EQUIVALENT DEFINITIONS OF FUSION SYSTEMS

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Abstract. In the literature there are two equivalent generalized local structures over a finite p-group, the *full Frobenius systems* introduced by Puig [Pu2] in 1990 and the *saturated fusion systems* introduced by Broto, Levi and Oliver [BLO] in 2000. We give a simplified description of the second approach and we will show the equivalence between the three notions. We simply call this notion *fusion system*.

1. Introduction

Let p be a prime number, G a finite group and $F_p(G)$ the Frobenius category of G related to p. It's a category whose objects are the nontrivial p-subgroups of G and whose morphisms are the morphisms given by conjugation by the elements of G. This category contains the p-local information of G. One can prove that the Frobenius category is equivalent to its full subcategory whose objects are the subgroups of a Sylow p-subgroup of G.

In the '90s, Puig, convinced of the richness of the p-local structures, as the Frobenius category or the Brauer category, gave an axiomatic description. The notes in which Puig introduced these new concepts have remained in the handwritten state, since recently, when the author wrote a paper [Pu2]. The new concepts that Puig introduced are the *full Frobenius systems* over a p-group P.

In a recent preprint, interested in solving the Martino-Priddy conjecture, Broto, Levi and Oliver [BLO] identified and studied a certain class of spaces which in many ways behave like p-completed classifying spaces of finite groups. They showed that these spaces occur as the "classifying spaces" of certain algebraic objects, which they called *p-local finite groups*. Roughly speaking a p-local finite group consists in a full Frobenius systems on which we put a supplementary structure, called *centric linking system*. In fact, in this paper, they don't speak about full Frobenius systems but about *saturated fusion systems* and the definition they give is slightly different from those of Puig. The two definitions are proved to be equivalent.

After discussions with Radha Kessar and Markus Linckelmann we realized that some axioms in BLO's approach are redundant. We find a simplified set of axioms that are equivalent to the BLO's one, so to the Puig's one.

The paper is organized as follows. In the second section we will explain the Puig's approach and give some properties of the fusion systems. All the results in this sections are due to Puig. The third section is dedicated to BLO's approach. We will also prove that the two approaches are equivalent. The last section consit

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in our simplification of BLO's set of axioms and the proof of the equivalence of our set of axioms and the BLO's one.

2. Full Frobenius systems

All the results and the proofs in this section are due to Puig [Pu2]. Let us start with a more general definition.

Definition 2.1. *A category $\mathcal{C}$ over P is a category whose objects are the subgroups of P and whose set of morphisms between the subgroups Q and R of P, is a set $\mathcal{C}(Q,R)$ of injective group morphisms from Q to R, that contains the morphisms by conjugation by the elements of P.*

A full Frobenius system $\mathcal{F}$ over P is a category over P whose morphisms satisfy the three following axioms.

A1 *If Q and R are subgroups of P, then, for any morphism $\phi \in \mathcal{F}(Q,R)$ there exists a morphism $\psi \in \mathcal{F}(\phi(Q),Q)$, such that $\psi\phi(u) = u$ for all $u \in Q$.*

If $Q \leq R$, the inclusion of Q in R is also in $\mathcal{F}(Q,R)$ as it is induced by conjugation by 1. In particular, the morphisms in a full Frobenius system $\mathcal{F}$ over P are determined by the sets $\mathcal{F}(Q,P)$, where Q runs over all the subgroups of P. An other important remark is the fact that if we have a morphism $\phi \in \mathcal{F}(Q,R)$ and $T \leq Q$ then the restriction of ϕ to T is a morphism of $\mathcal{F}(T,R)$. Indeed, let $\iota \in \mathcal{F}(T,Q)$ be the inclusion of T in Q. Then $\phi\iota$ is a morphism in $\mathcal{F}(T,R)$. It's the restriction of ϕ to T.

Before writing down Axiom 2 we give some notations. Let Q be a subgroup of P. We denote by $\mathcal{F}(Q) := \mathcal{F}(Q,Q)$ the set of automorphisms of Q which belong to $\mathcal{F}$. By Axiom A1, $\mathcal{F}(Q)$ is a group. Let $R \leq P$ and denote by $\mathcal{I}_R(Q) := N_R(Q)/C_R(Q)$ the set of morphisms of Q, given by the conjugation by the elements of R. As all the morphisms by conjugation by the elements of P are in $\mathcal{F}$, it is straight forward that $\mathcal{I}_R(Q) \leq \mathcal{F}(Q)$. Finally, put $\mathcal{I}(Q) := \mathcal{I}_Q(Q)$, which is the group of inner automorphisms of Q.

A2 *The group $\mathcal{I}(P)$, of inner automorphisms of P, is a Sylow p-subgroup of $\mathcal{F}(P)$.*

If G is a finite group and S a Sylow p-subgroup of G, then we say that a subgroup Q of S is *fully normalized*, respectively *fully centralized* in S if $N_S(Q)$, respectively $C_S(Q)$ is a Sylow p-subgroup of $N_G(Q)$, respectively $C_G(Q)$. We give now a definition equivalent to the notions of fully normalized and fully centralized in a p-group for the full Frobenius systems. For doing this in the same time for the two notions, we denote by $N_P^K(Q)$ the K-normalizer of Q in P, which is the inverse image of K in $N_P(Q)$, where K is a subgroup of $\mathrm{Aut}(Q)$. For any morphism $\phi \in \mathcal{F}(Q \cdot N_P^K(Q), P)$, we have $\phi(N_P^K(Q)) \leq N_P^{{}^{\phi}K}(\phi(Q))$, where ${}^{\phi}K = \{\phi\theta\phi^{-1} | \theta \in K\}$ is a subgroup of $\mathrm{Aut}(\phi(Q))$.

