ON NORMAL FUSION SUBSYSTEMS

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ABSTRACT. Let P be a finite p-group and $\mathcal{F}$ a fusion system on P. Let Q be a $\mathcal{F}$-stongly closed subgroup of P and $\mathcal{G}$ be the fusion system on Q canonicaly generated by Q. We will prove that a necessary and sufficient condition for $\mathcal{G}$ to be normal in $\mathcal{F}$ is that $N_{\mathcal{F}}(Q) = \mathcal{F}$.

1. Introduction

The notion of *fusion systems* was introduced by Puig [Pu] in 1990. Puig called them Full Frobenius Systems. Their name came from the fact that they appeared as a natural generalization of the notion of Frobenius category of a finite group. Ten years latter, Broto, Levi and Oliver [BLO] studied a certain class of topological space which in many way behave like p-completed classifying spaces of finite groups. They proved that those spaces occur as classifying space of certain algebraic objects, which they called *p-local finite groups*. If fact, a p-local finite group is a fusion system admiting an additional structure *centric linking system*. Their definition of fusion systems is slyghtly different from the Puig's one but they proove that the definitions are equivalent. The definition we use came from our approach which we present in [St2] and that we prove to be equivalent to the others two.

2. Definition and Basic Properties of Fusion Systems

Let us start with a more general definition.

Definition 2.1. *A category $\mathcal{F}$ on P is a category whose objects are the subgroups of P and whose set of morphisms between the subgroups Q and R of P, is a set $\mathrm{Hom}_{\mathcal{F}}(Q, R)$ of injective group morphisms from Q to R, with the following properties:*
(1) if $Q \leq R$ then the inclusion $Q \to R$ is a morphism in $\mathcal{F}$;
(2) for any $\phi \in \mathrm{Hom}_{\mathcal{F}}(Q, R)$ the induced isomorphism $Q \simeq \phi(Q)$ and its inverse are morphisms in $\mathcal{F}$.
(3) composition of morphisms in $\mathcal{F}$ is the usual composition of group homomorphisms.

Let's define what it means for a subgroup Q of P to be 'well placed'.

Definition 2.2. *A subgroup Q of P is fully $\mathcal{F}$-centralized, respectively fully $\mathcal{F}$-normalized if $|C_P(Q)| \leq |C_P(Q')|$, respectively $|N_P(Q)| \leq |N_P(Q')|$, for all $Q' \leq P$ which is $\mathcal{F}$-conjugated to Q.*

We are able now to give the definition of a fusion system.

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Definition 2.3. *A fusion system on a finite p-group P is a category $\mathcal{F}$ on P satisfying the following properties:*
FS1. $\mathrm{Hom}_P(Q,R) \subset \mathrm{Hom}_{\mathcal{F}}(Q,R)$ *for all* $Q,R \leq P$.
FS2. $\mathrm{Aut}_P(P)$ *is a Sylow p-subgroup of* $\mathrm{Aut}_{\mathcal{F}}(P)$
FS3. Every $\phi : Q \to P$ such that $\phi(Q)$ is fully normalized extends to a morphism $\bar{\phi} : N_\phi \to P$ where

$$N_\phi = \{x \in N_P(Q) \mid \exists y \in N_P({}^gQ),\, \phi({}^xu) =^y \phi(u),\, \forall u \in Q\}\,.$$

From now on P will denote a finite p-group and $\mathcal{F}$ a fusion system on P. Here is a equivalent caracterization of beeing fully $\mathcal{F}$-normalized for a subgroup Q of P.

Proposition 2.4 ([Li], Prop. 2.7)**.** *Let Q be a subgroup of P. Then Q is fully $\mathcal{F}$-normalized if and only if Q is fully $\mathcal{F}$-centralized in and $\mathrm{Aut}_P(Q)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(Q)$.*

In a fusion system we have an equivalent notions for the normaliser and the centralizer in a finite group:

Definition 2.5. *Let $\mathcal{F}$ be a fusion system on P and let Q be a subgroup of P. The normalizer $N_{\mathcal{F}}(Q)$ is the category on $N_P(Q)$ having as morphisms $\psi \in \mathrm{Hom}_{N_{\mathcal{F}}(Q)}(R,T)$ sucht that there exists a morphism $\phi \in \mathrm{Hom}_{\mathcal{F}}(QR,QT)$ satisfying $\phi|_Q \in \mathrm{Aut}_{\mathcal{F}}(Q)$ and $\phi|_R = \psi$. The centralizer $C_{\mathcal{F}}(Q)$ is the category on $C_P(Q)$ having as morphisms $\psi \in \mathrm{Hom}_{C_{\mathcal{F}}(Q)}(R,T)$ sucht that there exists a morphism $\phi \in \mathrm{Hom}_{\mathcal{F}}(QR,QT)$ satisfying $\phi|_Q = \mathrm{id}_Q$ and $\phi|_R = \psi$.*

