SPRINGER CORRESPONDENCE IN NON CONNECTED REDUCTIVE GROUPS

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Abstract:
Let G be an algebraic connected group and let σ be a quasi-semi-simple unipotent automorphism of G, i.e. an automorphism of G which stabilises a pair (T, B) where T is a maximal torus included in a Borel subgroup B of G. The purpose of this article is to show that there is a Springer correspondence in the component $G.\sigma$ of the non connected reductive group $\tilde{G} = G.\langle\sigma\rangle$ between some pairs $(C, \mathcal{E})$, where C is a unipotent conjugacy class of $G.\sigma$ and $\mathcal{E}$ a G-equivariant irreducible local system on C, and the irreducible representations of the Weyl group of the group G^σ of σ-fixed points of G.

Introduction

In this article, we consider a non connected reductive algebraic group of the form $\tilde{G} = G.\langle\sigma\rangle$, where G is a connected reductive group on an algebraic closed field of characteristic $p > 0$ and σ is a unipotent quasi-semi-simple automorphism of G. An automorphism σ of G is said to be quasi-semi-simple if there is a σ-stable pair (T, B), where T is a maximal torus included in a Borel subgroup B of G. If $G = GL(n)$, the subgroup B of upper triangular matrices is a Borel subgroup of G and the subgroup of diagonal matrices is a maximal torus of G included in B, moreover let σ be the automorphism of G defined by $A \mapsto J^tA^{-1}J^{-1}$, where J is the matrix

$$J = \begin{pmatrix} & & 1 \\ & \ldots & \\ 1 & & \end{pmatrix}$$

Then σ stabilises both T and B, so it is quasi-semi-simple.

The purpose of this article is to define the analogous of the Springer correspondence for the connected component $G^1 = G.\sigma$. In the connected case, the Springer correspondence was often studied. In the case of the connected reductive group G, it is a map from the set of irreducible representations of the Weyl group of G to the set of pairs $(C, \mathcal{E})$, where C is a unipotent G-conjugacy class in G and $\mathcal{E}$ is a G-equivariant irreducible local system on C. For example, it is well-known that irreducible characters of the symmetric group of degree n, which is the Weyl group of $GL(n)$, are in bijection with unipotent conjugacy classes of $GL(n)$.

The Springer correspondence for G^1 is a map from the set of irreducible representations of the Weyl group of the group G^σ of σ-fixed elements of G (which is then a connected reductive group) to the set of pairs $(C, \mathcal{E})$, where C is a unipotent G-class in G^1 and $\mathcal{E}$ is a G-equivariant irreducible local system on C. This correspondence induces an injective map from the set of unipotent conjugacy classes in G^1 to the set of irreducible representations of the Weyl group of G^σ.

In section 1, we recall some results from [7] about quasi-semi-simple automorphisms of a connected reductive group. The subject of section 2 is the study of regular quasi-semi-simple elements of G^1 which play the same role in the Springer correspondence for G^1 as regular semi-simple elements in the Springer correspondence for G. In section 3, we give some preliminary

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results on the variety $\mathcal{B}_{g.\sigma}$ of the Borel subgroups of G which are stable under conjugation by an element $g.\sigma$ of G^1. In section 4, we construct representations of the group W^σ of σ-fixed elements of the Weyl group W of G on l-adic groups $\mathrm{H}^i(\mathcal{B}^G_{g.\sigma}, \overline{\mathbb{Q}}_l)$, where $g.\sigma \in G^1$ (proposition 4.7), which we call Springer representations of W^σ from G^1.

As σ is unipotent quasi-semi-simple, W^σ is also the Weyl group of G^σ and $\mathcal{B}^G_\sigma$ is isomorphic to the variety $\mathcal{B}^{G^\sigma}$ of Borel subgroups of G^σ. In section 5, we compare the Springer representation of W^σ from G^1 on $\mathrm{H}^*(\mathcal{B}^G_\sigma, \overline{\mathbb{Q}}_l)$ constructed in section 4 with the Springer representation of W^σ from the connected reductive group G^σ on $\mathrm{H}^*(\mathcal{B}^{G^\sigma}, \overline{\mathbb{Q}}_l)$ (proposition 5.1), both of them coincide with the regular representation of W^σ (corollaire 5.4). This result is necessary for the proof of theorem 6.3 which defines the Springer correspondence for G^1.

The existence of a Springer correspondence is then proved in theorem 6.3 but the correspondence is not explicitely known. We have to use its properties to calculate it in some exemples. Two of these properties are proved at the end of this article. Section 6 ends with the proof of a property whose analogous is true in the connected case (proposition 6.7): for any unipotent element $u.\sigma$ in G^1, let ρ be the irreducible representation of W^σ which corresponds to the pair $(\mathrm{Cl}_G(u.\sigma), \overline{\mathbb{Q}}_l)$ by the Springer correspondence, then $\dim(\mathcal{B}_{u.\sigma}) = a_\rho$, where a_ρ is the valuation of the fake degree of ρ. The purpose of the last section is to prove a second property of the Springer correspondence for G^1. Let L be a σ-stable Levi subgroup of a σ-stable parabolic subgroup of G. Let C^L be a unipotent L-conjugacy class in $L^1 = L.\sigma$ and let C be the unipotent G-conjugacy class in G^1 obtained by inducing C^L from L^1 to G^1 (definition 7.1). Then, the image $\rho_{(C^L, \overline{\mathbb{Q}}_l)}$ of $(C^L, \overline{\mathbb{Q}}_l)$ by the Springer correspondence for L^1 is an irreducible component of the restriction to $W(L)^\sigma$ of the image $\rho_{(C, \overline{\mathbb{Q}}_l)}$ of $(C, \overline{\mathbb{Q}}_l)$ under the Springer correspondence for G^1 (proposition 7.13).

1. Automorphims of connected reductive groups

Let G be a connected reductive group on an algebraic closed field K of characteristic p. According to [7] (théorème 1.8), when an automorphism is quasi-semi-simple, the group G^σ of fixed points of G under σ is a reductive group, and there is a one-to-one correspondence between σ-stable maximal tori of G included in a σ-stable Borel subgroup of G and maximal tori of G^σ defined by $T \mapsto T \cap (G^\sigma)^0$.

A quasi-semi-simple automorphism σ is said to be quasi-central if the following equivalent properties are true:

Theorem 1.1. *([7] théorème 1.15) Let G be a connected reductive group and σ a quasi-semi-simple automorphism of G. The following properties are equivalent:*

(i) $\dim(G^\sigma) > \dim(G^{\sigma'})$ for any quasi-semi-simple automorphism $\sigma' = \sigma \circ \mathrm{ad}(g)$, where $\mathrm{ad}(g)$ is the conjugation by g for all $g \in G$.

(ii) There is a bijection between σ-stable Borel subgroups of G and maximal Borel subgroups of G^σ defined by $B \mapsto B \cap (G^\sigma)^0$.

(iii) There exists a σ-stable maximal torus T included in a σ-stable Borel subgroup B such that the Weyl group $W(G^\sigma)$ of G^σ is isomorphic to the group $W(G)^\sigma$ of fixed points by σ of the Weyl group of G.

(iii') Statement (iii) is true for every pair (T, B) where T is a σ-stable maximal torus T included in a σ-stable Borel subgroup B.

Until the end of the article, we will consider a reductive group of the form $\tilde{G} = G.\langle\sigma\rangle$ where G is a connected reductive group and σ is a unipotent quasi-semi-simple automorphism of G. Such a σ is quasi-central by (i) of the following result:

Proposition 1.2. *([7] corollaire 1.33)*
Let H be a connected reductive group and let σ be a unipotent quasi-semi-simple automorphism of H. Then,
(i) σ is quasi-central,
(ii) H^σ is connected,
(iii) For any σ-stable maximal torus T of H, we have $T = T^\sigma \mathcal{L}_\sigma(T)$ where $\mathcal{L}_\sigma(T)$ is the map from T to T defined by $t \mapsto t^{-1}\sigma(t)$.

We say that an element of $\tilde{G}$ is quasi-semi-simple if it induces a quasi-semi-simple automorphism of G. Like in [7], we will call quasi-borel of $\tilde{G}$ the normalizer in $\tilde{G}$ of a Borel subgroup of G and quasi-torus of $\tilde{G}$ the normalizer in $\tilde{G}$ of a pair (T, B) where T is a maximal torus included in a Borel subgroup B of G.

We will use the following notation: we first fix a pair (T, B) where T is a σ-stable maximal torus included in a σ-stable Borel subgroup B of G. Then, T^σ is a maximal torus and B^σ a Borel subgroup of the connected reductive group G^σ. Let U be the unipotent radical of B. We will denote by G^1 the connected component $G.\sigma$ and by $\mathcal{U}(G^1)$ the subvariety of unipotent elements of G^1. Let W be the Weyl group of G then W^σ is the Weyl group of G^σ. For all connected reductive group H, we will denote by $\mathrm{rk}(H)$ the rank of H and $\mathcal{B}^H$ the variety of Borel subgroups of H. Moreover, for all $x \in \tilde{G}$, $\mathcal{B}^G_x$ will be the variety of Borel subgroups of G which are stable by conjugation under x.