Definition 2.2. *We say that Q is K-fully normalized in $\mathcal{F}$ if, for any morphism $\phi \in \mathcal{F}(Q \cdot N_P^K(Q), P)$, we have $\phi(N_P^K(Q)) = N_P^{{}^{\phi}K}(\phi(Q))$.*

If $K = 1$ or $K = \mathrm{Aut}(Q)$, we say that Q is fully centralized or, respectively, fully normalized in $\mathcal{F}$. Moreover, we have the next property.

Lemma 2.3 ([Pu2], 2.3.2)**.** *Let R be a subgroup of $Q \cdot N_P^K(Q)$ containing Q and $\phi \in \mathcal{F}(R,P)$, satisfying that, for any other morphism $\phi' \in \mathcal{F}(R,P)$, we have*

$$|N_P^{{}^{\phi'}K}(\phi'(Q))| \leq |N_P^{{}^{\phi}K}(\phi(Q))| . \qquad (*)$$

Then $\phi(Q)$ is ${}^{\phi}K$-fully normalized in $\mathcal{F}$.

Proof. Note $Q' := \phi(Q)$ and $K' := {}^{\phi}K$. With these notations, for any morphism $\psi \in \mathcal{F}(Q' \cdot N_P^{K'}(Q'), P)$, we have a morphism $\phi' \in \mathcal{F}(R, P)$ mapping $v \in R$ to $\psi(\phi(v))$. Moreover, note $Q'' := \phi'(Q)$ and $K'' := {}^{\phi'}K$. By $(*)$, we obtain that $|N_P^{K''}(Q'')| \leq |N_P^{K'}(Q')|$. But we always have $\psi(N_P^{K'}(Q')) \leq N_P^{K''}(Q'')$. As

$$|\psi(N_P^{K'}(Q'))| = |N_P^{K'}(Q')| \geq |N_P^{K''}(Q'')|\,,$$

we conclude that $\psi(N_P^{K'}(Q')) = N_P^{K''}(Q'')$. So Q' is K'-fully normalized. □

In particular, there always exists a morphism $\phi \in \mathcal{F}(Q, P)$ such that $\phi(Q)$ is ${}^{\phi}K$-fully normalized in $\mathcal{F}$.

We can give now the axiom allowing us to have the equivalent of Sylow's theorem, in the field of full Frobenius systems.

A3 *For any subgroup Q of P, for any subgroup K of* $\mathrm{Aut}(Q)$ *and for any morphism $\phi \in \mathcal{F}(Q, P)$, such that $\phi(Q)$ is ${}^{\phi}K$-fully normalized in P, there exists a morphism $\psi \in \mathcal{F}(Q \cdot N_P^K(Q), P)$ and $\chi \in K$, such that $\psi(u) = \phi(\chi(u))$ for all $u \in Q$.*

The approach between Axiom A3 and the Sylow's theorem is a little bit hidden. We see that being K-fully normalized for a subgroup Q of P is equivalent to say that Q is the best placed between all its images by morphisms in $\mathcal{F}(Q, P)$. The Sylow's theorem is equivalent to say that we can bring any p-subgroup of a group G, by conjugation by an element of G, inside a given Sylow p-subgroup of G. In the full Frobenius systems, Axiom A3 allow us to bring the K-normalizer in P of a image of Q by a morphism of $\mathcal{F}$, inside $N_P^K(Q)$, if Q is K-fully normalized in $\mathcal{F}$.

By the way, the following property, which will be very useful in the following, shows us that, behind the fact that a group is K-fully normalized is hidden, somewhere, a Sylow p-subgroup.

Proposition 2.4 ([Pu2], Prop. 2.7)**.** *Let Q be a subgroup of P and K a subgroup of* $\mathrm{Aut}(Q)$. *Then Q is K-fully normalized in $\mathcal{F}$ if and only if Q is fully centralized in $\mathcal{F}$ and $K \cap \mathcal{I}_P(Q)$ is a Sylow p-subgroup of $K \cap \mathcal{F}(Q)$.*

Proof. If Q is fully centralized in $\mathcal{F}$ and $\phi \in \mathcal{F}(Q \cdot N_P^K(Q), P)$, then the restriction of ϕ to $Q \cdot C_P(Q)$ is in $\mathcal{F}(Q \cdot C_P(Q), P)$, so we have $\phi(C_P(Q)) = C_P(\phi(Q))$. If, moreover, $K \cap \mathcal{I}_P(Q) = N_P^K(Q)/C_P(Q)$ is a Sylow p-subgroup of $K \cap \mathcal{F}(Q)$, we obtain

$$\begin{aligned}|N_P^K(Q)| &= |C_P(Q)| \cdot |K \cap \mathcal{I}_P(Q)| = \\ &= |C_P(Q)| \cdot |K \cap \mathcal{F}(Q)|_p = |C_P(\phi(Q))| \cdot |\,{}^{\phi}K \cap \mathcal{F}(\phi(Q))|_p \geq \\ &\geq |C_P(\phi(Q))| \cdot |\,{}^{\phi}K \cap \mathcal{I}_P(\phi(Q))| = |N_P^{{}^{\phi}K}(\phi(Q))|\,.\end{aligned}$$

where by $|G|_p$ we denote the order of a Sylow p-subgroup of G. But we always have that $\phi(N_P^K(Q)) \leq N_P^{{}^{\phi}K}(\phi(Q))$, so $|N_P^K(Q)| = |\phi(N_P^K(Q))| \leq |N_P^{{}^{\phi}K}(\phi(Q))|$. In this way we obtain $\phi(N_P^K(Q)) = N_P^{{}^{\phi}K}(\phi(Q))$ and so Q is K-fully normalized in $\mathcal{F}$.

