# Completed Iwahori-Hecke algebras and parahorical Hecke algebras for Kac-Moody groups over local fields

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**Abstract**

Let $G$ be a split Kac-Moody group over a non-archimedean local field. We define a completion of the Iwahori-Hecke algebra of $G$. We determine its center and prove that it is isomorphic to the spherical Hecke algebra of $G$ using the Satake isomorphism. This is thus similar to the situation of reductive groups. Our main tool is the masure $\mathcal{I}$ associated to this setting, which is the analogue of the Bruhat-Tits building for reductive groups. Then, for each special and spherical facet $F$, we associate a Hecke algebra. In the Kac-Moody setting, this construction was known only for the spherical subgroup and for the Iwahori subgroup.

## 1 Introduction

Let $\mathbf{G}_0$ be a split reductive group over a non archimedean local field $\mathcal{K}$ and $G_0 = \mathbf{G}_0(\mathcal{K})$. An important tool in the study of the representation theory of $G$ are Hecke algebras. They are attached to each compact open subgroup of $G$: if $K$ is such a group, the Hecke algebra $\mathcal{H}_K$ associated to $K$ is the set of $K$-bi-invariant functions on $G_0$, with compact support; this set being equipped with some convolution product. Two choices of $K$ are of particular interest. The first one is when $K = K_s$ is a maximal compact open subgroup of $G$. Then $\mathcal{H}_s = \mathcal{H}_{K_s}$ is commutative and is called the spherical algebra of $G_0$. This algebra can be described explicitly thanks to the Satake isomorphism: if $W^v$ is the Weyl group of $G$ and $Q^\vee$ is the coweight lattice of $G_0$, $\mathcal{H}_s$ is isomorphic to $\mathbb{C}[Q^\vee]^{W^v}$, which is the sub-algebra of $W^v-$invariant elements of the algebra of the group $(Q^\vee, +)$. The second is when $K = K_I$ is the Iwahori subgroup of $G_0$. Then $\mathcal{H} := \mathcal{H}_{K_I}$ is called the Iwahori-Hecke algebra of $G_0$. It has a basis indexed by the affine Weyl group of $G$, and the product of two elements of this basis can be expressed with the Bernstein-Lusztig presentation. It enables to determine the center of $\mathcal{H}$ and one sees that it is isomorphic to the spherical Hecke algebra of $G_0$. We summarize this results as follows:

$$\mathcal{H}_s \underset{S}{\xrightarrow{\backsim}} \mathbb{C}[Q^\vee]^{W^v} \underset{g}{\hookrightarrow} \mathcal{H}, \text{ and } \mathrm{Im}(g) = \mathcal{Z}(\mathcal{H})$$

where $S$ is the Satake isomorphism and $g$ comes from the Bernstein-Lusztig basis. The aim of this article is to study the extension of this theory to the Kac-Moody setting.

Kac-Moody groups are interesting generalizations of reductive groups. There are several definitions of Kac-Moody groups but we will study the groups defined by Tits in [Tit87]. Let $\mathbf{G}$ be a split Kac-Moody group over $\mathcal{K}$ and $G = \mathbf{G}(\mathcal{K})$. In order to study $G$, Gaussent and Rousseau constructed an object $\mathcal{I} = \mathcal{I}(G)$ called "masure" (or affine ordered hovel) in [GR08], and it was generalized in [Rou16] and [Rou12]. This masure is a generalization of a Bruhat-Tits building (introduced in [BT72] and [BT84]) and when $G$ is reductive, $\mathcal{I}$ is the usual Bruhat-Tits building of $G$. The set $\mathcal{I}$ is a union of apartments, all isomorphic to a standard one $\mathbb{A}$ and $G$ acts on $\mathcal{I}$. There is still an arrangement of hyperplanes, called walls, but it is no more locally finite in general. For this reason the faces in $\mathbb{A}$ are no more sets but filters. An other important difference with buildings is that there can be two points which are not included in any apartment.

There is up to now no topology on $G$ generalizing the usual topology of reductive groups. We can nevertheless define analogs of $K_s$ and $K_I$ in this setting. They are fixers of some faces in $\mathcal{I}$. However, as we shall see, there exists no topology of topological group on $G$ such that $K_s$ is open compact, and the same result holds for $K_I$ (this is Proposition 3.5). Braverman, Kazhdan and Patnaik extended the definition of the spherical Hecke algebra and of the Iwahori-Hecke algebra of $G$ when $G$ is affine in [BK11] and [BKP16]. They obtained a Satake isomorphism and Bernstein-Lusztig relations. These definitions and results where extended to the general case ($G$ no more assumed affine) by Bardy-Panse, Gaussent and Rousseau in [GR14] and [BPGR16]. In this framework, the Satake isomorphism is an isomorphism between $\mathcal{H}_s := \mathcal{H}_{K_s}$ and $\mathbb{C}[[Y]]^{W^v}$, where $Y$ is a lattice which can be thought of as the coroot lattice in a first approximation (but it can be different, notably when $G$ is affine) and $\mathbb{C}[[Y]]$ is the Looijenga's algebra of $Y$, which is some completion of the group algebra $\mathbb{C}[Y]$ of $Y$. Let $\mathcal{H}$ be the Iwahori-Hecke algebra of $G$. As we shall see (Theorem 4.23), the center of $\mathcal{H}$ is more or less trivial. Moreover, $\mathbb{C}[[Y]]^{W^v}$ is a set of infinite formal series and there is no obvious injection from $\mathbb{C}[[Y]]$ to $\mathcal{H}$. For these reasons, we define a "completion" $\hat{\mathcal{H}}$ of $\mathcal{H}$. More precisely, let $(Z^\lambda H_w)_{\lambda\in Y^+, w\in W^v}$, where $Y^+$ is a sub-monoid of $Y$, be the Bernstein-Lusztig basis of $\mathcal{H}$. Then $\hat{\mathcal{H}}$ is the set of formal series $\sum_{w\in W^v,\lambda\in Y^+} c_{w,\lambda} Z^\lambda H_w$, whose support satisfy some conditions similar to what appears in the definition of $\mathbb{C}[[Y]]$. We equip it with a convolution compatible with the inclusion $\mathcal{H}\subset\hat{\mathcal{H}}$. The fact that this product is well defined is not obvious and this is our main result: Theorem 4.14. We then determine the center of $\hat{\mathcal{H}}$ and we show that it is isomorphic to $\mathbb{C}[[Y]]^{W^v}$ (Theorem 4.23), which is similar to the classical case. We thus get the following diagram:

$$\mathcal{H}_s \xrightarrow[S]{\backsimeq} \mathbb{C}[[Y]]^{W^v} \underset{g}{\hookrightarrow} \hat{\mathcal{H}}, \text{ and } \mathrm{Im}(g) = \mathcal{Z}(\hat{\mathcal{H}}),$$

where $S$ is the Satake isomorphism (see Section 8 of [BK11] or Theorem 5.4 of [GR14]), and $g$ comes from the Bernstein-Lusztig basis.

In a second part, we associate Hecke algebras to subgroups more general than $K_I$. The group $K_s$ is the fixer of $\{0\}$ and $K_I$ is the fixer of some chamber $C_0^+$ based at zero. When $G$ is reductive, for all faces $F$ between $\{0\}$ and $C_0^+$, the fixer $K_F$ (the parahoric group associated to $F$) of $F$ in $G$ is compact open. Therefore it seems natural to try to associate a Hecke algebra to the fixer $K_F$ of $F$ in $G$ for all facet $F$ between $\{0\}$ and $C_0^+$. We succeed in defining such an algebra when $F$ is spherical, which means that its fixer in the Weyl group is finite. Our construction is very close to the construction of the Iwahori-Hecke algebra of [BPGR16]. When $F$ is no more spherical and different from $\{0\}$ (this case does not occur when $G$ is affine), we prove that this construction fails: the structure constants are infinite.

Actually, this article is written in a more general framework explained in Section 2: we ask $\mathcal{I}$ to be an abstract masure and $G$ to be a strongly transitive group of (positive, type-preserving) automorphisms of $\mathcal{I}$. This applies in particular to almost-split Kac-Moody groups over local fields.

In Section 2, one recalls the definition of masures. The reader only interested in Section 4 can read only Subsection 2.1.

In Section 3 we prove that there exists no topology of topological group on $G$ for which $K_s$ or $K_I$ are compact and open.

In Section 4, we define the completed Iwahori-Hecke algebra of $G$ and determine its center.

In Section 5, we associate Hecke algebras to each spherical facet between $\{0\}$ and $C_0^+$ and prove that this construction fails when $F$ is not spherical and different from $\{0\}$.

# Contents




**Acknowledgement** We thank Stéphane Gaussent for suggesting our collaboration, discussions on the subject and comments on previous versions of this manuscript. We thank Nicole Bardy-Panse and Guy Rousseau for discussions on this topic.

**Funding** The first author was supported by the ANR grant ANR-16-CE40-0010-01 and the second by the ANR grant ANR-15-CE40-0012.

# 2 General framework, Masure

## 2.1 Standard apartment

### 2.1.1 Root generating system

A reference for this section is [Rou11], Section 1 and 2.

A Kac-Moody matrix (or generalized Cartan matrix) is a square matrix $C = (c_{i,j})_{i,j\in I}$ with integers coefficients, indexed by a finite set $I$ and satisfying:

1. $\forall i \in I,\ c_{i,i} = 2$
2. $\forall (i,j) \in I^2 | i \neq j,\ c_{i,j} \leq 0$
3. $\forall (i,j) \in I^2,\ c_{i,j} = 0 \Leftrightarrow c_{j,i} = 0$.

A root generating system is a 5-tuple $\mathcal{S} = (C, X, Y, (\alpha_i)_{i\in I}, (\alpha_i^\vee)_{i\in I})$ made of a Kac-Moody matrix $C$ indexed by $I$, of two dual free $\mathbb{Z}$-modules $X$ (of characters) and $Y$ (of cocharacters) of finite rank $\mathrm{rk}(X)$, a family $(\alpha_i)_{i\in I}$ (of simple roots) in $X$ and a family $(\alpha_i^\vee)_{i\in I}$ (of simple coroots) in $Y$. They have to satisfy the following compatibility condition: $c_{i,j} = \alpha_j(\alpha_i^\vee)$ for all $i, j \in I$. We also suppose that the family $(\alpha_i)_{i\in I}$ is free in $X$ and that the family $(\alpha_i^\vee)_{i\in I}$ is free in $Y$.

We now fix a Kac-Moody matrix $C$ and a root generating system with matrix $C$.

Let $\mathbb{A} = Y \otimes \mathbb{R}$. Every element of $X$ induces a linear form on $\mathbb{A}$. We will consider $X$ as a subset of the dual $\mathbb{A}^*$ of $\mathbb{A}$: the $\alpha_i$, $i \in I$ are viewed as linear form on $V$. For $i \in I$, we define an involution $r_i$ of $V$ by $r_i(v) = v - \alpha_i(v)\alpha_i^\vee$ for all $v \in V$. Its space of fixed points is $\ker \alpha_i$. The subgroup of $\mathrm{GL}(\mathbb{A})$ generated by the $\alpha_i$ for $i \in I$ is denoted by $W^v$ and is called the Weyl group of $\mathcal{S}$. The system $(W^v, \{r_i | i \in I\})$ is a Coxeter system. For $w \in W^v$, we denote by $l(w)$ the length of $w$ with respect to $\{r_i | i \in I\}$.

For $x \in \mathbb{A}$ one sets $\underline{\alpha}(x) = (\alpha_i(x))_{i\in I} \in \mathbb{R}^I$.

Let $Q^\vee = \bigoplus_{i\in I} \mathbb{Z}\alpha_i^\vee$ and $P^\vee = \{v \in \mathbb{A} | \underline{\alpha}(v) \in \mathbb{Z}^I\}$. We call $Q^\vee$ the *coroot-lattice* and $P^\vee$ the *co-weight-lattice* (but if $\bigcap_{i\in I} \ker \alpha_i \neq \{0\}$, this is not a lattice). Let $Q^\vee_+ = \bigoplus_{i\in I} \mathbb{N}\alpha_i^\vee$, $Q^\vee_- = -Q^\vee_+$ and $Q^\vee_\mathbb{R} = \bigoplus_{i\in I} \mathbb{R}\alpha_i^\vee$. This enables us to define a pre-order $\leq_{Q^\vee}$ on $\mathbb{A}$ by the following way: for all $x, y \in \mathbb{A}$, one writes $x \leq_{Q^\vee} y$ if $y - x \in Q^\vee_+$.

One defines an action of the group $W^v$ on $\mathbb{A}^*$ by the following way: if $x \in \mathbb{A}$, $w \in W^v$ and $\alpha \in \mathbb{A}^*$ then $(w.\alpha)(x) = \alpha(w^{-1}.x)$. Let $\Phi = \{w.\alpha_i | (w,i) \in W^v \times I\}$, $\Phi$ is the set of *real roots*. Then $\Phi \subset Q$, where $Q = \bigoplus_{i\in I} \mathbb{Z}\alpha_i$. Let $W^a = Q^\vee \rtimes W^v \subset \mathrm{GA}(\mathbb{A})$ the *affine Weyl group* of $\mathcal{S}$, where $\mathrm{GA}(\mathbb{A})$ is the group of affine isomorphisms of $\mathbb{A}$.

### 2.1.2 Vectorial faces

Define $C_f^v = \{v \in \mathbb{A} |\ \alpha_i(v) > 0,\ \forall i \in I\}$. We call it the *fundamental chamber*. For $J \subset I$, one sets $F^v(J) = \{v \in \mathbb{A} |\ \alpha_i(v) = 0\ \forall i \in J, \alpha_i(v) > 0\ \forall i \in J \backslash I\}$. Then the closure $\overline{C_f^v}$ of $C_f^v$ is the union of the $F^v(J)$ for $J \subset I$. The *positive* (resp. *negative*) *vectorial faces* are the sets $w.F^v(J)$ (resp. $-w.F^v(J)$) for $w \in W^v$ and $J \subset I$. A *vectorial facet* is either a positive vectorial facet or a negative vectorial facet. We call *positive chamber* (resp. *negative*) every cone of the shape $w.C_f^v$ for some $w \in W^v$ (resp. $-w.C_f^v$). For all $x \in C_f^v$ and for all $w \in W^v$,

$w.x = x$ implies that $w = 1$. In particular the action of $w$ on the positive chambers is simply transitive. The *Tits cone* $\mathcal{T}$ is defined by $\mathcal{T} = \bigcup_{w \in W^v} w.\overline{C^v_f}$. We also consider the negative cone $-\mathcal{T}$. We define a $W^v$ invariant relation $\leq$ on $\mathbb{A}$ by: $\forall (x, y) \in V^2$, $x \leq y \Leftrightarrow y - x \in \mathcal{T}$.

In the next subsection, we define masures. The reader only interested in the completion of Iwahori-Hecke algebras can skip it and go directly to Section 4.

## 2.2 Masure

In this section, we define masures. They were introduced in [GR08] for symmetrizable split Kac-Moody groups over a valuated field whose residue field contains $\mathbb{Q}$, axiomatized in [Rou11] and developed and generalized to almost-split Kac-Moody groups over non-archimedean local fields in [Rou16] and [Rou12]. We consider semi-discrete masures which are thick of finite thickness.

### 2.2.1 Filters

**Definition 2.1.** *A filter in a set $E$ is a nonempty set $F$ of nonempty subsets of $E$ such that, for all subsets $S$, $S'$ of $E$, if $S$, $S' \in F$ then $S \cap S' \in F$ and, if $S' \subset S$, with $S' \in F$ then $S \in F$.*

If $F$ is a filter in a set $E$, and $E'$ is a subset of $E$, one says that $F$ contains $E'$ if every element of $F$ contains $E'$. If $E'$ is nonempty, the set $F_{E'}$ of subsets of $E$ containing $E'$ is a filter. By abuse of language, we will sometimes say that $E'$ is a filter by identifying $F_{E'}$ and $E'$. If $F$ is a filter in $E$, its closure $\overline{F}$ (resp. its convex envelope) is the filter of subsets of $E$ containing the closure (resp. the convex envelope) of some element of $F$. A filter $F$ is said to be contained in an other filter $F'$: $F \subset F'$ (resp. in a subset $Z$ in $E$: $F \subset Z$) if and only if any set in $F'$ (resp. if $Z$) is in $F$.

