QUOTIENTS OF FUSION SYSTEMS

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Abstract. let G be a finite group, P a Sylow p-subgroup of G, H a normal subgroup of G and Q is a Sylow p-subgroup of H contained in P. Then Q is strongly $F_P(G)$-closed and Puig's contruction $F_P(G)/Q$ is the same as the Frobenius category $F_{P/Q}(G/H)$ of the quotient G/H.

1. Introduction

Let p be a prime number. In the '90s, Puig gave an axiomatic description for the p-local structures, generalizing the notions of Frobenius category and Brauer category [Pu]. The new concepts that Puig introduced are the 'full Frobenius systems' on a p-group P. We call them simply 'fusion systems'. Recall that Frobenius category $F_p(G)$ of a finite group G at p is the category whose objects are the p-subgroups of G and whose morphisms are the morphisms given by conjugation by the elements of G. This category contains the p-local information of G. One can prove that Frobenius category is equivalent to its full subcategory $F_P(G)$ whose objects are the subgroups of a Sylow p-subgroup P of G.

In a recent article, Broto, Levi and Oliver [BLO] identified and studied a certain class of spaces which in many ways behave like p-completed classifying spaces of finite groups. They showed that these spaces occur as the "classifying spaces" of fusion systems. In fact, in the paper, they use yet another terminology which is 'saturated fusion systems' and the definition they give is slightly different from that of Puig for full Frobenius systems, but they prove that the two definition are equivalent. In this paper we use a simplification of the definition in Broto, Levi and Oliver's paper, equivalent to the latter.

The contruction of the quotient of a fusion system by a strongly $\mathcal{F}$-closed subgoup was introduced by Puig. When the fusion system it given by the fusion in a finite group G having a normal subgroup H we have also the natural construction of the fusion system given by the fusion in the quotient group G/H. In this paper we show that the two constructions are the same.

2. Definition and Basic Properties of Fusion Systems

Let us start with a more general definition.

Definition 2.1. *A category $\mathcal{F}$ on a p-group P is a category whose objects are the subgroups of P and whose set of morphisms between the subgroups Q and R of P, is a set $\mathrm{Hom}_{\mathcal{F}}(Q,R)$ of injective group homomorphisms from Q to R, with the following properties:*
(1) if $Q \leq R$ then the inclusion of Q in R is a morphism in $\mathrm{Hom}_{\mathcal{F}}(Q,R)$;

Date: November 23, 2003.
* The author is supported by Swiss National Science Fundation Grant no PBEL2-100846.

(2) for any $\phi \in \mathrm{Hom}_{\mathcal{F}}(Q,R)$ *the induced isomorphism* $Q \simeq \phi(Q)$ *and its inverse are morphisms in* $\mathcal{F}$*.*
(3) the composition of morphisms in $\mathcal{F}$ *is the usual composition of group homomorphisms.*

Note that the above definition of a category on P differs from what Puig calls 'divisible Frobenius system' and what, equivalently, Broto, Levi and Oliver call 'fusion system' by the fact that we do not ask for the inner automorphisms of P to be in the category.

In a finite group G having P as a Sylow p-subgroup, every G-conjugation class of subgroups in P contains an element Q such that a Sylow p-subgroup of the G-normalizer of Q is contained in P. Let's see what is the analog in a category $\mathcal{F}$ on P. If there exists an isomorphism $\phi \in \mathrm{Hom}_{\mathcal{F}}(Q,R)$ we say that Q and R are $\mathcal{F}$-conjugate.

Definition 2.2. *We say that a subgroup* Q *of* P *is fully* $\mathcal{F}$*-centralized, respectively fully* $\mathcal{F}$*-normalized if* $|C_P(Q)| \geq |C_P(Q')|$*, respectively* $|N_P(Q)| \geq |N_P(Q')|$*, for all* $Q' \leq P$ *which are* $\mathcal{F}$*-conjugated to* Q*.*

For Q, R, $T \leq P$ we denote $\mathrm{Hom}_T(Q,R) := \{u \in T \mid {}^uQ \leq R\}/C_T(Q)$ and $\mathrm{Aut}_T(Q) := \mathrm{Hom}_T(Q,Q)$. Other useful notations are $\mathrm{Aut}_{\mathcal{F}}(Q) := \mathrm{Hom}_{\mathcal{F}}(Q,Q)$ and $\mathrm{Out}_{\mathcal{F}}(Q) := \mathrm{Aut}_{\mathcal{F}}(Q)/\mathrm{Aut}_Q(Q)$. We are now able to give the definition of a fusion system.