Suppose now that Q is K-fully normalized in $\mathcal{F}$. First, we'll prove that Q is fully centralized in $\mathcal{F}$. Let $\phi \in \mathcal{F}(Q \cdot C_P(Q), P)$. By Axiom A1, there exists $\phi^{-1} \in \mathcal{F}(\phi(Q), Q)$ such that $\phi^{-1}\phi(u) = u$ for all $u \in Q$. Now, the group Q is K-fully normalized, so, by Axiom A3, applied to ϕ^{-1}, it follows that there exists $\rho \in \mathcal{F}(\phi(Q) \cdot N_P^{{}^{\phi}K}(\phi(Q)), P)$ and $\chi \in {}^{\phi}K$ such that $\rho(v) = \phi^{-1}(\chi(v))$ for all $v \in \phi(Q)$. Moreover, we have $\rho(C_P(\phi(Q))) \leq C_P(\rho\phi(Q)) = C_P(Q)$, as

$\rho(\phi(u)) = \phi^{-1}(\chi(\phi(u)) \in Q$ for all $u \in Q$. This gives $|C_P(\phi(Q))| \leq |C_P(Q)|$, so $\phi(C_P(Q)) = C_P(\phi(Q))$. As this result is true for all $\phi \in \mathcal{F}(Q \cdot C_P(Q), P)$, we obtain that Q is fully centralized in $\mathcal{F}$.

We show now, by induction on the index $|P : Q|$, that $K \cap \mathcal{I}_P(Q)$ is a Sylow p-subgroup of $K \cap \mathcal{F}(Q)$. For $P = Q$, by Axiom A2 the group $\mathcal{I}(P)$ is a Sylow p-subgroup of $\mathcal{F}(P)$ and, as $\mathcal{I}(P)$ is a normal subgroup of $\mathrm{Aut}(P)$, we obtain that $K \cap \mathcal{I}(P)$ is a Sylow p-subgroup of $K \cap \mathcal{F}(P)$ for all $K \leq \mathrm{Aut}(P)$.

Let now $Q < P$ and let us prove our assertion first in the case where $K = \mathrm{Aut}(Q)$, so, when Q is fully normalized. Put $R := N_P(Q)$ and let J be the subgroup of $\mathrm{Aut}(R)$ stabilizing Q. It is straight forward that $N_P^J(R) = N_P(Q) \cap N_P(R) = R$. Moreover, R is J-fully normalized in $\mathcal{F}$. Indeed, as Q is fully normalized, for all $\phi \in \mathcal{F}(R, P)$, we have $\phi(R) = N_P(\Phi(Q))$. Also, ${}^{\phi}J$ is the subgroup of $\mathrm{Aut}(\phi(R))$ stabilizing $\phi(Q)$ so $N_P^{{}^{\phi}J}(\phi(R)) = N_P(\phi(Q)) \cap N_P(\phi(R)) = \phi(R)$. We obtain that, for all $\phi \in \mathcal{F}(R \cdot N_P^J(R), P) = \mathcal{F}(R, P)$, we have $N_P^{{}^{\phi}J}(\phi(R)) = \phi(R) = \phi(N_P^J(R))$, so R is J-fully normalized in $\mathcal{F}$. As Q is a proper subgroup of R, by induction hypothesis, we obtain that $J \cap \mathcal{I}_P(R) = \mathcal{I}_R(R)$ is a Sylow p-subgroup of $J \cap \mathcal{F}(R)$.

We have clearly $\mathcal{I}_P(Q) = \mathcal{I}_R(Q)$. For showing that $\mathcal{I}_R(Q)$ is a Sylow p-subgroup of $\mathcal{F}(Q)$, it suffices to prove that $\mathcal{I}_R(Q)$ is a Sylow p-subgroup of $N_{\mathcal{F}(Q)}(\mathcal{I}_R(Q))$, because, if a p-subgroup of a finite group G is a Sylow p-subgroup of its normalizer, then it is a Sylow p-subgroup of G.

Let us show now that $\mathcal{I}_R(Q)$ is a Sylow p-subgroup of $N_{\mathcal{F}(Q)}(\mathcal{I}_R(Q))$. For this, we'll show that any morphism $\phi \in N_{\mathcal{F}(Q)}(\mathcal{I}_R(Q))$ can be lifted in $J \cap \mathcal{F}(R)$, which means that the group morphism $f : J \cap \mathcal{F}(R) \longrightarrow N_{\mathcal{F}(Q)}(\mathcal{I}_R(Q))$, given by the restriction to Q of the morphisms in $J \cap \mathcal{F}(R)$, is surjective. As, moreover, $f(\mathcal{I}(R)) = \mathcal{I}_R(Q)$, it's following that $\mathcal{I}_R(Q)$ is a Sylow p-subgroup of $N_{\mathcal{F}(Q)}(\mathcal{I}_R(Q))$. We used here the fact that the image of a Sylow p-subgroup of $J \cap \mathcal{F}(R)$ by the surjective morphism f is a Sylow p-subgroup of $N_{\mathcal{F}(Q)}(\mathcal{I}_R(Q))$.