2. Regular quasi-semi-simple elements of G^1

Regular quasi-semi-simple elements of G^1 play the same role for the Springer correspondence in G^1 than that played by regular semi-simple elements for the Springer correspondence in G.

Definition 2.1. An element x of $\tilde{G}$ is said regular if it is regular in the connected component $G.x$ of $\tilde{G}$ which it belongs to, that is to say, if its centralizer $\mathrm{C}_G(x)$ in G has minimal dimension among the $\mathrm{C}_G(y)$ for $y \in G.x$.

Proposition 2.1.
(i) An element $g.\sigma$ of G^1 is regular if and only if $\dim(C_G(g.\sigma)) = \mathrm{rk}(G^\sigma)$.
(ii) An element $g.\sigma$ of G^1 is regular if and only if $\dim(\mathcal{B}^G_{g.\sigma}) = 0$.
(iii) Let t be a semi-simple element of G^σ. Then, t is regular in the connected reductive group G^σ if and only if $t.\sigma$ is regular in G^1.

Proof: Let t be a semi-simple element of G^σ. Then, t (resp. σ) is the semi-simple (resp. unipotent) part of $t.\sigma$. So, for all $x \in \mathrm{C}_G(t.\sigma)$, we have $(xtx^{-1})(x\sigma x^{-1}) = t.\sigma$, hence, by unicity of the Jordan decomposition, $xtx^{-1} = x$ and $x\sigma x^{-1} = \sigma$. So, we have $\mathrm{C}_G(t.\sigma) = \mathrm{C}_G(t) \cap \mathrm{C}_G(\sigma) = \mathrm{C}_{G^\sigma}(t)$. Then, we have $\dim(\mathrm{C}_G(t.\sigma)) = \dim(\mathrm{C}_{G^\sigma}(t))$.

But, t is semi-simple regular in G^σ if and only if $\dim(\mathrm{C}_{G^\sigma}(t)) = \mathrm{rk}(G^\sigma)$ ([6] proposition 14.2). So, if t is semi-simple regular in G^σ then $\dim(\mathrm{C}_G(t.\sigma)) = \mathrm{rk}(G^\sigma)$.

Moreover, according to [14] (théorème II.10.15), we have $\dim(\mathrm{C}_{\tilde{G}}(x)) = \mathrm{rk}_x(G) + 2\dim(\mathcal{B}^G_x)$ for all $x \in \tilde{G}$, where $\mathrm{rk}_x(G) = \max\{\mathrm{rk}(\mathrm{C}_G(y)) | y \in G.x\}$. But, as σ is quasi-central, we have $\mathrm{rk}_{g.\sigma}(G) = \mathrm{rk}(G^\sigma)$ and $\dim(\mathrm{C}_G(g.\sigma)) \geq \mathrm{rk}(G^\sigma)$ for all $g.\sigma \in G^1$. Hence, if t is semi-simple regular in G^σ then $\dim(\mathrm{C}_G(t.\sigma)) = \mathrm{rk}(G^\sigma)$ and that means that $t.\sigma$ is regular in G^1. Whence, we have proved assertion (i) and the direct implication of (iii). Assertion (ii) is a direct consequence of (i) and of the above result of [14] (théorème II.10.15).

Let us prove the converse of (iii). Let t be a semi-simple element of G^σ such that $t.\sigma$ is regular. Then, we have $\dim(C_G(t.\sigma)) = \mathrm{rk}(G^\sigma)$ which implies that $\dim(C_{G^\sigma}(t)) = \mathrm{rk}(G^\sigma)$, that is to say that t is regular in G^σ.

End of the proof.

Theorem 2.2. *Regular quasi-semi-simple elements are dense in G^1.*

Proof: We have shown in the proof of the previous proposition that if $t \in T^\sigma$ is a regular element of G^σ, then $t.\sigma$ is regular in $\tilde{G}$. Moreover $t.\sigma$ is quasi-semi-simple since T and B are stable by conjugation under $t.\sigma$. As regular elements of G^σ are dense in T^σ ([6] proposition 14.7), so quasi-semi-simple regular elements of G^1 are dense in $T^\sigma.\sigma$.

We will then need the following lemma:

Lemma 2.3. *Let σ' be a regular quasi-semi-simple element of $\tilde{G}$ stabilizing T and B; then the map $f : U \to U$ defined by $u \mapsto u^{\sigma'}u^{-1}$ is surjective.*

Proof of lemma 2.3: Let $\sigma' = su$ be the Jordan decomposition of σ'. Then, we have $\dim(\mathrm{C}_G(\sigma')) = \dim(\mathrm{C}_{\mathrm{C}_G(s)}(u))$. But, as σ' is regular, we have $\dim(\mathrm{C}_G(\sigma')) = \mathrm{rk}_{\sigma'}(G)$, and according to [14] (lemma II.1.14), $\mathrm{rk}_{\sigma'}(G) = \mathrm{rk}_u(\mathrm{C}_G(s))$. So, u is a regular element of $\mathrm{C}_G(s)^0.u$. Moreover, according to [7] (lemma 1.14), u is quasi-semi-simple in $\mathrm{C}_{\tilde{G}}(s)^0.u$. We set $H = \mathrm{C}_{\tilde{G}}(s)^0$. Let W_H be the Weyl group of H, then the action of u on H induces an action of u on W_H. We set $W_H^u = \{w \in W_H | u.w = w\}$. Then, according to [14] (proposition II.2.20), as u is unipotent quasi-semi-simple in $H.u$, we have $\dim(\mathcal{B}_u^H) = l_u(w^H)$ where $l_u(w^H)$ is the length in W_H^u of the longest element of W^H. Moreover, by [14] (proposition II.10.2), as u is regular in $H.u$, we have $|\mathcal{B}_u^H| = 1$. Thus, we have $W_H^u = \{1\}$ and H is a torus.

So, we have proved that $(G^{\sigma'})^0 = ((G^s)^u)^0$ is a torus which leads to the fact that $U^{\sigma'} = \{1\}$. Thus, σ' induces a surjective endomorphism of U with a finite number of fixed points. According to Lang-Steinberg's theorem ([15] theorem 10), we can then conclude that f is surjective.

End of the proof of lemma 2.3.

Applying lemma 2.3, with $\sigma' = t.\sigma$ where t is any regular element of T^σ, we get that for all $v \in U$ there exists $u \in U$ such that $v = u^{t.\sigma}u^{-1}$, that is to say, such that $tv.\sigma = {}^tu(t.\sigma)({}^tu)^{-1}$. Hence, $tv.\sigma$ is quasi-semi-simple (${}^{tu}T$ and ${}^{tu}B$ are stable under conjugation by $tv.\sigma$) and regular ($\mathrm{C}_G(tv.\sigma) = {}^{tu}\mathrm{C}_G(t.\sigma)$). So regular quasi-semi-simple elements of G^1 are dense in $T^\sigma U.\sigma$.

By proposition 1.2, we have $B = \mathcal{L}_\sigma(T)T^\sigma U$. Moreover, if $b.\sigma \in T^\sigma U.\sigma$ is regular quasi-semi-simple, then, for all $t \in T$, we have $\mathcal{L}_\sigma(t)b.\sigma = {}^{t^{-1}}b.\sigma$, so $\mathcal{L}_\sigma(t)b.\sigma$ is also regular quasi-semi-simple. Hence, regular quasi-semi-simple elements of G^1 are dense in $B.\sigma$.

Let B' be a Borel subgroup of G, then there is $x \in G$ such that $B' = {}^xB$ and we have $\mathrm{N}_G(B') \cap G^1 = {}^x(B.\sigma)$. Hence, for any quasi-borel $\mathrm{N}_G(B')$, quasi-semi-simple regular elements of G^1 are dense in $\mathrm{N}_G(B') \cap G^1$. Then, $\tilde{G}$ is the union of its quasi-borels ([15] proposition 7.2).

End of the proof of theorem 2.2.

Proposition 2.4.

(i) Let $g.\sigma$ be a regular quasi-semi-simple element of G^1, then $g.\sigma$ is conjugate under G to an element $t.\sigma$, where t is a regular element of G^σ.

(ii) Let $g.\sigma$ be a regular quasi-semi-simple element of G^1 then $g.\sigma$ belongs to a unique quasi-torus of $\tilde{G}$.

(iii) Let $t.\sigma$ be a regular quasi-semi-simple element of $T^\sigma.\sigma$ then $\mathcal{B}_{t.\sigma}^G = \{{}^wB; w \in W^\sigma\}$.