If $x \in V$ and $\Omega$ is a subset of $V$ containing $x$ in its closure, then the *germ* of $\Omega$ in $x$ is the filter $germ_x(\Omega)$ of subsets of $V$ containing a neighborhood in $\Omega$ of $x$.

A *sector* in $V$ is a set of the shape $\mathfrak{s} = x + C^v$ with $C^v = \pm w.C^v_f$ for some $x \in V$ and $w \in W^v$. The point $x$ is its *base point* and $C^v$ is its *direction*. The intersection of two sectors of the same direction is a sector of the same direction.

The *sector-germ* of a sector $\mathfrak{s} = x + C^v$ is the filter $\mathfrak{S}$ of subsets of $V$ containing a $V$-translate of $\mathfrak{s}$. It only depends on the direction $C^v$. We denote by $+\infty$ (resp. $-\infty$) the sector-germ of $C^v_f$ (resp. of $-C^v_f$).

A ray $\delta$ with base point $x$ and containing $y \neq x$ (or the interval $]x, y] = [x, y] \backslash \{x\}$ or $[x, y]$) is called *preordered* if $x \leq y$ or $y \leq x$ and *generic* if $y - x \in \pm \mathring{\mathcal{T}}$, the interior of $\pm\mathcal{T}$.

### 2.2.2 Definitions of enclosures, faces, chimneys and related notions

Let $\Delta = \Phi \cup \Delta^+_{im} \cup \Delta^-_{im} \subset Q$ be the set of all roots (recall that $Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i$), defined in [Kac94]. The group $W^v$ stabilizes $\Delta$. For $\alpha \in \Delta$, and $k \in \mathbb{Z} \cup +\infty$, let $D(\alpha, k) = \{v \in V | \alpha(v) + k \geq 0\}$ (and $D(\alpha, +\infty) = V$ for all $\alpha \in \Delta$) and $D^\circ(\alpha, k) = \{v \in V | \alpha(v) + k > 0\}$ (for $\alpha \in \Delta$ and $k \in \mathbb{Z} \cup \{+\infty\}$).

Given a filter $F$ of subsets of $V$, its *enclosure* $\mathrm{cl}_V(F)$ is the filter made of the subsets of $V$ containing an element of $F$ of the shape $\bigcap_{\alpha \in \Delta} D(\alpha, k_\alpha)$ where $k_\alpha \in \mathbb{Z} \cup \{+\infty\}$ for all $\alpha \in \Delta$.

**faces** A *facet* $F$ in $V$ is a filter associated to a point $x \in V$ and a vectorial facet $F^v \subset V$. More precisely, a subset $S$ of $V$ is an element of the facet $F = F(x, F^v)$ if and only if, it contains an intersection of half-spaces $D(\alpha, k_\alpha)$ or open half-spaces $D^\circ(\alpha, k_\alpha)$, with $k_\alpha \in \mathbb{Z}$ for all $\alpha \in \Delta$, that contains $\Omega \cap (x + F^v)$, where $\Omega$ is an open neighborhood of $x$ in $V$.

There is an order on the faces: if $F \subset \overline{F'}$ we say that "$F$ is a facet of $F'$" or "$F'$ contains $F$". The dimension of a facet $F$ is the smallest dimension of an affine space generated by some $S \in F$. Such an affine space is unique and is called its support. A facet is said to be *spherical* if the direction of its support meets the open Tits cone $\mathring{\mathcal{T}}$ or its opposite $-\mathring{\mathcal{T}}$; then its point-wise stabilizer $W_F$ in $W^v$ is finite.

We have $W^a \subset P^\vee \rtimes W^v$. As $\alpha(P^\vee) \subset \mathbb{Z}$ for all $\alpha$ in $\bigoplus_{i \in I} \mathbb{Z}\alpha_i$, if $\tau$ is a translation of $V$ of a vector $p \in P^\vee$, then for all $\alpha \in Q$, $\tau$ permutes the sets of the shape $D(\alpha, k)$ where $k$ runs over $\mathbb{Z}$. As $W^v$ stabilizes $\Delta$, any element of $W^v$ permutes the sets of the shape $D(\alpha, k)$ where $\alpha$ runs over $\Delta$. Therefore, $W^a$ permutes the sets $D(\alpha, k)$, where $(\alpha, k)$ runs over $\Delta \times \mathbb{Z}$ and thus $W^a$ permutes the enclosures, faces, ... of $V$.

Let $V_{in} = \bigcap_{i \in I} \ker \alpha_i$. We denote by $F_0$ the facet $F(0, V_{in})$. Actually, when $V_{in}$ is not reduced to $\{0\}$, $\{0\}$ is not a facet but its fixer in $G$ equals the fixer of $F_0$, as we will see in Subsection 2.2.4.

A *chamber* (or alcove) is a maximal facet, or equivalently, a facet such that all its elements contains a nonempty open subset of $V$. We denote by $C_0^+$ the chamber $F(0, C_f^v)$.

A *panel* is a spherical facet maximal among faces that are not chambers or, equivalently, a spherical facet of dimension $n - 1$.

**Chimneys** In [Rou11], Rousseau defines chimneys and uses it in its axiomatization of masures. We do not define this notion, we only precise that each sector-germ is a splayed, solid chimney-germ, that each spherical facet is included in a solid chimney and that $W^a$ permutes the chimneys of $V$ (and preserve their properties: splayed, solid, ...).

### 2.2.3 Masure

An apartment of type $\mathbb{A}$ is a set $A$ with a nonempty set $\mathrm{Isom}(\mathbb{A}, A)$ of bijections (called Weyl-isomorphisms) such that if $f_0 \in \mathrm{Isom}(\mathbb{A}, A)$ then $f \in \mathrm{Isom}(\mathbb{A}, A)$ if and only if, there exists $w \in W^a$ satisfying $f = f_0 \circ w$. We will say isomorphism instead of Weyl-isomorphism in the sequel. An isomorphism between two apartments $\phi : A \to A'$ is a bijection such that ($f \in \mathrm{Isom}(\mathbb{A}, A)$ if, and only if, $\phi \circ f \in \mathrm{Isom}(\mathbb{A}, A')$). We extend all the notions that are preserved by $W^a$ to each apartment. Thus sectors, enclosures, faces and chimneys are well defined in any apartment of type $\mathbb{A}$.

**Definition 2.2.** *A masure of type $\mathbb{A}$ is a set $\mathcal{I}$ endowed with a covering $\mathcal{A}$ of subsets called apartments such that:*

*(MA1) Any $A \in \mathcal{A}$ admits a structure of an apartment of type $\mathbb{A}$.*

*(MA2) If $F$ is a point, a germ of a preordered interval, a generic ray or a solid chimney in an apartment $A$ and if $A'$ is another apartment containing $F$, then $A \cap A'$ contains the enclosure $\mathrm{cl}_A(F)$ of $F$ and there exists an isomorphism from $A$ onto $A'$ fixing $\mathrm{cl}_A(F)$.*

*(MA3) If $\mathfrak{R}$ is the germ of a splayed chimney and if $F$ is a facet or a germ of a solid chimney, then there exists an apartment that contains $\mathfrak{R}$ and $F$.*

*(MA4) If two apartments $A$, $A'$ contain $\mathfrak{R}$ and $F$ as in (MA3), then there exists an isomorphism from $A$ to $A'$ fixing $\mathrm{cl}_A(\mathfrak{R} \cup F)$.*

*(MAO) If $x$, $y$ are two points contained in two apartments $A$ and $A'$, and if $x \leq_A y$ then the two segments $[x, y]_A$ and $[x, y]_{A'}$ are equal.*

In this definition, one says that an apartment contains a germ of a filter if it contains at least one element of this germ. One says that a map fixes a germ if it fixes at least one element of this germ.

Until the end of this article, $\mathcal{I}$ will be a masure. We suppose that $\mathcal{I}$ is thick of *finite thickness*: the number of chambers (=alcoves) containing a given panel has to be finite, greater or equal to 3.

We assume that $\mathcal{I}$ has a strongly transitive group of automorphisms $G$, which means that all isomorphisms involved in the above axioms are induced by elements of $G$. We choose in $\mathcal{I}$ a fundamental apartment, that we identify with $\mathbb{A}$. As $G$ is strongly transitive, the apartments of $\mathcal{I}$ are the sets $g.\mathbb{A}$ for $g \in G$. The stabilizer $N$ of $\mathbb{A}$ induces a group $\nu(N)$ of affine automorphisms of $\mathbb{A}$ and we suppose that $\nu(N) = W^v \ltimes Y$.

An example of such a masure $\mathcal{I}$ is the masure associated to a split Kac-Moody group over a non-archimedean local field constructed in [GR08], and [Rou12], see Subsection 2.2.4.

**Definition 2.3.** (Pre-order on $\mathcal{I}$) As the pre-order $\leq$ on $\mathbb{A}$ (induced by the Tits cone) is invariant under the action of $W^a$, we can equip each apartment $A$ with a pre-order $\leq_A$. Let $A$ be an apartment of $\mathcal{I}$ and $x, y \in A$ such that $x \leq_A y$. Then by Proposition 5.4 of [Rou11], if $B$ is an apartment containing $x$ and $y$, $x \leq_B y$. This defines a relation $\leq$ on $\mathcal{I}$. By Théorème 5.9 of [Rou11], this defines a $G$-invariant pre-order on $\mathcal{I}$.

### 2.2.4 Example: masure associated to a Kac-Moody group

We consider the group functor $\mathbf{G}$ associated to the root generating system $\mathcal{S}$ in [Tit87] and in Chapitre 8 of [Ŕ02]. This functor is a functor from the category of rings to the category of groups satisfying axioms (KMG1) to (KMG 9) of [Tit87]. When $R$ is a field, $\mathbf{G}(R)$ is uniquely determined by these axioms by Theorem 1' of [Tit87]. This functor contains a toric functor $\mathbf{T}$, from the category of rings to the category of commutative groups (denoted $\mathcal{T}$ in [Ŕ02]) and two functors $\mathbf{U}^+$ and $\mathbf{U}^-$ from the category of rings to the category of groups.

Let $\mathcal{K}$ be a non-archimedean local field, $\mathcal{O}$ its ring of integers, $q$ the residue cardinal and $G = \mathbf{G}(\mathcal{K})$ (and $U^+ = \mathbf{U}^+(\mathcal{K})$, ...). For all $\epsilon \in \{-, +\}$, and all $\alpha \in \Phi^\epsilon$, we have an isomorphism $x_\alpha$ from $\mathcal{K}$ to a group $U_\alpha$. For all $k \in \mathbb{Z}$, one defines a subgroup $U_{\alpha,k} := x_\alpha(\pi^k\mathcal{O})$, where $\pi$ is a uniformizer of $\mathcal{O}$ (see 3.1 of [GR08]). Let $\mathcal{I}$ be the masure associated to $G$ constructed in [Rou12]. Then we have:

- the fixer of $\mathbb{A}$ in $G$ is $H = \mathbf{T}(\mathcal{O})$ (by remark 3.2 of [GR08])
- the fixer of $\{0\}$ in $G$ is $K_s = \mathbf{G}(\mathcal{O})$. By Lemma 5.2 of [Héb16b], $K_s$ is also the fixer of $F_0$ in $G$,
- for all $\alpha \in \Phi$ and $k \in \mathbb{Z}$, the fixer of $D(\alpha, k)$ in $G$ is $H.U_{\alpha,k}$ (by 4.2 7) of [GR08])
- for all $\epsilon \in \{-, +\}$, $U^\epsilon$ is the fixer of $\epsilon\infty$ (by 4.2 4) of [GR08]).

Each panel is contained in $1 + q$ chambers and thus $\mathcal{I}$ is thick of finite thickness.

**Remark 2.4.** The group $G$ is reductive if and only if $W^v$ is finite. In this case, $\mathcal{I}$ is the usual Bruhat-Tits building of $G$. In this case one has $\mathcal{T} = \mathbb{A}$ and thus $Y^+ = Y$.

# 3 A topological restriction on parahoric subgroups

In this section, we prove that there exists no topology of topological group on $G$ for which $K_s$ or $K_I$ are compact and open.

Let $F$ be a *special facet* of $\mathbb{A}$, i.e a facet whose vertex is in $Y$. Maybe considering $h.F$ for some $h \in G$, one can suppose that $F \subset \pm C_0^+$. One supposes that $F \subset C_0^+$ as the other case is similar. Let $K_F$ be the fixer of $F$ in $G$. In this subsection, we show that if $W^v$ is infinite, there exists no topology of topological group on $G$ such that $K_F$ is open and compact. For this, we show that there exists $g \in G$ such that $K_F/(K_F \cap g.K_F.g^{-1})$ is infinite.

Let $\alpha \in \Phi^+$ and $i \in I$ such that $\alpha = w.\alpha_i$, for some $w \in W^v$. For $l \in \mathbb{N}$, one sets $M_l = \{t \in \mathbb{A} | \alpha(t) = l\}$ and $D_l = \{t \in \mathbb{A} | \alpha(t) \leq l\}$ and one denotes by $K_{\alpha,l}$ the fixer of $D_l$ in $G$. For $l \in \mathbb{Z}$, one chooses a panel $P_l$ in $M_l$ and a chamber $C_l$ dominating $P_l$ and included in $\mathrm{conv}(M_l, M_{l+1})$.

For $i \in I$, one denotes by $1 + q_i$ the number of chambers containing $P_0$ and by $1 + q_i'$ the number of chambers containing $P_1$. By Proposition 2.9 of [Rou11] and Lemma 3.2 of [Héb16a], this does not depend on the choices of $P_0$ and $P_1$ (this reasoning will be detailed in the proof of Lemma 3.1). As $\alpha_i(\alpha_i^\vee) = 2$ and as there exists an element of $G$ inducing a translation of vector $\alpha_i^\vee$ on $\mathbb{A}$ (because the stabilizer $N$ of $\mathbb{A}$ induces the group $W^v \ltimes Y$ on $\mathbb{A}$), $1 + q_i$ is the number of chambers containing $P_{2l}$ and $1 + q_i'$ is the number of half-apartments containing $P_{2l+1}$ for all $l \in \mathbb{Z}$.

Let us explain the basic idea of the proof. Let $g \in G$ such that $g.0 \in C_f^v$ and $F' = g.F$. Then $K_F/(K_F \cap K_{F'})$ is in bijection with $K_F.F'$. Let $\tilde{K_\alpha} = \bigcup_{l\in\mathbb{Z}} K_{\alpha,l}$, then $\tilde{K_\alpha}.\mathbb{A}$ is a semi-homogeneous extended tree with parameters $q_i$ and $q_i'$. Let $K_\alpha = \bigcup_{l\geq 1} K_{\alpha,l}$. We deduce from the thickness of $\mathcal{I}$ that if $n_\alpha$ is the number of walls parallel to $\alpha^{-1}(\{0\})$ between $a$ and $g.a$, $|K_\alpha.g.a| \geq 2^{n_\alpha}$ and hence that $|K_F.g.a| \geq 2^{n_\alpha}$. As $W^v$ is infinite, $n_\alpha$ can be made arbitrarily large by changing $\alpha$.

**Lemma 3.1.** *Let $x \in \mathbb{A}$ and $M_x = \{t \in \mathbb{A} | \alpha(t) = \lceil \alpha(x) \rceil\}$. Then the map* $f : \begin{array}{c} K_\alpha.x \to K_\alpha.M_x \\ u.x \mapsto u.M_x \end{array}$ *is well defined and is a bijection.*

*Proof.* Let $u, u' \in K_\alpha$ such $u.x = u'.x$. Then $u.\mathbb{A} \cap u'.\mathbb{A} \supset D_0$ and by Lemma 3.2 of [Héb16a], $u.\mathbb{A} \cap u'.\mathbb{A}$ is a true half-apartment or $u.\mathbb{A} = u'.\mathbb{A}$. In both cases, $u.\mathbb{A} \cap u'.\mathbb{A}$ contains $u'.M_x$ and $u.M_x$. Let $\mathfrak{q}$ be a sector-germ of $D$ and $\rho : \mathcal{I} \to \mathbb{A}$ be the retraction centered at $\mathfrak{q}$ defined in 2.6 of [Rou11]. Then $u'.M_x$ is a wall of $u.\mathbb{A}$ retracting on $M_x$: $u.M_x = u'.M_x$. Therefore $f$ is well defined. ☐

If $l \in \mathbb{N}$, one denotes by $\mathcal{C}_l$ the set of chambers $C$ dominating an element of $K_\alpha.P_l$ and satisfying $P_l \subset \mathrm{conv}(D_0, C)$. Let $C_l$ be the chamber of $\mathbb{A}$ dominating $P_l$ and not included in $D_l$.