We'll show now that any morphism $\phi \in N_{\mathcal{F}(Q)}(\mathcal{I}_R(Q))$ can be lifted in $J \cap \mathcal{F}(R)$. Take $\phi \in N_{\mathcal{F}(Q)}(\mathcal{I}_R(Q))$ so we have $\phi(Q) = Q$ and ${}^{\phi}(\mathcal{I}_R(Q)) = \mathcal{I}_R(Q)$. Moreover, R contains $Q \cdot C_P(Q)$, so $N_P^{\mathcal{I}_R(Q)}(Q) = R$. We have already seen that Q is fully centralized in $\mathcal{F}$. We have also that Q is $\mathcal{I}_R(Q)$-fully normalized in $\mathcal{F}$. Indeed, for any morphism $\eta \in \mathcal{F}(Q \cdot N_P^{\mathcal{I}_R(Q)}(Q), P) = \mathcal{F}(R, P)$, we have $C_P(\eta(Q)) = \eta(C_P(Q))$, as Q is fully normalized and so $C_P(\eta(Q)) \leq \eta(R)$. This gives that $\eta(Q)C_P(\eta(Q)) \leq \eta(R)$ so, $N_P^{\mathcal{I}_{\eta(R)}(\eta(Q))}(\eta(Q)) = \eta(R) = \eta(N_P^{\mathcal{I}_R(Q)}(Q))$. It follows that Q is $\mathcal{I}_R(Q)$-fully normalized in $\mathcal{F}$. By Axiom A3 applied to the morphism ϕ, there exists $\psi \in \mathcal{F}(Q \cdot N_P^{\mathcal{I}_R(Q)}(Q), P) = \mathcal{F}(R, P)$ and $\chi \in \mathcal{I}_R(Q)$ such that $\psi(u) = \phi\chi(u)$, for all $u \in Q$. Let $\widetilde{\chi}$ be a preimage of χ in $\mathcal{I}_R(R)$. We have then that $\psi\widetilde{\chi}^{-1}$ is a preimage of ϕ in $\mathcal{F}(R, P)$. In fact $\psi\widetilde{\chi}^{-1} \in \mathcal{F}(R)$. Indeed, ψ and $\widetilde{\chi}$ normalize $\mathcal{I}_R(Q)$ and $\psi\widetilde{\chi}^{-1}(C_P(Q)) = C_P(\psi\widetilde{\chi}^{-1}(Q))) = C_P(Q)$. So, $R/C_P(Q) = \mathcal{I}_R(Q) = {}^{\psi\widetilde{\chi}^{-1}}\mathcal{I}_R(Q) = \psi\widetilde{\chi}^{-1}(R)/C_P(Q)$ which gives that that $R = \psi\widetilde{\chi}^{-1}(R)$ and, so, $\psi\widetilde{\chi}^{-1} \in \mathcal{F}(R)$. This finishes the the proof in the case where $K = \mathrm{Aut}(Q)$.

In the general case, let $\theta \in \mathcal{F}(Q, P)$ such that $\theta(Q)$ is fully normalized in $\mathcal{F}$. We have seen that such a morphism θ always exists. Now, $\theta(Q)$ is fully normalized, so by the proof in the case $K = \mathrm{Aut}(Q)$, applied to $\theta(Q)$ in the place of Q, we obtain that $\mathcal{I}_P(\theta(Q))$ is a Sylow p-subgroup of $\mathcal{F}(\theta(Q))$. As ${}^{\theta}(K \cap \mathcal{F}(Q))$ is a subgroup of ${}^{\theta}(\mathcal{F}(Q)) = \mathcal{F}(\theta(Q))$, there exists $\rho \in \mathcal{F}(\theta(Q))$ such that a Sylow p-subgroup of

${}^{\rho\theta}(K \cap \mathcal{F}(Q))$ is contained in $\mathcal{I}_P(\theta(Q))$. So ${}^{\rho\theta}K \cap \mathcal{I}_P(\theta(Q))$ is a Sylow p-subgroup of ${}^{\rho\theta}(K \cap \mathcal{F}(Q))$.

We take now $\theta' := \rho\theta$, $Q' := \theta(Q)$, $K' := {}^{\theta'}K$ and we apply Axiom A3 to θ'^{-1}, as Q is K-fully normalized. So there exists $\psi \in \mathcal{F}(Q' \cdot N_P^{K'}(Q'), P)$ and $\chi' \in K'$ such that $\psi(u) = \theta'^{-1}(\chi'(u))$ for all $u \in Q'$. Let $\chi := {}^{\theta'^{-1}}\chi'$, which is an element of K. We have then $\psi^{-1} = \theta'\chi^{-1}$ which implies that ${}^{\psi^{-1}}K = K'$. But $\psi(N_P^{K'}(Q')) \leq N_P^K(Q)$ and, as Q' is fully centralized, we obtain that $\psi(C_P(Q')) = C_P(Q)$, so $\psi(K' \cap \mathcal{I}_P(Q')) \leq K \cap \mathcal{I}_P(Q)$. Moreover, $K' \cap \mathcal{I}_P(Q')$ is a Sylow p-subgroup of $K' \cap \mathcal{F}(Q')$ and $\phi(K' \cap \mathcal{F}(Q')) = K \cap \mathcal{F}(Q)$. We deduce that $K \cap \mathcal{I}_P(Q)$ is a Sylow p-subgroup of $K \cap \mathcal{F}(Q)$. □