Proof: (i) Let $g.\sigma$ be a regular quasi-semi-simple element of G^1. Then, there is a pair (T', B') where T' is a maximal torus included in a Borel subgroup B' of G such that $g.\sigma$ belongs to the quasi-torus $\mathrm{N}_{\tilde{G}}(T', B')$. Let x be in G so that $T' = {}^xT$ and $B' = {}^xB$, then $\mathrm{N}_{\tilde{G}}(T', B') = {}^x\mathrm{N}_{\tilde{G}}(T, B)$. So, ${}^{x^{-1}}(g.\sigma)$ is a regular element of $\mathrm{N}_{\tilde{G}}(T, B) \cap G.\sigma = T.\sigma$. So, in order to prove (i), we can suppose that $g.\sigma$ is a regular element of $T.\sigma$.

By proposition 1.2, every regular element of $T.\sigma$ is conjugate to a regular element of $T^\sigma.\sigma$. Moreover, we have seen in 2.1 that $t.\sigma \in T^\sigma.\sigma$ is regular in G^1 if and only if t is regular in G^σ, whence the first assertion.

(ii) Let $g.\sigma$ be a regular quasi-semi-simple element of G^1. According to (i), $g.\sigma$ is then conjugate to $t.\sigma \in T^\sigma.\sigma$, where t is regular in G^σ. Moreover, $t.\sigma$ belongs to a unique quasi-torus if and only if this is also true for $g.\sigma$. Let $\mathrm{N}_{\tilde{G}}(T', B')$ be a quasi-torus containing $t.\sigma$. As $t.\sigma$ is a Jordan decomposition, t and σ are also in $\mathrm{N}_{\tilde{G}}(T', B')$. But, t is a regular semi-simple element of G^σ, so T^σ is the unique maximal torus of G^σ which contains t and $\mathcal{B}_t^{G^\sigma} = \{{}^wB^\sigma, w \in W^\sigma\}$ ([6] proposition 14.6).

So, $T'^\sigma = T^\sigma$ and $B'^\sigma = {}^wB^\sigma$. But, σ is quasi-central, hence $B' = {}^wB$ and $T' = T$, whence $\mathrm{N}_{\tilde{G}}(T', B') = \mathrm{N}_{\tilde{G}}(T, B)$, which proves (ii).

Let us prove (iii). Let $t.\sigma$ be a regular quasi-semi-simple element of $T^\sigma.\sigma$ then $t.\sigma$ is a Jordan decomposition. So, if $t.\sigma$ is in the quasi-borel $\mathrm{N}_{\tilde{G}}(B')$ then both t and σ are in $\mathrm{N}_{\tilde{G}}(B')$. Hence, the fact that $\mathcal{B}_t^{G^\sigma} = \{{}^wB^\sigma, w \in W^\sigma\}$ is sufficient to prove (ii) because σ is quasi-central and because the map $B' \mapsto (B')^\sigma$ is a bijection between σ-stable Borel subgroups of G and Borel subgroups of G^σ.

End of the proof.

3. Borel subgroups stabilised by an element of G^1

The purpose of this section is to study the variety $\mathcal{B}_{g.\sigma}^G$ of the Borel subgroups stabilised by conjugation under $g.\sigma \in G^1$. We need the following varieties:

$$Z = \{(g.\sigma, B_1, B_2) \in G^1 \times \mathcal{B}^G \times \mathcal{B}^G | g.\sigma \in N_1 \cap N_2\}$$

$$Z' = \{(g.\sigma, B_1, B_2) \in \mathcal{U}(G^1) \times \mathcal{B}^G \times \mathcal{B}^G | g.\sigma \in N_1 \cap N_2\},$$

where N_i is the normalizer of B_i in $\tilde{G}$.

Theorem 3.1.

(i) Let C be the conjugacy class of $g.\sigma$ in G^1, then

$$2\dim(\mathcal{B}_{g.\sigma}^G) + \dim(C) = \dim(G) - \mathrm{rk}(G^\sigma)$$

(ii) Irreductible components of Z (resp. Z') are parametrised by W^σ. Moreover, $\dim(Z) = \dim(G)$ *and* $\dim(Z') = \dim(G) - \mathrm{rk}(G^\sigma)$.

Proof:

(i) According to [14] (proposition II.10.15) we have $\dim(\mathrm{C}_G(g.\sigma)) = \mathrm{rk}(G^\sigma) + 2\dim(\mathcal{B}_{g.\sigma})$, for all $g.\sigma \in G^1$, whence the result.

(ii) Let $p : Z' \to \mathcal{B}^G \times \mathcal{B}^G$ be the projection on the second and the third factors. Let $\mathcal{O}_w$ be the G-orbit of $(B, {}^wB)$ in $\mathcal{B} \times \mathcal{B}$. Then we set $Z'_w = p^{-1}(\mathcal{O}_w)$.

Let us show that Z'_w is empty when w is not σ-stable. Let w be in W such that Z'_w is not empty. This means that there is $x \in G$ such that $\mathrm{N}_{\tilde{G}}({}^xB) \cap \mathrm{N}_{\tilde{G}}({}^{xw}B) \cap \mathcal{U}(G^1)$ is not empty, i.e. that $E = \mathrm{N}_{\tilde{G}}(B) \cap \mathrm{N}_{\tilde{G}}({}^wB) \cap \mathcal{U}(G^1)$ is not empty. Let $u.\sigma$ be in E then both u and $w^{-1}u\sigma(w)$ belong to U, so that $BwB \cap B\sigma(w)B$ is not empty and by the Bruhat decomposition we can conclude that $\sigma(w) = w$. So we have a partition of Z' into subsets Z'_w such that $w \in W^\sigma$.

Let us show that when w is σ-stable then Z'_w is an irreducible variety. Let (B_1, B_2) be in $\mathcal{O}_w$; then there is $x \in G$ such that $(B_1, B_2) = ({}^{xw_0}B, {}^{xw_0w}B)$, where we denote by w_0 the longest element of W. Let $V(B_1, B_2)$ be the subset of $\mathcal{O}_w$ defined by $\{({}^{xuw_0}B, {}^{xuw_0w}B); u \in U\}$. Then $(B_1, B_2) \in V(B_1, B_2)$ and $V(B_1, B_2)$ is isomorphic to U and is dense open in $\mathcal{O}_w$: $\{{}^{uw_0}B; u \in U\} \simeq Uw_0B/B$ is indeed a dense open subset of $\mathcal{B}^G = G/B$ as, according to [5], Bw_0B is an open dense subset of G. Let $\mathcal{L}_\sigma(T)$ be the image of T by the map $t \mapsto t^{-1}\sigma(t)$.

Let $\phi : U \times \mathcal{L}_\sigma(U \cap {}^wU) \to p^{-1}((B_1, B_2))$ be the morphism of varieties defined by $(u, b) \mapsto ({}^{xuw_0}(b.\sigma), {}^{xuw_0}B, {}^{xuw_0w}B)$. It is an isomorphism of varieties and the inverse morphism is defined by $(g.\sigma, B', B'') \mapsto (u, (xuw_0)^{-1}(g.\sigma)(xuw_0))$ where u is the unique element of U such that $B' = {}^{xuw_0}B$. So Z'_w is a locally trivial fibration over $\mathcal{O}_w \simeq G/(B \cap wBw^{-1})$ with fiber isomorphic to $\mathcal{L}_\sigma(T)(U \cap {}^wU)$. Whence Z'_w is an irreducible variety and $\dim(Z'_w) = \dim(G) - \dim(T^\sigma)$. All the varieties Z'_w have the same dimension so $\dim(Z') = \dim(G) - \dim(T^\sigma)$ and the varieties $\overline{Z'_w}$ for $w \in W^\sigma$ are the irreducible components of Z'.

In the case of Z, the proof is similar if we take, instead of Z'_w, the inverse image Z_w of $\mathcal{O}_w$ by the projection $Z \to \mathcal{B}^G \times \mathcal{B}^G$. The varieties Z_w for $w \in W^\sigma$ are then locally trivial fibration over $\mathcal{O}_w$ with fiber isomorphic to $B \cap {}^wB$.

End of the proof.

Corollary 3.2.

