TATE'S AND YOSHIDA'S THEOREM ON CONTROL OF TRANSFER FOR FUSION SYSTEMS

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ABSTRACT. We prove analogues of results of Tate and Yoshida on control of transfer for fusion systems. This requires the notions of p-group residuals and transfer maps in cohomology for fusion systems. As a corollary we obtain a p-nilpotency criterion due to Tate.

1. INTRODUCTION

In the theory of finite groups, the focal subgroup of a Sylow p-subgroup is determined entirely by p-fusion and detects whether the whole group G has a nontrivial p-group quotient. Moreover, under certain conditions, some subgroups of G containing its Sylow p-subgroup determine the focal subgroup and hence whether G has a nontrivial p-group quotient. This phenomenon is traditionally called control of transfer; indeed these results can be obtained by using transfer maps in group cohomology.

A fusion system is a category $\mathcal{F}$ whose objects are the subgroups of a fixed finite p-group S and whose morphisms behave like conjugation maps in finite groups having S as a Sylow p-subgroup. First introduced by Puig [13][14] and further developed by Broto, Levi and Oliver [4], it is a useful framework for studying local theory of (blocks of) finite groups and p-local homotopy theory. Hence it is a natural question whether and how classical results of local group theory can be extended to fusion systems.

Given a fusion system, one can define the focal subgroup (and other related subgroups like the hyperfocal subgroup) analogously to the group case. Moreover, it has been shown, with substantial effort, that they have the same key properties as their group theoretic counterparts.([3], see also appendix.) On the other hand, using the characteristic elements of a fusion system, which were introduced in [4] and refined in [15], we can define an appropriate notion of transfer maps in cohomology of fusion systems.

In this paper, using these tools at hand, we generalize two classical theorems on control of transfer in finite groups by Tate and Yoshida to fusion systems.

Tate's theorem, reformulated as in [8], concerns three types of residuals of a finite group G: the elementary abelian p-group residual, the abelian p-group residual and the p-group residual. It states that, for a given subgroup H of G containing a p-Sylow of G, H has isomorphic residual to that of G of one of these types if and only if H does so for the three types.

Date: November 27, 2009.

For the generalization to fusion systems we first need appropriate notions of residuals. The case of the p-group residual is solved in [3], where a notion of fusion subsystems of p-power index is given. This motivates the following

Definition 1.1. Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S. We define

(1) $O^p_{\mathcal{F}}(S) = \langle [P, O^p(\mathrm{Aut}_{\mathcal{F}}(P))] \mid P \leq S \rangle$. (The hyperfocal subgroup of $\mathcal{F}$).
(2) $A^p_{\mathcal{F}}(S) = [S, \mathcal{F}] = \langle [P, \mathrm{Aut}_{\mathcal{F}}(P)] \mid P \leq S \rangle$. (The focal subgroup of $\mathcal{F}$).
(3) $E^p_{\mathcal{F}}(S) = \Phi(S)[S, \mathcal{F}] = \Phi(S) O^p_{\mathcal{F}}(S)$.

Tate's theorem for fusion systems follows.

Theorem 1.2. *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S, and let $\mathcal{H}$ be a saturated subsystem of $\mathcal{F}$ on S. The following are equivalent.*

(1) $E^p_{\mathcal{F}}(S) = E^p_{\mathcal{H}}(S)$.
(2) $A^p_{\mathcal{F}}(S) = A^p_{\mathcal{H}}(S)$.
(3) $O^p_{\mathcal{F}}(S) = O^p_{\mathcal{H}}(S)$.

Instead of Tate's original cohomological proof, we follow the strategy of Gagola and Isaacs in [8], which uses the transfer maps. As a corollary we obtain a fusion version of the p-nilpotency criterion [20] suggested by Atiyah:

Corollary 1.3. *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S. If the restriction map $H^1(\mathcal{F}; \mathbb{F}_p) \to H^1(S; \mathbb{F}_p)$ is an isomorphism then $\mathcal{F} = \mathcal{F}_S(S)$.*

In [17] the authors prove this and other p-nilpotency criterions for fusion systems by other methods. Another interesting result on control of transfer is Yoshida's theorem [21, Theorem 4.2]. The author proves that if G is a finite group with Sylow p-subgroup S, then $\mathrm{N}_G(S)$ controls transfer unless $C_p \wr C_p$ is a quotient of S. We generalize this result to fusion systems following the transfer argument given by Issacs in [12, Chapter 10], instead of the original character-theoretical proof.

Theorem 1.4. *Let $\mathcal{F}$ be a fusion system on a finite p-group S and let $\mathcal{H} = N_{\mathcal{F}}(S)$. If $[S, \mathcal{F}] \neq [S, \mathcal{H}]$, then S has $C_p \wr C_p$ as a homomorphic image.*

Organization of the paper: In section 2 we introduce double Burnside rings and characteristic elements in order to define the transfer later in the same section. In the next section we prove Yoshidas's theorem. In section 4 we prove a deeper property of the the p-power index trasnfer that is needed for Tate's theorem, which is then proved in section 5. In the appendix we recall normal subsystems and quotient systems, and prove some of their properties in the p-power index case, which are used in this paper.

Acknowledgments: The authors are grateful to Department of Mathematical Sciences of University of Copenhagen for funding and hosting a workshop on fusion systems, where the basis of this work were established. Antonio Díaz and Radu Stancu visited Mathematisches Forschungsinstitut Oberwolfach twice while Sejong Park was an Oberwolfach Leibniz Fellow, and worked on this project. We thank the institute and the staff for their hospitality and excellent working environment. We also sincerely thank Kári Ragnarsson for supplying us with a proof of a key property of the transfer in Theorem 4.1.

2. Characteristic elements and transfer for fusion systems

2.1. **Double Burnside ring.** We begin this section with a brief review of the (p-localized) double Burnside ring of a finite group, following closely [15]. In particular, for finite groups G and H, we mean by a (G,H)-*biset* a finite set with commuting right G-action and left H-action. The *Burnside module* $A(G,H)$ of G and H is the Grothendieck group of the monoid of isomorphism classes of (G,H)-bisets with free left H-action, under disjoint union. If G, H and K are finite groups there is a bilinear map

$$A(K,H) \times A(G,K) \to A(G,H)$$

given by

$$(\Omega, \Lambda) \mapsto \Omega \circ \Lambda := \Omega \times_K \Lambda.$$

As an abelian group, $A(G,H)$ is free with one generator for each isomorphism class of transitive (G,H)-bisets with free left H-action. These generators are represented by bisets of the form $H \times_{(K,\psi)} G$, where $K \leq G$, $\psi \in \mathrm{Hom}(K,H)$ and

$$H \times_{(K,\psi)} G = (H \times G)/\sim \text{, where } (x, uy) \sim (x\psi(u), y) \text{ for } x \in H,\ y \in G,\ u \in K.$$

We use the notation $[K,\psi]_G^H$ to denote the generator corresponding to $H \times_{(K,\psi)} G$, and we write $[K,\psi]$ if G and H are clear from the context. In case $G = H$, $A(G,G)$ becomes a ring, called the *double Burnside ring* of the group G. We will also consider its p-localization

$$A(G,G)_{(p)} := A(G,G) \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}.$$

Note that $A(G,G)$ is a subring of $A(G,G)_{(p)}$.

For any $\mathbb{Z}G$-module A there is a linear map

$$H^*(-;A)\colon A(G,G) \to \mathrm{End}(H^*(G;A))$$

that takes the generator $[K,\psi]$ to

$$\mathrm{tr}_K^G \circ \psi^* : H^*(G;A) \to H^*(G;A),$$

where $\mathrm{tr}_K^G : H^*(K;A) \to H^*(G;A)$ is the usual transfer map and $\psi^* : H^*(G;A) \to H^*(K;A)$ is restriction via ψ. It turns out that $H^*(-;A)$ is a ring homomorphism: for $\Omega, \Lambda \in A(G,G)$ we have

$$H^*(\Omega \circ \Lambda; A) = H^*(\Omega;A) \circ H^*(\Lambda;A).$$

If A is a $\mathbb{Z}_{(p)}G$-module, the ring homomorphism

$$H^*(-;A)\colon A(G,G)_{(p)} \to \mathrm{End}(H^*(G;A)).$$

is defined analogously.

Now let $\mathcal{F}$ be a saturated fusion system over a finite p-group S. It is a remarkable result in the theory of fusion systems that there exist certain elements in $A(S,S)_{(p)}$, called *characteristic elements*, which reflects all the properties of $\mathcal{F}$. See [4] and [15]. We discuss them below, and they are at the core of our definition of transfer for fusion systems.