**Lemma 3.2.** *The map* $g : \begin{array}{c} K_\alpha.M_{l+1} \to \mathcal{C}_l \\ u.M_{l+1} \to u.C_l \end{array}$ *is well defined and is a bijection.*

*Proof.* The same reasoning as in Lemma 3.1 shows that $g$ is well defined and injective.

It remains to show that $\mathcal{C}_l = K_\alpha.C_l$. Let $C \in \mathcal{C}_l$. Then $C$ dominates $u.P_l$ for some $u \in K_\alpha$. By Proposition 2.9 1 of [Rou11], there exists an apartment $A$ containing $u.D_l$ and $C$. Let $\phi : A \to \mathbb{A}$ fixing $A \cap \mathbb{A}$ (such an isomorphism exists by Section 2.6 of [Rou11]) and $g \in G$ inducing $\phi$. Then $g.C$ is included in the half-apartment of $\mathbb{A}$ opposite to $D_l$ and dominates $P_l$ and thus $g.C = C_l$, which concludes the proof. ☐

By combining Lemma 3.1 and Lemma 5.5, we get the following corollary:

**Corollary 3.3.** *Let $x \in \mathbb{A}$. Then if $l = \max(0, \lceil \alpha(x) \rceil)$, $|K_\alpha.x| = q_i q_i' q_i \dots$ ($l$ factors)*

We now suppose that $W^v$ is infinite.

**Lemma 3.4.** *Let $F$ be a special facet of $\mathbb{A}$. Then there exists $g \in G$ such that if $F' = g.F$, $K_F/K_F \cap K_{F'}$ is infinite.*

*Proof.* Let $g \in G$ such $a := g.0 \in C^v_f$. Let $(\alpha_k) \in (\Phi^+)^{\mathbb{N}}$ be an injective sequence. For all $k \in \mathbb{N}$, $K_{\alpha_k} \subset K_F$ and thus $|K_F.F'| \geq |K_{\alpha_k}.a|$. By Corollary 3.3, it suffices to show that $\alpha_k(a) \to +\infty$ (by thickness of $\mathcal{I}$).

One has $\alpha_k = \sum_{i \in I} \lambda_{i,k} \alpha_i$, with $\lambda_{i,k} \in \mathbb{N}$ for all $(i,k) \in I \times \mathbb{N}$. By injectivity of $(\alpha_k)$, $\sum_{i \in I} \lambda_{i,k} \to +\infty$. Therefore, $\alpha_k(a) \to +\infty$, which proves the lemma. □

**Proposition 3.5.** *Let $F$ be a special facet of $\mathcal{I}$. Then there exists no topology on $G$, inducing a structure of topological group on $G$ such that $K_F$ is open and compact.*

*Proof.* Suppose that such a topology exists. Let $g \in G$ and $F' = g.F$. The group $K_{F'} = g.K_F.g^{-1}$ is open and compact and thus $K_F \cap K_{F'}$ is open and compact. Therefore $K_F/K_F \cap K_{F'}$ is finite: a contradiction with Lemma 3.4. □

This proposition applies to $K = K_s = K_{F_0}$ and to the Iwahori group $K_I = K_{C_0^+}$, which shows that reductive groups and (non-reductive) Kac-Moody groups are very different from this viewpoint.

# 4 Completed Iwahori-Hecke algebra

## 4.1 Iwahori-Hecke algebra

Let us recall briefly the construction of the Iwahori-Hecke algebra of [BPGR16]. We give here the construction by generators and relations. In [BPGR16], this algebra is first defined as an algebra of functions on pairs of chambers in a masure. This definition is recalled in Section 5. The definition we give is less general and imposes restrictions on the ring of scalars. It authorizes nevertheless $\mathbb{C}$ and $\mathbb{Z}[\frac{1}{\sqrt{q}}, \sqrt{q}]$ if $G$ is a split Kac-Moody group over $\mathcal{K}$.

Let $(\sigma_i)_{i \in I}, (\sigma_i')_{i \in I}$ be indeterminates satisfying the following relations:

- if $\alpha_i(Y) = \mathbb{Z}$, then $\sigma_i = \sigma_i'$
- if $r_i, r_j$ $(i, j \in I)$ are conjugate (i.e if $\alpha_i(\alpha_j^\vee) = \alpha_j(\alpha_i^\vee) = -1$), $\sigma_i = \sigma_j = \sigma_i' = \sigma_j'$.

When $G$ is a split Kac-Moody group over $\mathcal{K}$, $\sigma_i = \sigma_i' = \sqrt{q}$, for all $i \in I$. Let $\mathcal{R}_1 = \mathbb{Z}[\sigma_i, \sigma_i' | i \in I]$.

In order to define the Iwahori-Hecke algebra $\mathcal{H}$ associated to $\mathbb{A}$ and $(\sigma_i)_{i \in I}$, $(\sigma_i')_{i \in I}$, we first introduce the Bernstein-Lusztig-Hecke algebra ${}^{BL}\mathcal{H}$. Let ${}^{BL}\mathcal{H}$ be the free $\mathcal{R}_1$-module with basis $(Z^\lambda H_w)_{\lambda \in Y, w \in W^v}$. For short, we write $H_i = H_{r_i}$, $H_w = Z^0 H_w$ and $Z^\lambda H_1 = Z^\lambda$, for $i \in I$, $\lambda \in Y^+$ and $w \in W^v$. The Iwahori-Hecke algebra ${}^{BL}\mathcal{H}$ is the module ${}^{BL}\mathcal{H}$ equipped with the unique product $*$ which makes it an associative algebra and satisfying the following relations (the Bernstein-Luztig's relations):

1. $\forall \lambda \in Y$, $Z^\lambda * H_w = Z^\lambda H_w$,

2. $\forall i \in I$, $\forall w \in W^v$, $H_i * H_w = \begin{cases} H_{r_i w} & \text{if } l(r_i w) = l(w)+1 \\ (\sigma_i - \sigma_i^{-1})H_w + H_{r_i w} & \text{if } l(r_i w) = l(w)-1 \end{cases}$,

3. $\forall (\lambda,\mu) \in Y^2$, $Z^\lambda * Z^\mu = Z^{\lambda+\mu}$,

4. $\forall \lambda \in Y$, $\forall i \in I$, $H_i * Z^\lambda - Z^{r_i(\lambda)} * H_i = b(\sigma_i, \sigma_i'; Z^{-\alpha_i^\vee})(Z^\lambda - Z^{r_i(\lambda)})$, where $b(t,u;z) = \frac{(t+t^{-1})+(u-u^{-1})z}{1-z^2}$.

The existence and unicity of such a product is Theorem 6.2 of [BPGR16]. The Iwahori-Hecke algebra $\mathcal{H}_{\mathcal{R}_1}$ associated to $\mathbb{A}$ and $(\sigma_i)_{i\in I}$, $(\sigma_i')_{i\in I}$ over $\mathcal{R}_1$ is the submodule spanned by $(Z^\lambda H_w)_{\lambda\in Y^+, w\in W^v}$, where $Y^+ = Y \cap \mathcal{T}$ (where $\mathcal{T}$ is the Tits cone). When $G$ is reductive, we find the usual Iwahori-Hecke algebra of $G$.

**Extension of scalars** Let $(\mathcal{R},\phi)$ be a a couple such that $\mathcal{R}$ is a ring containing $\mathbb{Z}$, $\phi : \mathcal{R}_1 \to \mathcal{R}$ is a ring morphism and the $\sigma_i$ and $\sigma_i'$ are invertible in $\mathcal{R}$ for all $i \in I$. The Iwahori-Hecke algebra associated to $\mathbb{A}$ and $(\sigma_i)_{i\in I}$, $(\sigma_i')_{i\in I}$ over $\mathcal{R}$ is $\mathcal{H}_\mathcal{R} = \mathcal{R} \otimes_{\mathcal{R}_1} \mathcal{H}_{\mathcal{R}_1}$.

## 4.2 Looijanga's algebra and almost finite sets

We fix a ring $\mathcal{R}$ as in the above paragraph of Subsection 4.1. In this subsection, we introduce almost finite sets. We use them to define the Looijanga algebra $\mathcal{R}[[Y]]$ and we will use them in Subsection 4.3 to define $\hat{\mathcal{H}} = \hat{\mathcal{H}}_\mathcal{R}$.

**Definition 4.1.** A set $E \subset Y$ is said to be almost finite if there exists a finite set $J \subset Y$ such that for all $\lambda \in E$, there exists $\nu \in J$ such that $\lambda \leq_{Q^\vee} \nu$.

**Remark 4.2.** We will also use almost finite sets included in $Y^+$ and thus we could define an almost finite set of $Y^+$ as follows: a set $E \subset Y^+$ is almost finite if there exists a finite set $J \subset Y^+$ such that for all $\lambda \in E$, there exists $\nu \in J$ such that $\lambda \leq_{Q^\vee} \nu$. This is actually the same definition by the lemma below applied to $F = Y^+$.

**Lemma 4.3.** *Let $E \subset Y$ be an almost finite set and $F \subset Y$. Then there exists a finite set $J \subset F$ such that $F \cap E \subset \bigcup_{j\in J} j - Q^\vee_+$.*

*Proof.* One can suppose that $E \subset y - Q^\vee_+$, for some $y \in Y$. Let $J$ be the set of elements of $F \cap E$ which are maximal in $F \cap E$ for $\leq_{Q^\vee}$. As $E$ is almost finite, for all $x \in E$, there exists $\nu \in J$ such that $x \leq_{Q^\vee} \nu$. It remains to prove that $J$ is finite. Let $J' = \{u \in Q^\vee | y - u \in J\}$. One identifies $Q^\vee$ and $\mathbb{N}^I$. If $x = (x_i)_{i\in I}$ and $x' = (x_i')_{i\in I}$ one says that $x \prec x'$ if $x_i \leq x_i'$ for all $i \in I$ and $x \neq x'$. Then the elements of $J'$ are pairwise non comparable. Therefore $J'$ is finite by Lemma 2.2 of [Héb16b], which completes the proof. □

**Definition 4.4.** The Looijanga algebra $\mathcal{R}[[Y]]$ of $Y$ over $\mathcal{R}$ (defined in [Loo80]) is the set of formal series $\sum_{\lambda\in Y} a_\lambda e^\lambda$, with $(a_\lambda) \in \mathcal{R}^Y$ such that $\mathrm{supp}((a_\lambda)) \subset Y$ is almost finite and the $e^\lambda$ are symbols satisfying $e^\lambda e^\mu = e^{\lambda+\mu}$ for all $\lambda, \mu \in Y$.

For all $\lambda \in Y$, one defines $\pi_\lambda : \mathcal{R}[[Y]] \to \mathcal{R}$ by $\pi_\lambda(\sum_{\mu\in Y} a_\mu e^\mu) = a_\lambda$. One sets $\mathcal{R}[[Y^+]] = \{a \in \mathcal{R}[[Y]] \mid \pi_\lambda(a) = 0\ \forall \lambda \in Y\backslash Y^+\}$. One also sets $\mathcal{R}[[Y]]^{W^v} = \{a \in \mathcal{R}[[Y]] \mid \pi_\lambda(a) = \pi_{w(\lambda)}(a)\ \forall (\lambda, w) \in Y \times W^v\}$. Then $\mathcal{R}[[Y^+]]$ and $\mathcal{R}[[Y]]^{W^v}$ are sub-algebras of $\mathcal{R}[[Y]]$.

One denotes by $\mathrm{AF}_\mathcal{R}(Y^{++})$ the set of $a \in \mathcal{R}^{Y^{++}}$ with almost finite support. A family $(a_j)_{j\in J} \in (\mathcal{R}[[Y]])^J$ is said to be summable if:

- for all $\lambda \in Y$, $\{j \in J | \pi_\lambda(a_j) \neq 0\}$ is finite

- the set $\{\lambda \in Y \mid \exists j \in J | \pi_\lambda(a_j) \neq 0\}$ is almost finite.

In this case, one sets $\sum_{j\in J} a_j = \sum_{\lambda \in Y} b_\lambda e^\lambda \in R[[Y]]$, where $b_\lambda = \sum_{j\in J} \pi_\lambda(a_j)$ for all $\lambda \in Y$.

For all $\lambda \in Y^{++}$, one sets $E(\lambda) = \sum_{\mu \in W^v.\lambda} e^\mu \in \mathcal{R}[[Y]]$ (this is well defined by Lemma 2.4 a) of [GR14]. Let $\lambda \in \mathcal{T}$. There exists a unique $\mu \in \overline{C^v_f}$ such that $W^v.\lambda = W^v.\mu$. One defines $\lambda^{++} = \mu$. The following two results are proved (but not stated) in the proof of Theorem 5.4 of [GR14].

**Lemma 4.5.** *Let $x \in Y$. Then $W^v.x$ is majorized for $\leq_{Q^\vee}$ if and only if $x \in Y^+$.*

*Proof.* If $x \in Y^+$, then $W.x$ is majorized by $x^{++}$ by Lemma 2.4 of [GR14].

Let $x \in Y$ such that $W^v.x$ is majorized. Let $y \in W^v.x$ be maximal for $\leq_{Q^\vee}$ and $i \in I$. One has $r_i(y) \leq_{Q^\vee} y$ and thus $\alpha_i(y) \geq 0$. Therefore $y \in \overline{C^v_f}$, which proves that $x \in Y^+$. □

**Proposition 4.6.** *The map $E : \mathrm{AF}_\mathcal{R}(Y^{++}) \to \mathcal{R}[[Y]]^{W^v}$ defined by $E((x_\lambda)) = \sum_{\lambda \in Y^{++}} x_\lambda E(\lambda)$ is well defined and is a bijection. In particular, $\mathcal{R}[[Y]]^{W^v} \subset \mathcal{R}[[Y^+]]$.*

*Proof.* Let $(x_\lambda) \in \mathrm{AF}_\mathcal{R}(Y^{++})$ and $J$ be a finite set such that for all $\mu \in \mathrm{supp}((x_\lambda))$ there exists $j \in J$ such that $\mu \leq_{Q^\vee} j$. Let us prove that $(x_\lambda E(\lambda))_{\lambda \in Y^+}$ is summable. Let $\nu \in Y$ and $F_\nu = \{\lambda \in Y^{++} | \ \pi_\nu(x_\lambda E(\lambda)) \neq 0\}$. Let $\lambda \in F_\nu$. Then $\nu \in W^v.\lambda$ and by Lemma 2.4 a) of [GR14] $\nu \leq_{Q^\vee} \lambda$. Moreover, $\lambda \leq_{Q^\vee} j$ for some $j \in J$, which proves that $F_\nu$ is finite.

Let $F = \{\nu \in Y | \ \exists \lambda \in Y^{++} | \ \pi_\nu(x_\lambda E(\lambda)) \neq 0\}$. If $\nu \in F$ then $\nu \leq_{Q^\vee} j$ for some $j \in J$ and thus $F$ is almost finite: the family $(x_\lambda E(\lambda))_{\lambda \in Y^{++}}$ is summable.

As for all $\lambda \in Y^{++}$, $E(\lambda) \in \mathcal{R}[[Y]]^{W^v}$, $\sum_{\lambda \in Y^{++}} x_\lambda E(\lambda) \in \mathcal{R}[[Y]]^{W^v}$. Therefore, $E$ is well-defined.

Let $(x_\lambda) \in \mathrm{AF}_\mathcal{R}(Y^{++})$ such that $E((x_\lambda)) = 0$. Suppose that $(x_\lambda) \neq 0$. Let $\lambda \in Y^{++}$ be maximal for the $Q^\vee$ order such that $x_\lambda \neq 0$. Then $\pi_\lambda\big(E((x_\lambda))\big) = x_\lambda \neq 0$: this is absurd and hence $(x_\lambda) = 0$. Therefore $E$ is injective.