(i) For all $i > 0$, we have

$$\dim\{g.\sigma \in G^1 | \dim(\mathcal{B}^G_{g.\sigma}) \geq i/2\} < \dim(G) - i.$$

(ii) For all $i \geq 0$, we have

$$\dim\{g.\sigma \in \mathcal{U}(G^1) | \dim(\mathcal{B}^G_{g.\sigma}) \geq i/2\} \leq \dim(G) - i.$$

Proof: Let us prove (i). We suppose there exists $i > 0$ such that (i) is false. We consider $V = \{g.\sigma \in G^1 | \dim(\mathcal{B}^G_{g.\sigma}) \geq i/2\}$. We have $\dim(V) \geq \dim(G) - i$ and V is a closed subvariety of G^1 which is stable by conjugation under G. We consider the variety Z of theorem 3.1. Let $p_1 : Z \to G^1$ be the first projection. We have $\dim(p_1^{-1}(V)) \geq \dim(G) - i + 2 \times i/2 = \dim(G)$. But, according to 3.1, $\dim(Z) = \dim(G)$, so $p_1^{-1}(V)$ is a subvariety of Z of the same dimension as Z. Hence, $p_1^{-1}(V)$ contains a irreducible component of Z, that is to say, there is $w \in W^\sigma$ such that $\overline{Z_w} \subset p_1^{-1}(V)$, where

$$Z_w = \{(g.\sigma, xB, xwB) \in G^1 \times (G/B) \times (G/wB) | g.\sigma \in \mathrm{N}_{\tilde{G}}({}^xB) \cap \mathrm{N}_{\tilde{G}}({}^{xw}B)\}.$$

Let $t.\sigma$ be a regular element of $T.\sigma$, then $(t.\sigma, B, wB) \in Z_w$ because $T.\sigma \subset \mathrm{N}_{\tilde{G}}(B) \cap \mathrm{N}_{\tilde{G}}({}^wB)$. So, we have $(t.\sigma, B, wB) \in p_1^{-1}(V)$, which means that $t.\sigma \in V$. But, as $t.\sigma$ is regular then $\dim(\mathcal{B}^G_{t.\sigma}) = 0$ (proposition 2.1) which contradicts the definition of V.

The proof of the second assertion is similar. We suppose that there exists $i \geq 0$ such that (ii) is false. We consider $V' = \{g.\sigma \in G^1 | \dim(\mathcal{B}^G_{g.\sigma}) \geq i/2\}$. and $p'_1 : Z' \to \mathcal{U}(G^1)$ be the projection on the first factor. Then, we have $\dim({p'_1}^{-1}(V')) > \dim(G) - \dim(T^\sigma) = \dim(Z')$ which is obviously false.

End of the proof.

4. Springer representations

Let us consider the following variety:

$$\tilde{G}^1 = \{(g.\sigma, xB) \in G^1 \times (G/B) | x^{-1}(g.\sigma)x \in B\},$$

and $\pi : \tilde{G}^1 \to G^1$ the projection on the first factor. In this section, we construct a representation of W^σ on the direct image $\pi_*\overline{\mathbb{Q}}_l$ of the constant sheaf $\overline{\mathbb{Q}}_l$ on $\tilde{G}^1$ (proposition 4.5), which we call the Springer representation of W^σ from G^1. Moreover, by proposition 4.7, it induces an action of W^σ on any cohomology group $\mathrm{H}^i(\mathcal{B}^G_{g.\sigma}, \overline{\mathbb{Q}}_l)$.

Proposition 4.1.
(i) $\tilde{G}^1$ *is a smooth variety.*
(ii) π *is a proper morphism.*
(iii) $\pi^{-1}(\mathcal{U}(G^1))$ *is a smooth variety and* $\dim(\pi^{-1}(\mathcal{U}(G^1))) = \dim(G) - \mathrm{rk}(G^\sigma)$.

Proof:
(i) The projection $p_2 : \tilde{G}^1 \to G/B$ is a locally trivial fibration with fiber isomorphic to B. Hence, $\tilde{G}^1$ is a smooth variety.

(ii) $\tilde{G}^1$ is a closed subset of $G^1 \times (G/B)$. Hence, $\tilde{G}^1 \to G^1 \times (G/B)$ is a proper morphism. Moreover, the projection on the first factor $G^1 \times (G/B) \to G^1$ is proper (it is a universally closed morphism because G/B is complete and it is obviously separate). Hence, $\tilde{G}^1 \to G^1$ is proper.

(iii) $\pi^{-1}(\mathcal{U}(G^1))$ is a locally trivial fibration over G/B with fiber isomorphic to $\mathcal{L}_\sigma(T)U$ via the second projection, whence the result.

End of the proof.

We denote by G^1_{reg} the set of regular quasi-semi-simple elements of G^1. Let us consider the following variety:

$$\tilde{G}^1_{reg} = \{(g.\sigma, xT) \in G^1 \times (G/T) | x^{-1}(g.\sigma)x \in T^1_{reg}\}.$$

Proposition 4.2. *The map* $\phi : \tilde{G}^1_{reg} \to \pi^{-1}(G^1_{reg})$ *defined by* $(g.\sigma, xT) \mapsto (g.\sigma, xB)$ *is an isomorphism of varieties.*

Proof: Let us first show that ϕ is injective. Let $(g.\sigma, xT)$ and $(g.\sigma, x'T)$ be in $\tilde{G}^1_{reg}$ such that $xB = x'B$. Let us show that $xT = x'T$. The regular quasi-semi-simple element $g.\sigma$ is in both quasi-tori $\mathrm{N}_{\tilde{G}}({}^xT, {}^xB)$ and $\mathrm{N}_{\tilde{G}}({}^{x'}T, {}^{x'}B)$. So, $\mathrm{N}_{\tilde{G}}({}^xT, {}^xB) = \mathrm{N}_{\tilde{G}}({}^{x'}T, {}^{x'}B)$ (proposition 2.4), which means $x^{-1}x' \in \mathrm{N}_G(T)$. But, we also have $xB = x'B$, so $x^{-1}x' \in T$, which prove the injectivity of ϕ.

Let us now show that ϕ is surjective. Let $(g.\sigma, xB)$ be in $\pi^{-1}(G^1_{reg})$, let us show that there exists $z \in xB$ such that $(g.\sigma, zT) \in \tilde{G}^1_{reg}$. By hypothesis, $g.\sigma$ is a quasi-semi-simple element of $\mathrm{N}_{\tilde{G}}({}^xB)$ which is regular in G^1. By proposition 2.4, there are $t \in T^\sigma$ regular in G^σ and $y \in G$ such that ${}^y(g.\sigma) = t.\sigma$. Thus, $t \in T^\sigma$ is regular in G^σ and ${}^{y^{-1}x}B \in \mathcal{B}^G_{t.\sigma}$. So, there exists $w \in W^\sigma$ such that ${}^{y^{-1}x}B = {}^wB$ (proposition 2.4, (iii)). Let us set $z = y\tilde{w}$ where $\tilde{w}$ is a representative of w in $\mathrm{N}_G(T)^\sigma$. Then $z^{-1}g.\sigma z = w^{-1}y^{-1}g.\sigma yw \in T.\sigma$, hence, $(g.\sigma, zT) \in \tilde{G}^1_{reg}$.

Let ψ be the inverse map of ϕ. Let us show that ψ is a morphism of varieties. Let $(g.\sigma, xB)$ be in $\tilde{G}^1_{reg}$ and let $(g.\sigma, zT)$ be the image of $(g.\sigma, xB)$ by ψ. Then, $\psi(({}^{x^{-1}}(g.\sigma), B)) = ({}^{x^{-1}}(g.\sigma), x^{-1}zT)$, that is to say that ${}^{x^{-1}z}T$ is the unique torus of B which is normalised by ${}^{x^{-1}}(g.\sigma)$. Thus, the problem is to prove that the map f, which associates to each element of $B.\sigma \cap G^1_{reg}$ the unique torus of B it normalises, is a morphism of varieties.

Let $b.\sigma$ be a regular quasi-semi-simple element of $B.\sigma$. Then, there is a unique pair $(u,t) \in T \times U$ such that $ut.\sigma = b.\sigma$. Let us show that $t.\sigma$ is also regular. Applying lemma 2.3, the map $U \to U$ defined by $v \mapsto v^{-1}\,{}^{b.\sigma}v$ is surjective. So, there is $v \in U$ such that $v^{-1}\,{}^{b.\sigma}v = u^{-1}$ and we have $t.\sigma = u^{-1}b.\sigma = v^{-1}\,{}^{b.\sigma}v b.\sigma = v^{-1}b.\sigma v$. Hence, $t.\sigma$ is regular. We then need the following lemma:

Lemma 4.3. *Let* $t.\sigma$ *a regular element of* $T.\sigma$*, then the map* $h_t : U \to U$ *defined by* $u \mapsto u^{-1}\,{}^{t.\sigma}u$ *is an isomorphism of varieties.*