We denote by $A_{\mathcal{F}}(S,S)$ and $A_{\mathcal{F}}(S,S)_{(p)}$ the subrings of $A(S,S)$ and $A(S,S)_{(p)}$, respectively, generated by $[P,\varphi]$ with $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$. Let $\Omega \in A(S,S)_{(p)}$. We say that Ω *is right $\mathcal{F}$-stable* if for $P \leq S$ and every morphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$ we have

$$\Omega \circ [P,\varphi]_P^S = \Omega \circ [P,\mathrm{incl}]_P^S,$$

where $\mathrm{incl} : P \hookrightarrow S$ is the inclusion. Left $\mathcal{F}$-stability is defined analogously using the condition

$$[\varphi(P), \varphi^{-1}]_S^P \circ \Omega = [P, \mathrm{id}]_S^P \circ \Omega,$$

where $\mathrm{id} : P \to P$ is the identity. There is a unique linear extension ϵ to $A(S,S)_{(p)}$ of the map sending every generator $[P,\varphi]_S^S$ to its number of right S-orbits:

$$\epsilon([P,\varphi]_S^S) = |S|/|P|.$$

It is easy to see that, in fact, $\epsilon : A(S,S)_{(p)} \to \mathrm{Z}_{(p)}$ is a ring homomorphims and that it restricts to $\epsilon : A(S,S) \to \mathrm{Z}$.

Definition 2.1. Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S. A *characteristic element* for $\mathcal{F}$ is $\Omega \in A(S,S)_{(p)}$ satisfying the following properties:

(a) $\Omega \in A_{\mathcal{F}}(S,S)_{(p)}$,
(b) Ω is right and left $\mathcal{F}$-stable, and
(c) $\epsilon(\Omega) \not\equiv 0 \mod p\mathrm{Z}_{(p)}$.

These three properties were first formulated by Linckelmann and Webb. In [4, 5.5] Broto, Levi and Oliver proved that for any saturated fusion system $\mathcal{F}$ there exists such a characteristic element Ω. In fact, the element Ω they constructed is contained in $A_{\mathcal{F}}(S,S)$ and has nonnegative coefficients; that is, it is an isomorphism class of an actual (S,S)-biset. We call such a characteristic element a *characteristic biset* for $\mathcal{F}$; more generally, if negative integral coefficients are allowed, we call it a *virtual characteristic biset.* In case $\mathcal{F}$ is the fusion system induced by a finite group G on its Sylow p-subgroup S, i.e., if $\mathcal{F} = \mathcal{F}_S(G)$, then G, viewed as an (S,S)-biset in the obvious way, is a characteristic biset for $\mathcal{F}_S(G)$. See also Example 2.3 for more details.

Characteristic elements of a given saturated fusion system $\mathcal{F}$ are far from unique. Indeed, one can simply multiply a p'-number to a given characteristic element to get a new one. But there is one special characteristic element introduced by Ragnarsson, which plays a key role in the theory.

Definition 2.2. Let $\mathcal{F}$ be a saturated fusion system over the p-group S. A *characteristic idempotent* for $\mathcal{F}$ is a characteristic element ω for $\mathcal{F}$ which is an idempotent in the ring $A(S,S)_{(p)}$.

Note that the idempotent condition implies that $\epsilon(\omega) = 1$. In [15] Ragnarsson shows that there exists a unique characteristic idempotent $\omega_{\mathcal{F}}$ for every saturated fusion system $\mathcal{F}$. We briefly recall here Ragnarsson's construction of ω as it will be needed later.([15, 4.9, 5.8]) Given any virtual characteristic biset $\Omega \in A_{\mathcal{F}}(S,S)$ for $\mathcal{F}$, there is a large enough integer M such that Ω is an idempotent modulo p. Then the sequence Ω^M, Ω^{Mp}, Ω^{Mp^2}, ... converges in the p-adic topology to an idempotent ω in $A(S,S)_p^\wedge := A(S,S) \otimes_{\mathrm{Z}} \mathrm{Z}_p^\wedge$. It turns out that ω is a characteristic element lying in $A(S,S)_{(p)}$, i.e. $\omega = \omega_{\mathcal{F}}$.

2.2. **Transfer.** We devote the rest of the section to define the transfer map for fusion systems using characteristic elements and prove some basic properties. In particular, we will show that the definition is essentially unique in spite of the non-uniqueness of characteristic elements.

Fix a saturated fusion system $\mathcal{F}$ on a finite p-group S. Let A be a $\mathrm{Z}_{(p)}S$-module and consider a characteristic element $\Omega \in A_{\mathcal{F}}(S,S)_{(p)}$ for $\mathcal{F}$ expressed as

$$\Omega = \sum c_{[P,\varphi]}[P,\varphi],$$

where the sum runs over the generators $[P,\varphi]$ of $A(S,S)$ and $c_{[P,\varphi]} \in \mathbb{Z}_{(p)}$. The endomorphism $H^*(\Omega;A)$ of $H^*(S;A)$ can be explicitly described as

$$H^*(\Omega;A) = \sum c_{[P,\varphi]} \cdot \mathrm{tr}_P^S \circ \varphi^*. \tag{1}$$

The following example explains which feature of finite groups is $H^*(\Omega;A)$ modeling:

Example 2.3. Let G be a finite group with Sylow p-subgroup S and let $\mathcal{F} = \mathcal{F}_S(G)$. The biset $\Omega = G$, where the (S,S)-biset structure is given by left and right multiplication in the group G, is a characteristic biset for $\mathcal{F}$. An easy calculation shows that

$$\Omega \cong \coprod_{g\in[S\backslash G/S]} S \times_{(S\cap {}^gS, c_{g^{-1}})} S,$$

and hence we get

$$H^*(\Omega;A) = \sum_{g\in[S\backslash G/S]} \mathrm{tr}^S_{S\cap {}^gS} \circ c^*_{g^{-1}}.$$

But this is just the Mackey decomposition formula for the double cosets SgS in G. Therefore

$$H^*(\Omega;A) = \mathrm{res}^G_S \circ \mathrm{tr}^G_S.$$

where $\mathrm{res}^G_S : H^*(G;A) \to H^*(S;A)$ is restriction via the inclusion $S \hookrightarrow G$.

Assume that Ω is a characteristic element for $\mathcal{F}$ and A is an abelian p-group with trivial S-action. Then the argument in [4, Proposition 5.5] shows that $H^*(\Omega;A)$ is an idempotent in $\mathrm{End}(H^*(S;A))$ up to p'-factor $\epsilon(\Omega)$ and that the image of $H^*(\Omega;A)$ is exactly

$$H^*(\mathcal{F};A) := \{z \in H^*(S;A) \mid \varphi^*(z) = \mathrm{res}^S_P(z)\, \forall\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)\}.$$

Hence, for characteristic elements Ω and Λ for $\mathcal{F}$, $H^*(\Omega;A)$ and $H^*(\Lambda;A)$ are projections (up to p'-factor) in $\mathrm{End}(H^*(S;A))$ which have the same image. But more is true:

Corollary 2.4. *Let $\mathcal{F}$ be a saturated fusion system on the p-group S and let A be an abelian p-group with trivial S-action. If Ω and Λ are characteristics elements for $\mathcal{F}$ then there is a p'-number r such that $H^*(\Omega;A) = r \cdot H^*(\Lambda;A)$.*

Proof. By multiplying suitable p'-numbers, we may assume that Ω and Λ lie in $A(S,S)$. Let p^e be the exponent of A. As remarked after Definition 2.2, there is a large enough positive integer k such that $\Lambda^k - \Omega^k = p^e\Upsilon$ for some $\Upsilon \in A(S,S)$. Because both $H^*(\Lambda;A)$ and $H^*(\Omega;A)$ are idempotents up to a p'-factor, we get $H^*(\Lambda;A)^k = q_1 \cdot H^*(\Lambda;A)$ and $H^*(\Omega;A)^k = q_2 \cdot H^*(\Omega;A)$, where q_1 and q_2 are p'-numbers. On the other hand, p^e is the exponent of A and therefore $H^*(p^e\Upsilon;A) = p^eH^*(\Upsilon;A) = 0$. As $H^*(-;A)$ is a ring homomorphism we finally obtain

$$0 = H^*(\Lambda^k - \Omega^k;A) = H^*(\Lambda;A)^k - H^*(\Omega;A)^k = q_1 \cdot H^*(\Lambda;A) - q_2 \cdot H^*(\Omega;A).$$

□

Now we are ready to define the transfer map. We work on degree 1, and identify $H^1(S;A) = \mathrm{Hom}(S,A)$. Note that $H^1(\mathcal{F};A) = \mathrm{Hom}(S/[S,\mathcal{F}],A)$.

