of the space $\mathrm{Ext}^1_{G_S}(\pi_{i,r},\pi_{j,r})$ is 2.*

Proof. Observe that for $0<r<(p-1)/2$ the modules $M(i,r)$ and $M(j,s)$ are not isomorphic. Now using the exact sequence (3) and Proposition 3.1 we get that

$$\mathrm{Ext}^1_{G_S}(\pi_{i,r},\pi_{j,r})\simeq\mathrm{Ext}^1_{\mathcal{H}}(M(i,r),M(j,r)).$$

The corollary follows from the Lemma 2.1. □

Remark 3.4. *The results of Corollary 3.3 remain valid for $r=0$ but we prove this later. It is interesting to note that for $0<r<(p-1)/2$ and $i\neq j$ any extension E of $\pi_{i,r}$ by $\pi_{j,r}$ for $i\neq j$ is generated by its $I(1)_S$ invariants, i.e, $E=< G_SE^{I(1)_S} >$.*

4 Calculation of degree one self extensions.

Let us first consider the case when $0<r\leq(p-1)/2$. In order to determine the dimensions of $\mathrm{Ext}^1(\pi_{i,r},\pi_{i,r})$ we first show that the map

$$\mathrm{Ext}^1_{G_S}(\pi_{i,r},\pi_{i,r})\to\mathrm{Hom}_{\mathcal{H}}(\mathcal{I}(\pi_{i,r}),\mathbb{R}^1\mathcal{I}(\pi_{i,r})))\tag{11}$$

is non-zero. Explicitly the above map takes an extension E, with $0\to\pi_{i,r}\to E\to\pi_{i,r}\to 0$, to the delta map in the associated long exact sequence, given by: $\mathcal{I}(\pi_{i,r})\xrightarrow{\delta_E}\mathbb{R}^1\mathcal{I}(\pi_{i,r})$. Note that the dimension of $E^{I(1)}$ is one if and only if $\delta_E\neq 0$.

Lemma 4.1. *For $0<r\leq(p-1)/2$ then map (11) is non-zero.*

Proof. For $0 < r \leq (p-1)/2$ there exists a self extension E of $\tilde{\pi}_r$ such that the map $\mathcal{I}(\tilde{\pi}_r) \xrightarrow{\delta_E} \mathbb{R}^1\mathcal{I}(\tilde{\pi}_r)$ is non-zero. We fix an extension E such that $\delta_E \neq 0$. Since δ_E is a Hecke-equivariant map and $\mathcal{I}(\tilde{\pi}_r)$ is an irreducible Hecke-module of dimension 2 we get that the inclusion map of $\mathcal{I}(\tilde{\pi}_r)$ in $\mathcal{I}(E)$ is an isomorphism i.e, $\dim E^{I(1)} = 2$. Now consider the pullback diagram

$$\begin{tikzcd} 0 \arrow[r] & \tilde{\pi}_r \arrow[r] \arrow[d] & E_1 \arrow[r] \arrow[d] & \pi_{i,r} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \tilde{\pi}_r \arrow[r] & E \arrow[r] & \tilde{\pi}_r \arrow[r] & 0. \end{tikzcd} \tag{12}$$

The long exact sequences in $I(1)$-group cohomology attached to (12) gives us:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{I}(\tilde{\pi}_r) \arrow[r, "f"] \arrow[d] & \mathcal{I}(E_1) \arrow[r] \arrow[d] & \mathcal{I}(\pi_{i,r}) \arrow[r, "\delta_2"] \arrow[d] & \mathbb{R}^1\mathcal{I}(\tilde{\pi}_r) \arrow[d] \\ 0 \arrow[r] & \mathcal{I}(\tilde{\pi}_r) \arrow[r] & \mathcal{I}(E) \arrow[r] & \mathcal{I}(\tilde{\pi}_r) \arrow[r, "\delta_1"'] & \mathbb{R}^1\mathcal{I}(\tilde{\pi}_r). \end{tikzcd}$$