Let $u = \sum_{\lambda \in Y} u_\lambda e^\lambda \in \mathcal{R}[[Y]]^{W^v}$ and $\lambda \in \mathrm{supp}\ u$. As $\mathrm{supp}\ u$ is almost finite, $W^v.\lambda$ is majorized and by Lemma 4.5, $\lambda \in Y^+$. Consequently $\mathrm{supp}\ u \subset Y^+$. One has $u = E((u_\lambda)_{\lambda \in (\mathrm{supp}\ u)^{++}})$, and the proof is complete. □

### 4.3 Completed Iwahori-Hecke algebra

In this subsection, we define the completed Iwahori-Hecke algebra $\hat{\mathcal{H}}$. We equip $W^v$ with its Bruhat order $\leq$. One has $1 \leq w$ for all $w \in W^v$. If $u \in W^v$, one sets $[1,u] = \{w \in W^v | w \leq u\}$.

Let $\mathcal{B} = \prod_{w \in W^v, \lambda \in Y^+} \mathcal{R}$. If $f = (a_{\lambda,w}) \in \mathcal{B}$, the set $\{(\lambda, w) \in W^v \times Y^+ | a_{\lambda,w} \neq 0\}$ is called the support of $f$ and is denoted by $\mathrm{supp} f$, the set $\{w \in W^v | \exists \lambda \in Y^+ | a_{\lambda,w} \neq 0\}$ is called the *support of $f$ along* $W^v$ and denoted $\mathrm{supp}_{W^v} f$, and the set $\{\lambda \in Y^+ | \exists w \in W^v | a_{\lambda,w} \neq 0\}$ is called the *support of $f$ along* $Y$ and denoted $\mathrm{supp}_Y f$. A set $Z \subset Y^+ \times W^v$ is said to be almost finite if $\{w \in W^v | \exists \lambda \in Y^+ | (\lambda, w) \in Z\}$ is finite and for all $w \in W^v$, $\{\lambda \in Y^+ | (\lambda, w) \in Z\}$ is almost finite.

Let $\hat{\mathcal{H}}$ be the set of $a \in \mathcal{B}$ such that $\mathrm{supp}\ a$ is almost finite. If $a = (a_{\lambda,w}) \in \hat{\mathcal{H}}$, one writes $a = \sum_{(\lambda,w) \in Y^+ \times W^v} a_{\lambda,w} Z^\lambda H_w$. For $(\lambda, w) \in Y^+ \times W^v$, we define $\pi_{\lambda,w} : \hat{\mathcal{H}} \to \mathcal{R}$ by $\pi_{\lambda,w}(\sum a_{\lambda',w'} Z^{\lambda'} H_{w'}) = a_{\lambda,w}$. In order to extend $*$ to $\hat{\mathcal{H}}$, we prove that if $\sum a_{\lambda,w} Z^\lambda H_w$, $\sum b_{\lambda,w} Z^\lambda H_w \in \hat{\mathcal{H}}$ and $(\mu, v) \in Y^+ \times W^v$,

$$\sum_{(\lambda,w),(\lambda',w') \in Y^+ \times W^v} \pi_{\mu,v}(a_{\lambda,w} b_{\lambda',w'} Z^\lambda H_w * Z^{\lambda'} H_{w'})$$

is well defined, i.e that only a finite number of $\pi_{\mu,v}(a_{\lambda,w}b_{\lambda',w'}Z^\lambda H_w * Z^{\lambda'}H_{w'})$ are non zero. The key ingredient to prove this is the fact that if $w \in W^v$ and $\lambda \in Y^+$, the support of $H_w * Z^\lambda$ along $Y^+$ is in the convex hull of the $u.\lambda$, for $u \leq w$ for the Bruhat order (this is Lemma 4.7).

For $E \subset Y$ and $i \in I$, one sets $R_i(\lambda) = \mathrm{conv}(\{E, r_i(E)\}) \cap Q^\vee$. If $E = \{\lambda\}$, one writes $R_i(\lambda)$ for short. Let $w \in W^v$, one sets $R_w(\lambda) = \bigcup R_{i_1}(R_{i_2}(\ldots(R_{i_k}(\lambda)\ldots))$ where the union is taken over all the reduced writings of $w$.

**Lemma 4.7.** *Let $u, v \in W^v$ and $\mu \in Y$. Then there exists $(z_{\nu,t}^{u,\mu,v})_{\nu\in R_u(\mu),t\in[1,u].[1,v]} \in \mathcal{R}^{R_u(\mu)\times([1,u].[1,v])}$ such that*

$$H_u * Z^\mu H_v = \sum_{\nu\in R_u(\mu),t\in[1,u].[1,v]} z_{\nu,t}^{u,\mu,v} Z^\nu H_t$$

*Proof.* We do it by induction on $l(u)$. Let $k \in \mathbb{N}^*$ and suppose that for all $w \in W^v$ such that $l(w) \leq k-1$, there exists $(z_{\nu,t}^{w,\mu,v})_{\nu\in R_w(\mu),t\in[1,w].[1,v]} \in \mathcal{R}^{R_w(\mu)\times[1,w].[1,v]}$ such that $H_w * Z^\mu H_v = \sum_{\nu\in R_w(\mu),t\in[1,w].[1,v]} z_{\nu,t}^{w,\mu,v} Z^\nu H_t$. Let $u \in W^v$ such that $l(u) = k$. One writes $u = r_i w$, with $i \in I$ and $w \in W^v$ such that $l(w) = k-1$. One has

$$H_u * Z^\mu H_v = H_i * H_w Z^\mu H_v = \sum_{\nu\in R_w(\mu),t\in[1,w].[1,v]} z_{\nu,t}^{w,\mu,v} H_i * Z^\nu H_t.$$

Let $\nu \in R_w(\mu)$. Suppose $\sigma_i = \sigma_i'$. Then by Theorem 6.2 of [BPGR16], one has

$$H_i * Z^\nu = Z^{r_i(\nu)} * H_i + (\sigma_i - \sigma_i^{-1}) Z^\nu \frac{1 - Z^{-\alpha_i(\nu)\alpha_i^\vee}}{1 - Z^{-\alpha_i^\vee}}.$$

If $\alpha_i(\nu) = 0$, $H_i * Z^\nu = Z^\nu * H_i$.

If $\alpha_i(\nu) > 0$, $H_i * Z^\nu = Z^{r_i(\nu)} * H_i + (\sigma_i - \sigma_i^{-1}) \sum_{h=0}^{\alpha_i(\nu)-1} Z^{\nu - h\alpha_i^\vee}$ and $r_i(\nu), \nu - h\alpha_i^\vee \in R_i(\nu)$ for all $h \in [\![0, \alpha_i(\nu) - 1]\!]$.

If $\alpha_i(\nu) < 0$, $H_i * Z^\nu = Z^{r_i(\nu)} * H_i + (\sigma_i - \sigma_i^{-1}) \sum_{h=1}^{-\alpha_i(\nu)} Z^{\nu + h\alpha_i^\vee}$ and $r_i(\nu), \nu + h\alpha_i^\vee \in R_i(\nu)$ for all $h \in [\![1, -\alpha_i(\nu)]\!]$.

Suppose $\sigma_i \neq \sigma_i'$. Then $\alpha_i(Y) = 2\mathbb{Z}$. One has

$$H_i * Z^\lambda = Z^{r_i(\lambda)} * H_i + Z^\lambda \big((\sigma_i - \sigma_i^{-1}) + (\sigma_i' - \sigma_i'^{-1}) Z^{-\alpha_i^\vee}\big) \frac{1 - Z^{-\alpha_i(\lambda)\alpha_i^\vee}}{1 - Z^{-2\alpha_i^\vee}}$$

The same computations as above complete the proof. □

**Lemma 4.8.** *Let $u \in W^v$ and $\mu \in Y$. Then for all $\nu \in R_u(\mu)$, there exists $(\lambda_{u'})_{u'\leq u} \in [0,1]^{\{u'\in W^v | u' \leq u\}}$ such that $\sum_{u'\leq u} \lambda_{u'} = 1$ and $\nu = \sum_{u'\leq u} \lambda_{u'} u'.\mu$.*

*Proof.* We do it by induction on $l(u)$. Let $k \in N$ and suppose this is true for all $u$ having length $k$. Let $\tilde{u} \in W^v$ such that $l(\tilde{u}) = k+1$. Let $\nu \in R_{\tilde{u}}(\mu)$. Then $\nu \in R_i(\nu')$ for some $i \in I$, and $\nu' \in R_u(\mu)$, for some $u \in W^v$ having length $k$. One writes $\nu = s\nu' + (1-s) r_i.\nu'$, with $s \in [0,1]$. One writes $\nu' = \sum_{u'\leq u} \lambda_{u'} u'.\mu$. One has $\nu = s \sum_{u'\leq u} \lambda_{u'} u'.\mu + (1-s) \sum_{u'\leq u} \lambda_{u'} r_i.u'.\mu$. As $r_i.u' \leq \tilde{u}$ for all $u' \leq u$, one gets the lemma. □

Let $\lambda \in \mathcal{T}$. There exists a unique $\mu \in \overline{C_f^v}$ such that $W^v.\lambda = W^v.\mu$. One defines $\lambda^{++} = \mu$.

**Lemma 4.9.** 1. *Let $\lambda, \mu \in Y^+$. Then $(\lambda + \mu)^{++} \leq_{Q^\vee} \lambda^{++} + \mu^{++}$.*

2. *Let $\mu \in Y^+$ and $v \in W^v$. Then for all $\nu \in R_v(\mu)$, $\nu^{++} \leq_{Q^\vee} \mu^{++}$.*

*Proof.* Let $\lambda, \mu \in Y^+$. Let $w \in W^v$ such that $(\lambda+\mu)^{++} = w.(\lambda+\mu)$. By Lemma 2.4 a) of [GR14], $w.\lambda \leq_{Q^\vee} \lambda^{++}$ and $w.\mu \leq_{Q^\vee} \mu^{++}$ and thus we get 1.

The point 2 is a consequence of Lemma 4.8, Lemma 2.4 a) of [GR14] and point 1. □

If $x = \sum_{i\in I} x_i \alpha_i^\vee \in Q^\vee$, one sets $h(x) = \sum_{i\in I} x_i$. If $\lambda \in Y^+$, we denote by $w_\lambda$ the element $w$ of $W^v$ of minimal length such that $w.\lambda \in \overline{C_f^v}$.

**Lemma 4.10.** *Let $\lambda \in Y^{++}$ and $(\mu_n) \in (W^v.\lambda)^{\mathbb{N}}$ such that $l(w_{\mu_n}) \to +\infty$. Then $h(\mu_n - \lambda) \to -\infty$.*

*Proof.* The fact that $h(\mu_n - \lambda)$ is well defined is a consequence of Lemma 2.4 a) of [GR14]. Suppose that $h(\mu_n - \lambda)$ does not converge to $-\infty$. By Lemma 2.4 a) of [GR14], $h(\mu_n - \lambda) \leq 0$ for all $n \in \mathbb{N}$ and thus, maybe considering a subsequence of $(\mu_n)$, one can suppose that $(\mu_n)$ is injective and that $h(\mu_n - \lambda) \to k$, for some $k \in \mathbb{Z}$. Let $(h_i)_{i\in I}$ be the dual basis of the basis $(\alpha_i^\vee)$ of $Q^\vee$ and $\underline{h} = (h_i)_{i\in I}$. For all $i \in I$ and $n \in \mathbb{N}$, $h_i(\mu_n - \lambda) \leq 0$ and $(\underline{h}(\mu_n - \lambda))$ is injective. This is absurd and thus $h(\mu_n - \lambda) \to -\infty$. □

**Lemma 4.11.** *Let $\lambda \in Y^{++}$, $\nu \in Y^+$ and $u \in W^v$. Then $F = \{\mu \in W^v.\lambda | \nu \in R_u(\mu)\}$ is finite.*

*Proof.* Let $N \in \mathbb{N}$ such that for all $\nu' \in W^v.\lambda$ satisfying $l(w_{\nu'}) \geq N$, $h(\nu' - \lambda) < h(\nu - \lambda)$ ($N$ exists by Lemma 4.10).

Let $\mu \in F$ and $w = w_\mu$. One writes $\nu = \sum_{x\leq u} \lambda_x x.\mu$, with $\lambda_x \in [0,1]$ for all $x \leq u$ and $\sum_{x\leq u} \lambda_x = 1$, which is possible by Lemma 4.8.

If $x \leq u$, one sets $v(u') = w_{u'.\mu}$. Suppose that for all $u' \leq u$, $l(v(u')) \geq N$. One has

$$\nu - \lambda = \sum_{u'\leq u} \lambda_{u'}(u'.\mu - \lambda) = \sum_{u'\leq u} \lambda_{u'}(v(u').\mu - \lambda)$$

and thus

$$h(\nu - \lambda) = \sum_{u'\leq u} \lambda_{u'} h(v(u') - \lambda) < \sum_{u'\leq u} \lambda_{u'} h(\nu - \lambda) = h(\nu - \lambda),$$

which is absurd. Therefore, $l(v(u')) < N$, for some $u' \leq u$.

One has $u'.\mu = v(u').\mu^{++}$, thus $u'^{-1}.v(u').\mu^{++} = \mu$ and hence $l(u'^{-1}v(u')) \geq l(w)$, by definition of $w$. Therefore, $l(v(u')) + l(u) \geq l(v(u')) + l(u') \geq l(w)$. As a consequence, $l(w) \leq N + l(u)$ and thus $F$ is finite. □

**Definition 4.12.** A family $(a_j)_{j\in J} \in \hat{\mathcal{H}}^J$ is said to be summable if:

- $\bigcup_{j\in J} \mathrm{supp}_{W^v} a_j$ is finite
- for all $\lambda \in Y^+$, $\{j \in J | \exists w \in W^v | \pi_{w,\lambda}(a_j) \neq 0\}$ is finite
- $\bigcup_{j\in J} \mathrm{supp}\ a_j$ is almost finite.

When $(a_j)_{j\in J} \in \hat{\mathcal{H}}^J$ is summable, one defines $\sum_{j\in J} a_j \in \hat{\mathcal{H}}$ as follows: $\sum_{j\in J} a_j = \sum_{\lambda,w} x_{\lambda,w} Z^\lambda H_w$ where $x_{\lambda,\lambda} = \sum_{j\in J} \pi_{\lambda,w}(a_j)$ for all $(\lambda, w) \in Y^+ \times W^v$.

**Lemma 4.13.** *Let $(a_j)_{j\in J} \in (\mathcal{H})^J$, $(b_k)_{k\in K} \in (\mathcal{H})^K$ be two summable families. Then $(a_j * b_k)_{(j,k)\in J\times K}$ is summable. Moreover $\sum_{(j,k)} a_j * b_k$ depends only on $\sum_{j\in J} a_j$ and $\sum_{k\in K} b_k$ and we denote it by $a * b$, if $a = \sum_{j\in J} a_j$ and $b = \sum_{k\in K} b_k$.*

*Proof.* For $j \in J$, $k \in K$, one writes $a_j = \sum_{(\lambda,u)\in Y^+\times W^v} x_{j,\lambda,u} Z^\lambda H_u$, $b_k = \sum_{(\mu,v)\in Y^+\times W^v} y_{k,\mu,v} Z^\mu H_v$.
By Lemma 4.7, one has

$$a_j * b_k = \sum_{(\lambda,u),(\mu,v)\in Y^+\times W^v,\ \nu\in R_u(\mu),\ t\in[1,u].[1,v]} x_{j,\lambda,u} y_{k,\mu,v} z^{u,\mu,v}_{\nu,t} Z^{\lambda+\nu} H_t.$$

As a consequence, if $S^a_j = \bigcup_{u\in \mathrm{supp}_{W^v} a_j} [1,u]$ and $S^b_k = \bigcup_{v\in \mathrm{supp}_{W^v} b_k} [1,v]$, $\mathrm{supp}_{W^v}(a_j * b_k) \subset S^a_j . S^b_k$.
Thus

$$S_{W^v} = \bigcup_{(j',k')\in J\times K} \mathrm{supp}_{W^v}(a_{j'} * b_{k'}) \subset (\bigcup_{j'\in J} S^a_{j'}).(\bigcup_{k'\in K} S^b_{k'})$$

is finite.