Let Q a subgroup of P and K a subgroup of $\mathrm{Aut}(Q)$. We define, based on $\mathcal{F}$, a category $N_{\mathcal{F}}^K(Q)$ over $N_P^K(Q)$, whose objects are the subgroups of $N_P^K(Q)$ and whose morphisms between two subgroups R and T of $N_P^K(Q)$ are the morphisms $\phi \in \mathcal{F}(R, T)$, satisfying that there exists a morphism $\psi \in \mathcal{F}(Q \cdot R, Q \cdot T)$ and $\chi \in K$ such that $\chi(u) = \psi(u)$ for all $u \in Q$ and $\psi(v) = \phi(v)$ for all $v \in R$; in other words, all the morphisms which can be extended to $Q \cdot R$ and such that this extension acts over Q by a morphism of K. In the case where $K = \mathrm{Aut}(Q)$, we denote this category, simply, $N_{\mathcal{F}}(Q)$.

Proposition 2.5 ([Pu2], Prop. 2.8)**.** *Let $\mathcal{F}$ be a full Frobenius system over P. If $Q \leq P$ is K-fully normalized in $\mathcal{F}$, then, $N_{\mathcal{F}}^K(Q)$ is a full Frobenius system over $N_P(Q)$.*

For the proof of this proposition we'll need the next technical lemma:

Lemma 2.6 ([Pu2], Lemma 2.9)**.** *Let $\mathcal{F}$ be a full Frobenius system over P, Q a subgroup of P and K a subgroup of* $\mathrm{Aut}(Q)$. *Let R be a subgroup of $N_P^K(Q)$, L a subgroup of $Aut(R)$ and M the subgroup of* $\mathrm{Aut}(Q \cdot R)$ *stabilizing Q and R and acting on them by the morphisms in K, respectively L.*

Then $N_P^M(Q \cdot R) = N_P^K(Q) \cap N_P^L(R)$ and, if R is fully normalized in $N_{\mathcal{F}}^K(Q)$, the group $Q \cdot R$ is M-fully normalized in $\mathcal{F}$.

Proof. The first equality is straight forward as we have $u \in N_P^M(Q \cdot R)$ if and only if $u \in P$ stabilize Q and R and u acts on Q, respectively R by elements of K, respectively L which is equivalent with $u \in N_P^K(Q)$, respectively $u \in N_P^L(R)$, or, in other words, with $u \in N_P^K(Q) \cap N_P^L(R)$. We remark that this is independent on the existence of $\mathcal{F}$.

For the second part of the lemma, let's put $T := Q \cdot R$, $P' := N_P^K(Q)$ and $\mathcal{F}' := N_{\mathcal{F}}^K(Q)$. Let's consider $\phi \in \mathcal{F}(T \cdot N_P^M(T), P)$. We want to show that $\phi(N_P^M(Q)) = N_P^{{}^{\phi}M}(\phi(T))$. Note $Q' := \phi(Q)$, $T' := \phi(T)$ and $K' := {}^{\phi}K$. Now, we have $\phi|_Q \in \mathcal{F}(Q, P)$, so, by Axiom A3 applied to ϕ^{-1}, using that Q is K-fully normalized in $\mathcal{F}$, there exists $\psi \in \mathcal{F}(Q' \cdot N_P^{K'}(Q'), P)$ and $\chi' \in \mathcal{F}(Q')$ such that $\psi(u) = \phi^{-1}\chi'(u)$ for all $u \in Q'$. In particular, $\psi(N_P^{K'}(Q)') \leq N_P^K(Q)$. Now, we have $\phi(T \cdot N_P^M(T)) \leq \phi(T) \cdot N_P^{{}^{\phi}M}(T') \leq Q' \cdot N_P^{K'}(Q')$ so, if we note $\eta := \psi\phi$, we obtain $\eta \in \mathcal{F}(T \cdot N_P^M(T), QP') = \mathcal{F}(Q \cdot R \cdot N_P^M(T), QP')$. Moreover $\eta(R \cdot N_P^M(T)) \leq P'$, which implies that $\eta|_{R \cdot N_P^M(T)} \in \mathcal{F}'(R \cdot N_P^M(T), P')$, by the very definition of $\mathcal{F}'$.

By a simple computation, we have $N_P^M(T) = N_P^K(Q) \cap N_P^L(R) = N_{P'}^L(R)$ (1) and, using the fact that R is L-fully normalized in $\mathcal{F}'$, we obtain $\eta(N_{P'}^L(R)) = N_{P'}^{{}^{\eta}L}(\eta(R))$ (2). Moreover, we remark that $\eta(Q) = Q$ and ${}^{\eta}K = K$ and ${}^{\eta}M$

is a subgroup of the automorphisms group of $Q \cdot \eta(R)$ stabilizing Q and $\eta(R)$ and acting on Q, respectively $\eta(R)$ by morphisms of K, respectively $\eta(L)$. So, $N_P^{\,^{\eta}M}(Q \cdot \eta(R)) = N_P^K(Q) \cap N_P^{\,^{\eta}L}(\eta(R)) = N_{P'}^{\,^{\eta}L}(\eta(R))$ (3), by the first part of the lemma. From (1), (2) and (3) we deduce that $\eta(N_P^M(T)) = N_P^{\,^{\eta}M}(Q \cdot \eta(R))$. So, we have

$$\psi(\phi(N_P^M(T))) = \eta(N_P^M(T)) = N_P^{\,^{\eta}M}(Q \cdot \eta(R)) \geq \psi(N_P^{\,^{\phi}M}(\phi(T)))$$

We also have that $|\phi(N_P^M(T))| \geq |N_P^{\,^{\phi}M}(\phi(T))|$. Now, as $\phi(N_P^M(T)) \leq N_P^{\,^{\phi}M}(\phi(T))$, we obtain that $\phi(N_P^M(T)) = N_P^{\,^{\phi}M}(\phi(T))$.