Since the dimension of $\mathcal{I}(E)$ is 2 we get that δ_1 is injective and hence the map δ_2 is non-zero. The dimension of the space $\mathcal{I}(\pi_{i,r})$ is one hence f is an isomorphism. This shows that the space $\mathcal{I}(E_1)$ has dimension 2. For $r = (p-1)/2$ the representations $\pi_{1,r} \simeq \pi_{0,r}$. We assume without loss of generality $\operatorname{img} \delta_2$ is contained in $\mathbb{R}^1\mathcal{I}(\pi_{i,r})$. For any r such that $0 < r \leq (p-1)/2$ consider the pushout of $\tilde{\pi}_r$ by $\pi_{i,r}$

$$\begin{tikzcd} 0 \arrow[r] & \pi_{i,r} \arrow[r] \arrow[d] & E_2 \arrow[r] \arrow[d] & \pi_{i,r} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \tilde{\pi}_r \arrow[r] & E_1 \arrow[r] & \pi_{i,r} \arrow[r] & 0 \end{tikzcd} \tag{13}$$

The self extension E_2 of $\pi_{i,r}$ is non-split and the induced map δ_{E_2} is non-zero. To see this consider the long exact sequence in cohomology attached to (13):

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{I}(\pi_{i,r}) \arrow[r, "g"] & \mathcal{I}(E_2) \arrow[r] & \mathcal{I}(\pi_{i,r}) \arrow[r, "\delta_3"] & \mathbb{R}^1\mathcal{I}(\pi_{i,r}) \\ 0 \arrow[r] & \mathcal{I}(\tilde{\pi}_r) \arrow[r, "\simeq"] \arrow[u] & \mathcal{I}(E_1) \arrow[r, "0"] \arrow[u] & \mathcal{I}(\pi_{i,r}) \arrow[r, "\delta_2"] \arrow[u] & \mathbb{R}^1\mathcal{I}(\tilde{\pi}_r) \arrow[u, "\mathbb{R}^1\mathcal{I}(p_2)"'] \end{tikzcd}$$

Note that $\mathbb{R}^1\mathcal{I}(\tilde{\pi}_r)$ is isomorphic to $\mathbb{R}^1\mathcal{I}(\pi_{0,r}) \oplus \mathbb{R}^1\mathcal{I}(\pi_{1,r})$ and $\mathbb{R}^1\mathcal{I}(p_2)$ is the projection map. This shows that $\mathbb{R}^1\mathcal{I}(p_2)\delta_2 \neq 0$ and hence $\delta_3 \neq 0$ using which we get that g is an isomorphism. This shows that E_2 is a non-split self-extension of $\pi_{i,r}$ by $\pi_{i,r}$. ☐

Corollary 4.2. *For any integer r such that $0 < r < (p-1)/2$ we have* $\dim_{\bar{\mathbb{F}}_p} \operatorname{Ext}^1_G(\pi_{i,r}, \pi_{i,r}) \geq 1$.

Theorem 4.3. *Let $p \geq 5$ and $0 \leq r < (p-1)/2$ then the dimension of $\operatorname{Ext}^1_{G_S}(\pi_{i,r}, \pi_{i,r})$ is 1 and dimension of $\operatorname{Ext}^1_{G_S}(\pi_{i,r}, \pi_{j,r})$ is 2 for $i \neq j$. For $r = (p-1)/2$ the dimension of $\operatorname{Ext}^1_{G_S}(\pi_{0,r}, \pi_{0,r})$ is 3.*

Proof. The subgroup $G_S Z$ is an index 2 subgroup of G and id and Π are two double coset representatives for $K\backslash G/G_S Z$. We note that $K \cap G_S$ and $K^{\Pi} \cap G_S$ are representatives for the two distinct classes of maximal compact subgroups of G_S and we denote them by K_1 and K_2 respectively. Let σ_r' be the representation σ_r^{Π} of K^{Π}. Using Mackey-decomposition we get that

$$\operatorname{res}_{G_S}\operatorname{ind}_{KZ}^{G}\sigma_r = \operatorname{ind}_{K^{\Pi}\cap G_S}^{G_S}\sigma_r^{\Pi}\oplus\operatorname{ind}_{K\cap G_S}^{G_S}\sigma_r = \operatorname{ind}_{K_1}^{G_S}\sigma_r\oplus\operatorname{ind}_{K_2}^{G_S}\sigma_r'. \tag{14}$$

using the long exact sequence of Ext groups for the exact sequence,

$$0\to\operatorname{ind}_{ZK}^{G}\sigma_r\xrightarrow{T}\operatorname{ind}_{ZK}^{G}\to\tilde{\pi}_r\to 0$$

we get that an exact sequence

$$\operatorname{Hom}_G(\operatorname{ind}_{ZK}^{G}\sigma_r,\tilde{\pi}_r)\to\operatorname{Ext}^1_G(\tilde{\pi}_r,\tilde{\pi}_r)\to\operatorname{Ext}^1_G(\operatorname{ind}_{ZK}^{G}\sigma_r,\tilde{\pi}_r)\xrightarrow{T}\operatorname{Ext}^1_G(\operatorname{ind}_{ZK}^{G}\sigma_r,\tilde{\pi}_r). \tag{15}$$