Proof of lemma 4.3: According to lemma 2.3, h_t is surjective. Moreover, applying proposition 1.2, we can write $t = t_1t_2^{-1}\sigma(t_2)$ where $t_1 \in T^\sigma$ and $t_2 \in T$. So, we have $\mathrm{C}_{\tilde{G}}(t.\sigma) = {}^{t_2^{-1}}\mathrm{C}_{\tilde{G}}(t_1.\sigma) = T^\sigma$. Then, $U^{t.\sigma} = 1$, which shows that h_t is injective. As h_t is a bijective morphism $U \to U$ and U is an irreducible variety, we only have to check that h_t is separable and that U is normal. As U is smooth, U is normal. Let us show that h_t is separable. According to [9] (page 44), as h_t is a dominant morphism of irreducible varieties, it is sufficient to check that $(dh_t)_0 : \mathcal{T}(U)_0 \to \mathcal{T}(U)_0$ is surjective. The tangent space $\mathcal{T}(U)_0$ is isomorphic to $k^{|\phi^+|}$. Let $u = \prod_{\alpha\in\Phi^+} x_\alpha(k_\alpha)$ be an element of U. For all $\alpha \in \Phi^+$, there is $c_\alpha \in K^\times$ such that $\sigma(x_\alpha(k_\alpha)) = x_{\sigma(\alpha)}(c_\alpha k_\alpha)$ for all $k_\alpha \in K$. Then, $h_t(u) = \prod_{\alpha\in\Phi^+} x_\alpha(k_\alpha - c_{\sigma^{-1}(\alpha)}\alpha(t)k_{\sigma^{-1}(\alpha)})u_*$, where u_* is a product of commutators. Hence, $(dh_t)_0$ is the linear map defined by $(k_\alpha)_{\alpha\in\Phi^+} \mapsto (k_\alpha - c_{\sigma^{-1}(\alpha)}\alpha(t)k_{\sigma^{-1}(\alpha)})_{\alpha\in\Phi^+}$. Let us reorganize the coefficients k_α in σ-orbits. The matrix corresponding to h_t is then decomposed into blocks which look like:

$$A(\alpha) = \begin{pmatrix} 1 & & & & \dots & -c_{\sigma^{-1}(\alpha)}\alpha(t_1) \\ -c_\alpha\alpha(t_1) & 1 & & & \dots & \\ & -c_{\sigma(\alpha)}\alpha(t_1) & 1 & & \dots & \\ & & & & \dots & \\ & & & -c_{\sigma^{l-2}(\alpha)}\alpha(t_1) & & 1 \end{pmatrix},$$

where l is the cardinal number of the σ-orbit of α, and $t = t_1\mathcal{L}_\sigma(t_2)$, with $t_1 \in T^\sigma$ regular in G^σ. Then, $\det(A(\alpha)) = 1 - c_\alpha ... c_{\sigma^{-1}(\alpha)}\alpha(t_1)^l$, and, as σ is quasi-central, we have $c_\alpha ... c_{\sigma^{-1}(\alpha)} = 1$, according to [7] (theorem 1.15). However, t_1 is a regular semi-simple element of G^σ, hence, $\alpha(t_1) \neq 1$ ([6] proposition 14.16). Moreover, t_1 is semi-simple, σ is unipotent and l divides the order of σ. If r is the order of t_1 then l and r are relatively prime, and according to Bezout, there are two integers a and b such that $al + br = 1$. Thus, we have $\alpha(t_1)^{al} = \alpha(t_1)^{1-br} = \alpha(t_1) \neq 1$, so, $\alpha(t_1)^l \neq 1$. We have then proved that the determinant of $A(\alpha)$ is not equal to zero. Then, the jacobian at 0 of h_t is a product of matrices with non zero determinant, so that $(dh_t)_0$ is surjective.

End of the proof of lemma 4.3.

Let $\pi_{reg} : \pi^{-1}(G^1_{reg}) \to G^1_{reg}$ be the restriction of π to $\pi^{-1}(G^1_{reg})$. The group W^σ of σ-stable elements in the Weyl group W of G acts on $\tilde{G}^1_{reg}$ on the right by $w : (g.\sigma, xT) \mapsto (g.\sigma, xwT)$ and then acts on $\pi^{-1}(G^1_{reg})$ via the isomorphism of proposition 4.2.

Proposition 4.4. *$\pi_{reg} : \pi^{-1}(G^1_{reg}) \to G^1_{reg}$ is an unramified covering with Galois group W^σ.*

Proof: The map $\pi_{reg} : \pi^{-1}(G^1_{reg}) \to G^1_{reg}$ is a finite morphism of varieties whose fibers have the same cardinality as W^σ. Indeed, if $g.\sigma \in G^1_{reg}$, then $\pi^{-1}(g.\sigma)$ is conjugated to $\{(t_0.\sigma, {}^wB); w \in W^\sigma\}$, where t_0 is a regular element of T^σ. Thus, according to [11] (proposition 2.9), $\pi_{reg} : \pi^{-1}(G^1_{reg}) \to G^1_{reg}$ is a flat morphism of varieties. As π_{reg} is also surjective, it is faithfully flat.

The group W^σ acts on $\pi^{-1}(G^1_{reg})$ via the isomorphism $\tilde{G}^1_{reg} \to \pi^{-1}(G^1_{reg})$ of proposition 4.2 and the action of W^σ on $\tilde{G}^1_{reg}$. This is an action on the right over G^1_{reg} (indeed π_{reg} is W^σ-invariant with respect to the trivial action of W^σ on G^1_{reg}). Moreover, the map ϕ which sends $(w, (g.\sigma, xT)) \in W^\sigma \times \tilde{G}^1_{reg}$ to $((g.\sigma, xT), (g.\sigma, xwT))$ is an isomorphism from $W^\sigma \times \tilde{G}^1_{reg}$ to $\tilde{G}^1_{reg} \times_{G^1_{reg}} \tilde{G}^1_{reg}$. Indeed, for all $g.\sigma \in G^1_{reg}$, there exists $x \in G$ such that $x^{-1}g.\sigma x \in T^1_{reg}$ and $\{yT | y^{-1}g.\sigma y \in T^1_{reg}\} = \{xwT | w \in W^\sigma\}$. Moreover, W^σ acts faithfully and transitively on this set.

End of the proof.

Let us denote by $\mathcal{L} = (\pi_{reg})_*\overline{\mathbb{Q}}_l$ the direct image of the constant sheaf $\overline{\mathbb{Q}}_l$ on $\pi^{-1}(G^1_{reg})$. We can deduce from the previous proposition that W^σ acts on the local system $\mathcal{L} = (\pi_{reg})_*\overline{\mathbb{Q}}_l$ on G^1_{reg}.

By theorem 2.2, the set G^1_{reg} of regular quasi-semi-simple elements is dense in G^1. Let us now consider the intersection cohomology complex $\mathrm{IC}(G^1,\mathcal{L})$ on $\overline{G^1}_{reg} = G^1$ associated with $\mathcal{L}$.

Definition 4.1. Let X be an algebraic variety over K. Let $Y \subset X$ be a locally closed smooth irreducible subvariety of dimension d of X. We can associate to each local system $\mathcal{L}$ on Y an intersection cohomology complex $\mathrm{IC}(\overline{Y},\mathcal{L})[d]$ and $S = \mathrm{IC}(\overline{Y},\mathcal{L})[d]$ is then a perverse sheaf on Y which is characterized by the following properties:
(i) $\mathcal{H}^iS = 0$ if $i < -d$,
(ii) $\mathcal{H}^{-d}S|_Y = \mathcal{L}$,
(iii) $\dim \mathrm{Supp}\mathcal{H}^iS < -i$ if $i > -d$,
(iv) $\dim \mathrm{Supp}\mathcal{H}^iDS < -i$ if $i > -d$.

As IC is a faithful functor from the category of local systems on G^1_{reg} to the category of perverse sheaves, the action of W^σ on $\mathcal{L}$ induces an action of W^σ on $\mathrm{IC}(G^1,\mathcal{L})[\dim(G)]$. Then, by the following proposition we get an action of W^σ on $\pi_*(\overline{\mathbb{Q}}_l)$ which we call the Springer representation of W^σ from G^1.

Proposition 4.5. *Let us set $n = \dim(G)$, then,*

$$\mathrm{IC}(G^1,\mathcal{L})[n] \simeq \pi_*(\overline{\mathbb{Q}}_l)[n] \text{ in } \mathcal{M}G^1,$$

where, $\pi_*(\overline{\mathbb{Q}}_l)$ is the direct image of the constant sheaf $\overline{\mathbb{Q}}_l$ on $\tilde{G^1}$ by π.

In order to prove the above proposition, we will need the following definition and the following proposition:

Definition 4.2. Let $\phi : Z \to X$ be a proper morphism between two varieties of same dimension n. If X is smooth, ϕ is said semi-small, if for all i, we have

$$\dim\{a \in X | \dim(\phi^{-1}(a)) \geq i/2\} \leq \dim(X) - i,$$

The morphism ϕ is said small if the last inequality is strict.

Proposition 4.6. *Let $\phi : Z \to X$ be a morphism between two varieties of same dimension n. We suppose that X and Z are smooth. Let us consider the constant sheaf $\overline{\mathbb{Q}}_l$ on Z, then, if ϕ is semi-small, $\phi_*\overline{\mathbb{Q}}_l[n]$ is a perverse sheaf on X, if ϕ is small, $M = \phi_*\overline{\mathbb{Q}}_l[n]$ verify the following properties:*
(i) $\mathcal{H}^iM = 0$ if $i < -n$,
(ii) $\dim \mathrm{Supp}\mathcal{H}^iM < -i$ *if $i > -n$,*
(iii) $\dim \mathrm{Supp}\mathcal{H}^iDM < -i$ *if $i > -n$.*

Proof: As the functor ϕ_* is left exact and preserves injectives, the sheaves $\mathcal{H}^i(\phi_*(\overline{\mathbb{Q}}_l))$ and $R^i\phi_*(\overline{\mathbb{Q}}_l)$ are isomorphic.