This is true for all $\phi \in \mathcal{F}(T \cdot N_P^M(T), P)$, so T is M-fully centralized in $\mathcal{F}$.

□

Proof of Proposition 2.5. As in the preceding lemma put $\mathcal{F}' := N_{\mathcal{F}}^K(Q)$ and $P' := N_P^K(Q)$. We will verify Axioms A1, A2 and A3 for $\mathcal{F}'$. Axiom A1 is trivially satisfied.

Let's verify now Axiom A2. Clearly P' is fully normalized in $\mathcal{F}'$, and taking N the subgroup of $\mathrm{Aut}(Q \cdot P')$ which stabilizes Q and P' and acts on Q by morphisms of K, as Q is K-fully normalized, by the lemma 2.6, the group $Q \cdot P'$ is N-fully normalized in $\mathcal{F}$. By Proposition 2.4, the group $N \cap \mathcal{I}_P(Q \cdot P')$ is a Sylow p-subgroup of $N \cap \mathcal{F}$. But, by the definition of $\mathcal{F}'$, any morphism of $\mathcal{F}'(P')$ lifts in a morphism of $\mathcal{F}(Q \cdot P') \cap N$. So the restriction morphism $N \cap \mathcal{F}(Q \cdot P') \to \mathcal{F}'(P')$ is surjective. Moreover this morphism maps $N \cap \mathcal{I}_P(Q \cdot P')$ onto $\mathcal{I}(P')$. Thus $\mathcal{I}(P')$ is a Sylow p-subgroup of $\mathcal{F}'(P')$ and Axiom A2 is satisfied.

For verifying Axiom A3, let $R \leq P'$, let L a subgroup of $\mathrm{Aut}(R)$ and $\phi \in \mathcal{F}'(R, P')$ such that $\phi(R)$ is $^{\phi}L$-fully normalized in $\mathcal{F}'$. By the definition of $\mathcal{F}'$ we have that there exists $\psi \in \mathcal{F}(Q \cdot R, Q \cdot P')$ and $\chi \in K$ such that $\psi(v) = \phi(v)$, for all $v \in R$ and $\psi(u) = \chi(u)$, for all $u \in Q$. Let's take again the notations in the lemma 2.6: let $T := Q \cdot R$ and M be the subgroup of $\mathrm{Aut}(T)$ stabilizing Q and R and acting on Q, respectively R by morphisms of K, respectively L. As Q is K-fully normalized in $\mathcal{F}$ and $\psi(R)$ is $^{\psi}L$-fully normalized in $\mathcal{F}'$, by the lemma 2.6, we obtain that $\psi(T)$ is M-fully normalized in $\mathcal{F}$.

We apply Axiom A3 to $\psi \in \mathcal{F}(T, P)$ and we obtain that there exists $\rho \in \mathcal{F}(T \cdot N_P^M(T), P)$ and $\nu \in M$ such that $\rho(w) = \psi(\nu(w))$, for all $w \in T$. In particular, we have $\rho(u) = \psi\nu(u)$ and so ρ determines an element of K, as it was already the case for ψ and ν. Now, R and $N_P^M(T)$ are subgroups of $N_P^K(Q)$ and as ρ determines an element of K, the images $\rho(R)$ and $\rho(N_P^M(T))$ remain subgroups of $N_P^K(Q)$. On the other hand, the action of ν over R determines an element λ of L and, by the lemma 2.6, we have $N_P^M(T) = N_{P'}^L(R)$. We obtain, in consequence, that the restriction of ρ to $R \cdot N_{P'}^L(R)$ is a morphism of $\mathcal{F}'(R \cdot N_{P'}^L(R), P')$. Moreover we have $\rho(v) = \psi(\nu(v)) = \phi(\lambda(v))$ for all $v \in R$, so Axiom A3 is also satisfied. □

3. Saturated fusion systems

Here is the approach of Broto Levi and Oliver.

Definition 3.1. *A fusion system over a finite p-group P is a category whose objects are the subgroups in P and whose morphism set $\mathrm{Hom}_{\mathcal{F}}(P, Q)$ satisfy the following*

(1) $\mathrm{Hom}_P(Q, R) \subset \mathrm{Hom}_{\mathcal{F}}(Q, R) \subset \mathrm{Inj}(Q, R)$ *for all* $Q, R \leq P$ *.*

(2) Every morphism in $\mathcal{F}$ factors as an isomorphism in $\mathcal{F}$ followed by an inclusion.