Now using (14) the exact sequence (15) becomes

$$0\to\operatorname{Hom}_{K_1}(\sigma_r,\tilde{\pi}_r)\oplus\operatorname{Hom}_{K_2}(\sigma_r',\tilde{\pi}_r)\to\operatorname{Ext}^1_G(\tilde{\pi}_r,\tilde{\pi}_r)\to\operatorname{Ext}^1_{K_1}(\sigma_r,\tilde{\pi}_r)\oplus\operatorname{Ext}^1_{K_2}(\sigma_r',\tilde{\pi}_r). \tag{16}$$

The groups K_1 is contained in K/Z_1. For all $i\geq 0$ we note that

$$\operatorname{Ext}^i_{K_1}(\sigma_r,\tilde{\pi}_r)\simeq\operatorname{Ext}^i_{K/Z_1}(\operatorname{ind}_{K_1}^{K/Z_1}(\sigma_r),\tilde{\pi}_r)\simeq\bigoplus_{0\leq a<p-1}\operatorname{Ext}^i_{K/Z_1}(\sigma_r\otimes\det^a,\tilde{\pi}_r).$$

The spaces $\operatorname{Ext}^1_{K/Z_1}(\sigma_r\otimes\det^a,\tilde{\pi}_r)$ can be calculated from the work of Paškūnas. We recall his calculations as needed. There exists a G smooth representation Ω such that $\operatorname{res}_K\Omega$ is an injective envelope of $\operatorname{Soc}_K(\tilde{\pi}_r)$ in the category of smooth representations of K. In particular we get that $\tilde{\pi}_r$ is contained in Ω. The restriction $\operatorname{res}_K\Omega$ is isomorphic to $\operatorname{inj}\sigma_r\oplus\operatorname{inj}\sigma_{p-1-r}$. Now $\operatorname{Ext}^1_{K/Z_1}(\sigma_r\otimes\det^a,\tilde{\pi}_r)$ is isomorphic to $\operatorname{Hom}_{K/Z_1}(\sigma_r\otimes\det^a,\Omega/\tilde{\pi}_r)$.

We now use the notations from [Paš10, Notations, Section 9]. We make one modification. Paškūnas uses the notation χ for the character

$$\begin{pmatrix}[\lambda] & 0\\ 0 & [\mu]\end{pmatrix}\mapsto(\lambda)^r(\lambda\mu)^a$$

for all $\lambda,\mu\in\mathbb{F}_p^{\times}$ and $[\ ]$ is the Teichmuller lift. For convenience we use the notation $\chi_{a,r}$ instead of χ. The idempotent e_χ in [Paš10, Section 9] will be denoted $e_{\chi_{a,r}}$. The space $\operatorname{Hom}_{K_1}(\sigma_r\otimes\det^a,\Omega/\tilde{\pi}_r)$ is the same as

$$\ker(\mathcal{I}(\Omega/\tilde{\pi}_r)e_{\chi_{r,a}}\xrightarrow{T_{n_s}}\mathcal{I}(\Omega/\tilde{\pi}_r)e_{\chi^s_{r,a}}) \tag{17}$$

and from [Paš10, Proposition 10.10] has dimension less than or equal to 2. For $0\leq r\leq(p-1)/2$ the space $\operatorname{Hom}_{K/Z_1}(\sigma_r\otimes\det^a,\tilde{\pi}_r)$ is non-zero if and only if $a=0$ and has dimension 1 if $r<(p-1)/2$ and 2 otherwise. Using (17) for $0\leq r<(p-1)/2$ the space $\operatorname{Ext}^1_{K/Z_1}(\sigma_r\otimes\det^a,\tilde{\pi}_r)$ is non-zero if and only if $a=0$ and has dimension at most 2 (see [Paš10, Proposition 10.10] for $0<r<(p-1)/2$ and [Paš13, Corollary 6.13 and Corollary 6.16] for $r=0$). When $r=(p-1)/2$ the space $\operatorname{Ext}^1_{K/Z_1}(\sigma_r\otimes\det^a,\tilde{\pi}_r)$ is non-zero for $a=0$ and $a=(p-1)/2$ and in each of these cases the dimension of the space $\operatorname{Ext}^1_{K/Z_1}(\sigma_r\otimes\det^a,\tilde{\pi}_r)$ is less than or equal to 2.