Let $(\rho, s) \in Y^+ \times W^v$. One has

$$\pi_{\rho,s}(a_j * b_k) = \sum_{(\lambda,u)\in Y^+\times W^v,(\mu,v)\in Y^+\times W^v,\ \nu\in R_u(\mu),\ \lambda+\nu=\rho} x_{j,\lambda,u} y_{k,\mu,v} z^{u,\mu,v}_{\nu,s}$$

Let $S = \bigcup_{j\in J} \mathrm{supp}\ a_j \cup \bigcup_{k\in K} \mathrm{supp}\ b_k$ and $S_Y = \pi_Y(S)$, where $\pi_Y : Y \times W^v \to Y$ is the projection on the first coordinate.

Let $k \in \mathbb{N}$ and $\kappa_1, \dots, \kappa_k \in Y^{++}$ such that for all $\lambda \in S_Y$, $\lambda^{++} \leq_{Q^\vee} \kappa_i$, for some $i \in [\![1, k]\!]$.

Let $F(\rho) = \{(\lambda, \nu) \in S_Y \times Y^+ |\ \exists(\mu, u) \in S_Y \times S_{W^v} | \nu \in R_u(\mu)$, and $\lambda + \nu = \rho\}$. Let $(\lambda, \nu) \in F(\rho)$, $(\mu, u) \in S_Y \times S_{W^v}$ such that $\nu \in R_u(\mu)$. By Lemma 4.9, one has $\lambda \leq_{Q^\vee} \lambda^{++} \leq_{Q^\vee} \kappa_i$ and $\nu \leq \mu^{++} \leq \kappa_j$ for some $i, j \in [\![1, k]\!]$. Therefore, $F(\rho)$ is finite.

Let $F'(\rho) = \{\mu \in S_Y |\ \exists(u, \lambda, \nu) \in S_{W^v} \times F(\rho) | \nu \in R_u(\mu)\}$. Let $\mu \in F'(\rho)$ and $(u, \lambda, \nu) \in S_{W^v} \times F(\rho)$ such that $\nu \in R_u(\mu)$. Then by Lemma 4.9, $\nu^{++} \leq_{Q^\vee} \mu^{++} \leq_{Q^\vee} \kappa_i$, for some $i \in [\![1, k]\!]$. As a consequence, $F'(\rho)^{++}$ is finite and by Lemma 4.11, $F'(\rho)$ is finite.

If $\lambda \in Y^+$, one sets $J(\lambda) = \{j \in J |\ \exists u \in W^v |\ x_{j,\lambda,u} \neq 0\}$ and $K(\lambda) = \{k \in K |\ \exists u \in W^v |\ y_{k,\lambda,u} \neq 0\}$. Let $F_1(\rho) = \{\lambda \in Y^+ |\ \exists \nu \in Y^+ |\ (\lambda, \nu) \in F(\rho)\}$ and $L(\rho) = \bigcup_{(\lambda,\mu)\in F_1(\rho)\times F'(\rho)} J(\lambda) \times K(\mu)$. Then $L(\rho)$ is finite and for all $(j, k) \in J \times K$, $\pi_{\rho,s}(a_j * b_k) \neq 0$ implies that $(j, k) \in L(\rho)$.

Let $(\rho, s) \in \bigcup_{(j,k)\in J\times K} \mathrm{supp}(a_j * b_k)$. Then there exist $(\lambda, \mu) \in S_Y^2$, $u \in S_{W^v}$ and $\nu \in R_u(\mu)$ such that $\lambda + \nu = \rho$. Thus $\rho^{++} \leq_{Q^\vee} \lambda^{++} + \mu^{++} \leq_{Q^\vee} \kappa_i + \kappa_{i'}$ for some $i, i' \in [\![1, k]\!]$. Consequently, $\bigcup_{(j,k)\in J\times K} \mathrm{supp}(a_j * b_k)$ is almost finite and $(a_j * b_k)$ is summable.

Moreover,

$$\begin{aligned}\pi_{\rho,s}(\sum_{(j,k)\in J\times K} a_j * b_k) &= \sum_{(\lambda,u)\in Y^+\times W^v,(\mu,v)\in Y^+\times W^v,\nu\in R_u(\mu),\lambda+\nu=\rho} (\sum_{(j,k)\in J\times K} x_{j,\lambda,u} y_{k,\mu,v} z^{u,\mu,v}_{s,\nu}) \\ &= \sum_{(\lambda,u)\in Y^+\times W^v,(\mu,v)\in Y^+\times W^v,\nu\in R_u(\mu),\lambda+\nu=\rho} x_{\lambda,u} y_{\mu,v} z^{u,\mu,v}_{s,\nu}\end{aligned},$$

where $a = \sum_{(\lambda,u)\in Y^+\times W^v} x_{\lambda,u} Z^\lambda H_u$ and $b = \sum_{(\mu,v)\in Y^+\times W^v} y_{\mu,v} Z^\mu H_v$, which completes the proof. □

**Theorem 4.14.** *The convolution $*$ equips $\hat{\mathcal{H}}$ with a structure of associative algebra.*

*Proof.* By Lemma 4.13, $(\hat{\mathcal{H}}, *)$ is an algebra. The associativity comes from Lemma 4.13 and from the associativity of $\mathcal{H}$. □

**Definition 4.15.** The algebra $\hat{\mathcal{H}} = \hat{\mathcal{H}}_{\mathcal{R}}$ is the completed Iwahori-Hecke algebra of $(\mathbb{A}, (\sigma_i)_{i\in I}, (\sigma'_i)_{i\in I})$ over $\mathcal{R}$.

**Example 4.16.** Let $\mathcal{I}$ be a masure and suppose that $\mathcal{I}$ is thick of finite thickness and that a group $G$ acts strongly transitively on $\mathcal{I}$. Let $i \in I$ and $P_i$ (resp. $P_i'$) be a panel of $\{x \in \mathbb{A}|\ \alpha_i(x) = 0\}$ (resp.$\{x \in \mathbb{A}|\ \alpha_i(x) = 1\}$). One denotes by $1 + q_i$ (resp. $1 + q_i'$) the number of chambers containing $P_i$ (resp. $P_i'$). One sets $\sigma_i = \sqrt{q_i}$ and $\sigma_i' = \sqrt{q_i'}$ for all $i \in I$. Then $(\sigma_i)_{i\in I}$, $(\sigma_i')_{i\in I}$ satisfy the conditions of the beginning of Section 4 and the completed Iwahori-Hecke algebra over $\mathcal{R}$ associated to $(\mathbb{A}, (\sigma_i)_{i\in I}, (\sigma_i')_{i\in I})$ is the completed Iwahori-Hecke algebra of $\mathcal{I}$ over $\mathcal{R}$.

## 4.4 Center of Iwahori-Hecke algebras

In this subsection, we determine the center $\mathcal{Z}(\hat{\mathcal{H}})$ of $\hat{\mathcal{H}}$. For this we adapt the proof of Theorem 1.4 of [NR03].

### 4.4.1 Completed Bernstein-Lusztig bimodule

In order to determine $\mathcal{Z}(\hat{\mathcal{H}})$, we would like compute $Z^\mu * z * Z^{-\mu}$ if $z \in \mathcal{Z}(\hat{\mathcal{H}})$ and $\mu \in Y$. However, left and right multiplication by $Z^\mu$ is defined in $\hat{\mathcal{H}}$ only when $\mu \in Y^+$. We need to extend this multiplication to arbitrary $\mu \in Y$ in a compatible way with the multiplication in $\hat{\mathcal{H}}$. Obviously, multiplication by $Z^\mu$ cannot stabilize $\hat{\mathcal{H}}$ (because $Z^\mu * 1 = Z^\mu \notin \hat{\mathcal{H}}$ if $\mu \in Y\backslash Y^+$). Thus we define a "completion" ${}^{BL}\overline{\mathcal{H}}$ of ${}^{BL}\mathcal{H}$ containing $\hat{\mathcal{H}}$. We do not equip ${}^{BL}\overline{\mathcal{H}}$ with a structure of algebra but we equip it with a structure of $Y$-bimodule compatible with the convolution product on $\hat{\mathcal{H}}$.

If $a = (a_{\lambda,w}) \in \mathcal{R}^{Y\times W^v}$, one writes $a = \sum_{(\lambda,w)\in Y\times W^v} a_{\lambda,w} Z^\lambda H_w$. The *support of a along* $W^v$ is $\{w \in W^v|\ \exists \lambda \in Y|\ a_{\lambda,w} \neq 0\}$ and is denoted $\mathrm{supp}_{W^v}(a)$.

Let ${}^{BL}\overline{\mathcal{H}} = \{a \in \mathcal{R}^{W^v\times Y}|\ \mathrm{supp}_{W^v}(a)$ is finite $\}$. If $(\rho, s) \in Y \times W^v$, one defines $\pi_{\rho,s}$ : ${}^{BL}\overline{\mathcal{H}} \to \mathcal{R}$ by $\pi_{\rho,s}(\sum_{(\lambda,w)\in Y\times W^v} a_{\lambda,w} Z^\lambda H_w) = a_{\rho,s}$ for all $\sum_{(\lambda,w)\in Y\times W^v} a_{\lambda,w} Z^\lambda H_w) \in {}^{BL}\overline{\mathcal{H}}$.

One considers $\hat{\mathcal{H}}$ as a subspace of ${}^{BL}\overline{\mathcal{H}}$.

**Definition 4.17.** A family $(a_j)_{j\in J} \in ({}^{BL}\overline{\mathcal{H}})^J$ is said to be summable if:

- for all $(s, \rho) \in W^v \times Y, \{j \in J|\ \pi_{s,\rho}(a_j) \neq 0\}$ is finite
- $\bigcup_{j\in J} \mathrm{supp}_{W^v}(a_j)$ is finite.

When $(a_j)_{j\in J}$ is summable, one defines $\sum_{j\in J} a_j \in {}^{BL}\overline{\mathcal{H}}$ by $\sum_{j\in J} a_j = \sum_{(\lambda,w)\in Y\times W^v} a_{\lambda,w} Z^\lambda H_w$, with $a_{\lambda,w} = \sum_{j\in J} \pi_{\lambda,w}(a_j)$ for all $(\lambda, w) \in Y \times W^v$.

**Lemma 4.18.** *Let $(a_j) \in ({}^{BL}\mathcal{H})^J$ be a summable family of ${}^{BL}\overline{\mathcal{H}}$, $\mu \in Y$ and $a = \sum_{j\in J} a_j$. Then $(a_j * Z^\mu)$ and $(Z^\mu * a_j)$ are summable and $\sum_{j\in J} a_j * Z^\mu$, $\sum_{j\in J} Z^\mu * a_j$ depends on a (and $\mu$), but not on the choice of the family $(a_j)_{j\in J}$.*

*One sets $a \mathbin{\overline{*}} Z^\mu = \sum_{j\in J} a_j * Z^\mu$ and $Z^\mu \mathbin{\overline{*}} a = \sum_{j\in J} Z^\mu * a_j$. Then this convolution equips ${}^{BL}\overline{\mathcal{H}}$ with a structure of $Y$-bimodule.*

*Proof.* Let $S = \bigcup_{j\in J} \mathrm{supp}_{W^v}(a_j)$. If $(\lambda, w) \in Y \times W^v$, one sets $J(\lambda, w) = \{j \in J|\pi_{\lambda,w}(a_j) \neq 0\}$.

Let $(\rho, s) \in Y \times W^v$. Let $j \in J$. One has $\pi_{\rho,s}(Z^\mu * a_j) = \pi_{\rho+\mu,s}(a_j)$, therefore $\bigcup_{j\in J} \mathrm{supp}_{W^v}(Z^\mu * a_j) = \bigcup_{j\in J} \mathrm{supp}_{W^v}(a_j) = S$ is finite and $\{j \in J|\ \pi_{\rho,s}(Z^\mu * a_j) \neq 0\} = J(\rho+\mu, s)$ is finite. Consequently $(Z^\mu * a_j)$ is summable. Moreover $\pi_{\rho,s}(\sum_{j\in J} Z^\mu * a_j) = \pi_{\rho+\mu,s}(a)$, which depends only on $a$.

Let $w \in W^v$. By Lemma 4.7, there exists $(z^w_{\nu,t})_{(\nu,t)\in R_w(\mu)\times[1,w]} \in \mathcal{R}^{R_w(\mu)\times[1,w]}$ such that

$$H_w * Z^\mu = \sum_{\nu\in R_w(\mu),t\in[1,w]} z^w_{\nu,t} Z^\nu H_t.$$

Let $j \in J$. One writes $a_j = \sum_{(\lambda,w)\in Y\times W^v} a_{j,\lambda,w} Z^\lambda H_w$, with $(a_{j,\lambda,w}) \in \mathcal{R}^{Y\times W^v}$.
One has

$$\begin{aligned}\pi_{\rho,s}(a_j * Z^\mu) &= \pi_{\rho,s}\Big(\sum_{(\lambda,w)\in Y\times S} a_{j,\lambda,w} Z^\lambda H_w * Z^\mu\Big)\\ &= \pi_{\rho,s}\Big(\sum_{(\lambda,w)\in Y\times S} \big(\sum_{\nu\in R_w(\mu),t\in[1,w]} a_{j,\lambda,w} z^w_{\nu,t} Z^{\nu+\lambda} H_t\big)\Big)\\ &= \sum_{(\lambda,w)\in Y\times S}\Big(\sum_{\nu\in R_w(\mu),\nu+\lambda=\rho} a_{j,\lambda,w} z^w_{\nu,s}\Big).\end{aligned}$$

Let $F_{\rho,s} = \{j \in J | \pi_{\rho,s}(a_j * Z^\mu) \neq 0\}$. Then $F_{\rho,s} \subset \bigcup_{w\in S,\nu\in R_w(\mu)} J(\rho-\nu, w)$, which is finite. Moreover $\mathrm{supp}_{W^v}(a_j * Z^\mu) \subset \bigcup_{w\in S}[1, w]$ and thus $\bigcup_{j\in J} \mathrm{supp}_{W^v}(a_j * Z^\mu)$ is finite: $(a_j * Z^\mu)$ is summable. One has

$$\begin{aligned}\pi_{\rho,s}(\sum_{j\in J} a_j * Z^\mu) &= \sum_{j\in J}\Big(\sum_{(\lambda,w)\in Y\times S}\big(\sum_{\nu\in R_w(\mu),\nu+\lambda=\rho} a_{j,w,\lambda} z^w_{\nu,s}\big)\Big)\\ &= \sum_{(\lambda,w)\in Y\times S}\Big(\sum_{\nu\in R_w(\mu),\nu+\lambda=\rho}\big(\sum_{j\in J} a_{j,\lambda,w} z^w_{\nu,s}\big)\Big)\\ &= \sum_{(\lambda,w)\in Y\times S}\sum_{\nu\in R_w(\mu),\nu+\lambda=\rho} a_{\lambda,w} z^w_{\nu,s},\end{aligned}$$

if $a = \sum_{(\lambda,w)\in Y\times W^v} a_{\lambda,w} Z^\lambda H_w$.

Let $b$, $\mu, \mu' \in Y$. It remains to show that $Z^\mu \overline{*} (Z^{\mu'} \overline{*} b) = (Z^{\mu+\mu'}) \overline{*} b$, $(b \overline{*} Z^\mu) \overline{*} Z^{\mu'} = b \overline{*} (Z^{\mu+\mu'})$ and $Z^\mu \overline{*} (b \overline{*} Z^{\mu'}) = (Z^\mu \overline{*} b) \overline{*} Z^{\mu'}$. One writes $b = \sum_{(w,\lambda)\in W^v\times Y} b_{w,\lambda} Z^\lambda H_w$ and one applies the first part of the lemma with $J = W^v \times Y$, using the fact that if $x \in {}^{BL}\mathcal{H}$, $Z^\mu * (Z^{\mu'} * x) = (Z^{\mu+\mu'}) * x$, $(x * Z^\mu) * Z^{\mu'} = x * (Z^{\mu+\mu'})$ and $Z^\mu * (x * Z^{\mu'}) = (Z^\mu * x) * Z^{\mu'}$, which is a consequence of the associativity of $({}^{BL}\mathcal{H}, *)$. □

**Corollary 4.19.** *Let $a \in \hat{\mathcal{H}}$ and $\mu \in Y^+$. Then $Z^\mu * a = Z^\mu \overline{*}\, a$ and $a * Z^\mu = a \overline{*}\, Z^\mu$.*

### 4.4.2 Center of Iwahori-Hecke algebras

We now write $*$ instead of $\overline{*}$. Let $\mathcal{Z}(\hat{\mathcal{H}})$ be the center of $\hat{\mathcal{H}}$.