Definition 2.5. Let $\mathcal{F}$ be a saturated fusion system on the p-group S and let $\mathcal{H}$ be a saturated fusion subsystem of $\mathcal{F}$. Set $A = S/[S,\mathcal{H}]$ and consider the canonical projection $\pi\colon S \to S/[S,\mathcal{H}]$. Given a characteristic element Ω for $\mathcal{F}$, the *transfer map from $\mathcal{H}$ to $\mathcal{F}$ with respect to Ω* is

$$\tau_{\mathcal{H}}^{\mathcal{F}} = \tau_{\mathcal{H},\Omega}^{\mathcal{F}} = H^1(\Omega; A)(\pi)\colon S \to S/[S,\mathcal{H}].$$

When $\mathcal{H}$ is the trivial fusion system $\mathcal{F}_S(S)$ on S then $[S,\mathcal{H}] = [S,S] = S'$, the derived subgroup of S. In this case we write $\tau_S^{\mathcal{F}}$ instead of $\tau_{\mathcal{H}}^{\mathcal{F}}$ and we call it the *transfer map from S to $\mathcal{F}$ (with respect to Ω)*. The transfer $\tau_S^{\mathcal{F}}$ was successfully used in [6] by three of the authors and Nadia Mazza to study control of transfer and weak closure in fusion systems. Note that $[S,\mathcal{F}]$ is contained in the kernel of $\tau_{\mathcal{H}}^{\mathcal{F}}$ because $\tau_{\mathcal{H}}^{\mathcal{F}} \in H^1(\mathcal{F}; S/[S,\mathcal{H}])$. In particular, $\tau_{\mathcal{H}}^{\mathcal{F}}$ can be viewed as a map from $S/[S,\mathcal{H}]$ to itself.

Now we deal with how $\tau_{\mathcal{H}}^{\mathcal{F}}$ depends on the choice of characteristic elements, and determine its kernel.

Lemma 2.6. *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S and let $\mathcal{H}$ be a saturated fusion subsystem of $\mathcal{F}$ on S. Let Σ and Ω be characteristic elements for $\mathcal{F}$.*

(t1) $\tau_{\mathcal{H},\Sigma}^{\mathcal{F}} = r \cdot \tau_{\mathcal{H},\Omega}^{\mathcal{F}}$ *for some p'-number r.*
(t2) $\mathrm{Im}(\tau_{\mathcal{H},\Sigma}^{\mathcal{F}}) = \mathrm{Im}(\tau_{\mathcal{H},\Omega}^{\mathcal{F}})$.
(t3) $\mathrm{Ker}(\tau_{\mathcal{H},\Sigma}^{\mathcal{F}}) = \mathrm{Ker}(\tau_{\mathcal{H},\Omega}^{\mathcal{F}}) = [S,\mathcal{F}]$.
(t4) $\tau_{\mathcal{H},\Omega}^{\mathcal{F}} \circ \tau_{\mathcal{H},\Omega}^{\mathcal{F}} = \epsilon(\Omega) \cdot \tau_{\mathcal{H},\Omega}^{\mathcal{F}}$.

Proof. The statements (t1) and (t4) are direct consequences of Corollary 2.4 and that $H^1(\Omega, S/[S,\mathcal{H}])$ is an idempotent up to $\epsilon(\Omega)$ multiplication. From (t1) we obtain (t2) and the two first equalities in (t3). That $[S,\mathcal{F}]$ lies in the kernel of the transfer follows from the discussion before the lemma. To prove that $\mathrm{Ker}(\tau_{\mathcal{H},\Omega}^{\mathcal{F}})$ is not larger than $[S,\mathcal{F}]$ we might as well assume that Ω is a characteristic biset for $\mathcal{F}$. Then Ω has the form $\Omega = \coprod_{i\in I} S \times_{(P_i,\varphi_i)} S$ and

$$\tau_{\mathcal{H}}^{\mathcal{F}} = \sum_{i\in I} t_{P_i}^S(\pi \circ \varphi_i). \tag{2}$$

For $x \in \mathrm{Ker}(\tau_{\mathcal{H},\Omega}^{\mathcal{F}})$ we have, by (2),

$$\begin{aligned} \tau_{\mathcal{H}}^{\mathcal{F}}(x) &= \sum_{i\in I} t_{P_i}^S(\pi \circ \varphi_i)(x) \\ &= \sum_{i\in I} \sum_{t\in[S/P_i]} (\pi \circ \varphi_i)((x \cdot t)^{-1} x t), \end{aligned}$$

where $[S/P_i]$ denotes a set of representatives of the left cosets of P_i in S, and for $t \in [S/P_i]$, $x \cdot t \in [S/P_i]$ such that $(x \cdot t)P_i = xtP_i$. Considering a set W of $\langle x\rangle$-orbit representatives of $[S/P_i]$ we obtain

$$\tau_{\mathcal{H}}^{\mathcal{F}}(x) = \sum_{i\in I} \sum_{w\in W} \pi(\varphi_i(w^{-1} x^{r(w)} w)),$$

where $r(w)$ denotes the length of the $\langle x\rangle$- orbit of $[S/P_i]$ containg $w \in W$. Because $\varphi_i(w^{-1}x^{r(w)}w)[S,\mathcal{F}] = w^{-1}x^{r(w)}w[S,\mathcal{F}]$, we find that

$$\begin{aligned}\tau^{\mathcal{F}}_{\mathcal{H}}(x) + \pi([S,\mathcal{F}]) &= \pi(\sum_{i\in I}\sum_{w\in W} x^{r(w)}) + \pi([S,\mathcal{F}]) \\ &= \pi(\sum_{i\in I} x^{|S:P_i|}) + \pi([S,\mathcal{F}]) \\ &= \pi(x^{|\Omega|/|S|}) + \pi([S,\mathcal{F}]) \\ &= |\Omega|/|S|\cdot\pi(x) + \pi([S,\mathcal{F}]).\end{aligned}$$

Recall that we are assuming that $x \in \mathrm{Ker}(\tau^{\mathcal{F}}_{\mathcal{H},\Omega})$, i.e., that $\tau^{\mathcal{F}}_{\mathcal{H}}(x) = 0$ in $S/[S,\mathcal{H}]$. We conclude that $|\Omega|/|S|\cdot\pi(x) \in \pi([S,\mathcal{F}])$ and, $|\Omega|/|S|$ being a p'-number, that $\pi(x) \in \pi([S,\mathcal{F}])$ too. Hence $x \in [S,\mathcal{F}]$ as $[S,\mathcal{H}] \le [S,\mathcal{F}]$. □

Throughout the paper we will refer to the transfer map without making an explicit choice of characteristic elements. By Lemma 2.6 this transfer map is well defined up to multiplication by some p'-number, and its kernel and image do not depend on the choice.

3. Yoshida's theorem

First, we state a useful lemma that helps detect a homomorphic image isomorphic to $C_p \wr C_p$. This appears as Lemma 6.4 in [9].

Lemma 3.1. *Let R be a finite p-group having an elementary abelian subgroup E of index p. Suppose that there are $x \in E$ and $z \in R - E$ such that*

$$\prod_{i=0}^{p-1} z^{-i}xz^{i} \neq 1.$$

Then R has $C_p \wr C_p$ as a homomorphic image.

Theorem 3.2. *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S and let $\mathcal{H} = N_{\mathcal{F}}(S)$. If $[S,\mathcal{F}] \neq [S,\mathcal{H}]$, then S has $C_p \wr C_p$ as a homomorphic image.*

Proof. Fix a characteristic biset Ω for $\mathcal{F}$ and write

$$\Omega = \sum_{i\in I}[P_i,\varphi_i].$$

Let $I_0 = \{i \in I \mid P_i = S\}$. Then

$$\epsilon(\Omega) = \sum_{i\in I}|S:P_i| = |I_0| + \sum_{i\in I-I_0}|S:P_i| \equiv |I_0| \mod p.$$

By (c) of Definition 2.1, it follows that $|I_0| \not\equiv 0 \mod p$.