Let us first suppose that ϕ is small. Let us show that M verify (i). For all $x \in X$, the fiber $R^i\phi_*(\overline{\mathbb{Q}}_l)_x$ at x of $R^i\phi_*(\overline{\mathbb{Q}}_l)$ is isomorphic to $\mathrm{H}^i(\phi^{-1}(x),\overline{\mathbb{Q}}_l)$, so we have $R^i\phi_*(\overline{\mathbb{Q}}_l)_x[n] \simeq \mathrm{H}^{i+n}(\phi^{-1}(x),\overline{\mathbb{Q}}_l)$, for all $x \in X$. Now, we have $\mathrm{H}^i(\phi^{-1}(x),\overline{\mathbb{Q}}_l) = 0$ for all $i < 0$ and all $x \in X$. So, $\mathcal{H}^{i+n}(\phi_*(\overline{\mathbb{Q}}_l)_x) = 0$ for all $i < -n$ and for all $x \in X$, which leads to $\mathcal{H}^{i+n}(\phi_*(\overline{\mathbb{Q}}_l)) = 0$ for all $i < -n$.

Let us now check (ii) and (iii). As Z is a smooth variety of dimension n, we have $D(\overline{\mathbb{Q}}_l[n]) = \overline{\mathbb{Q}}_l[n]$, where D is the Verdier duality. Moreover, as ϕ is proper, D commute with ϕ_* ([4] théoréme 10.11). Thus, we have $D(M) = M$. So, we just need to check (ii). Let us recall that $R^i\phi_*(\overline{\mathbb{Q}}_l)_x \simeq \mathrm{H}^i(\phi^{-1}(x),\overline{\mathbb{Q}}_l)$. If $\mathrm{H}^i(\phi^{-1}(x),\overline{\mathbb{Q}}_l) \neq 0$, then, $i \leq 2\dim(\phi^{-1}(x))$. So, $\mathrm{Supp}\mathcal{H}^i(\phi_*(\overline{\mathbb{Q}}_l)[n])$ is a subset of $\{x \in X | i \leq 2\dim(\phi^{-1}(x))\}$. Thus, (ii) and (iii) are direct consequences of the fact that ϕ is small.

The same proof shows that if ϕ is semi-small, then $\phi_*\overline{\mathbb{Q}}_l[n]$ is a perverse sheaf.

End of the proof.

Proof of proposition 4.5: Let us check that $S' = \pi_*(\overline{\mathbb{Q}}_l)[n]$ satisfies properties (i) to (iv) which characterize $S = \mathrm{IC}(G^1, \mathcal{L})[\dim(G)]$. The map $\pi : \tilde{G}^1 \to G^1$ is a proper morphism between two varieties of the same dimension and $\tilde{G}^1$ is smooth. Moreover, for all $x.\sigma \in G^1$, we have $\pi^{-1}(x.\sigma) \simeq \mathcal{B}^G_{x.\sigma}$, so, by corollary 3.2, π is a small morphism. Moreover, $\tilde{G}^1$ is a smooth variety (proposition 4.1). The hypotheses of proposition 4.6 are satisfied, which proves (i), (iii) and (iv).

Let us show (ii). Let us consider the following cartesian diagram:

$$\begin{array}{ccc} \pi^{-1}(G^1_{reg}) & \xrightarrow{\pi_{reg}} & G^1_{reg} \\ \downarrow i & & \downarrow i' \\ \tilde{G}^1 & \xrightarrow{\pi} & G^1 \end{array},$$

where i and i' are inclusions. According to the proper base change theorem applied to the proper morphism $\pi : \tilde{G}^1 \to G^1$, we have $i'^* R^0\pi_*(\overline{\mathbb{Q}}_l) \simeq \pi_{reg*}(\overline{\mathbb{Q}}_l)$. Moreover, $\mathcal{H}^{-n}K|_{G^1_{reg}}$ is isomorphic to $i'^* R^0\pi_*(\overline{\mathbb{Q}}_l) = \pi_{reg*}(\overline{\mathbb{Q}}_l) = \mathcal{L}$.

End of the proof of proposition 4.5.

Proposition 4.7. *For all $g.\sigma \in G^1$, the group W^σ acts on $\mathrm{H}^i(\mathcal{B}^G_{g.\sigma}, \overline{\mathbb{Q}}_l)$.*

Proof: According to the previous proposition, W^σ acts on $\pi_*(\overline{\mathbb{Q}}_l)[n]$ and then on any stalk $\mathcal{H}^i(\pi_*(\overline{\mathbb{Q}}_l)[n])_{g.\sigma}$, which is isomorphic to $\mathrm{H}^i(\mathcal{B}_{g.\sigma}, \overline{\mathbb{Q}}_l)$.

End of the proof.

5. The action of W^σ on $\mathrm{H}^*(\mathcal{B}_\sigma, \overline{\mathbb{Q}}_l)$

From the previous section, we have an action of W^σ on $\mathrm{H}^*(\mathcal{B}^G_\sigma, \overline{\mathbb{Q}}_l)$, where $\mathcal{B}^G_\sigma$ is the variety of σ-stable Borel subgroups of G. We will call this representation the Springer representation from G^1.

On another part, we can apply to G^σ the construction of the Springer representation for a connected reductive group described in [13] (section 4), we then have an action of the Weyl group $W(G^\sigma)$ of G^σ on $\mathrm{H}^*(\mathcal{B}^{G^\sigma}, \overline{\mathbb{Q}}_l)$, where $\mathcal{B}^{G^\sigma}$ is the variety of Borel subgroups of G^σ. We will call this representation the Springer representation from G^σ.

As σ is unipotent quasi-semi-simple, we have $W(G^\sigma) \simeq W^\sigma$ and $\mathcal{B}^G_\sigma \simeq \mathcal{B}^{G^\sigma}$. Till the end of this article, we will identify the groups $\mathrm{H}^*(\mathcal{B}^G_\sigma, \overline{\mathbb{Q}}_l)$ and $\mathrm{H}^*(\mathcal{B}^{G^\sigma}, \overline{\mathbb{Q}}_l)$ and the Weyl groups $W(G^\sigma)$ and W^σ. Hence, we get two Springer representations of W^σ on $\mathrm{H}^*(\mathcal{B}^G_\sigma, \overline{\mathbb{Q}}_l)$, one from G^1 and one from G^σ. We show in proposition 5.1 below that these representations coincide.

We set

$$\tilde{G}^\sigma = \{(g, B') \in G^\sigma \times \mathcal{B}^\sigma | g \in B'\},$$

and we denote by $\pi' : \tilde{G}^\sigma \to G^\sigma$ the projection on the first factor. Let us consider the set G^σ_{reg} of regular semi-simple elements of G^σ. Let $\tilde{G}^\sigma_{reg}$, π'_0 and $\tilde{i}'$ be defined by the following cartesian diagram:

$$\begin{array}{ccc} \tilde{G}^\sigma_{reg} & \xrightarrow{\tilde{i}'} & \tilde{G}^\sigma \\ \downarrow \pi'_0 & & \downarrow \pi' \\ G^\sigma_{reg} & \xrightarrow{i'} & (G^\sigma)^0 \end{array},$$

Let $\mathcal{L}'$ be the local system $(\pi'_0)_*\overline{\mathbb{Q}}_l$ on G^σ_{reg}. As in the previous section, we have

$$\mathrm{IC}(G^\sigma, \mathcal{L}')[n'] \simeq \pi'_*(\overline{\mathbb{Q}}_l)[n'] \text{ in } \mathcal{M}(\mathrm{G}^\sigma),$$

where $n' = \dim(G^\sigma)$. This induces the Springer representation of W^σ from G^σ on $\mathrm{H}^*(\mathcal{B}^G_\sigma, \overline{\mathbb{Q}}_l)$.