Definition 2.3. *A fusion system on a finite* p*-group* P *is a category* $\mathcal{F}$ *on* P *satisfying the following properties:*
FS1. $\mathrm{Hom}_P(Q,R) \subset \mathrm{Hom}_{\mathcal{F}}(Q,R)$ *for all* $Q, R \leq P$.
FS2. $\mathrm{Aut}_P(P)$ *is a Sylow* p*-subgroup of* $\mathrm{Aut}_{\mathcal{F}}(P)$
FS3. Every $\phi : Q \to P$ *such that* $\phi(Q)$ *is fully* $\mathcal{F}$*-normalized extends to a morphism* $\bar{\phi} : N_\phi \to P$ *where* $N_\phi = \{x \in N_P(Q) \mid \exists y \in N_P({}^gQ),\ \phi({}^xu) = {}^y\phi(u),\ \forall u \in Q\}$*.*

We remark that N_ϕ is the largest subgroup of $N_P(Q)$ such that ${}^\phi(N_\phi/C_P(Q)) \leq \mathrm{Aut}_P(\phi(Q))$. Thus we have $QC_P(Q) \leq N_\phi$.

Through the rest of the section P denotes a finite p-group, Q a subgroup of P and $\mathcal{F}$ a fusion system on P. Here is an equivalent characterization of beeing fully $\mathcal{F}$-normalized.

Proposition 2.4 ([Li], Prop. 2.7)**.** *We have that* Q *is fully* $\mathcal{F}$*-normalized if and only if* Q *is fully* $\mathcal{F}$*-centralized and* $\mathrm{Aut}_P(Q)$ *is a Sylow* p*-subgroup of* $\mathrm{Aut}_{\mathcal{F}}(Q)$*.*

In a fusion system we have analogous notions for the normalizer and the centralizer in a finite group:

Definition 2.5. *The normalizer* $N_{\mathcal{F}}(Q)$ *is the category on* $N_P(Q)$ *having as morphisms, those morphisms* $\psi \in \mathcal{F}(R,T)$ *satisfying that there exists a morphism* $\phi \in \mathrm{Hom}_{\mathcal{F}}(QR,QT)$ *such that* $\phi|_Q \in \mathrm{Aut}_{\mathcal{F}}(Q)$ *and* $\phi|_R = \psi$*. The centralizer* $C_{\mathcal{F}}(Q)$ *is the category on* $C_P(Q)$ *having as morphisms those morphisms* $\psi \in \mathcal{F}(R,T)$ *satisfying that there exists a morphism* $\phi \in \mathrm{Hom}_{\mathcal{F}}(QR,QT)$ *such that* $\phi|_Q = \mathrm{id}_Q$ *and* $\phi|_R = \psi$*.*

In general $N_{\mathcal{F}}(Q)$ is not a fusion system on $N_P(Q)$ because Property FS2 fails to be satisfied, but it becomes one if Q is fully $\mathcal{F}$-normalized. It is the same for $C_{\mathcal{F}}(Q)$ when Q is fully $\mathcal{F}$-centralized.

Proposition 2.6 ([Pu], Prop. 2.8)**.** *If Q is fully $\mathcal{F}$-normalized then $N_{\mathcal{F}}(Q)$ is a fusion system on $N_P(Q)$. If Q is fully $\mathcal{F}$-centralized then $C_{\mathcal{F}}(Q)$ is a fusion system on $C_P(Q)$.*

Here is some more terminology in a fusion system.