Suppose that $[S,\mathcal{F}] \neq [S,\mathcal{H}]$. By Lemma 2.6, the transfer map $\tau^{\mathcal{F}}_{\mathcal{H}}$ has kernel $[S,\mathcal{F}]$ and hence induces the map

$$\overline{\tau}^{\mathcal{F}}_{\mathcal{H}}\colon S/[S,\mathcal{F}] \to S/[S,\mathcal{H}],$$

which is not surjective. Let $\mathrm{Im}(\tau^{\mathcal{F}}_{\mathcal{H}}) = X/[S,\mathcal{H}]$ where $[S,\mathcal{H}] \le X < S$. Take a maximal subgroup Y of S containing X, and take an element $x \in S - Y$ of minimal order. We have

$$\begin{aligned}\tau_{\mathcal{H}}^{\mathcal{F}}(x) &= \sum_{i\in I_0} t_S^S(\pi\circ\varphi_i)(x) + \sum_{j\in I-I_0} t_{P_j}^S(\pi\circ\varphi_j)(x)\\ &= \sum_{i\in I_0}\varphi_i(x)[S,\mathcal{H}] + \sum_{j\in I-I_0} t_{P_j}^S(\pi\circ\varphi_j)(x) \in Y/[S,\mathcal{H}].\end{aligned}$$

Also, since $\varphi_i \in \mathrm{Aut}_{\mathcal{H}}(S)$ whenever $i \in I_0$,

$$\begin{aligned}\sum_{i\in I_0}\varphi_i(x)[S,\mathcal{H}] &= \sum_{i\in I_0} xx^{-1}\varphi_i(x)[S,\mathcal{H}]\\ &= |I_0|x[S,\mathcal{H}] \notin Y/[S,\mathcal{H}],\end{aligned}$$

because $x \notin Y$ and $|I_0|$ is not divisible by p. Thus there is a proper subgroup $P = P_j < S$ and $\varphi = \varphi_j \in \mathrm{Hom}_{\mathcal{F}}(P,S)$ for some $j \in I - I_0$ such that

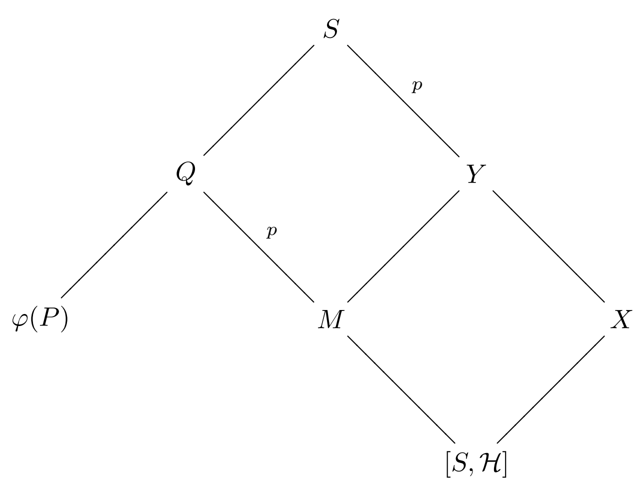
$$t_P^S(\pi\circ\varphi)(x) \notin Y/[S,\mathcal{H}]. \tag{3}$$

Note that for every $u \in S$,

$$t_P^S(\pi\circ\varphi)(u) = \sum_{t\in[S/P]} (\pi\circ\varphi)((u\cdot t)^{-1}ut) \in \varphi(P)[S,\mathcal{H}]/[S,\mathcal{H}]$$

Thus we can view $t_P^S(\pi\circ\varphi)$ as a map from S to $Q/[S,\mathcal{H}]$ where $Q = \varphi(P)[S,\mathcal{H}]$. By (3), we have $Q \nleq Y$ and hence $M := Y \cap Q < Q$. Since $|S : Y| = p$, it follows that

$$|Q : M| = p. \tag{4}$$



Let A be a maximal subgroup of S containing P. Suppose $x \notin A$. Then we can take $[S/A] = \{x^i \mid 0 \le i \le p-1\}$ and

$$x\cdot x^i = \begin{cases} x^{i+1} & \text{if } i < p-1,\\ 1 & \text{if } i = p-1.\end{cases}$$

Using the transitivity of the transfer maps we get

$$\begin{aligned}
t_P^S(\pi\circ\varphi)(x) &= t_A^S(t_P^A(\pi\circ\varphi))(x)\\
&= \sum_{i=0}^{p-1} t_P^A(\pi\circ\varphi)((x\cdot x^i)^{-1}xx^i)\\
&= t_P^A(\pi\circ\varphi)(x^p)\\
&= \sum_{v\in[A/P]} (\pi\circ\varphi)((x^p\cdot v)^{-1}x^pv)\\
&= \sum_{w\in W} (\pi\circ\varphi)(w^{-1}x^{pr(w)}w)\\
&\notin Y/[S,\mathcal{H}]
\end{aligned}$$

where W denotes a set of $\langle x^p\rangle$-orbit representatives of $[A/P]$ and $r(w)$ denotes the length of the $\langle x^p\rangle$- orbit containg $w\in W$. So there is a $w\in W$ such that $\varphi(w^{-1}x^{pr(w)}w)\notin Y$. But by the minimality of the order $o(x)$ of x, we get

$$o(x)\le o(\varphi(w^{-1}x^{pr(w)}w)) = o(w^{-1}x^{pr(w)}w) = o(x^{pr(w)}) < o(x),$$

a contradiction. Thus $x\in A$.

Let $z\in S-A$. Then $[S/A]=\{z^i \mid 0\le i\le p-1\}$ and $x\cdot z^i = z^i$ for all i because $x\in A$ and $A\lhd S$. Thus we get

$$\begin{aligned}
t_P^S(\pi\circ\varphi)(x) &= t_A^S(t_P^A(\pi\circ\varphi))(x)\\
&= \sum_{i=0}^{p-1} t_P^A(\pi\circ\varphi)((x\cdot z^i)^{-1}xz^i)\\
&= \sum_{i=0}^{p-1} t_P^A(\pi\circ\varphi)(z^{-i}xz^i)\\
&= t_P^A(\pi\circ\varphi)(\prod_{i=0}^{p-1} z^{-i}xz^i).
\end{aligned}$$

Suppose $\prod_{i=0}^{p-1} z^{-i}xz^i\in\Phi(A)$. Since $\Phi(A)=A^p[A,A]$, we have $t_P^S(\pi\circ\varphi)(x)\in\Phi(Q/[S,\mathcal{H}])$. But by (4), we have $\Phi(Q/[S,\mathcal{H}])\le M/[S,\mathcal{H}]$. Thus $t_P^S(\pi\circ\varphi)(x)\in Y/[S,\mathcal{H}]$, contradicting (3). Hence

$$\prod_{i=0}^{p-1} z^{-i}xz^i\notin\Phi(A).$$

Now by Lemma 3.1 applied to $R=S/\Phi(A)$ and $E=A/\Phi(A)$, we have that $S/\Phi(A)$ has $C_p\wr C_p$ as a homomorphic image, and so does S. □

Recall that if $\mathcal{F}$ is a fusion system on a finite p-group S and $\mathcal{H}$ is a subsystem of $\mathcal{F}$, then we say that $\mathcal{H}$ *controls transfer* in $\mathcal{F}$ if $[S,\mathcal{F}]=[S,\mathcal{H}]$.

Corollary 3.3. *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S. If any of the following conditions hold, then $\mathrm{N}_{\mathcal{F}}(S)$ controls transfer in $\mathcal{F}$.*

1. *S has nilpotence class less than p;*
2. *The exponent of S is less than or equal to p^p;*
3. *S is a regular p-group;*

(4) *If p is odd and S is metacyclic.*

Proof. The first two conditions imply the result since $C_p \wr C_p$ has nilpotence class p and contains an element of order p^2. The third statement is immediate since a regular p-group does not have a homomorphic image isomorphic to $C_p \wr C_p$ and the last statement follows from the third as every metacyclic p-group is regular if p is odd (cf. [11, Satz III.10.2]). □

Note that the last statement of the corollary is also a consequence of [19, Proposition 5.4] or [7, Theorem 4.1] and that it cannot be extended to $p = 2$ since $C_2 \wr C_2 \cong D_8$ is metacyclic. Also, since a regular p-group can have an arbitrarily large nilpotence class, so that 3.3.3 is considerably different than 3.3.1.