**Lemma 4.20.** *Let $a \in \mathcal{Z}(\hat{\mathcal{H}})$ and $\mu \in Y$. Then $a * Z^\mu = Z^\mu * a$.*

*Proof.* One writes $\mu = \mu_+ - \mu_-$, with $\mu_+, \mu_- \in Y^+$.

One has $Z^{\mu_-} * (Z^{-\mu_-} * a) = a$ and $Z^{\mu_-} * (a * Z^{-\mu_-}) = a$. Therefore $Z^{-\mu_-} * a = a * Z^{-\mu_-}$. Consequently, $Z^\mu * a = Z^{\mu_+} * a * Z^{-\mu_-} = a * Z^\mu$. □

Let $w \in W^v$. Let ${}^{BL}\overline{\mathcal{H}}_{\not\geq w} = \{\sum_{(\lambda,v)\in Y\times W^v} a_{\lambda,v} Z^\lambda H_v \in {}^{BL}\overline{\mathcal{H}} \mid a_{\lambda,v} \neq 0 \Rightarrow v \not\geq w\}$, $\hat{\mathcal{H}}_{\not\geq w} = \hat{\mathcal{H}} \cap {}^{BL}\overline{\mathcal{H}}_{\not\geq w}$, ${}^{BL}\overline{\mathcal{H}}_{=w} = \{\sum_{(\lambda,v)\in Y\times W^v} a_{\lambda,v} Z^\lambda H_v \in {}^{BL}\overline{\mathcal{H}} \mid a_{\lambda,v} \neq 0 \Rightarrow w = v\}$ and $\hat{\mathcal{H}}_{=w} = \hat{\mathcal{H}} \cap {}^{BL}\overline{\mathcal{H}}_{=w}$.

**Lemma 4.21.** *Let $w \in W^v$. Then :*

1. *For all $\lambda \in Y$,*
   - ${}^{BL}\overline{\mathcal{H}}_{\not\geq w} * Z^\lambda \subset {}^{BL}\overline{\mathcal{H}}_{\not\geq w}$
   - $Z^\lambda * {}^{BL}\overline{\mathcal{H}}_{\not\geq w} \subset {}^{BL}\overline{\mathcal{H}}_{\not\geq w}$
   - $Z^\lambda * {}^{BL}\overline{\mathcal{H}}_{=w} \subset {}^{BL}\overline{\mathcal{H}}_{=w}$
2. *Let $\lambda \in Y$. Then there exists $S \in {}^{BL}\overline{\mathcal{H}}_{\not\geq w}$ such that $H_w * Z^\lambda = Z^{w(\lambda)} H_w + S$.*

*Proof.* This is a consequence of Theorem 6.2 of [BPGR16] or of Lemma 4.7 and of Lemma 4.18. □

**Lemma 4.22.** *One has $\mathcal{Z}(\mathcal{H}) = \mathcal{Z}(\hat{\mathcal{H}}) \cap \mathcal{H}$.*

*Proof.* Let $a \in \mathcal{Z}(\mathcal{H})$. Then $a * Z^\lambda H_w = Z^\lambda H_w * a$ for all $(\lambda, w) \in Y \times W^v$. By Lemma 4.13, $a \in \mathcal{Z}(\hat{\mathcal{H}})$. The other inclusion is clear. □

Let $Y_{in} = Y \cap \mathbb{A}_{in}$, where $\mathbb{A}_{in} = \bigcap_{i \in I} \ker\ \alpha_i$. The following theorem is a generalization of a well-known theorem of Bernstein, whose first published version seems to be Theorem 8.1 of [Lus83].

**Theorem 4.23.** 1. *The center $\mathcal{Z}(\hat{\mathcal{H}})$ of $\hat{\mathcal{H}}$ is $R[[Y]]^{W^v}$.*

2. *If $W^v$ is infinite, the center $\mathcal{Z}(\mathcal{H})$ of $\mathcal{H}$ is $R[Y_{in}]$.*

*Proof.* We first prove 1. Let $z \in R[[Y]]^{W^v} \subset \hat{\mathcal{H}}$, $z = \sum_{\lambda \in Y^+} a_\lambda Z^\lambda$. Let $i \in I$. One has $z = x + y$, with $x = \sum_{\lambda \in Y^+ | \alpha_i(\lambda) = 0} a_\lambda Z^\lambda$ and $y = \sum_{\lambda \in Y^+ | \alpha_i(\lambda) > 0} a_\lambda (Z^\lambda + Z^{r_i(\lambda)})$. As $H_i * x = x * H_i$ and $H_i * y = y * H_i$, we get that $z \in \mathcal{Z}(\hat{\mathcal{H}})$ and thus $R[[Y]]^{W^v} \subset \mathcal{Z}(\hat{\mathcal{H}})$.

Let $z \in \mathcal{Z}(\hat{\mathcal{H}})$. One writes $z = \sum_{\lambda \in Y, w \in W^v} c_{\lambda,w} Z^\lambda H_w \in {}^{BL}\overline{\mathcal{H}}$. Suppose that there exists $w \in W^v \backslash \{1\}$ such that for some $\lambda \in Y$, $\pi_{\lambda,w}(z) \neq 0$. Let $m \in W^v$ maximal (for the Bruhat order) for this property. One writes $z = x + y$ with $x \in \hat{\mathcal{H}}_{=m}$ and $y \in \hat{\mathcal{H}}_{\not\geq m}$. One writes $x = \sum_{\lambda \in Y} c_{\lambda,m} Z^\lambda H_m$. By and Lemma 4.20 and Lemma 4.21, if $\mu \in Y$,

$$z = Z^\mu * z * Z^{-\mu} = \sum_{\lambda \in Y} c_{\lambda,m} Z^{\lambda + \mu - m(\mu)} H_m + y',$$

with $y' \in {}^{BL}\overline{\mathcal{H}}_{\not\geq m}$.

By projecting on ${}^{BL}\overline{\mathcal{H}}_{=m}$, we get that $x = \sum_{\lambda \in Y} c_{\lambda,m} Z^{\lambda+\mu-m(\mu)} H_m \in \hat{\mathcal{H}}_{=m}$. Let $J \subset Y$ finite such that for all $(w, \lambda) \in W^v \times Y$, $c_{\lambda,w} \neq 0$ implies that there exists $\nu \in J$ such that $\lambda \leq \nu$. Let $\gamma \in Y$ such that $c_{\gamma,m} \neq 0$. For all $\mu \in Y$, one has $\pi_{\gamma+\mu-m(\mu),m}(z) \neq 0$ therefore $\gamma + \mu - m(\mu) \leq_{Q^\vee} \nu(\mu)$ for some $\nu(\mu) \in J$ for all $\mu \in Y$. Let $\mu \in Y \cap C^v_f$. Let $\nu \in J$ such that for some $\sigma : \mathbb{N} \to \mathbb{N}$ such that $\sigma(n) \to +\infty$, $\gamma + \sigma(n)(\mu - m(\mu)) \leq_{Q^\vee} \nu$ for all $n \in \mathbb{N}$. In particular $\gamma + \sigma(1)(\mu - m(\mu)) - \nu \in Q^\vee$. By Lemma 2.4 a) of [GR14], $\mu - m(\mu) \in Q^\vee_+ \backslash \{0\}$ and thus for $n$ large enough $\gamma + \sigma(n)(\mu - m(\mu)) = \gamma + \sigma(1)(\mu - m(\mu)) + (\sigma(n) - \sigma(1))(\mu - m(\mu)) >_{Q^\vee} \nu$, which is absurd. Therefore $\mathcal{Z}(\hat{\mathcal{H}}) \subset R[[Y]]$.

Let $z \in \mathcal{Z}(\hat{\mathcal{H}})$. One writes $z = \sum_{\lambda \in Y} c_\lambda Z^\lambda$. Let $w \in W^v$. By Lemma 4.21, one has $H_w z = \sum_{\lambda \in Y} Z^{w(\lambda)} H_w + y$, with $y \in {}^{BL}\overline{\mathcal{H}}_{\not\geq w}$. But $H_w * z = z * H_w = \sum_{\lambda \in Y} c_\lambda Z^\lambda H_w$. By projecting on $\hat{\mathcal{H}}_{=w}$, we get that $\sum_{\lambda \in Y} c_\lambda Z^{w(\lambda)} H_w = \sum_{\lambda \in Y} c_\lambda Z^\lambda H_w$. Therefore, $z \in R[[Y]]^{W^v}$.

To prove 2, Lemma 4.22 shows that $\mathcal{Z}(\mathcal{H}) = \mathcal{H} \cap R[[Y]]^{W^v}$. We then use Corollary 5.18 to conclude. □

## 4.5 Case of a reductive group

In this subsection, we study the case where $G$ is reductive.

In [GR14], an almost finite set is a set $E$ such that $E \subset (\bigcup_{i=1}^{k} y_i - Q_+^\vee) \cap Y^{++}$ for some $y_1, \dots, y_k \in Y$. If $G$ is reductive, then such a set is finite. Indeed the Kac-Moody matrix $C$ of Subsubsection 2.1.1 is a Cartan matrix: it satisfies condition (FIN) of Theorem 4.3 of [Kac94]. In particular, $Y^{++} \subset Q_+^\vee \oplus \mathbb{A}_{in}$, which proves our claim.

However, the algebra $\hat{\mathcal{H}}$ that we define is different from $\mathcal{H}$ even in the reductive case. If $G$ is reductive, $\mathcal{T} = \mathbb{A}$ and thus $Y^+ = Y$. For instance, $\sum_{\mu \in Q_+^\vee} Z^{-\mu} \in \hat{\mathcal{H}} \backslash \mathcal{H}$.

**Proposition 4.24.** *Let $R$ be a ring. Then $R[[Y]]^{W^v} = R[Y]^{W^v}$ if and only if $W^v$ is finite.*

*Proof.* Suppose that $W^v$ is infinite. Let $y \in Y \cap C_f^v$. Then $\sum_{w \in W^v} e^{w.y} \in R[[Y]]^{W^v} \backslash R[Y]^{W^v}$.

Suppose that $W^v$ is finite. Let $w_0$ be the longest element of $W^v$. By the paragraph after Theorem of Section 1.8 of [Hum92], $w_0.Q_+^\vee = Q_-^\vee$. Let $E \subset Y$ be an almost finite set invariant under the action of $W^v$. One has $E \subset \bigcup_{j \in J} y_j - Q_+^\vee$ for some finite set $J$. Therefore $E = w_0.E \subset \bigcup_{j \in J} w_0.y_j + Q_+^\vee$. Consequently, for all $x \in E$, there exists $j, j' \in J$ such that $w_0.y_{j'} \leq_{Q^\vee} x \leq_{Q^\vee} y_j$ and hence $E$ is finite, which completes the proof. ☐

By Theorem 8.1 of [Lus83] and Theorem 4.23, when $W^v$ is finite, one has:

$$\mathcal{Z}(\hat{\mathcal{H}}) = R[Y]^{W^v} = \mathcal{Z}(\mathcal{H}).$$

# 5 Hecke algebra associated to a parahoric subgroup

In this section, we associate Hecke algebras to subgroups more general than $K = K_I$ (the Iwahori subgroup). This generalizes constructions of [BKP16] and [BPGR16].

## 5.1 Motivation from the reductive case

This subsection uses I 3.3 of [Vig96]. Assume that $G$ is reductive and let $K$ be an open compact subgroup of $G$. Let $\mathbb{Z}_c(G/K)$ be the space of functions from $G$ to $\mathbb{Z}$ which are $K$-invariant under right multiplication and have compact support. One defines an action of $G$ on this set as follows: $g.f(x) = f(g.x)$ for all $g \in G$, $f \in \mathbb{Z}_c(G/K)$ and $x \in G/K$. The *Hecke algebra of $G$ relative to $K$* is the algebra $H(G,K) = \mathrm{End}_G \mathbb{Z}_c(G/K)$ of $G$-equivariant endomorphisms of $\mathbb{Z}_c(G/K)$. Let $\mathbb{Z}_c(G//K)$ be the ring of functions from $G$ to $\mathbb{Z}$, with compact support, which are invariant under the action of $K$ on the left and on the right. We have an isomorphism of $\mathbb{Z}$-modules $\Upsilon : H(G,K) \to \mathbb{Z}_c(G//K)$ defined by $\Upsilon(\phi) = \phi(\mathbb{1}_K)$ for all $\phi \in H(G,K)$. Therefore, $H(G,K)$ is a free $\mathbb{Z}$-algebra, with canonical basis $(e_g)_{g \in K \backslash G/K}$, where $e_g = \mathbb{1}_{KgK}$ for all $g \in G$. If $\mathcal{R}$ is a commutative ring, one defines $H_{\mathcal{R}}(G,K) = H(G,K) \otimes_{\mathbb{Z}} \mathcal{R}$: this is the *Hecke algebra over $\mathcal{R}$ of $G$ relative to $K$*.

If $g, g' \in K \backslash G/K$, $e_g e'_g = \sum_{g'' \in K \backslash G/K} m(g,g';g'') e_{g''}$, where

$$m(g,g';g'') = |(KgK \cap g''Kg'^{-1}K)/K|$$

for all $g'' \in K \backslash G/K$ ($m(g,g';g'') \neq 0$ implies $Kg''K \subset KgKg'K$ for all $g,g',g'' \in K \backslash G/K$).

We no more suppose $G$ to be reductive. We want to define Hecke algebras relative to some subgroups of $G$. As there is (up to now ?) no topology on $G$ similar to the topology of reductive groups, we cannot define "open compact" in $G$. However we can still define special parahoric subgroups, which are fixers of special faces (whose vertices are in $G.0$) in the masure $\mathcal{I}$. Let $K = K_F$ be the fixer of some face$F$ such that $F_0 \subset \overline{F} \subset \overline{C_0^+}$, where $F_0 = F(0, \mathbb{A}_{in})$ and $C_0^+ = F(0, C_f^v)$. Using the method of Bardy-Panse, Gaussent and Rousseau of [GR14] and [BPGR16], we view the $(KgK \cap g''Kg'^{-1}K)/K$ as intersection of "spheres" in $\mathcal{I}$. We prove that when $F$ is spherical, these intersections are finite. We then define the Hecke ${}^F\mathcal{H}$ algebra of $G$ relative to $K$ as follows: ${}^F\mathcal{H}$ is the free module over $\mathbb{Z}$ with basis $e_g = \mathbb{1}_{KgK}$, $g \in G^+$, where $G^+ = \{g \in G | g.0 \geq 0\}$, with convolution product $e_g * e'_g = \sum_{g'' \in K\backslash G^+/K} m(g, g'; g'')e_{g''}$, where $m(g, g'; g'') = |(KgK \cap g''Kg'^{-1}K)/K|$ for all $g'' \in K\backslash G^+/K$. To prove that this formula indeed defines an algebra, we need to prove finiteness results. We prove these results by using the fact that they are true when $F$ is a chamber, which was proved by Bardy-Panse, Gaussent and Rousseau to define the Iwahori-Hecke algebra, and the fact that the number of chamber containing $F$ is finite. The reason why one uses $G^+$ instead of $G$ is linked to the fact that two points are not always in a same apartment. This was already done in [BK11], [GR14], [BKP16] and [BPGR16].

We also prove that when $F$ is a facesuch that $F_0 \subset \overline{F} \subset \overline{C_0^+}$, $F \neq F_0$ is non-spherical, there exists $g \in G$ such that $(KgK \cap g''Kg'^{-1}K)/K$ is infinite and thus this method fails to associate a Hecke algebra to $F$.

## 5.2 Distance and spheres associated to a spherical facet

In this subsection, we define an "$F$-distance" (or a $W_F$-distance, where $W_F$ is the fixer of $F$ in $W^v$) for each spherical face$F$ between $F_0$ and $C_0^+$ , generalizing the $W^v$-distance of [GR14] and the $W$-distance of [BPGR16].

If $E_1, \dots, E_k, E'_1, \dots, E'_k$ are subsets or filters of apartments, the notation $\phi : (E_1, \dots, E_k) \mapsto (E'_1, \dots, E'_k)$ means that $\phi$ is an isomorphism of apartments such that $\phi(E_i) = E'_i$ for all $i \in [\![1, k]\!]$.