In general $N_{\mathcal{F}}(Q)$ is not a fusion system on $N_P(Q)$, but it becomes one if Q is fully $\mathcal{F}$-normalized. It is the same for $C_{\mathcal{F}}(Q)$ when Q is fully $\mathcal{F}$-centralized

Proposition 2.6 ([Pu], Prop. 2.8)**.** *Let $\mathcal{F}$ be a fusion system on P. If $Q \leq P$ is fully $\mathcal{F}$-normalized then $N_{\mathcal{F}}(Q)$ is a fusion system on $N_P(Q)$. If $Q \leq P$ is fully $\mathcal{F}$-centralized then $C_{\mathcal{F}}(Q)$ is a fusion system on $C_P(Q)$.*

Here is some more terminology in a fusion system.

Definition 2.7. *Let $\mathcal{F}$ be a fusion system on a finite p-group P and Q a subgroup of P. We say that*
(i) Q is $\mathcal{F}$-centric if $C_P(\phi(Q)) \subset \phi(Q)$, for all $\phi \in \mathrm{Hom}_{\mathcal{F}}(Q,P)$.
(ii) Q is $\mathcal{F}$-radical if $O_p(\mathrm{Aut}_{\mathcal{F}}(Q)/\mathrm{Aut}_Q(Q)) = 1$.
(iii) Q is $\mathcal{F}$-essential if Q is $\mathcal{F}$-centric and $\mathrm{Aut}_{\mathcal{F}}(Q)/\mathrm{Aut}_Q(Q)$ has a strongly p-embedded subgroup.
(iv) Q is strongly $\mathcal{F}$-closed if for any subgroup R of Q and any morphism $\phi \in \mathrm{Hom}_{\mathcal{F}}(R,P)$ we have $\phi(R) \leq Q$.
(v) Q is weakly $\mathcal{F}$-closed if for any morphism $\phi \in \mathrm{Hom}_{\mathcal{F}}(Q,R)$ we have $\phi(Q) = Q$.

A $\mathcal{F}$-centric subgroup Q of P is fully $\mathcal{F}$-centralized in $\mathcal{F}$. Indeed, for any morphism $\phi \in \mathrm{Hom}_{\mathcal{F}}(Q \cdot C_P(Q), P) = \mathcal{F}(Q,P)$, we have

$$\phi(C_P(Q)) = \phi(Z(Q)) = Z(\phi(Q)) = C_P(\phi(Q))\,.$$

From the definition we also deduce that if Q is $\mathcal{F}$-essential if and only if the Quillen complex of the outer automorphism group $\mathcal{S}_p(\mathrm{Out}(Q))$ is disconnected implying that Q is $\mathcal{F}$-radical. Recall that for a finite group G the Quillen complex of G has vertices the objects in $F_p(G)$ and simplices are given by chains of groups ordered by inclusion.

Another easy remark is that Q is strongly $\mathcal{F}$-closed if and only if $\mathcal{F}(R,P) = \mathcal{F}(R,Q)$, for any subgroup R of Q, and is weakly $\mathcal{F}$-closed if and only if $\mathcal{F}(Q,P) = \mathcal{F}(Q)$. It's clear that, if Q is strongly $\mathcal{F}$-closed, then Q is weakly $\mathcal{F}$-closed.

The Alperin's theorem on local control, also hold in the fusion systems. In this theorem P is a finite p-group, Q a subgroup of P and $\mathcal{F}$ a fusion system on P. If $\phi \in \mathrm{Aut}_{\mathcal{F}}(P)$, we say that ϕ is a maximal morphism; if $\phi \in \mathrm{Aut}_{\mathcal{F}}(E)$, where E is $\mathcal{F}$-essential, we say that ϕ is an essential morphism.

Theorem 2.8. [St1] *Any morphism $\phi \in \mathrm{Hom}_{\mathcal{F}}(Q,P)$ can be written as the composition of essential morphisms, followed by a maximal morphism. More precisely, there exists a positive integer n and, for all $1 \leq i \leq n$, $\mathcal{F}$-essential subgroups E_i fully $\mathcal{F}$-normalized and morphisms $\psi_i \in \mathrm{Aut}_{\mathcal{F}}(E_i)$, $\psi_{n+1} \in \mathrm{Aut}_{\mathcal{F}}(P)$ such that we have*

$$\phi(u) = \psi_{n+1}\psi_n \dots \psi_1(u), \text{ for all } u \in Q\,.$$

In the middle of the '80s, Gilotti and Serena [GS] gave necessary and sufficient conditions for the normalizer of a p-subgroup of a finite group G to control the p-fusion in G. Here is the generalization to fusion systems.