Theorem 3.4 (Huppert)**.** *Let p be an odd prime and let $\mathcal{F}$ be a saturated fusion system on a finite p-group S. If S is nonabelian and metacyclic, then $[S, \mathcal{F}] < S$.*

Proof. By Corollary 3.3.4, we may assume that $\mathrm{N}_S(\mathcal{F}) = \mathcal{F}$. In this case, $\mathcal{F}$ is constrained and so, by [2, Proposition 4.3], $\mathcal{F} = \mathcal{F}_S(G)$ for some finite group G with Sylow p-subgroup S. Thus, $[S, \mathcal{F}] = [S, \mathcal{F}_S(G)] < S$ by [11, Hilfssatz IV.8.5]. □

4. p-power index transfer

In this section, we prove a deeper property of characteristic idempotents and transfer maps of fusion systems. The elementary situation in group theory we want to mimic is the following: let G be a finite group with two subgroups S and L such that $G = SL$, i.e. $G = \{xy \mid x \in S, y \in L\}$, and let $N = S \cap L$. Then we have a bijection between left coset spaces

$$L/N \xrightarrow{\cong} G/S$$

induced by the inclusion $L \hookrightarrow G$. As a consequence, we get a commutative diagram

$$\begin{array}{ccccc} S & \longrightarrow & G & \xrightarrow{t_S^G} & S/S' \\ \uparrow & & \uparrow & & \uparrow{\scriptstyle p} \\ N & \longrightarrow & L & \xrightarrow{t_N^L} & N/N'. \end{array}$$

where t_S^G, t_N^L are group transfer maps, p is the map induced by the inclusion $N \hookrightarrow S$, and all other arrows are inclusions. Furthermore, if S is a Sylow p-subgroup of G and $L \trianglelefteq G$, the outer rectangle in the above gives a commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{\tau_{S,G}^{\mathcal{F}_S(G)}} & S/S' \\ {\scriptstyle \mathrm{incl}}\uparrow & & \uparrow{\scriptstyle p} \\ N & \xrightarrow{\tau_{N,L}^{\mathcal{F}_N(L)}} & N/N'. \end{array}$$

In particular, we have $p(\mathrm{Im}(\tau_{N,L}^{\mathcal{F}_N(L)})) \subseteq \mathrm{Im}(\tau_{S,G}^{\mathcal{F}_S(G)})$. This inclusion between images of transfers is the result we want to generalize to fusion systems. As cosets do not make sense in the fusion system setting we need an alternative approach on how to prove this inclusion in the group case.

One can view G as an (S,S)-biset and L as an (N,N)-biset in the obvious way. Then we have an isomorphism of (N,S)-bisets

$$S \times_N L \xrightarrow{\cong} G$$

induced by the product map $(x,y) \in S \times L \mapsto xy \in G$. Note that G and L are characteristic bisets for the fusion systems $\mathcal{F}_S(G)$ and $\mathcal{F}_N(L)$, respectively. We can rewrite this isomorphism of (N,S)-bisets as the following equality in $A(N,S)$:

$$[N, \mathrm{incl}]_N^S \circ L = G \circ [N, \mathrm{incl}]_N^S.$$

We give below an analogous equality in terms of characteristic idempotents, whose proof was provided by Kári Ragnarsson through private communication. As we show in Corollary 4.3, this is enough to deduce the inclusion between the images of the transfers.

Theorem 4.1. *Let $\mathcal{F}$ be a saturated fusion system on the p-group S. Let N be a normal subgroup of S containing $O^p_{\mathcal{F}}(S)$, and let $\mathcal{F}_N$ be the unique saturated subsystem of $\mathcal{F}$ on N of p-power index. (See appendix.) Then we have*

(1) $[N, \mathrm{incl}]_N^S \circ \omega_{\mathcal{F}_N} = \omega_{\mathcal{F}} \circ [N, \mathrm{incl}]_N^S$ *and*
(2) $\omega_{\mathcal{F}_N} \circ [N, \mathrm{id}]_S^N = [N, \mathrm{id}]_S^N \circ \omega_{\mathcal{F}}$.

Corollary 4.2. *In the situation of Theorem 4.1, we have*

$$\mathrm{tr}_N^S \circ H^*(\omega_{\mathcal{F}_N}; A) = H^*(\omega_{\mathcal{F}}; A) \circ \mathrm{tr}_N^S : H^*(N;A) \to H^*(\mathcal{F};A) \tag{5}$$

$$H^*(\omega_{\mathcal{F}_N}; A) \circ \mathrm{res}_N^S = \mathrm{res}_N^S \circ H^*(\omega_{\mathcal{F}}; A) : H^*(S;A) \to H^*(\mathcal{F}_N;A) \tag{6}$$

for any $\mathrm{Z}_{(p)}S$-module A.

Proof. This follows from Theorem 4.1 and the fact that $H^*([N, \mathrm{incl}]_N^S) = \mathrm{res}_N^S$, $H^*([N, \mathrm{id}]_S^N) = \mathrm{tr}_N^S$. □

Corollary 4.3. *In the situation of Theorem 4.1, we have a commutative diagram*

$$\begin{array}{ccc}
S & \xrightarrow{\tau^{\mathcal{F}}_{S,\omega_{\mathcal{F}}}} & S/S' \\
{\scriptstyle \mathrm{incl}}\uparrow & & \uparrow{\scriptstyle p} \\
N & \xrightarrow{\tau^{\mathcal{F}_N}_{N,\omega_{\mathcal{F}_N}}} & N/N'
\end{array}$$

where p is the map induced by the inclusion $N \hookrightarrow S$. In particular, we have

$$p(\mathrm{Im}(\tau_N^{\mathcal{F}_N})) \subseteq \mathrm{Im}(\tau_S^{\mathcal{F}}).$$

Proof. By Corollary 4.2, we have the following commutative diagram

$$\begin{array}{ccc}
H^1(S, S/S') & \xrightarrow{H^1(\omega_{\mathcal{F}}, S/S')} & H^1(S, S/S') \\
{\scriptstyle \mathrm{res}_N^S}\downarrow & & \downarrow{\scriptstyle \mathrm{res}_N^S} \\
H^1(N, S/S') & \xrightarrow{H^1(\omega_{\mathcal{F}_N}, S/S')} & H^1(N, S/S') \\
{\scriptstyle p_*}\uparrow & & \uparrow{\scriptstyle p_*} \\
H^1(N, N/N') & \xrightarrow{H^1(\omega_{\mathcal{F}_N}, N/N')} & H^1(N, N/N').
\end{array}$$

For a group H, let $\pi_H\colon H \to H/H'$ denote the canonical surjection. Since $\mathrm{res}^S_N(\pi_S) = p_*(\pi_N)$, by chasing the arrows we get

$$\tau^{\mathcal{F}}_{S,\omega_{\mathcal{F}}} \circ \mathrm{incl}^S_N = p \circ \tau^{\mathcal{F}_N}_{N,\omega_{\mathcal{F}_N}},$$

as desired. □

Now we turn to the proof of Theorem 4.1. First we need several lemmas.

Lemma 4.4 ([15])**.** *Let $\omega_{\mathcal{F}} \in A(S,S)_{(p)}$ be the characteristic idempotent of a saturated fusion system $\mathcal{F}$ on a finite p-group S. Let T be another p-group and let $X \in A(S,T)$. The following are equivalent:*

1. *$X \circ \omega_{\mathcal{F}} = X$.*
2. *X is right $\mathcal{F}$-stable, in the sense that for all $P \leq S$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$ we have $X \circ [P,\varphi]^S_P = X \circ [P,\mathrm{incl}]^S_P$.*

Proof. This is proved for stable maps in [15, Corollary 6.4], but the same argument works for $X \in A(S,T)$, or indeed elements in any Mackey functor evaluated at S. □

For the definition of normal subsystem used in the next Lemma we refer the reader to Definition A.1 in the appendix.

Lemma 4.5 ([16, Theorem 8.2])**.** *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S and $\mathcal{N}$ a saturated fusion subsystem of $\mathcal{F}$ on a strongly $\mathcal{F}$-closed subgroup N of S. Let $\omega_{\mathcal{N}}$ be the characteristic idempotent of $\mathcal{N}$. Then the following are equivalent:*

1. *$\mathcal{N}$ is a normal fusion subsystem of $\mathcal{F}$.*
2. *For every subgroup Q of N and every morphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q,S)$, the following identity in $A(Q,Q)_{(p)}$ holds:*

$$[\varphi(Q),\varphi^{-1}]^Q_N \circ \omega_{\mathcal{N}} \circ [Q,\varphi]^N_Q = [Q,\mathrm{id}]^Q_N \circ \omega_{\mathcal{N}} \circ [Q,\mathrm{incl}]^N_Q\,.$$

Proof of Theorem 4.1. First we remark that parts (1) and (2) of the proposition are equivalent by applying the opposite homomorphism [16, Definition 3.19]. We proceed to prove part (1). Note that by Proposition A.9, N is a strongly $\mathcal{F}$-closed subgroup of $\mathcal{F}$ and $\mathcal{F}_N$ is a normal subsystem of $\mathcal{F}$.