Let $F$ be a spherical faceof $\mathcal{I}$ such that $F \subset \overline{C_0^+}$ or $F = F_0$. Let $W_F$ be the fixer of $F$ in $W^v$. Let $\Delta_F = G.F$. We have a bijection $\Upsilon : G/K_F \to \Delta_F$ mapping each $g.K_F$ to $g.F$.

If $F_1, F_2 \in \Delta_F$, one writes $F_1 \leq F_2$ if $a_1 \leq a_2$, where $a_1$ and $a_2$ are the vertices of $F_1$ and $F_2$. One denotes by $\Delta_F \times_\leq \Delta_F$ the set $\{(F_1, F_2) \in \Delta_F^2 | F_1 \leq F_2\}$.

**Definition/Proposition 5.1.** Let $(F_1, F_2) \in \Delta_F \times_\leq \Delta_F$. Then there exists an apartment $A$ containing $F_1$ and $F_2$ and a isomorphism $\phi : (A, F_1) \mapsto (\mathbb{A}, F)$. One sets $d^F(F_1, F_2) = [\phi(F_2)] := W_F.\phi(F_2)$. This does not depend on the choices we made.

*Proof.* Proposition 5.1 of [Rou11] yields the existence of $A$.

By definition, $F_1 = g.F$ for some $g \in G$. Let $A' = g.\mathbb{A}$. By (MA2), there exists $\psi : (A, F_1) \mapsto (A', F_1)$ and if $\psi' = g_{|\mathbb{A}}^{|A'}$, then $\phi := \psi'^{-1} \circ \psi : (A, F_1) \mapsto (\mathbb{A}, F)$: $\phi$ has the desired property.

Suppose $A_1$ is an apartment containing $F_1, F_2$ and $\phi_1 : (A_1, F_1) \mapsto (\mathbb{A}, F)$. By Proposition 1.10 c) of [BPGR16], there exists $f : (A, F_1, F_2) \mapsto (A_1, F_1, F_2)$. Then one has the following diagram:

$$\begin{array}{ccc} (\mathbb{A}, F_1, F_2) & \xrightarrow{f} & (A_1, F_1, F_2) \\ \big\downarrow{\scriptstyle \phi} & & \big\downarrow{\scriptstyle \phi_1} \\ (\mathbb{A}, F, \phi(F_2)) & \longrightarrow & (\mathbb{A}, F, \phi_1(F_2)), \end{array}$$

and the lower horizontal arrow is in $W_F$, which completes the proof. □

**Remark 5.2.** Suppose that $F = F_0$. Using the natural bijection $\Delta_{F_0} \simeq \mathcal{I}_0$, where $\mathcal{I}_0 = G.0$ and $Y^{++} \simeq Y^+/W^v$, we get that $d^{F_0}$ is the "vectorial distance" $d^v$ of [GR14].

Suppose that $F = C_0^+$. Then $W_{C_0^+} = \{1\}$. One has $[C] = \{C\}$ for all $C \in \Delta_{C_0^+}$. Therefore the distance $d^{C_0^+}$ is the distance $d^W$ of [BPGR16], by identifying each element $w$ of $W$ to the chamber $w.C_0^+$.

Let $\Delta_{\geq F}^{\mathbb{A}} = \{E \in \Delta_F | E \subset \mathbb{A} \text{ and } E \geq F\}$. Let $[\Delta_F] = \{[F'] | F' \in \Delta_{\geq F}^{\mathbb{A}}\}$. If $E \in \Delta_F$ and $[R] \in [\Delta_F]$, one sets $\mathcal{S}^F(E,[R]) = \{E' \in \Delta_F | E' \geq E \text{ and } d^F(E,E') = [R]\}$ and $\mathcal{S}_{op}^F(E,[R]) = \{E' \in \Delta_F | E' \leq E \text{ and } d^F(E',E) = [R]\}$. If $E \in \Delta_{\geq F}^{\mathbb{A}}$, one chooses $g_E \in N$ such that $E = g_E.F$. Such a $g_E$ exists: let $g \in G$ such that $E = g.F$ and $A = g.\mathbb{A}$. By (MA2) and 2.2.1) of [Rou11], there exists $\phi : (A, g.F) \mapsto (\mathbb{A}, g.F)$. Let $\psi = g_{|\mathbb{A}}^{|A}$. Then $\phi \circ \psi \in N$ and $\phi \circ \psi(F) = \phi(E) = E$.

**Lemma 5.3.** *Let $[R] \in [\Delta_F]$ and $\Upsilon : G/K_F \xrightarrow{\sim} \Delta_F$. Then $\Upsilon^{-1}(\mathcal{S}^F(F,[R])) = K_F g_R K_F/K_F$ and $\Upsilon^{-1}(\mathcal{S}_{op}^F(F,[R])) = K_F g_R^{-1} K_F/K_F$.*

*Proof.* Let $E \in \mathcal{S}^F(F,[R])$. Then there exists $g \in K_F$ such that $g.E = R = g_R.F$. Thus $\Upsilon^{-1}(E) \in K_F g_R K_F/K_F$. Let $x \in K_F g_R K_F$, $x = k_1 g_R k_2$, with $k_1, k_2 \in K_F$. Then $\Upsilon(x) = k_1 g_R.F = k_1.R$. As $d^F$ is $G$-invariant, $d^F(k_1.F, k_1.R) = d^F(F,R) = d^F(F, k_1.R)$, and thus $x \in \Upsilon^{-1}(S^F(F,[R]))$. The proof of the second statement is similar. □

## 5.3 Hecke algebra associated to a spherical facet

In this subsection we define the Hecke algebra associated to a spherical face$F$ between $F_0$ and $C_0^+$ (or to $K_F$).

Let $C, C'$ be two positive chambers based at some $x \in \mathcal{I}_0$. One identifies the elements of $C_0^+$ and $W$. Then $d^W(C,C')(=d^{C_0^+}(C,C'))$ is in $W^v$. One sets $d(C,C') = l(d^W(C,C'))$.

**Lemma 5.4.** *Let $C$ be a chamber of $\mathcal{I}$ based at some $x \in \mathcal{I}_0$ and $n \in \mathbb{N}$. Let $B_n(C)$ be the set of chambers $C'$ of $\mathcal{I}$ based at $x$ and such that $d(C,C') \leq n$. Then $B_n(C)$ is finite.*

*Proof.* We do it by induction on $n$. The set $B_1(F')$ is finite for all $F' \in G.C_0^+$ by the fact that $\mathcal{I}$ is of finite thickness. Let $n \in \mathbb{N}$. Suppose that $B_k(F')$ is finite for all $k \leq n$ and $F' \in G.C_0^+$. Let $C' \in B_{n+1}(C)$. Let $\phi$ be an isomorphism of apartments such that $\phi(C) = C_0^+$. One has $\phi(C') = w.C_0^+$, with $w \in W^v$ and $l(w) = n+1$. Let $\tilde{w} \in W^v$ such that $l(\tilde{w}) = n$ and $d(\tilde{w}.C_0^+, \phi(C')) = 1$. Let $\tilde{C} = \phi^{-1}(\tilde{w}.C_0^+)$. Then $d(C,\tilde{C}) = 1$ and thus $B_{n+1}(C) \subset \bigcup_{C'' \in B_n(C)} B_1(C'')$, which is finite. □

**Type of a special facet** If $\mathcal{E}$ is a filter of $A$, one sets $\mathbb{R}_+^*\mathcal{E} = \{\mathbb{R}_+^* E | \ E \in \mathcal{E}\}$. Let $\mathcal{F}_{\mathbb{A}}^v$ be the set of positive vectorial faces of $\mathbb{A}$ and $\mathcal{F}_{\mathbb{A}}^0$ be the set of positive faces of $\mathbb{A}$ based at 0.

**Lemma 5.5.** *The map $f : \mathcal{F}_{\mathbb{A}}^v \to \mathcal{F}_{\mathbb{A}}^0$ mapping each $F^v \in \mathcal{F}_{\mathbb{A}}^v$ on $F(0,F^v)$ is a bijection and the inverse of $f$ is the map $\mathcal{F}_{\mathbb{A}}^0 \to \mathcal{F}^v$ sending each $F \in \mathcal{F}_{\mathbb{A}}^v$ on $\mathbb{R}_+^* F$.*

*Proof.* By definition of the faces, $f$ is surjective. Let $F \in \mathcal{F}_{\mathbb{A}}^0$. One writes $F = F(0,F^v)$, with $F^v \in \mathcal{F}_{\mathbb{A}}^v$. By definition of faces, $\mathbb{R}_+^* F \supset F^v$.

By definition of $F^v$, there exist half-apartments $D_1, \ldots, D_k$ of $\mathbb{A}$ such that $F^v = D_1 \cap \ldots \cap D_k$ and thus $F^v \in F(0,F^v) = F$. Therefore $F^v \in \mathbb{R}_+^* F$, which proves the lemma. □

Let $F$ be a positive special faceof $\mathcal{I}$. One has $F = g_1.F_1$ for some face$F_1 \leq C_0^+$. Let $J \subset I$ such that $F_1 = F(0, F^v(J))$ (see Subsection 2.1.2 for the definition of $F^v(J)$). The *type of* $F$, denoted type$(F)$ is $J$ and it is well-defined. Indeed, suppose $F = g_2.F(0,F^v(J_2))$, for some $g_2 \in G$ and $J_2 \subset I$. Let $g = g_2^{-1}.g_1$. One has $F(0,F^v(J)) = g.F(0,F^v(J_2))$. By

(MA2) and 2.2.1) of [Rou11], one can suppose that $g \in N$, thus $g_{|\mathbb{A}} \in W^v$ and by Lemma 5.5, $F^v(J) = g.F^v(J_2)$. By Section 1.3 of [Rou11] $J = J_2$ and the type is well defined.

The type is invariant under the action of $G$ and if $C$ is a special chamber, there exists exactly one sub-faceof $C$ of type $J$ for each $J \subset I$.

**Lemma 5.6.** *Let $F' \in \Delta_F$ and $\mathcal{C}_{F'}$ be the set of chambers of $\mathcal{I}$ containing $F'$. Then $\mathcal{C}_{F'}$ is finite.*

*Proof.* We fix $C \in \mathcal{C}_{F'}$. Let $x$ be the vertex of $C$ and $C' \in \mathcal{C}_{F'}$. Let $A$ be an apartment containing $C$ and $C'$ (such an apartment exists by Proposition 5.1 of [Rou11]). We fix the origin of $A$ in $x$. Let $N_A$ be the stabilizer of $A$ and $W_A^v$ be the fixer of $x$ in $N_A$. There exists $w \in W_A^v$ such that $C' = w.C$. Let $J$ be the type of $F'$. Then $w.F'$ is the faceof $C'$ of type $J$ and thus $w.F' = F'$. Therefore $w \in W_{F',A}$ where $W_{F',A}$ is the fixer of $F'$ in $W_A^v$. Therefore $d(C, C') \leq |W_{F',A}| = |W_F|$ ($W_F$ is finite by definition of spherical). Lemma 5.4 concludes the proof. □

We now fix a spherical facebetween $F_0$ and $C_0^+$.

**Lemma 5.7.** *Let $(E_1, E_2), (E_1', E_2') \in \Delta_F \times_{\leq} \Delta_F$. Then $d^F(E_1, E_2) = d^F(E_1', E_2')$ if and only if there exists an isomorphism $\phi : (E_1, E_2) \mapsto (E_1', E_2')$.*

*Proof.* Suppose that $d^F(E_1, E_2) = d^F(E_1', E_2') = [R]$. Let $\psi : (E_1, E_2) \mapsto (F, R)$ and $\psi' : (E_1', E_2') \mapsto (F, R)$. Then $\phi = \psi'^{-1} \circ \psi : (E_1, E_2) \mapsto (E_1', E_2')$.

Suppose that there exists an isomorphism $\phi : (E_1, E_2) \mapsto (E_1', E_2')$. Let $\psi : (E_1, E_2) \mapsto (F, R)$. Then $\phi^{-1} \circ \psi : (E_1', E_2') \mapsto (F, R)$. □

**Lemma 5.8.** *Let $(F_1, F_2) \in \Delta_F \times_{\leq} \Delta_F$ and $[R] = d^F(F_1, F_2)$. Then if $C_1, C_2$ are two chambers containing $F_1$ and $F_2$, $d^W(C_1, C_2) \in \mathcal{C}_{\mathbb{A}}([R])$ where $\mathcal{C}_{\mathbb{A}}([R])$ is the set of chambers of $\mathbb{A}$ an element of $W_F.R$. Moreover $\mathcal{C}_{\mathbb{A}}([R])$ is finite.*

*Proof.* Let $A$ be an apartment containing $C_1$ and $C_2$ and $\phi : (A, C_1) \mapsto (\mathbb{A}, C_0^+)$. Then $\phi(F_1)$ is the faceof $C_0^+$ of type type$(F)$: $\phi(F_1) = F$. Therefore $\phi(F_2) \in W_F.R$ and thus $d^W(C_1, C_2) \in \mathcal{C}_{\mathbb{A}}([R])$.

Using the type, we get that if $w \in W_F$, the set of chambers containing $w.R$ is in bijection with the fixer of $R$ in $W$, which is conjugated to $W_F$ and the lemma follows. □

**Lemma 5.9.** *Let $(F_1, F_2) \in \Delta_F \times_{\leq} \Delta_F$ and $R_1, R_2 \in [\Delta_F]$. Then the set $\mathcal{S}^F(F_1, R_1) \cap \mathcal{S}^F_{op}(F_2, R_2)$ is finite. Its cardinal depends only on $R_1, R_2$ and $R := d^F(F_1, F_2)$ and we denote it by $a^R_{R_1,R_2}$.*

*Proof.* Let $\mathcal{S}$ be the set of chambers containing an element of $\mathcal{S}^F(F_1, R_1) \cap \mathcal{S}^F_{op}(F_2, R_2)$. Let $C_1$ (resp. $C_2$) be a chamber containing $F_1$ (resp. $F_2$).

By Lemma 5.8, if $C \in \mathcal{S}$, one has $d^W(C_1, C) \in \mathcal{C}_{\mathbb{A}}(R_1)$ and $d^W(C, C_2) \in \mathcal{C}_{\mathbb{A}}(R_2)$. Consequently,

$$\mathcal{S} \subset \bigcup_{w_1 \in \mathcal{C}_{\mathbb{A}}(R_1), w_2 \in \mathcal{C}_{\mathbb{A}}(R_2)} \{C \in \mathcal{C}_0^+ |\ C_1 \leq C \leq C_2,\ d^W(C_1, C) = w_1 \text{ and } d^W(C, C_2) = w_2\}.$$

By Lemma 5.8 and Proposition 2.3 of [BPGR16], $\mathcal{S}$ is finite. Hence $\mathcal{S}^F(F_1, R_1) \cap \mathcal{S}^F_{op}(F_2, R_2)$ is finite.

It remains to prove the invariance of the cardinal. Let $(F_1', F_2') \in \Delta_F \times_{\leq} \Delta_F$ such that $d^F(F_1', F_2') = [R]$ and $\phi : (F_1, F_2) \mapsto (F_1', F_2')$, which exists by Lemma 5.7. Then $\mathcal{S}^F(F_1', R_1) \cap \mathcal{S}^F_{op}(F_2', R_2) = \phi\big(\mathcal{S}^F(F_1, R_1) \cap \mathcal{S}^F_{op}(F_2, R_2)\big)$, which proves the lemma. □

**Lemma 5.10.** *Let $R_1, R_2 \in [\Delta_F]$ and*

$P_{R_1,R_2} := \{d^F(F_1,F_2) | (F_1,F',F_2) \in \Delta_F \times_{\leq} \Delta_F \times_{\leq} \Delta_F, d^F(F_1,F') = R_1 \text{ and } d^F(F',F_2) = R_2\}$.

*Then $P_{R_1,R_2}$ is finite.*

*Proof.* Let $\mathcal{E}$ be the set of triples $(C_1, C', C_2)$ of chambers such that for some faces $F_1, F'$ and $F_2$ (with $F_1 \subset C_1$, ...) of these chambers, $d^F(F_1, F') = R_1$ and $d^F(F', F_2) = R_2$.