Theorem 2.9. [St1] *Let P be a finite p-group, Q a subgroup of P and $\mathcal{F}$ a fusion system on P. Then $N_{\mathcal{F}}(Q) = \mathcal{F}$ if and only if Q is strongly $\mathcal{F}$-closed and Q admits a central series $Q = Q_n \geq Q_{n-1} \geq \ldots \geq Q_1$ where Q_i is weakly $\mathcal{F}$-closed, for all $1 \leq i \leq n-1$.*

3. Main Result

Recently, Linckelman [Li], motivated by the reduction of some problems on fusion systems introduced the notion of 'normal fusion subsystem'. Here his approach:

Definition 3.1. *Let $\mathcal{F}$ be a fusion system on a finite p-group P and $\mathcal{F}'$ a fusion subsystem of $\mathcal{F}$ on a subgroup P' of P. We say that $\mathcal{F}'$ is normal in $\mathcal{F}$ if P' is strongly $\mathcal{F}$-closed and if for every isomorphism $\phi : Q \to Q'$ in $\mathcal{F}$ and any two subgroups R, R' of $Q \cap P'$ we have*

$$\phi \circ \mathrm{Hom}_{\mathcal{F}'}(R,R') \circ \phi^{-1} \subseteq \mathrm{Hom}_{\mathcal{F}'}(\phi(R),\phi(R'))\,.$$

The deffinition of a simple fusion system came now in a natural way.

Definition 3.2. *A fusion system $\mathcal{F}$ on a finite p-group P is simple if it has no normal fusion subsystem other then itself and the trivial one.*

Here is now our theorem on normal fusion subsystems.

Theorem 3.3. *Let P be a finite p-group and $\mathcal{F}$ a fusion system on P. Let Q be a $\mathcal{F}$-stongly closed subgroup of P and $\mathcal{G}$ be the fusion system on Q canonicaly generated by Q. Then $\mathcal{G}$ is be normal in $\mathcal{F}$ if and only if $N_{\mathcal{F}}(Q) = \mathcal{F}$.*

Proof. If $N_{\mathcal{F}}(Q) = \mathcal{F}$ then any morphism in $\mathcal{F}$, $\phi : R \to P$ where R is a subgroup of Q, can be lifted to a morphism $\tilde{\phi} : Q \to Q$ as Q is $\mathcal{F}$-strongly closed. To prove that $\mathcal{G}$ is normal in $\mathcal{F}$ is is sufficient to prove that for any $u \in Q$ and any morphism $\phi :< R, \phi(R) >\to P$ in $\mathcal{F}$ the morphism $\psi := \phi\mathrm{conj}_u\phi^{-1} : \phi(R) \to \phi({}^uR)$ is also in $\mathcal{G}$. But this is straight forward as ϕ lifts to $\tilde{\phi}$ so $\psi = \mathrm{conj}_{\tilde{\phi}(u)}$ which is a morphism in $\mathcal{G}$.

If $\mathcal{G}$ is a normal subsystem in $\mathcal{F}$ we apply Theorem 2.9 to prove that $N_{\mathcal{F}}(Q) = \mathcal{F}$. As Q is $\mathcal{F}$-strongly it suffices to prove that the components of the upper central series of Q are all $\mathcal{F}$-weakly closed. In general we will prove this property of any characteristic subgroup R of Q. Let $\phi : R \to \phi(R)$ be a morphism in $\mathcal{F}$ with $\phi(R)$ $\mathcal{F}$-fully normalised. As $\mathcal{G}$ is normal in $\mathcal{F}$ we have that for any $u \in Q$, $\psi := \phi \mathrm{conj}_u \phi^{-1}$ is a morphism in $\mathcal{G}$ so equal to conj_v for a $v \in Q$. So $u \in N_\phi$ for all $u \in Q$. This means that ϕ extends to $\tilde{\phi} : Q \to Q$ as Q is $\mathcal{F}$-strongly closed. We obtain that ϕ is the restriction to R of an automorphism of Q. As R is characteristic in Q we have that $\phi(R) = R$, so R is $\mathcal{F}$-weakly closed. □

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