Since $\mathcal{F}_N$ is a subsystem of $\mathcal{F}$, the $\mathcal{F}$-stability of $\omega_{\mathcal{F}}$ implies that $\omega_{\mathcal{F}} \circ [N,\mathrm{incl}]^S_N$ is $\mathcal{F}_N$-stable. Hence, by Lemma 4.4, we have

$$\omega_{\mathcal{F}} \circ [N,\mathrm{incl}]^S_N = \omega_{\mathcal{F}} \circ [N,\mathrm{incl}]^S_N \circ \omega_{\mathcal{F}_N}$$

and it suffices to show that

$$\omega_{\mathcal{F}} \circ [N,\mathrm{incl}]^S_N \circ \omega_{\mathcal{F}_N} = [N,\mathrm{incl}]^S_N \circ \omega_{\mathcal{F}_N}\,,$$

which, by applying the opposite homomorphism, is equivalent to

$$\omega_{\mathcal{F}_N} \circ [N,\mathrm{id}]^N_S \circ \omega_{\mathcal{F}} = \omega_{\mathcal{F}_N} \circ [N,\mathrm{id}]^N_S\,.$$

We prove this last equation by showing that $\omega_{\mathcal{F}_N} \circ [N,\mathrm{id}]^N_S$ is right $\mathcal{F}$-stable. Now, for $P \leq S$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$, the double coset formula gives

$$[N,\mathrm{id}]^N_S \circ [P,\varphi]^S_P = \sum_{x \in N\backslash S/\varphi(P)} [\varphi^{-1}(\varphi(P) \cap N^x), c_x \circ \varphi]^N_P\,.$$

Since N is normal in S we have $N^x = N$, and since N is strongly $\mathcal{F}$-closed we have $\varphi(P) \cap N = \varphi(P \cap N)$, so the equation simplifies to

$$[N, \mathrm{id}]_S^N \circ [P, \varphi]_P^S = \sum_{x \in [N \backslash S / \varphi(P)]} [P \cap N, c_x \circ \varphi]_P^N \,.$$

Applying (3) in Lemma A.9 to $(\varphi|_{P\cap N})^{-1}$ we find $t \in S$ and $\psi \in \mathrm{Hom}_{\mathcal{F}_N}(P \cap N, N)$ such that $\varphi|_{P \cap N} = c_t \circ \psi$.

Using that $[N, c_x]_N^N \circ [N, c_x^{-1}]_N^N = [N, \mathrm{id}]_N^N$ in Lemma 4.5 for the normal subsystem $\mathcal{F}_N$ we get that, for all $x \in S$, $\omega_{\mathcal{F}_N} \circ [N, c_x]_N^N = [N, c_x]_N^N \circ \omega_{\mathcal{F}_N}$. This result applied in the previous equation give

(7)
$$\begin{aligned}
\omega_{\mathcal{F}_N} \circ [N, \mathrm{id}]_S^N \circ [P, \varphi]_P^S &= \omega_{\mathcal{F}_N} \circ \sum_{x \in N \backslash S / \varphi(P)} [P \cap N, c_x \circ \varphi]_P^N \\
&= \omega_{\mathcal{F}_N} \circ \sum_{x \in N \backslash S / \varphi(P)} [P \cap N, c_x \circ c_t \circ \psi]_P^N \\
&= \omega_{\mathcal{F}_N} \circ \left(\sum_{x \in N \backslash S / \varphi(P)} [N, c_x \circ c_t]_N^N \right) \circ [P \cap N, \psi]_P^N \\
&= \left(\sum_{x \in N \backslash S / \varphi(P)} [N, c_x \circ c_t]_N^N \right) \circ \omega_{\mathcal{F}_N} \circ [P \cap N, \psi]_P^N \\
&= \left(\sum_{x \in N \backslash S / \varphi(P)} [N, c_x \circ c_t]_N^N \right) \circ \omega_{\mathcal{F}_N} \circ [P \cap N, \mathrm{incl}]_P^N \\
&= \omega_{\mathcal{F}_N} \circ \left(\sum_{x \in N \backslash S / \varphi(P)} [N, c_x \circ c_t]_N^N \right) \circ [P \cap N, \mathrm{incl}]_P^N \\
&= \omega_{\mathcal{F}_N} \circ \sum_{x \in N \backslash S / \varphi(P)} [P \cap N, c_x \circ c_t]_P^N \,.
\end{aligned}$$

On the other hand, the double coset formula gives

(8)
$$\omega_{\mathcal{F}_N} \circ [N, \mathrm{id}]_S^N \circ [P, \mathrm{incl}]_P^S = \omega_{\mathcal{F}_N} \circ \sum_{y \in N \backslash S / P} [P \cap N, c_y]_P^N,$$

and the result follows by showing that the expressions in (7) and (8) are equal. To be able to compare the two sums we try to have the summation over the same indices. Consider the maps $\alpha, \beta : S \to A(P, N)$ defined by $\alpha(x) = \omega_{\mathcal{F}_N} \circ [P \circ N, c_x \circ c_t]_P^N$ and $\beta(y) = \omega_{\mathcal{F}_N} \circ [P \circ N, c_y]_P^N$. For $y \in S$, the map β is constant on the double coset NyP. Since N is normal in S, NP is the subgroup of S and we have $NyP = yNP$. Hence

$$\omega_{\mathcal{F}_N} \circ \sum_{x \in N \backslash S / P} [P \cap N, c_y]_P^N = \frac{1}{|NP|} \sum_{y \in S} \beta(y) \,.$$

For $x \in S$, reversing the algebraic manipulations leading to (7) we obtain that $\alpha(x) = \omega_{\mathcal{F}_N} \circ [P \cap N, c_x \circ \varphi]_P^N$. Thus, α is constant on the double coset $Nx\varphi(P)$, and we get

$$\omega_{\mathcal{F}_N} \circ \sum_{x \in N \backslash S / \varphi(P)} [P \cap N, c_x \circ c_t]_P^N = \frac{1}{|N\varphi(P)|} \sum_{x \in S} \alpha(x) \,.$$

Observe that $\beta(xt) = \alpha(x)$ for all $x \in S$, so $\sum_{y \in S} \beta(y) = \sum_{x \in S} \alpha(x)$. Moreover $|NP| = |N\varphi(P)|$ since N is strongly $\mathcal{F}$-closed. Thus we conclude that

$$\omega_{\mathcal{F}_N} \circ [N, \mathrm{id}]_S^N \circ [P, \varphi]_P^S = \omega_{\mathcal{F}_N} \circ [N, \mathrm{id}]_S^N \circ [P, \mathrm{incl}]_P^S .$$

This shows that $\omega_{\mathcal{F}_N} \circ [N, \mathrm{id}]_S^N$ is $\mathcal{F}$-stable, completing the proof. □

5. Tate's theorem

For a saturated fusion system $\mathcal{F}$ on a finite p-group S, let $T_\mathcal{F}$ denote the subgroup of S containing S' such that $T_\mathcal{F}/S' = \mathrm{Im}(\tau_S^\mathcal{F})$. From Lemma 2.6 it is clear that

$$S/S' = [S, \mathcal{F}]/S' \times T_\mathcal{F}/S'. \tag{9}$$

Proposition 5.1. *Let $\mathcal{F}$ be a saturated fusion system on S and $\mathcal{N}$ a normal subsystem of $\mathcal{F}$ on a strongly $\mathcal{F}$-closed subgroup N of S. For every $s \in S$, we have*

$$c_s \circ \tau_{N,\omega_{\mathcal{F}_N}}^{\mathcal{F}_N} \circ c_{s^{-1}} = \tau_{N,\omega_{\mathcal{F}_N}}^{\mathcal{F}_N} .$$

In particular, $T_{\mathcal{F}_N} \trianglelefteq S$.

Proof. This follows immediately from Lemma 4.5 and the definition of $\tau_{N,\omega_{\mathcal{F}_N}}^{\mathcal{F}_N}$. □

Proposition 5.2. *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S, and let $[S, \mathcal{F}] \le N \le S$. Then we have $T_\mathcal{F} \cap N = T_{\mathcal{F}_N} S'$.*

Proof. Clearly $T_{\mathcal{F}_N} S' \le T_\mathcal{F} \cap N$, so it suffices to show that $T_\mathcal{F} \cap N \le T_{\mathcal{F}_N} S'$. From (9) we have $N/N' \cong [N, \mathcal{F}_N]/N' \times T_{\mathcal{F}_N}/N'$; in particular, $N = T_{\mathcal{F}_N}[N, \mathcal{F}_N]$. From (9) again we get $S/S' \cong [S, \mathcal{F}]/S' \times T_\mathcal{F}/S'$, and hence $T_\mathcal{F} \cap [S, \mathcal{F}] = S'$. Corollary 4.3 gives $T_{\mathcal{F}_N} \le T_\mathcal{F}$, hence we have $T_\mathcal{F} \cap N = T_\mathcal{F} \cap T_{\mathcal{F}_N}[N, \mathcal{F}_N] = T_{\mathcal{F}_N}(T_\mathcal{F} \cap [N, \mathcal{F}_N]) \le T_{\mathcal{F}_N} S'$, by Dedekind's lemma. □

The following gives a crucial inductive argument.