Definition 2.7. *We say that*
(i) Q is $\mathcal{F}$-centric if $C_P(\phi(Q)) \subset \phi(Q)$, for all $\phi \in \mathrm{Hom}_{\mathcal{F}}(Q,P)$.
(ii) Q is $\mathcal{F}$-radical if $O_p(\mathrm{Out}_{\mathcal{F}}(Q)) = 1$.
(iii) Q is $\mathcal{F}$-essential if Q is $\mathcal{F}$-centric and $\mathrm{Out}_{\mathcal{F}}(Q)$ has a strongly p-embedded subgroup M (that is M contains a Sylow p-subgroup S of $\mathrm{Out}_{\mathcal{F}}(Q)$ such that ${}^{\phi}S \cap S = \{1\}$ for every $\phi \in \mathrm{Out}_{\mathcal{F}}(Q) \setminus M$).
(iv) Q is strongly $\mathcal{F}$-closed if for any subgroup R of Q and any morphism $\phi \in \mathrm{Hom}_{\mathcal{F}}(R,P)$ we have $\phi(R) \leq Q$.
(v) Q is weakly $\mathcal{F}$-closed if for any morphism $\phi \in \mathrm{Hom}_{\mathcal{F}}(Q,P)$ we have $\phi(Q) = Q$.

An $\mathcal{F}$-centric subgroup Q of P is fully $\mathcal{F}$-centralized. Indeed, for any morphism $\phi \in \mathrm{Hom}_{\mathcal{F}}(Q,P)$, we have

$$\phi(C_P(Q)) = \phi(Z(Q)) = Z(\phi(Q)) = C_P(\phi(Q)) \,.$$

We can see if a subgroup Q of P is $\mathcal{F}$-essential only by studying the Quillen complex of the outer automorphism group of Q in $\mathcal{F}$. Recall that for a finite group G the Quillen complex of G at p, denoted by $\mathcal{S}_p(G)$ has vertices the non-trivial p-subgroups of G and simplices are given by chains of groups ordered by inclusion. Thevenaz showed [**?**, Theorem 48.8] that Q is $\mathcal{F}$-essential if and only if $\mathcal{S}_p(\mathrm{Out}_{\mathcal{F}}(Q))$ is disconnected. As any non-trivial normal subgroup of $\mathrm{Out}_{\mathcal{F}}(Q)$ would connect $\mathcal{S}_p(\mathrm{Out}_{\mathcal{F}}(Q))$, we have that if Q is $\mathcal{F}$-essential then $O_p(\mathrm{Out}_{\mathcal{F}}(Q)) = 1$ giving that Q is $\mathcal{F}$-radical.

Another easy remark is that Q is strongly $\mathcal{F}$-closed if and only if $\mathrm{Hom}_{\mathcal{F}}(R,P) = \mathrm{Hom}_{\mathcal{F}}(R,Q)$, for any subgroup R of Q, and is weakly $\mathcal{F}$-closed if and only if $\mathrm{Hom}_{\mathcal{F}}(Q,P) = \mathrm{Aut}_{\mathcal{F}}(Q)$. It is clear that if Q is strongly $\mathcal{F}$-closed then Q is weakly $\mathcal{F}$-closed.

3. Main result

For a fusion system $\mathcal{F}$ on P Puig defined [Pu] the quotient $\overline{\mathcal{F}}$ of $\mathcal{F}$ by a strongly $\mathcal{F}$-closed subgroup Q of P.

Definition 3.1. *Let $\mathcal{F}$ be a fusion system on P and Q a strongly $\mathcal{F}$-closed subgroup of P. We define $\overline{\mathcal{F}} := \mathcal{F}/Q$ as the category on P/Q whose objects are the subgroups of P/Q and whose morphisms are those induced by $\mathcal{F}$.*

We prove now that $\mathcal{F}/Q$ is a fusion system. This result is due to Puig. Note $\overline{} : P \to P/Q$ the canonical projection.