Proposition 5.1. *The Springer representation of W^σ on $\mathrm{H}^*(\mathcal{B}_\sigma, \overline{\mathbb{Q}}_l)$ from G^1 and that from G^σ coincide.*

Proof: Let us consider the following diagram:

$$\begin{tikzcd} \tilde{G^\sigma} \arrow[r, "\tilde{j}"] \arrow[d, "\pi'"] & \tilde{G^1} \arrow[d, "\pi"] \\ G^\sigma \arrow[r, "j"] & G^1 \end{tikzcd},$$

where j is defined by $g \mapsto g.\sigma$ and $\tilde{j}$ is defined by $(g, \tilde{B}) \mapsto (g.\sigma, B)$ where B is the only σ-stable Borel subgroup of G such that $B^\sigma = \tilde{B}$. It is not a cartesian diagram but we have $\tilde{G^\sigma} \simeq (\tilde{G^1})^\sigma$. As $\tilde{j}_*$ and $\tilde{j}^*$ are adjoint, there exists an isomorphism $\alpha : \mathrm{Hom}_{\mathcal{F}(\tilde{G^\sigma})}(\overline{\mathbb{Q}}_l, \overline{\mathbb{Q}}_l) \simeq \mathrm{Hom}_{\mathcal{F}(\tilde{G^1})}(\overline{\mathbb{Q}}_l, \tilde{j}_*\overline{\mathbb{Q}}_l)$ (where $\mathcal{F}(H)$ is the category of sheaves on H). Let us consider the adjunction morphism $\alpha(\mathrm{Id}_{\overline{\mathbb{Q}}_l}) : \overline{\mathbb{Q}}_l \to \tilde{j}_*\overline{\mathbb{Q}}_l$. By functoriality, it induces a morphism of sheaves $\phi : \pi_*\overline{\mathbb{Q}}_l \to \pi_*\tilde{j}_*\overline{\mathbb{Q}}_l$, and as $\pi \circ \tilde{j} = j \circ \pi'$, ϕ is a morphism of sheaves $\pi_*\overline{\mathbb{Q}}_l \to j_*\pi'_*\overline{\mathbb{Q}}_l$.

Let us consider the following cartesian diagram:

$$\begin{tikzcd} \tilde{G}^1_{reg} \arrow[r, "\tilde{i}"] \arrow[d, "\pi_{reg}"] & \tilde{G^1} \arrow[d, "\pi"] \\ G^1_{reg} \arrow[r, "i"] & G^1 \end{tikzcd},$$

We still denote by $\mathcal{L}$ the local system $\pi_{reg*}\overline{\mathbb{Q}}_l$. According to proposition 4.5, $\pi_*\overline{\mathbb{Q}}_l[n]$ is isomorphic to $i_{!*}\mathcal{L}[n]$ in $\mathcal{M}(G^1)$. In the following lemma, we prove a similar result for $j_*\pi'_*\overline{\mathbb{Q}}_l[n]$.

Lemma 5.2. *$j_*\pi'_*\overline{\mathbb{Q}}_l[n] \simeq i_{!*}j_{reg*}\mathcal{L}'[n]$ in $\mathcal{M}(G^1)$, where $\mathcal{L}' = \pi'_{0*}\overline{\mathbb{Q}}_l$.*

Proof of lemma 5.2: Let us consider the following diagram:

$$\begin{tikzcd} G^\sigma_{reg} \arrow[r, "i'"] \arrow[d, "j_{reg}"] & G^\sigma \arrow[d, "j"] \\ G^1_{reg} \arrow[r, "i"] & G^1 \end{tikzcd}.$$

The morphism j_{reg} is the open immersion defined by $g \mapsto g.\sigma$, indeed, according to 2.4, $g.\sigma \in (G^\sigma)^0$ is quasi-semi-simple regular if and only if g is semi-simple regular in $(G^\sigma)^0$. We need the following lemma:

Lemma 5.3. *The functors $T : \mathcal{D}((G^\sigma)^0) \to \mathcal{D}(G^1)$ and $T_{reg} : \mathcal{D}(G^\sigma_{reg}) \to \mathcal{D}(G^1_{reg})$, defined by $T(S) := j_*S[n-n']$ for all $S \in \mathcal{D}((G^\sigma)^0)$ and $T_{reg}(S) := j_{reg*}S[n-n']$ for all $S \in \mathcal{D}(G^\sigma_{reg})$, are t-exact.*

Proof of lemma 5.3: As j and j_{reg} are proper morphisms, j_* and j_{reg*} are exact functors and they commute with the Verdier duality. Thus, it is sufficient to check that for all perverse sheaf S on $(G^\sigma)^0$ (resp. G^σ_{reg}), we have $\dim(\mathrm{Supp}(\mathcal{H}^i(T(S)))) \leq -i$ (resp. $\dim(\mathrm{Supp}(\mathcal{H}^i(T_{reg}(S)))) \leq -i$) for all $i \geq -n$. We check below the case of the functor T, the case of T_{reg} is similar.

For all $i \geq -n$,

$$\begin{aligned}
\dim(\mathrm{Supp}(\mathcal{H}^i(T(S)))) &= \dim(\mathrm{Supp}(\mathcal{H}^i(j_*S[n-n']))) \\
&= \dim(\mathrm{Supp}(\mathcal{H}^{i+n-n'}(j_*S))) \\
&= \dim(\mathrm{Supp}(\mathcal{H}^{i+n-n'}(S))) \\
&\leq -i-(n-n') \\
&\quad (\text{because } \mathrm{i}+\mathrm{n}-\mathrm{n}' \geq -\mathrm{n}') \\
&\leq -i
\end{aligned}$$

End of the proof of lemma 5.3.

Thus, by the previous lemma, $j_*\pi'_*\overline{\mathbb{Q}}_l[n] = T(\pi'_*\overline{\mathbb{Q}}_l[n'])$ is a perverse sheaf on G^1 and T induces an exact functor from $\mathcal{M}((G^\sigma)^0)$ to $\mathcal{M}(G^1)$ denoted by ${}^pT$ and defined by ${}^p\mathrm{H}^0 \circ T \circ \epsilon$, where ${}^p\mathrm{H}^0 = \tau_{\geq 0}\tau_{\leq 0}$ and ϵ is the inclusion of the category of perverse sheaves into the derived category ([1] proposition 1.3.17).

$$\begin{aligned}
j_*\pi'_*\overline{\mathbb{Q}}_l[n] &= {}^pT(i'_{!*}\mathcal{L}'[n']) \\
&= {}^pT(\mathrm{Im}({}^pi'_!\mathcal{L}'[n'] \to {}^pi'_*\mathcal{L}'[n'])) \\
&= \mathrm{Im}({}^pT{}^pi'_!\mathcal{L}'[n'] \to {}^pT{}^pi'_*\mathcal{L}'[n'])
\end{aligned}$$

Thus, in order to prove lemma 5.2, it is sufficient to show that ${}^pT{}^pi'_!\mathcal{L}'[n'] = {}^pi_!(j_{reg*}\mathcal{L}'[n])$ and that ${}^pT{}^pi'_*\mathcal{L}'[n'] = {}^pi_*(j_{reg*}\mathcal{L}'[n])$.

Let us show the first equality, the proof of the second one is similar.

$$\begin{aligned}
{}^pT{}^pi'_!\mathcal{L}'[n'] &= {}^pT \circ {}^pi'_!(\mathcal{L}'[n']) \\
&= {}^p(T \circ i'_!)(\mathcal{L}'[n']) \\
&\quad ([1] \text{ proposition } 1.3.17) \\
&= {}^p(i_! \circ T_{reg})(\mathcal{L}'[n']) \\
&\quad (\text{because } j \circ i' = i \circ j_{reg}) \\
&= {}^pi_!{}^pT_{reg}\mathcal{L}'[n'] \\
&\quad ([1] \text{ proposition } 1.3.17) \\
&= {}^pi_!(j_{reg*}\mathcal{L}'[n])
\end{aligned}$$

End of the proof of lemma 5.2.

So, the morphism $\phi[n] : \pi_*\overline{\mathbb{Q}}_l[n] \to \pi_*\tilde{j}_*\overline{\mathbb{Q}}_l[n]$ is a morphism between two extensions of perverse sheaves on G^1_{reg}. But the functor $i_{!*}$ induces an equivalence of categories from $\mathcal{M}(G^1_{reg})$ to $\{i_{!*}A'; A' \in \mathcal{M}(G^1_{reg})\}$ whose reciprocal functor is i^* ([1] proposition 2.1.9). Thus, $i_{!*}$ is a fully faithful functor ($i_{!*} : \mathrm{Hom}(A', B') \to \mathrm{Hom}(i_{!*}A', i_{!*}B')$ is an isomorphism with reciprocal morphism i^*) and in order to show that $\phi[n]$ is W^σ-equivariant, it is sufficient to show that this is also true for its restriction to G^1_{reg}. Thus, we have to show that $i^*\pi_*\alpha(\mathrm{Id}_{\overline{\mathbb{Q}}_l}) : i^*\pi_*\overline{\mathbb{Q}}_l \to i^*\pi_*\tilde{j}_*\overline{\mathbb{Q}}_l$ is W^σ-equivariant, that is to say, by the proper base change theorem applied to proper morphism π, that the morphism $\pi_{reg*}\tilde{i}^*\alpha(\mathrm{Id}_{\overline{\mathbb{Q}}_l}) : \pi_{reg*}\overline{\mathbb{Q}}_l \to \pi_{reg*}\tilde{i}^*\tilde{j}_*\overline{\mathbb{Q}}_l$ is W^σ-equivariant. Nevertheless, π_{reg} is a galoisian covering with Galois group W^σ and the actions of W^σ on $\pi_{reg*}\overline{\mathbb{Q}}_l$ and on $\pi_{reg*}\tilde{i}^*\tilde{j}_*\overline{\mathbb{Q}}_l$ come from those on $\overline{\mathbb{Q}}_l$ and on $\tilde{i}^*\tilde{j}_*\overline{\mathbb{Q}}_l$. So we just have to check that $\tilde{i}^*\alpha(\mathrm{Id}_{\overline{\mathbb{Q}}_l})$ is