Proposition 5.3. *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S, and let $O_\mathcal{F}^p(S) \le U \trianglelefteq S$. If $T_{\mathcal{F}_U}[U, S] \le V \le U$, then $S/V \cong U/V \times T_\mathcal{F} V/V$.*

Proof. Suppose $T_{\mathcal{F}_U}[U, S] \le V \le U$. Then we have $[V, S] \le [U, S] \le V$. So $V \trianglelefteq S$ and $U/V \le Z(S/V)$. Thus the desired conclusion $S/V \cong U/V \times T_\mathcal{F} V/V$ is equivalent to $(T_\mathcal{F} V) \cap U = V$. By Dedekind's lemma, $(T_\mathcal{F} V) \cap U = (T_\mathcal{F} \cap U)V$, and hence it is also equivalent to $T_\mathcal{F} \cap U \le V$. We proceed by induction on $|S : U|$. The case $U = S$ being trivial, may assume $U < S$. Let $U \le W \triangleleft S$, $|S : W| = p$. Since $(\mathcal{F}_W)_U = \mathcal{F}_U$ and $O_{\mathcal{F}_W}^p(W) = O_\mathcal{F}^p(S)$ by Corollary A.12, we have $O_{\mathcal{F}_W}^p(W) \le U \trianglelefteq W$ and $T_{(\mathcal{F}_W)_U}[U, W] \le V \le U$. By induction, it follows that $W/V \cong U/V \times T_{\mathcal{F}_W} V/V$, or equivalently $(T_{\mathcal{F}_W} V) \cap U = V$. By Proposition 5.1, we have $J := T_{\mathcal{F}_W} V \trianglelefteq S$. Since $U/V \le Z(S/V)$, we have $W/J \le Z(S/J)$. Thus $(S/J)/Z(S/J)$ is cyclic (being a quotient of $S/W \cong C_p$), and hence S/J is abelian. Therefore $S' \le J$. Then by Proposition 5.2, we have $T_\mathcal{F} \cap W = T_{\mathcal{F}_W} S' \le J = T_{\mathcal{F}_W} V$. So $T_\mathcal{F} \cap U = T_\mathcal{F} \cap W \cap U \le (T_{\mathcal{F}_W} V) \cap U = V$, as desired. □

Theorem 5.4. *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S, and let $\mathcal{H}$ be a saturated fusion subsystem of $\mathcal{F}$ on S. The following are equivalent.*

(1) $E_\mathcal{F}^p(S) = E_\mathcal{H}^p(S)$.
(2) $A_\mathcal{F}^p(S) = A_\mathcal{H}^p(S)$.
(3) $O_\mathcal{F}^p(S) = O_\mathcal{H}^p(S)$.

Proof. (3) $\Rightarrow$ (2) $\Rightarrow$ (1): Follows from Lemma A.7.

(1) $\Rightarrow$ (3): Suppose (3). Then $O^p_{\mathcal{F}}(S) \le \Phi(S)O^p_{\mathcal{F}}(S) = \Phi(S)O^p_{\mathcal{H}}(S)$. Applying Proposition 5.3 to $U = O^p_{\mathcal{F}}(S)$ and $V = O^p_{\mathcal{H}}(S)[O^p_{\mathcal{F}}(S), \mathcal{H}]$, we get $S/V \cong U/V \times T_{\mathcal{F}}V/V$. Since $U/V \le \Phi(S)/V = \Phi(S/V)$, it follows that $S/V = T_{\mathcal{F}}V/V$, and hence $U/V = 1$, that is, $O^p_{\mathcal{F}}(S) = O^p_{\mathcal{H}}(S)[O^p_{\mathcal{F}}(S), \mathcal{H}]$. Let $\overline{\mathcal{H}} = \mathcal{H}/O^p_{\mathcal{H}}(S)$ and $\overline{S} = S/O^p_{\mathcal{H}}(S)$. Then $\overline{\mathcal{H}} = \mathcal{F}_{\overline{S}}(\overline{S})$, and hence $\overline{O^p_{\mathcal{F}}(S)} = [\overline{O^p_{\mathcal{F}}(S)}, \overline{\mathcal{H}}] = [\overline{O^p_{\mathcal{F}}(S)}, \overline{S}]$. Since $\overline{S}$ is a finite p-group, it follows that $\overline{O^p_{\mathcal{F}}(S)} = 1$, as desired. □

Appendix A. Normality and quotients in fusion systems

For the convenience of the reader, we recall definitions and some standard properties of normal subsystems, quotient systems, and p- power index subsystems of fusion systems used in this paper. Proofs of some claims are included.

We start with the notion of normal fusion subsystems.

Definition A.1. Let $\mathcal{F}$ be a fusion system on a finite p-group S and $\mathcal{H}$ a fusion subsystem of $\mathcal{F}$ on a subgroup T of S. We say that $\mathcal{H}$ is *normal* in $\mathcal{F}$ if T is strongly $\mathcal{F}$-closed and if for every isomorphism $\varphi : Q \to P$ in $\mathcal{F}$ and any two subgroups R, U of $Q \cap P$ we have

$$\varphi \circ \mathrm{Hom}_{\mathcal{H}}(R, U) \circ \varphi^{-1} \subseteq \mathrm{Hom}_{\mathcal{H}}(\varphi(R), \varphi(U))\,.$$

In presence of saturation, we have a very useful characterization of normal fusion subsystems given by Aschbacher [1]. Note that Aschbacher uses a different terminology, calling $\mathcal{F}$- *invariant* fusion subsystem what we call normal fusion subsystem in $\mathcal{F}$ in this paper.

Lemma A.2 ([1])**.** *Let $\mathcal{F}$ be a saturated fusion system on S and $\mathcal{H}$ a saturated fusion subsystem on a strongly $\mathcal{F}$-closed subgroup T of S. We have that $\mathcal{N}$ is normal in $\mathcal{F}$ if and only if the following conditions are satisfied*

1) $\mathrm{Aut}_{\mathcal{F}}(T) \le \mathrm{Aut}(\mathcal{H})$,
2) *any morphism $\psi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ with $P, Q \le T$ decomposes as $\psi = \phi \circ \chi$ where $\phi \in \mathrm{Hom}_{\mathcal{N}}(\chi(P), Q)$ and $\chi \in \mathrm{Aut}_{\mathcal{F}}(T)$.*

Analogous to the case of finite groups one can define quotient fusion systems.

Definition A.3. Let $\mathcal{F}$ be a fusion system on S and let T be a strongly $\mathcal{F}$- closed subgroup of S. By the *quotient system* $\mathcal{F}/T$, we mean the fusion system on S/T, such that for any two subgroups U and V of S containing T, $\mathrm{Hom}_{\mathcal{F}/P}(U/P, V/P)$ is the set of homomorphisms induced by morphisms in $\mathrm{Hom}_{\mathcal{F}}(U, V)$.

When the fusion system is saturated Puig [13] proves that the saturation is inherited by the quotient system.

Theorem A.4 ([13])**.** *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S and let P be a strongly $\mathcal{F}$-closed subgroup of S. Then the quotient system $\mathcal{F}/P$ is saturated.*

Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S and let T be a strongly $\mathcal{F}$-closed subgroup of S. If P and Q are subgroups of S, then any $\mathcal{F}$-morphism $\varphi\colon P \to Q$ induces a group homomorphism $\overline{\varphi}\colon PT/T \to QT/T$ given by $\overline{\varphi}(uT) = \varphi(u)T$ for $u \in P$, because T is strongly $\mathcal{F}$-closed. The following theorem of Craven asserts that this induced map belongs to the quotient system $\mathcal{F}/T$.

Theorem A.5 ([5, 5.10])**.** *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S and let T be a strongly $\mathcal{F}$-closed subgroup of S. Let $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,Q)$ for some $P,Q \leq S$. Then the induced map $\overline{\varphi}\colon PT/T \to QT/T$ belongs to $\mathcal{F}/T$; that is, there is $\widetilde{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(PT,QT)$ such that $\overline{\varphi}(uT) = \widetilde{\varphi}(uT)$ for $u \in P$.*

Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S. We define for $\mathcal{F}$ analogous notions to focal and hyperfocal subgroups in a finite group.

(1) $O^p_{\mathcal{F}}(S) = \langle [P, O^p(\mathrm{Aut}_{\mathcal{F}}(P))] \mid P \leq S \rangle$.
(2) $A^p_{\mathcal{F}}(S) = [S,\mathcal{F}] = \langle [P, \mathrm{Aut}_{\mathcal{F}}(P)] \mid P \leq S \rangle$.
(3) $E^p_{\mathcal{F}}(S) = \Phi(S)[S,\mathcal{F}] = \Phi(S) O^p_{\mathcal{F}}(S)$.