Let $(C_1, C', C_2) \in \mathcal{E}$. By Lemma 5.8, $d^W(C_1, C') \in \mathcal{C}_{\mathbb{A}}(R_1)$ and $d^W(C', C_2) \in \mathcal{C}_{\mathbb{A}}(R_2)$. Therefore,

$$P := \{d^W(C_1,C_2) | (C_1,C',C_2) \in \mathcal{E}\} \subset \bigcup_{\mathbf{w_1} \in \mathcal{C}_{\mathbb{A}}(R_1), \mathbf{w_2} \in \mathcal{C}_{\mathbb{A}}(R_2)} P_{\mathbf{w_1},\mathbf{w_2}},$$

where the $P_{\mathbf{w_1},\mathbf{w_2}}$ are as in Proposition 2.2 of [BPGR16](or in the statement of this lemma). Thus $P$ is finite.

Let $(F_1, F', F_2) \in \Delta_F \times_{\leq} \Delta_F \times_{\leq} \Delta_F$ such that $d^F(F_1, F') = R_1$ and $d^F(F', F_2) = R_2$. Then $F_1$ and $F_2$ are some faces of $C_1$ and $C_2$, for some $(C_1, C', C_2) \in \mathcal{E}$. The distance $d^F(F_1, F_2)$ is $W_F.F''$ for some face$F''$ of $d^W(C_1, C_2)$, which proves the lemma. □

Let $\mathcal{R}$ be a unitary and commutative ring (we do not make the additional assumptions of Section 4 on $\mathcal{R}$) and let ${}^F\mathcal{H} = {}^F\mathcal{H}_{\mathcal{R}}^{\mathcal{I}}$ be the set of functions from $G\backslash\Delta_F \times_{\leq} \Delta_F$ to $\mathcal{R}$. Let $R \in [\Delta_F]$. One defines $T_R : \Delta_F \times_{\leq} \Delta_F \to \mathcal{R}$ by $T_R(F_1, F_2) = \delta_{d^F(F_1,F_2),R}$ for all $(F_1, F_2) \in \Delta_F \times_{\leq} \Delta_F$. Then ${}^F\mathcal{H}$ is a free $\mathcal{R}$-module with basis $T_R$, for $R \in [\Delta_F]$.

**Theorem 5.11.** *We equip ${}^F\mathcal{H}$ with a product $* : {}^F\mathcal{H} \times {}^F\mathcal{H} \to {}^F\mathcal{H}$ defined as follows: if $\phi_1, \phi_2 \in {}^F\mathcal{H}_{\mathcal{R}}^{\mathcal{I}}$,*

$$\phi_1 * \phi_2(F_1, F_2) = \sum_{F' \in \Delta_F | F_1 \leq F' \leq F_2} \phi_1(F_1, F')\phi_2(F', F_2)$$

*for all $(F_1, F_2) \in \Delta_F \times_{\leq} \Delta_F$. This product is well defined and equips ${}^F\mathcal{H}$ with a structure of associative algebra with identity element $T_F$. Moreover, if $R_1, R_2 \in [\Delta_F]$,*

$$T_{R_1} * T_{R_2} = \sum_{R \in P_{R_1,R_2}} a^R_{R_1,R_2} T_R.$$

*Proof.* The fact that $*$ is well defined and the expression of $T_{R_1} * T_{R_2}$ are consequences of Lemma 5.9 and of Lemma 5.10. The associativity is clear from the definition. The fact that $T_F$ is the identity element comes from the fact that $\mathcal{S}^F(F_1, [F]) = \{F_1\}$ for all $F_1 \in \Delta_F$. □

**Definition 5.12.** The algebra ${}^F\mathcal{H} = {}^F\mathcal{H}_{\mathcal{R}}^{\mathcal{I}}$ is the Hecke algebra of $\mathcal{I}$ associated to $F$ over $\mathcal{R}$.

**Remark 5.13.** Let $g \in G^+$, then $\{F' \in K_F g K_F.F | F' \subset \mathbb{A}\}$ is of the form $[R_g]$ for some $R_g \in \Delta^{\mathbb{A}}_{\geq F}$.

One has a bijection $f : G\backslash\Delta_F \times_{\leq} \Delta_F \xrightarrow{\sim} K_F\backslash G^+/K_F$. This map is defined as follows: let $(F_1, F_2) \in G\backslash\Delta_F \times_{\leq} \Delta_F$. One can suppose that $F_1 = F$. One has $F_2 = g.F$ and one sets $f(g) = K_F g K_F$. Then it is easy to see that $f$ is well defined and is a bijection. One identifies ${}^F\mathcal{H}$ and the set of functions from $K_F\backslash G^+/K_F$ to $\mathcal{R}$.

Through this identification, $e_g = \mathbb{1}_{K_F g K_F}$ corresponds to $T_{[R_g]}$ for all $g \in G^+$. If $g, g' \in K_F\backslash G^+/K_F$, one has $e_g * e_{g'} = \sum_{g'' \in K_F\backslash G^+/K_F} m(g, g'; g'')e_{g''}$, where $m(g, g'; g'') = a^{[R_{g''}]}_{[R_g],[R_{g'}]}$ for all $g'' \in K_F\backslash G^+/K_F$.

By Lemma 5.3 and Lemma 5.9, $m(g, g'; g'') = |(KgK \cap g''Kg'^{-1}K)/K|$ for all $g'' \in K_F\backslash G^+/K_F$.

## 5.4 Case of a non-spherical facet

In [GR14], Gaussent and Rousseau associated an algebra (the spherical Hecke algebra) to the face$F_0$. By Remark 5.2, their distance $d^v$ correspond to $d^{F_0}$. It seems natural to try to associate a Hecke algebra to each face$F$ between $F_0$ and $C_0^+$. Let $F$ be a non spherical facesuch that $F_0 \subsetneq F \subsetneq C_0^+$ (as a consequence $A$ is an indefinite Kac-Moody matrix of size at least 3 because if $A$ is of finite type, all faces are spherical and if $A$ is of affine type, $\{0\}$ is the only non-spherical faceof $C_0^+$). We do not know if the statement of Proposition 1.1.c) of [BPGR16] is still true. But we could define an $F$-distance on the set $\Delta_F \times_{\mathring{\leq}} \Delta_F$, where we say that $F_1, F_2 \in \Delta_F$ satisfy $F_1 \mathring{<} F_2$ if their vertices $a_1, a_2$ satisfy $a_1 \mathring{<} a_2$ (which means that for some $g \in G$, $g.a_1, g.a_2 \in \mathbb{A}$ and $g.a_2 - g.a_1 \in \mathring{\mathcal{T}}$) or $a_1 = a_2$. Then we can use Proposition 5.2 or Proposition 5.5 of [Rou11] instead of Proposition 1.1 c) of [BPGR16] and thus define a distance $d^F : \Delta_F \times_{\mathring{\leq}} \Delta_F \to W_F \backslash N.F$. But as we will see in this section, the definition of the product as above leads to infinite coefficients. To prove this, we use the fact that the restriction map which associates $w_{|Q^\vee}$ to each $w \in W^v$ is injective, which is proved in [Kac94]. As this is proved for less general realizations $\mathbb{A}$ of the Kac-Moody matrix $C$ than we use, we need to extend this result to our framework, which is the aim of the next subsection.

### 5.4.1 Realization of a Kac-Moody matrix

In this subsection, $\mathbb{A}$ is no more the standard apartment of $\mathbb{A}$.

Let $A = (a_{i,j})_{i,j \in [\![1,n]\!]}$ be a Kac-Moody matrix. A *realization* of $A$ (see chapter 1 of [Kac94]) is a triple $(\mathbb{A}, \Pi, \Pi^\vee)$ where $\mathbb{A}$ is a vectorial space over $\mathbb{R}$ (in [Kac94], Kac uses $\mathbb{C}$ instead of $\mathbb{R}$), $\Pi = \{\alpha_1, \ldots, \alpha_n\} \subset \mathbb{A}^*$ and $\Pi^v = \{\alpha_1^\vee, \ldots, \alpha_n^\vee\} \subset \mathbb{A}$ are sets with cardinality $n$ satisfying the following conditions:

(F): both sets $\Pi$ and $\Pi^\vee$ are linearly independent

(C): $\alpha_j(\alpha_i^\vee) = a_{i,j}$ for all $i, j \in [\![1, n]\!]$

(D): $n - \mathrm{rk}(A) = \dim \mathbb{A} - n$;

A *generalized free realization of* $A$ is a triple $(\mathbb{A}, \Pi, \Pi^\vee)$ as above satisfying (F) and (C). Two realizations $(\mathbb{A}_1, \Pi_1, \Pi_1^\vee)$, $(\mathbb{A}_2, \Pi_2, \Pi_2^\vee)$ are called isomorphic if there exists a vector space isomorphism $\phi : \mathbb{A}_1 \to \mathbb{A}_2$ such that $\phi(\Pi_1) = \Pi_2$ and $\phi^*(\Pi_1^\vee) = \Pi_2^\vee$. Proposition 1.1 of [Kac94] asserts in particular that up to isomorphism, $A$ admits a unique realization $(\mathbb{A}_0, \Pi_0, \Pi_0^\vee)$.

If $\mathbb{A}$ is a generalized free realization of $A$, the space $\bigcap_{i=1}^n \ker \alpha_i$ is the *inessential part of* $\mathbb{A}$ and is denoted $\mathbb{A}_{in}$. The following lemma is easy to prove.

**Lemma 5.14.** *Let $\mathbb{A}$ be a generalized free realization of $A$. Then there exists $\mathbb{A}' \subset \mathbb{A}$ and $B \subset \mathbb{A}_{in}$ such that $\mathbb{A}'$ is isomorphic to $\mathbb{A}_0$ (as a realization of $A$), $Q_{\mathbb{A}}^\vee \subset \mathbb{A}'$ and $\mathbb{A} = \mathbb{A}' \oplus B$.*

**Lemma 5.15.** *Let $\mathbb{A}$ be a generalized free realization of $A$. Then the map* $\begin{array}{c} W_{\mathbb{A}}^v \to \mathrm{Aut}_{\mathbb{Z}}(Q_{\mathbb{A}}^\vee) \\ w \mapsto w_{|Q^\vee} \end{array}$ *is injective.*

*Proof.* One writes $\mathbb{A} = \mathbb{A}' \oplus B$, with $\mathbb{A}'$ and $B$ as in Lemma 5.14. For all $x \in \mathbb{A}$ and $w \in W_{\mathbb{A}}^v$, $w(x) - x \in Q_{\mathbb{R},\mathbb{A}}^\vee$, where $Q_{\mathbb{R},\mathbb{A}}^\vee = \bigoplus_{i=1}^n \mathbb{R}\alpha_i^\vee$. Therefore, $\mathbb{A}'$ is stable by $W_{\mathbb{A}}^v$. Moreover, for all $x \in \mathbb{A}_{in}$, $w(x) = x$. Hence the restriction map $W_{\mathbb{A}}^v \to W_{\mathbb{A}'}^v$ is a an isomorphism. As a consequence, one can suppose that $\mathbb{A} = \mathbb{A}_0$. But by the assertion (3.12.1) of proof of Proposition 3.12 of [Kac94] (applied to $\Delta^\vee$ instead of $\Delta$), if $w \in W_{\mathbb{A}_0}^v$ satisfies $w_{|\Delta^\vee} = 1$ then $w = 1$ (where $\Delta^\vee$ is a set included in $Q^\vee$). This proves the lemma. □

### 5.4.2 Infinite intersection of spheres

In this subsection $\mathbb{A}$ denotes the standard apartment of $\mathcal{I}$.

We suppose that there exists a non-spherical face$F$ of $\mathbb{A}$ satisfying $F_0 \subsetneq F \subsetneq C_0^+$.

**Remark 5.16.** By 1.3 of [Rou11] the vectorial faces based at 0 form a partition of the Tits cone. Therefore, if $F^v$ is a vectorial faceand if for some $u \in F^v$ and $w \in W^v$, $w.u \in F^v$, then $w.F^v = F^v$. Therefore, if $W' \subset W$, $W'.F^v$ is infinite if and only if $W'.u$ is infinite for some $u \in F^v$ if and only if $W'.u$ is infinite for all $u \in F^v$.

For the next proposition, we use the graph of the matrix $A$, whose vertices are the $i \in I$ and whose arrows are the $\{i, j\}$ such that $a_{i,j} \neq 0$.

**Proposition 5.17.** *Let $F$ be a non-spherical faceF of $\mathbb{A}$ satisfying $F_0 \subsetneq F \subsetneq C_0^+$. Then there exists $w \in W^v$ such that $W_F.w.F$ is infinite.*

*Proof.* Let $F' = w.F$. One writes $F = F(0, F^v)$, with $F^v = \{u \in \mathbb{A} | \alpha_i(x) > 0 \ \forall i \in J \text{ and } \alpha_i(x) = 0 \ \forall i \in I \backslash J\}$ for some $J \subset I$. By Lemma 5.15, there exists $k \in I$ such that $W_F.\alpha_k^\vee$ is infinite.

First suppose that the Kac-Moody matrix $A$ is indecomposable. Let $i \in J$ ($J \neq \emptyset$ because $F(0, \mathbb{A}_{in}) \subsetneq F$). . By 4.7 of [Kac94], the graph of $A$ is connected. Therefore, there exists a sequence $j_1 = i, \ldots, j_l = k$ such that $a_{j_1,j_2} a_{j_2,j_3} \ldots a_{j_{l-1},j_l} \neq 0$.

Let $u \in F^v$. Let us show that there exists $w \in W^v$ such that $\alpha_k(w.u) \neq 0$. If $x \in \mathbb{A}$ and $m \in [\![1, l]\!]$, one says that $x$ satisfies $P_m$ if for all $m' \in [\![m+1, l]\!]$, $\alpha_{j_{m'}}(x) = 0$ and $\alpha_{j_m}(x) \neq 0$. Let $x \in \mathbb{A}$, $m \in [\![1, l-1]\!]$ and suppose that $x$ satisfies $P_m$. Let $x' = r_{j_m}(x) = x - \alpha_{j_m}(x)\alpha_{j_m}^\vee$. Then $\alpha_{j_{m+1}}(x') = -\alpha_{j_m}(x) a_{j_m, j_{m+1}} \neq 0$ and thus $x'$ satisfies $P_{m'}$ for some $m' \in [\![m+1, l]\!]$. As $i = j_1 \in J$, $\alpha_{j_1}(u) \neq 0$ and hence $u$ satisfies $P_m$ for some $m \in [\![1, l]\!]$. Therefore, there exists $w \in W^v$ such that $w(u)$ satisfies $P_l$: $\alpha_k(w(u)) \neq 0$.

If $W_F.w(u)$ is finite, $W_F.r_k(w(u)) = W_F.(u - \alpha_k(w(u))\alpha_k^\vee)$ is infinite and thus at least one of the sets $W_F.w(u)$ and $W_F.r_k(w(u))$ is infinite. This proves the lemma when $A$ is indecomposable by Remark 5.16.

We no more suppose that $A$ is indecomposable. Let $A_1, \ldots, A_r$ be the indecomposable components of $A$. One writes $\mathbb{A} = \mathbb{A}_1 \oplus \ldots \oplus \mathbb{A}_r$, where $\mathbb{A}_i$ is a realization of $A_i$ for all $i \in [\![1, r]\!]$. Then $F = F_1 \oplus \ldots \oplus F_r$ and $W_F = W_{F_1} \times \ldots \times W_{F_r}$. There exists $i \in [\![1, r]\!]$ such that $F_i$ is not spherical and thus for some $w \in W^v$, $W_F.(w.F)$ is infinite. ☐

Let $w \in W^v$ such that $W_F.F'$ is infinite, where $F' = w.F$. Then $W_F.F' \subset \mathcal{S}^F(F', [F']) \cap \mathcal{S}^F_{op}(F', [F'])$. Therefore, $\mathcal{S}^F(F', [F']) \cap \mathcal{S}^F_{op}(F', [F'])$ is infinite.

**Corollary 5.18.** *Let $\lambda \in Y^+ \backslash \mathbb{A}_{in}$. Then $W^v.\lambda$ is infinite.*

*Proof.* Let $F^v$ be the vectorial facecontaining $\lambda$. By Remark 5.16, the map $\begin{array}{c} W^v.F \to W^v.\lambda \\ w.F \to w.\lambda \end{array}$ is well defined and is a bijection. If $F^v$ is spherical, $W^v.F^v$ is infinite because the stabilizer of $F^v$ is finite and if $F^v$ is non-spherical, this is a consequence of the proof of Proposition 5.17 and of Remark 5.16. ☐

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