Proposition 3.2 ([Pu], Prop. 2.15)**.** *Let $\mathcal{F}$ be a fusion system on P and Q a strongly $\mathcal{F}$-closed subgroup of P. Then $\overline{\mathcal{F}} := \mathcal{F}/Q$ is a fusion system on P/Q.*

Proof. We verify the three properties for $\overline{\mathcal{F}}$. Property FS1 is trivially satisfied.

By construction of $\overline{\mathcal{F}}$, the canonical morphism $\mathrm{Aut}_{\mathcal{F}}(P) \longrightarrow \mathrm{Aut}_{\overline{\mathcal{F}}}(\overline{P})$ is surjective and the image of $\mathrm{Aut}_P(P)$ by this morphism is $\mathrm{Aut}_{\overline{P}}(\overline{P})$. As $\mathcal{F}$ is a fusion system, the group $\mathrm{Aut}_P(P)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(P)$, so $\mathrm{Aut}_{\overline{P}}(\overline{P})$ is a Sylow p-subgroup of $\mathrm{Aut}_{\overline{\mathcal{F}}}(\overline{P})$, thus Property FS2 is satisfied in $\overline{\mathcal{F}}$.

Let's verify now Propery FS3. Consider $\overline{\phi} \in \mathrm{Hom}_{\overline{\mathcal{F}}}(\overline{R}, \overline{P})$, R the inverse image of $\overline{R}$ in P and $\phi \in \mathrm{Hom}_{\mathcal{F}}(R, P)$ a representative of $\overline{\phi}$ in $\mathcal{F}$. Denote $T := \phi(R)$. Suppose that $\overline{T}$ is fully $\overline{\mathcal{F}}$-normalized. We have $Q \leq R$ so $Q \leq N_P(R)$ giving that $N_P(R)$ is the inverse image of $N_{\overline{P}}(\overline{R})$. The same is true for T as $Q = \phi(Q) \leq T$, and also for any subgroup R' of P, $\mathcal{F}$-conjugated to T. Thus, using that $\overline{T}$ is fully $\overline{\mathcal{F}}$-normalized we have $|N_P(R)'| = |N_{\overline{P}}(\overline{R}')| \cdot |Q| \leq |N_{\overline{P}}(\overline{T})| \cdot |Q| = |N_P(T)|$ so T is fully $\mathcal{F}$-normalized.

By Property FS3 applied to ϕ, there exists $\rho \in \mathrm{Hom}_{\mathcal{F}}(N_\phi, P)$ extending ϕ. As Q is strongly $\mathcal{F}$-closed, we obtain $\rho(Q) = Q$, so $\overline{\rho}$ is a morphism in $\mathrm{Hom}_{\overline{\mathcal{F}}}(\overline{N_\phi}, \overline{P})$. Moreover $\overline{N_\phi} = N_{\overline{\phi}}$. The inclusion of $\overline{N_\phi}$ in $N_{\overline{\phi}}$ is obvious. For the other inclusion denote by N the inverse image of $N_{\overline{\phi}}$ in P. The canonical projection on P/Q induces a morphism between $\mathrm{Aut}_P(R)$ and $\mathrm{Aut}_{\overline{P}}(\overline{R})$. We have that $\overline{{}^{\phi}(\mathrm{Aut}_N(R))} = {}^{\overline{\phi}}\mathrm{Aut}_{N_{\overline{\phi}}}(\overline{R}) \leq \mathrm{Aut}_{\overline{P}}(\overline{T})$ thus ${}^{\phi}(\mathrm{Aut}_N(R)) \leq \mathrm{Aut}_P(T)$ implying that $N \leq N_\phi$. So $\overline{\rho}$ extends $\overline{\phi}$ to $N_{\overline{\phi}}$ and Property FS3 is satisfied for $\overline{\mathcal{F}}$. □

In fact if G is a finite group having P as Sylow p-subgroup, and Q is a subgroup of P normal in G, then Q is strongly $F_P(G)$-closed and, moreover $F_P(G)/Q = F_{P/Q}(G/Q)$. The fact that Q is strongly $F_P(G)$-closed is straight forward, as any morphism in $F_P(G)$ given by conjugation by an element g of G mapping from a subgroup R of Q extends in a morphism by conjugation by g which maps from Q and we have ${}^gR \leq {}^gQ = Q$ as G normalizes Q. The equality $F_P(G)/Q = F_{P/Q}(G/Q)$ comes simply from the fact that the objects and the morphisms of the two categories are trivially the same.