As in [3, Definition 3.3], let $O^p_*(\mathcal{F})$ be the smallest restrictive subcategory of $\mathcal{F}$ whose morphism set contains $O^p(\mathrm{Aut}_{\mathcal{F}}(P))$ for all subgroups $P \leq S$. Using Alperin's fusion theorem and the fact that $\mathrm{Aut}_{\mathcal{F}}(P) = O^p(\mathrm{Aut}_{\mathcal{F}}(P))\mathrm{Aut}_S(P)$ for $P \leq S$ fully $\mathcal{F}$-normalized, one obtains the following decomposition lemma.

Lemma A.6. [3, Lemma 3.4] *Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S. Let $\psi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$ where $P \leq S$. Then there exist $s \in S$ and $\phi \in \mathrm{Hom}_{O^p_*(\mathcal{F})}(c_s(P),S)$ such that $\psi = \phi \circ c_s|_P$.*

Then we get the following relation.

Lemma A.7. *We have that $A^p_{\mathcal{F}}(S) = [S,S]O^p_{\mathcal{F}}(S)$.*

In the proofs of our transfer theorems we deal with a special class of fusion subsystems containing the hyperfocal subgroup.

Definition A.8. [3, Definition 3.1] Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S and $\mathcal{H}$ a fusion subsystem of $\mathcal{F}$ on a subgroup T of S. We say that $\mathcal{H}$ is a *p-power index subsystem of* $\mathcal{F}$ if T contains $O^p_{\mathcal{F}}(S)$ and $\mathrm{Aut}_{\mathcal{H}}(P)$ contains $O^p(\mathrm{Aut}_{\mathcal{F}}(P))$ for all the subgroups P of T.

Equivalently, a fusion subsystem $\mathcal{H} \subset \mathcal{F}$ on $T \geq O^p_{\mathcal{F}}(S)$ has p-power index if it contains all $\mathcal{F}$-automorphisms of order prime to p of subgroups of T. The p-power index saturated subsystems of a saturated fusion system on S are in bijection with the subgroups of S that contain $O^p_{\mathcal{F}}(S)$. This is given by Theorem 4.3 in [3]. Denote by $O^p(\mathcal{F})$ the unique saturated subsystem of $\mathcal{F}$ of p- power index on $O^p_{\mathcal{F}}(S)$ and by $\mathcal{F}_U$ the unique saturated fusion subsystem of p-power index of $\mathcal{F}$ on U with $O^p_{\mathcal{F}}(S) \leq U \leq S$. We can prove more when U is normal in S.

Proposition A.9. *Let $\mathcal{F}$ be a saturated fusion system on S and N be a normal subgroup of S containing $O^p_{\mathcal{F}}(S)$. Then*

(1) *N is strongly $\mathcal{F}$-closed.*
(2) *$\mathcal{F}_N$ is a normal fusion subsystem of $\mathcal{F}$.*
(3) *For every $P \leq N$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$, there exist $s \in S$ and $\psi \in \mathrm{Hom}_{\mathcal{F}_N}(c_s(P),S)$ such that $\varphi = \psi \circ c_s|_P$.*

Proof. Let $P \leq N$ and let $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$. By Lemma A.6, there exist $s \in S$ and $\psi \in \mathrm{Hom}_{O^p_*(\mathcal{F})}(c_s(P),S)$ such that $\varphi = \psi \circ c_s|_P$. Let $u \in P$. Then

$$\varphi(u) = \psi(c_s(u))c_s(u)^{-1}c_s(u),$$

where $\psi(c_s(u))c_s(u)^{-1} \in O^p_{\mathcal{F}}(S) \leq N$ and $c_s(u) \in N$ because $N \trianglelefteq S$. Thus $\varphi(u) \in N$. This shows that N is strongly $\mathcal{F}$-closed. Now that N is strongly $\mathcal{F}$-closed, ψ belongs to $\mathcal{F}_N$, proving (3). To finish the proof, it remains to show that $\mathrm{Aut}_{\mathcal{F}}(N) \leq$

$\mathrm{Aut}(\mathcal{F}_N)$. But this comes from the uniqueness of the saturated fusion subsystems of p-power index on a given subgroup of S containing $O^p_{\mathcal{F}}(S)$. Indeed, any morphism in $\alpha \in \mathrm{Aut}_{\mathcal{F}}(N)$ gives a fusion preserving isomorphism from $\mathcal{F}_N$ to ${}^{\alpha}\mathcal{F}_N$ which is another saturated fusion system on N containing $O^p(\mathrm{Aut}_{\mathcal{F}}(P))$ for any $P \leq N$. By the uniqueness of such systems, we have $\alpha \in \mathrm{Aut}(\mathcal{F}_N)$. □

Finally, we show that $O^p(\mathcal{F}) = O^p(O^p(\mathcal{F}))$. We use the following notations $S_1 := O^p_{\mathcal{F}}(S)$, $\mathcal{F}_1 := O^p(\mathcal{F})$, $S_2 := O^p_{\mathcal{F}_1}(S_1)$. By the definition of the hyperfocal subgroup we have that $\mathcal{F}/S_1$ and $\mathcal{F}_1/S_2$ are trivial fusion systems on S/S_1 and S_1/S_2, respectively.

Proposition A.10. *The subgroup S_2 is strongly $\mathcal{F}$-closed.*

Proof. Let $P \leq S_2$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$. By Proposition A.9, we have a decomposition $\varphi = \psi \circ \alpha$ with $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S_1)$ and $\psi \in \mathrm{Hom}_{\mathcal{F}_1}(\alpha(P), S_1)$. Since $\mathrm{Aut}_{\mathcal{F}}(S_1) \leq \mathrm{Aut}(\mathcal{F}_1)$, we have $\alpha(P) \leq S_2$. Then $\varphi(P) = \psi(\alpha(P)) \leq S_2$ because S_2 is strongly $\mathcal{F}_1$-closed. □

Proposition A.11. *We have $S_2 = S_1$. In particular $O^p(\mathcal{F}) = O^p(O^p(\mathcal{F}))$.*

Proof. **Step 1**. We show that $\mathcal{F}/S_2$ is a trivial fusion system on S/S_2. By Propositions A.4 and A.10, $\mathcal{F}/S_2$ is a saturated fusion system on S/S_2. Let $S_2 \leq Q \leq S$ and $\bar{\rho} \in \mathrm{Aut}_{\mathcal{F}/S_2}(Q/S_2)$ be a p'- automorphism. Then there exist $\rho \in \mathrm{Aut}_{\mathcal{F}}(Q)$ lifting $\bar{\rho}$ and, rising to the appropriate p-th power, we can suppose that ρ is also a p'-automorphism. Note that in particular ρ belongs to $\mathcal{F}_1$. Now ρ induces p'-automorphisms on $Q/(Q \cap S_1)$ and on $(Q \cap S_1)/S_2$. The induced p'-automorphism of $Q/(Q \cap S_1) \cong QS_1/S_1$ belongs to $\mathcal{F}/S_1$ by Theorem A.5, and so is the identity map because $\mathcal{F}/S_1$ is the trivial fusion system on S/S_1. Similarly, the induced p'-automorphism of $(Q \cap S_1)/S_2$ is the identity map. Hence ρ itself is the identity map by [10, 5.3.2], implying the claim in step one.

Step 2. We show that if T is a $\mathcal{F}$-strongly closed subgroup of S such that $\mathcal{F}/T$ is a trivial fusion system on S/T, then $O^p_{\mathcal{F}}(S) \leq T$. Let $T \leq Q \leq S$ and $\rho \in \mathrm{Aut}_{\mathcal{F}}(Q)$ be a p'- automorphism. Then ρ induces a trivial automorphism on Q/T implying that $u^{-1}\rho(u) \in T$ for any $u \in Q$. By Theorem A.5, those are the generators of $O^p_{\mathcal{F}}(S)$.

Step 1 and Step 2 imply the first claim in the proposition. The second follows by uniqueness of the saturated fusion subsystems of p-power index of $\mathcal{F}$ on subgroups of S containing $O^p_{\mathcal{F}}(S)$. □

This proposition implies that the hyperfocal subsystem of any p- power index subsystem of $\mathcal{F}$ is equal to the hyperfocal subsystem of $\mathcal{F}$.

Corollary A.12. *Let $O^p_{\mathcal{F}}(S) \leq T \leq S$ and $\mathcal{F}_T$ be the unique p-power index saturated fusion subsystem of $\mathcal{F}$ on T. Then $O^p_{\mathcal{F}_T}(T) = O^p_{\mathcal{F}}(S)$.*

Proof. As $O^p(\mathcal{F}) \subset \mathcal{F}_T \subset \mathcal{F}$ we have that $O^p_{O^p(\mathcal{F})}(O^p_{\mathcal{F}}(S)) \leq O^p_{\mathcal{F}_T}(T) \leq O^p_{\mathcal{F}}(S)$. Proposition A.11 tells us that the first and the last term in the inequality are equal so the corollary follows. □

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