Now using exact sequence (16) the space $\operatorname{Ext}^1_{G_S}(\tilde{\pi}_r,\tilde{\pi}_r)$ has dimension less than or equal to 6 for $0\leq r<(p-1)/2$ and its dimension is less than or equal to 12 if $r=(p-1)/2$. For $r\neq 0$ using this upper bound and the lower bounds from Corollary 4.2 and Corollary 3.3 we deduce the theorem in this case. When $r=0$ Paškūnas showed that (see [Paš10, Proposition 6.15]) the dimension of $\operatorname{Ext}^1_{G^+/Z}(\pi_{i,0},\pi_{j,0})$ is 2 when $i\neq j$ and 1 otherwise. Since $G_S/\{\pm 1\}$ has index a factor of 2 in G^+/Z and $G_S\cap Z$ acts trivially on $\pi_{i,0}$ we get that

$$\operatorname{Ext}^1_{G^+/Z}(\pi_{i,0},\pi_{j,0})\hookrightarrow\operatorname{Ext}^1_{G_S/\{\pm1\}}(\pi_{i,0},\pi_{j,0})=\operatorname{Ext}^1_{G_S}(\pi_{i,0},\pi_{j,0}). \tag{18}$$

From our upper bounds the inclusions (18) are strict and hence we prove the theorem. □

Corollary 4.4. *The Hecke module $\mathbb{R}^1\mathcal{I}(\pi_{i,0})$ is isomorphic to the module $\mathcal{I}(\pi_{i,0})\oplus\mathcal{I}(\pi_{j,0})$ for $i\neq j$.*

Proof. From the Theorem 4.3, exact sequence (3) and (10) we get that dimension of the space

$$\mathrm{Hom}_{\mathcal{H}}(\mathcal{I}(\pi_{0,0}), \mathbb{R}^1\mathcal{I}(\pi_{0,0}))$$

is 1 and using the Proposition 3.1 we get that $\mathbb{R}^1\mathcal{I}(\pi_{i,0}) \simeq \mathcal{I}(\pi_{i,0}) \oplus \mathcal{I}(\pi_{j,0})$. $\square$

The mod p reduction of the compact groups K_0 and K_1 is isomorphic to $\mathrm{SL}_2(\mathbb{F}_p)$. Let σ_r^0 and σ_r^1 be the representations of K_0 and K_1 obtained by inflation of $\mathrm{Sym}^r\bar{\mathbb{F}}_p{}^2$. The Satake transform identifies the Hekce algebra $\mathrm{End}_{G_S}(\mathrm{ind}_{K_i}^{G_S}\sigma_r^i)$ with a polynomial algebra in one variable and we choose a generator τ_r^i of this algebra. Let $\pi(K_i, r, \lambda)$ be the quotient

$$\frac{\mathrm{ind}_{K_i}^{G_S}\sigma_r^i}{(\tau_r^i - \lambda)(\mathrm{ind}_{K_i}^{G_S}\sigma_r^i)}.$$

It was shown by Abdellatif that $\pi(K_0, r, 0)$ and $\pi(K_1, p-1-r, 0)$ are non-split extensions of $\pi_{0,r}$ by $\pi_{1,r}$.

Corollary 4.5. *The extensions of G_S smooth representations $[\pi_{1,0} \to \pi(K_0,0,0) \to \pi_{0,0}]$ and $[\pi_{1,0} \to \pi(K_1, p-1, 0) \to \pi_{0,0}]$ are equivalent.*

Proof. For $r = 0$ there is exactly one extension class E in $\mathrm{Ext}^1_{G_S}(\pi_{i,0}, \pi_{j,0})$ for $i \neq j$ such that E is generated by its $I(1)_S$ invariants. $\square$

5 Extensions of non-supersingular modules.

In this section we are interested in calculating extensions of non-supersingular Hecke modules. The degree one extensions of $\mathrm{SL}_2(\mathbb{Q}_p)$ are calculated by (Julien...). However, he uses ordinary parts functor and higher derivatives defined by Emerton. We hope that results of this section help us undertand the $I(1)_S$ invariants of these extensions. To begin with we recall the structure of all non-supersingular Hecke modules. We follow an approach via representation theory. We describe the structure of $\mathcal{I}(\eta)$, $\mathcal{I}(\mathrm{St} \otimes \eta \circ \det)$ and finally $\mathcal{I}(\mathrm{ind}_{B_S}^{G_S}\eta)$ for $\eta \neq \mathrm{id}$.