Definition 3.2. *A subgroup Q of P is fully centralized if $|C_P(Q)| \leq |C_P(Q')|$ for all $Q' \leq P$ which is $\mathcal{F}$-conjugated to Q.*

Definition 3.3. *A subgroup Q of P is fully normalized if Q is fully centralized and if $\mathrm{Aut}_P(Q) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(Q))$.*

Definition 3.4. *$\mathcal{F}$ is a saturated fusion system if the two following conditions hold:*

(i) Each subgroup $Q \leq P$ is $\mathcal{F}$-conjugated to at least one fully normalized subgroup.

(ii) If $Q \leq P$ and $\phi \in \mathrm{Hom}_{\mathcal{F}}(Q, P)$ are such that $\phi(Q)$ is fully centralized and one set $N_\phi := \{g \in N_P(Q) | \phi c_g \phi^{-1} \in \mathrm{Aut}_P(\phi(Q))\}$, then there is $\bar{\phi} \in \mathrm{Hom}_{\mathcal{F}}(N_\phi, P)$ such that $\bar{\phi}|_P = \phi$.

We remark that the notions of fully normalized and fully centralized are the same in the BLO's and Puig's approach. To prove that Puig's definitions imply BLO's one we only have to prove the following.

Proposition 3.5. *Let $\mathcal{F}$ be a full Frobenius system ('à la Puig'). Then the definition 3.4 above is a consequence.*

Proof. We have that $\phi(Q)$ is fully centralized. In fact, $\phi(Q)$ is ${}^{\phi}(N_\phi/C_P(\phi(Q)))$-fully normalized. Indeed ${}^{\phi}(N_\phi/C_P(\phi(Q))) \subset \mathrm{Aut}_P(\phi(Q))$, so the equivalent condition in Proposition 2.4 are satisfied. Thus, by Axiom 3 there exists $\psi : N_\phi \to N_P(\phi(Q))$ such that $\psi({}^{g}u) = \phi(u)$. So we can put $\bar{\phi} = \psi \circ \mathrm{conj}(g)$. □

For the converse it is sufficient to prove the following.

Proposition 3.6. *Let $\mathcal{F}$ be a saturated fusion system ('à la BLO'). Then Puig's Axiom 3 is a consequence.*

Proof. Let Q be a subgroup of P, $\phi : Q \to P$ a morphism in $\mathcal{F}$ and $K \subset \mathrm{Aut}(Q)$ such as $Q' := \phi(Q)$ is $K' := {}^{\phi}K$-fully normalized. So Q' is fully centralized and $K' \cap \mathrm{Aut}_P(Q')$ is a Sylow p-subgroup of $K' \cap \mathrm{Aut}_{\mathcal{F}}(Q')$. Now ${}^{\phi}(K \cap \mathrm{Aut}_P(Q))$ is p-subgroup of $K' \cap \mathrm{Aut}_{\mathcal{F}}(Q')$. By conjugating by an appropriate element $g \in K$ we have that ${}^{g\,\phi}(K \cap \mathrm{Aut}_P(Q)) \subset {}^{\phi}(K \cap \mathrm{Aut}_P(Q))$. This means that $N_{{}^{g}\phi}$ contains $N_P^K(Q)$. By the definition of saturated fusion systems we have that ${}^{g}\phi$ extends to a morphism $\psi : N_{{}^{g}\phi} \to P$ and this can be restricted to $N_P^K(Q)$. The Axiom 3 comes straight forward. □

4. 'Simply' fusion systems

We have the folowing approach for the above notion.

Definition 4.1. *A subgroup Q of P is fully centralized, respectively fully normalized if $|C_P(Q)| \leq |C_P(Q')|$, respectively $|N_P(Q)| \leq |N_P(Q')|$, for all $Q' \leq P$ which is $\mathcal{F}$-conjugated to Q.*

Definition 4.2. *A fusion system on a finite p-group P is a category $\mathcal{F}$ on P satisfying the following properties:*

(1) $\mathrm{Aut}_P(P)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(P)$

(2) Every $\phi : Q \to P$ such that $\phi(Q)$ is fully normalized extends to a morphism $\bar{\phi} : N_\phi \to P$

It is obvious that the BLO's definition implies ours. We prove now the fact that our definition implies BLO's one.

Proposition 4.3. *Let $\mathcal{F}$ be a fusion system on a finite group P and let Q be a subgroup of P. Then Q is fully normalized if and only if Q is fully centralized and $\mathrm{Aut}_P(Q)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(Q)$.*

Proof. First, a fully normalized subgroup Q of P is also fully centralized as for any $\mathcal{F}$-isomorphic subgroup Q' we have that the morphism $\phi : Q' \to Q$ extends to a morphism $\bar{\phi} : N_\phi \to N_P(Q)$. But $C_P(Q') \subset N_\phi$ and $\bar{\phi}(C_P(Q')) \subset C_P(Q)$ giving that $|C_P(Q')| \leq |C_P(Q)|$. So Q is fully centralized.