We have something similar in a more general setting. Suppose that H is a normal subgroup of G and Q is a Sylow p-subgroup of H contained in P. We have the following.

Proposition 3.3. *let G be a finite group, P a Sylow p-subgroup of G, H a normal subgroup of G and Q a Sylow p-subgroup of H contained in P. Then Q is strongly $F_P(G)$-closed*

Proof. Denote $\mathcal{F} := F_P(G)$. Let R be a subgroup of Q and $\phi \in \mathrm{Hom}_{\mathcal{F}}(R, P)$. So $\phi = \mathrm{conj}_g$ for some $g \in G$. We have that H is a normal subgroup of G so ${}^gH = H$. Using that $\phi \in \mathrm{Hom}_{\mathcal{F}}(R, P)$ we have also ${}^gR \leq P$. As R is a subgroup of Q it is contained in H and thus ${}^gR \leq {}^gH$.

So we have ${}^gR \leq {}^gH \cap P = H \cap P = Q$. This gives that $\phi(R) \leq Q$ and thus $\phi \in \mathrm{Hom}_{\mathcal{F}}(R, Q)$. As this is true for all the subgroups R of Q, we obtain that Q is strongly $\mathcal{F}$-closed. □

So we can construct a fusion system by putting $\mathcal{G} := F_P(G)/Q$. But as H is a normal subgroup of G we can also consider the fusion system $F_{P/Q}(G/H)$. In fact these two constructions are the same.

Proposition 3.4. *let G be a finite group, P a Sylow p-subgroup of G, H a normal subgroup of G and Q is a Sylow p-subgroup of H contained in P. Then $F_P(G)/Q = F_{P/Q}(G/H)$.*

Proof.

Both fusion systems in the text of the theorem are on P/Q and $\mathcal{G} := F_P(G)/Q$ contains all the morhisms induced by conjugation with elemente of G. So, a priori,

$F_{P/Q}(G/H)$ is a subsystem of $\mathcal{G}$. To prove the equality we only need to show that for every subgroup of P containing Q, every morphism induced by H from R to P is trivial on R/Q. This would mean the taking the quotient by H does not affect the morphisms in $\mathrm{Hom}_{\mathcal{G}}(R/Q, P/Q)$.

Take $h \in H$ such that ${}^hR \leq P$. For every $u \in R$ we have that the comutator $c := huh^{-1}u^{-1}$ is in H, as H is normalised by R, and in P, as ${}^hu \in {}^hR \leq P$. So c is in the intersection $H \cap P = Q$ meaning that h acts trivially on R/Q. □

Markus Linckelman [Li] introduced the notion of *normal fusion system*.

Definition 3.5. *Let $\mathcal{F}$ be a fusion system on a finite p-group P and $\mathcal{F}'$ a fusion subsystem of $\mathcal{F}$ on a subgroup P' of P. We say that $\mathcal{F}'$ is normal in $\mathcal{F}$ if P' is strongly $\mathcal{F}$-closed and if for every isomorphism $\phi : Q \to Q'$ in $\mathcal{F}$ and any two subgroups R, R' of $Q \cap P'$ we have*

$$\phi \circ \mathrm{Hom}_{\mathcal{F}'}(R, R') \circ \phi^{-1} \subseteq \mathrm{Hom}_{\mathcal{F}'}(\phi(R), \phi(R'))\,.$$

When we have a normal subsystem $\mathcal{F}'$ on P' of a fusion system $\mathcal{F}$ on P it it natural to try to define the quotient $\mathcal{F}/\mathcal{F}'$. A definition suggested by the above result is that we should take Puig's quotient of $\mathcal{F}$ by P'. The only problem is that we don't know whether there is a normal fusion subsystem on every strongly $\mathcal{F}$-closed subgroup. This is still an open question.

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