The Hecke algebra $\mathcal{H}$ is generated by $\{T_t; t \in \Omega\}$, T_{s_0} and T_{s_1}. We now describe the action of these operators on $\mathcal{I}(\pi)$ where π is a non-supersingular representation of G_S. From [BP12, Corollary 6.4] we deduce the structure of $\mathfrak{m}(r, \lambda, \eta) := \pi(\sigma_r, \lambda, \eta)^{I(1)}$ as a module of $\mathcal{H}$. Note that the operator T_Π in the article and T_{s_1} are related as $T_{s_1} = T_\Pi T_{s_0} T_{\Pi^{-1}}$. Since we are looking at restriction to G_S we may assume that η is trivial and we will use the notation $\mathfrak{m}(r, \lambda)$ for $\mathfrak{m}(r, \lambda, \eta)$ The module $\mathfrak{m}(r, \lambda)$ has a basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ such that

1. $\mathbf{v}_1\, e_{\chi_{p-1-r}} = \mathbf{v}_1$ and $\mathbf{v}_2\, e_{\chi_r} = \mathbf{v}_2$,
2. If $0 < r < p-1$ then $\mathbf{v}_1\, T_{s_0} = 0$, $\mathbf{v}_1\, T_{s_1} = \lambda\, \mathbf{v}_2$, $\mathbf{v}_2\, T_{s_1} = 0$ and $\mathbf{v}_2\, T_{s_0} = \lambda\, \mathbf{v}_1$,
3. If $r = 0$ then $\mathbf{v}_1(1 + T_{s_1}) = \lambda\, \mathbf{v}_2$, $\mathbf{v}_2\, T_{s_1} = 0$, $\mathbf{v}_1\, T_{s_0} = 0$ and $\mathbf{v}_2(1 + T_{s_0}) = \lambda\, \mathbf{v}_1$
4. If $r = (p-1)$ then $\mathbf{v}_1\, T_{s_1} = \lambda\, \mathbf{v}_2$, $\mathbf{v}_2\, T_{s_1} = -\,\mathbf{v}_2$, $\mathbf{v}_1\, T_{s_0} = -\,\mathbf{v}_1$ and $\mathbf{v}_2\, T_{s_0} = \lambda\, \mathbf{v}_1$

Let κ be non-trivial character of B_S and we assume that $\kappa = \mu_\lambda \chi_{p-1-r}$. Observe that $(r, (\lambda)^{1/2}) \neq (0, \pm 1)$. This shows us that the representation $\mathrm{ind}_{B_S}^{G_S}(\kappa)$ is isomorphic to $\pi(p-1-r, \lambda^{-1/2}, \mathrm{id})$. Hence the module $\mathcal{I}(\mathrm{ind}_{B_S}^{G_S}(\mu_\lambda \chi_r))$ is isomorphic to $\mathfrak{m}(p-1-r, \lambda^{-1/2})$. Now the Hecke modules $\mathcal{I}(\mathrm{id})$ and $\mathcal{I}(\mathrm{St})$ are both characters and fit in an exact sequence of the form

$$0 \to \mathcal{I}(\mathrm{St}) \to \mathfrak{m}(0,1) \to \mathcal{I}(\mathrm{id}) \to 0.$$

It is clear that the vector $\mathbf{v}_3 := \mathbf{v}_1 - \mathbf{v}_2$ is stable under the action of T_{s_0} and T_{s_1} and moreover $T_{s_0}\,\mathbf{v}_3 = -\,\mathbf{v}_3$ and $T_{s_0}\,\mathbf{v}_3 = -\,\mathbf{v}_3$. The module $\mathfrak{m}(0, 1, \mathrm{id})$ is indecomposable and has quotient isomorphic to the character T_{s_0} and T_{s_1} to zero. We denote by $\mathfrak{m}(-1)$ and $\mathfrak{m}(0)$ for the subrepresentation and sub quotient of $\mathfrak{m}(0,1)$. This completes the classification of simple non-supersingular modules.