Second, if Q is fully normalized then $\mathrm{Aut}_P(Q)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(Q)$. To prove this, we will follow the proof of Proposition 1.5 in [Li]. Let Q be a subgroup of maximal order such that $\mathrm{Aut}_P(Q)$ is not a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(Q)$. Then Q is a proper subgroup of P by axiom 2. Choose a p-subgroup S of $\mathrm{Aut}_{\mathcal{F}}(Q)$ such that $\mathrm{Aut}_P(Q)$ is a proper normal subgroup of S. Let $\phi \in S \setminus \mathrm{Aut}_P(Q)$. Since ϕ normalizes $\mathrm{Aut}_P(Q)$, for every $y \in N_P(Q)$ there is $z \in N_P(Q)$ such that $\phi({}^y u) = {}^z\phi(u)$ for all $u \in Q$. Thus we have $N_\phi = N_P(Q)$. Since Q is fully normalized, by axiom 3 ϕ extends to $\bar{\phi} : N_\phi \to N_P(Q)$ so $\bar{\phi} \in \mathrm{Aut}_{\mathcal{F}}(N_P(Q))$. Since ϕ has p-power order, by decomposing $\bar{\phi}$ into its p-part and its p'-part we may assume that $\bar{\phi}$ has also p-power order. Let $\psi : N_P(Q) \to P$ be a morphism such that $N' := \psi(N_P(Q))$ is fully normalized. As the order on N' is greater then the order of Q, we have that $\mathrm{Aut}_P(N')$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(N')$. Now $\psi\bar{\phi}\psi^{-1}$ is a p-element of $\mathrm{Aut}_{\mathcal{F}}(N')$, thus conjugated to an element in $\mathrm{Aut}_P(N')$. Therefore we may choose ψ in a such way that there is $y \in N_P(N')$ satisfying $\psi\bar{\phi}\psi^{-1}(v) = {}^y v$ for all $v \in N'$. Since $\bar{\phi}|_Q = \phi$, the automorphism $\psi\bar{\phi}\psi^{-1}$ of N' stabilizes $\psi(Q)$. Thus $y \in N_P(()\psi(Q))$. Since Q is fully normalized and $\psi(N_P(Q)) \subset N_P(\psi(Q))$ we have that $\psi(N_P(Q)) = N_P(\psi(Q))$, hence $\bar{\phi}(u) = {}^{\tau^{-1}(y)}u$, for all $u \in N_P(Q)$. And, in particular, $\phi \in \mathrm{Aut}_P(Q)$, contradicting our first choice of ϕ.

The converse is straight forward as $|N_P(Q)| = |\mathrm{Aut}_P(Q)| \cdot |C_P(Q)|$.

□

We have to prove also that all the morphisms ϕ such that the image is fully $\mathcal{F}$-centralized extend to N_phi.

Proposition 4.4. *Every $\phi : Q \to P$ such that $\phi(Q)$ is fully $\mathcal{F}$-centralized extends to a morphism $\bar{\phi} : N_\phi \to P$*

Proof. We note $Q' := \phi(Q)$. Choose $\theta : Q' \to P$ such that $\theta(Q')$ is fully normalized and, as $\mathrm{Aut}_P(\theta(Q'))$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(\theta(Q'))$ we can modify θ by a morphism in $\mathrm{Aut}_{\mathcal{F}}(\theta(Q'))$ and suppose that $N_\theta = N_P(Q')$.

By the property (2) we have that θ extends to $\bar{\theta} : N_\theta \to P$. Note $\psi := \theta\phi$. By the same property (2) ψ extends to $\bar{\psi} : N_\psi \to P$.

Our aim in what follows is to prove that $N_\phi \subset N_\psi$ and $\bar{\psi}(N_\phi) \subset \bar{\theta}(N_\theta)$ so that $(\bar{\theta})^{-1}\bar{\psi}|_{N_\phi}$ would be the extension of ϕ to N_ϕ.

Both are simple verifications. Take $y \in N_\phi$ then by definition, there exists $z \in N_P(Q')$ such that $\phi({}^y u) = {}^z\phi(u)$ for all $u \in Q$. By composing with θ we obtain $\theta\phi({}^y u) = \theta({}^z\phi(u))$. But as $N_\theta = N_P(Q')$ we have that there exists $x \in N_P(\theta(Q'))$ such that $\theta({}^z\phi(u)) = {}^x\theta(\phi(u)) = {}^x\psi(u)$. By resuming, we have $\psi({}^y u) = {}^x\psi(u)$ which means that $y \in N_\psi$. As this is true for all $y \in N_\phi$ we obtain that $N_\phi \subset N_\psi$.

Take now $x \in \bar{\psi}(N_\phi)$. Suppose that $x = \bar{\psi}(y)$, $y \in N_\phi$. By definition, there exists $z \in N_P(Q')$ such that $\phi({}^y u) = {}^z\phi(u)$ for all $u \in Q$. We obtain $\psi({}^y u) = {}^x\psi(u)$, so $\theta({}^z\phi(u)) = {}^x\theta(\phi(u))$, which is equivalent to ${}^{\bar{\theta}(z)}\psi(u) = {}^x\theta(\phi(u))$ for all

$u \in Q$. This gives that $x = \bar{\theta}(z)c$ with $c \in C_P(\theta(Q))$. But as $C_P(Q') \subset N_\theta$ and $\bar{\theta}(C_P(Q')) \subset C_P(\theta(Q'))$ and using the fact that Q' is fully centralized we have that $\bar{\theta}(C_P(Q')) = C_P(\theta(Q'))$. This means that $c \in \bar{\theta}(N_\theta)$, so $x \in \bar{\theta}(N_\theta)$. Now this is true for all $x \in \bar{\psi}(N_\phi)$ so $\bar{\psi}(N_\phi) \subset \bar{\theta}(N_\theta)$.

Thus we showed that $(\bar{\theta})^{-1} \circ \bar{\psi}|_{N_\phi}$ extends ϕ to N_ϕ.

□

This finishes the proof of the equivalence of the three approaches given in the paper. Even if our description is simpler then the others two, it is very usefull to keep in mind as properties the other two definitions.

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