From the decomposition (see) we know that extensions can only exist between the subquotients of the modules $\mathfrak{m}(r,\lambda)$ and $\mathfrak{m}(p-1-r,\lambda')$. If $0<r<p-1$ then extensions can only exist between the simple modules $\mathfrak{m}(r,\lambda)$ and $\mathfrak{m}(p-1-r,\lambda')$. Another possibility is $\mathfrak{m}(0,\lambda)$ when $\lambda\neq\pm1$ can have extensions with $\mathfrak{m}(0,\lambda)$, $\mathfrak{m}(0,\lambda')$ with $\lambda'\neq\pm1$, $\mathfrak{m}(0)$ and $\mathfrak{m}(-1)$. In next few lemmas we determine these relations.

Lemma 5.1. *Let (r_1,λ_1) and (r_2,λ_2) such that $0<r_i<p-1$. Then,*

1. *If $r_1=r_2\neq(p-1)/2$ or $r_1\neq r_2$ and $r_1+r_2=p-1$ the dimension of $\mathrm{Ext}^1_{\mathcal{H}}(\mathfrak{m}(r_1,\lambda_1),\mathfrak{m}(r_2,\lambda_2))$ is 3 if $\lambda_1\neq\lambda_2$ and is 2 otherwise.*
2. *If $r_1=r_2=(p-1)/2$ then the dimension of $\mathrm{Ext}^1_{\mathcal{H}}(\mathfrak{m}(r_1,\lambda_1),\mathfrak{m}(r_2,\lambda_2))$ is 5 when $\lambda_1\neq\lambda_2$ and is 4 otherwise.*
3. *If $r_1\neq r_2$ and $r_1+r_2\neq p-1$ the dimension of $\mathrm{Ext}^1_{\mathcal{H}}(\mathfrak{m}(r_1,\lambda_1),\mathfrak{m}(r_2,\lambda_2))$ is zero.*

Proof. We will first calculate the dimensions of $\mathrm{Ext}^1_{\mathcal{H}_{v_i}}(\mathfrak{m}(r_1,\lambda_1),\mathfrak{m}(r_2,\lambda_2))$. Let E be an extension of $\mathfrak{m}(r_1,\lambda_1)$ by $\mathfrak{m}(r_2,\lambda_2)$. Let $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ and $\mathbf{v}_4$ be a basis of E such that $\mathbf{v}_1$ and $\mathbf{v}_2$ be basis for $\mathfrak{m}(r_2,\lambda_2)$ such that the relations 5 hold. Now we have

$$\begin{aligned}T_{s_i}\mathbf{v}_3&=a\,\mathbf{v}_1+b\,\mathbf{v}_2\\ T_{s_i}\mathbf{v}_4&=c\,\mathbf{v}_1+d\,\mathbf{v}_2+\lambda_2\,\mathbf{v}_3\end{aligned}$$

Now $0=T_{s_i}^2\mathbf{v}_3=b\lambda_1\mathbf{v}_1$ and hence b is zero. We also have $0=T_{s_i}^2\mathbf{v}_4=d\lambda_1\mathbf{v}_1+a\lambda_2\mathbf{v}_1$ and hence $(d\lambda_1+a\lambda_2)=0$. Moreover we have

$$\begin{aligned}t^{r_2}a\,\mathbf{v}_1=T_{s_i}T_t\,\mathbf{v}_3&=T_{t^{-1}}T_{s_i}\mathbf{v}_3=at^{p-1-r_1}\mathbf{v}_1\\ t^{p-1-r_2}(c\,\mathbf{v}_1-(a\lambda_2/\lambda_1)\,\mathbf{v}_2+\lambda_2\,\mathbf{v}_3)=T_{s_i}T_t\,\mathbf{v}_4&=T_{t^{-1}}T_{s_i}\mathbf{v}_4=T_{t^{-1}}(c\,\mathbf{v}_1+d\,\mathbf{v}_2+\lambda_2\,\mathbf{v}_3)\\ &=ct^{p-1-r_1}\mathbf{v}_1-(a\lambda_2/\lambda_1)t^{r_1}\mathbf{v}_2+\lambda_2t^{p-1-r_2}\mathbf{v}_3\,.\end{aligned}$$

Suppose $r_1=r_2\neq(p-1)/2$ then $a=0$ and c is arbitrary. If $(r_1=r_2=(p-1)/2)$ then both a and c can take arbitrary values. Suppose $r_1\neq r_2$ and $r_1+r_2\neq p-1$ then $a=0$ and also $c=0$ which is expected since the modules are in different blocks. Suppose $r_1+r_2=p-1$ then only $c=0$ and a is arbitrary. Hence when $r_1=r_2\neq(p-1)/2$ the space $\mathrm{Ext}^1_{\mathcal{H}_{v_i}}(\mathfrak{m}(r_1,\lambda_1),\mathfrak{m}(r_2,\lambda_2))$ has dimension 1 when $r_1=r_2=(p-1)/2$ and has dimension 2 when $r_1=r_2=(p-1)/2$. If $r_1+r_2=p-1$ and $r_1\neq r_2$ the dimension of $\mathrm{Ext}^1_{\mathcal{H}_{v_i}}(\mathfrak{m}(r_1,\lambda_1),\mathfrak{m}(r_2,\lambda_2))$ is 1. In the rest of the cases it is zero.

We return to the final calculations. Consider the case when $r_1=r_2\neq(p-1)/2$. The dimension of the space $\mathrm{Hom}_{\mathcal{H}_E}(\mathfrak{m}(r_1,\lambda_1),\mathfrak{m}(r_2,\lambda_2))$ is 2, the dimension of the space $\mathrm{Hom}_{\mathcal{H}_{v_i}}(\mathfrak{m}(r_1,\lambda_1),\mathfrak{m}(r_2,\lambda_2))$ is one since intertwining operators have to commute by T_t and each module is indecomposable. Now $\mathrm{Hom}_{\mathcal{H}}(\mathfrak{m}(r_1,\lambda_1),\mathfrak{m}(r_2,\lambda_2))$ has dimension 1 if $\lambda_1=\lambda_2$ and zero otherwise. This shows that the space $\mathrm{Ext}^1_{\mathcal{H}}(\mathfrak{m}(r_1,\lambda_1),\mathfrak{m}(r_2,\lambda_2))$ has dimension 3 if $\lambda_1=\lambda_2$ and has dimension 2 otherwise.

Assume that $r_1=r_2=(p-1)/2$. In this case the dimension of $\mathrm{Hom}_{\mathcal{H}_E}(\mathfrak{m}(r_1,\lambda_1),\mathfrak{m}(r_2,\lambda_2))$ is 4, the dimension of $\mathrm{Hom}_{\mathcal{H}_{v_i}}(\mathfrak{m}(r_1,\lambda_1),\mathfrak{m}(r_2,\lambda_2))$ is 2. Now $\mathrm{Hom}_{\mathcal{H}}(\mathfrak{m}(r_1,\lambda_1),\mathfrak{m}(r_2,\lambda_2))$ has dimension 1 if $\lambda_1=\lambda_2$ and zero otherwise. This shows that the space $\mathrm{Ext}^1_{\mathcal{H}}(\mathfrak{m}(r_1,\lambda_1),\mathfrak{m}(r_2,\lambda_2))$ has dimension 5 if $\lambda_1=\lambda_2$ and has dimension 4 otherwise.

If $r_1\neq r_2$ then extensions can only exist when $r_1+r_2=p-1$. In this case the dimension of $\mathrm{Hom}_{\mathcal{H}_E}(\mathfrak{m}(r_1,\lambda_1),\mathfrak{m}(r_2,\lambda_2))$ is 2, the dimension of $\mathrm{Hom}_{\mathcal{H}_{v_i}}(\mathfrak{m}(r_1,\lambda_1),\mathfrak{m}(r_2,\lambda_2))$ is 1 using similar calculations in $r_1=r_2$ case. Finally the space $\mathrm{Hom}_{\mathcal{H}}(\mathfrak{m}(r_1,\lambda_1),\mathfrak{m}(r_2,\lambda_2))$ has dimension 1 if $\lambda_1=\lambda_2$ and zero otherwise. This shows that $\mathrm{Ext}^1_{\mathcal{H}}(\mathfrak{m}(r_1,\lambda_1),\mathfrak{m}(r_2,\lambda_2))$ has dimension 3 if $\lambda_1=\lambda_2$ and has dimension 2 otherwise. ∎

Now we will study the extensions of $\mathfrak{m}(0)$ by non-supersingular Hecke modules.

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