W^σ-equivariant. Let us consider the following cartesian diagram:

$$\begin{array}{ccc} \tilde{G}^\sigma_{reg} & \xrightarrow{\tilde{i}'} & \tilde{G}^\sigma \\ \Big\downarrow \tilde{j}_{reg} & & \Big\downarrow \tilde{j} \\ \tilde{G}^1_{reg} & \xrightarrow{\tilde{i}} & \tilde{G}^1 \end{array}$$

The functors $\tilde{j}_{reg*}$ and $\tilde{j}^*_{reg}$ are adjoint; let α_{reg} be the isomorphism $\mathrm{Hom}_{\mathcal{F}(\tilde{G}^\sigma_{reg})}(\overline{\mathbb{Q}}_l, \overline{\mathbb{Q}}_l) \simeq \mathrm{Hom}_{\mathcal{F}(\tilde{G}^1_{reg})}(\overline{\mathbb{Q}}_l, \tilde{j}_{reg*}\overline{\mathbb{Q}}_l)$. Let us show that $\tilde{i}^*\alpha(\mathrm{Id}_{\overline{\mathbb{Q}}_l}) : \overline{\mathbb{Q}}_l \to \tilde{i}^*\tilde{j}_*\overline{\mathbb{Q}}_l$ is the adjunction morphism $\alpha_{reg}(\mathrm{Id}_{\overline{\mathbb{Q}}_l}) : \overline{\mathbb{Q}}_l \to \tilde{j}_{reg*}\overline{\mathbb{Q}}_l$. We will then be able to conclude that $\tilde{i}^*\alpha(\mathrm{Id}_{\overline{\mathbb{Q}}_l})$ is W^σ-equivariant: $\tilde{j}_{reg}$ is an equivariant morphism from $\tilde{G}^\sigma_{reg}$ to $\tilde{G}^1_{reg}$ with respect to the action of W^σ on $\tilde{G}^\sigma_{reg}$ (defined by $w : (g, xT) \mapsto (g, xwT)$) and on $\tilde{G}^1_{reg}$ (defined by $w : (g.\sigma, xT) \mapsto (g.\sigma, xwT)$).

According to the proper base change theorem applied to the proper morphism $\tilde{j}$, the sheaves $\tilde{i}^*\tilde{j}_*\overline{\mathbb{Q}}_l$ and $\tilde{j}_{reg*}\overline{\mathbb{Q}}_l$ are isomorphic. For any etale morphism $U \to \tilde{G}^1_{reg}$, we have $\alpha_{reg}(\mathrm{Id}_{\overline{\mathbb{Q}}_l})(U) : \overline{\mathbb{Q}}_l(U) \to \overline{\mathbb{Q}}_l(U \times_{\tilde{G}^1_{reg}} \tilde{G}^\sigma_{reg})$. Moreover, as $\tilde{i} : \tilde{G}^1_{reg} \to \tilde{G}^1$ is an open immersion, it is an etale morphism and $\tilde{i}^*$ is the restriction functor, so for any etale morphism $U \to \tilde{G}^1_{reg}$, $\tilde{i}^*\alpha(\mathrm{Id}_{\overline{\mathbb{Q}}_l})(U) = \alpha(\mathrm{Id}_{\overline{\mathbb{Q}}_l})(U) : \overline{\mathbb{Q}}_l(U) \to \tilde{j}_*\overline{\mathbb{Q}}_l(U)$, where $\tilde{j}_*\overline{\mathbb{Q}}_l(U) = \overline{\mathbb{Q}}_l(U \times_{\tilde{G}^1} \tilde{G}^\sigma)$ and $U \times_{\tilde{G}^1} \tilde{G}^\sigma = U \times_{\tilde{G}^1_{reg}} \tilde{G}^\sigma_{reg}$. Hence, we have $\tilde{i}^*\alpha(\mathrm{Id}_{\overline{\mathbb{Q}}_l}) = \alpha_{reg}(\mathrm{Id}_{\overline{\mathbb{Q}}_l})$.

The proposition is then proved if we prove that the stalk at σ of the W^σ-equivariant morphism ϕ is an isomorphism $\phi_\sigma : (\pi_*\overline{\mathbb{Q}}_l)_\sigma \to (\pi_*\tilde{j}_*\overline{\mathbb{Q}}_l)_\sigma$.

Let V be the open variety $\tilde{G}^1 - \tilde{G}^\sigma$ and $\tilde{i}_V$ the open immersion $V \to \tilde{G}^1$. Let us recall that $\tilde{j}$ is the closed immersion $\tilde{G}^\sigma \to \tilde{G}^1$. We have the following exact sequence:

$$0 \to \tilde{i}_{V!}\tilde{i}^*_V\overline{\mathbb{Q}}_l \to \overline{\mathbb{Q}}_l \to \tilde{j}_*\tilde{j}^*\overline{\mathbb{Q}}_l \to 0$$

The functor π_* is left-exact. Let us consider the right-derived functors $R^k\pi_*$ of π_*, we then obtain the following long exact sequence:

$$\ldots \to R^k\pi_*\tilde{i}_{V!}\tilde{i}^*_V\overline{\mathbb{Q}}_l \to R^k\pi_*\overline{\mathbb{Q}}_l \to R^k\pi_*\tilde{j}_*\tilde{j}^*\overline{\mathbb{Q}}_l \to R^{k+1}\pi_*\tilde{i}_{V!}\tilde{i}^*_V\overline{\mathbb{Q}}_l \to \ldots$$

Let us apply the exact functor $\mathcal{F} \mapsto \mathcal{F}_\sigma$, we then have the following long exact sequence:

$$\ldots \to (R^k\pi_*\tilde{i}_{V!}\tilde{i}^*_V\overline{\mathbb{Q}}_l)_\sigma \to (R^k\pi_*\overline{\mathbb{Q}}_l)_\sigma \to (R^k\pi_*\tilde{j}_*\tilde{j}^*\overline{\mathbb{Q}}_l)_\sigma \to (R^{k+1}\pi_*\tilde{i}_{V!}\tilde{i}^*_V\overline{\mathbb{Q}}_l)_\sigma \to \ldots$$

So, the fact that $(R^0\pi_*\overline{\mathbb{Q}}_l)_\sigma = (\pi_*\overline{\mathbb{Q}}_l)_\sigma$ is isomorphic to $(R^0\pi_*\tilde{j}_*\tilde{j}^*\overline{\mathbb{Q}}_l)_\sigma = (\pi_*\tilde{j}_*\overline{\mathbb{Q}}_l)_\sigma$ will be proved if we prove that $(R^0\pi_*\tilde{i}_{V!}\tilde{i}^*_V\overline{\mathbb{Q}}_l)_\sigma = 0$ and $(R^1\pi_*\tilde{i}_{V!}\tilde{i}^*_V\overline{\mathbb{Q}}_l)_\sigma = 0$, that is to say that $\mathrm{H}^0_c(V \cap \pi^{-1}(\sigma), \overline{\mathbb{Q}}_l) = 0$ and $\mathrm{H}^1_c(V \cap \pi^{-1}(\sigma), \overline{\mathbb{Q}}_l) = 0$. But this is clear as $\pi^{-1}(\sigma) \subset \tilde{G}^\sigma$ and $V = \tilde{G}^1 - \tilde{G}^\sigma$.

End of the proof of proposition 5.1.

Corollary 5.4. *The Springer representation of W^σ on $\mathrm{H}^*(\mathcal{B}^G_\sigma, \overline{\mathbb{Q}}_l)$ from G^1 is the regular representation of W^σ.*

Proof: According to [13] (corollary 5.5), the Springer representation of W^σ on $\mathrm{H}^*(\mathcal{B}^{G^\sigma}, \overline{\mathbb{Q}}_l)$ from G^σ is the regular representation of W^σ.

End of the proof.

6. Springer correspondence theorem

In this section, we define a Springer correspondence for G^1 between the irreducible representations of W^σ and some pairs $(C, \mathcal{E})$ where C is a unipotent G-conjugacy class of G^1 and $\mathcal{E}$ a G-equivariant irreducible local system on C. Let us denote by $\mathcal{N}$ the set of all such pairs $(C, \mathcal{E})$.

Proposition 6.1. *([14] proposition I.1.6)*
The variety $\mathcal{U}(G^1)$ of unipotent elements of G^1 is an irreducible closed subvariety of G^1 with codimension equal to the rank of G^σ.

We will need the following theorem in order to prove the main result of this section which is theorem 6.3:

Theorem 6.2. *Let $\phi : Z \to X$ be a semi-small morphism between two smooth varieties which have the same dimension equal to n. We suppose there is an action of a group H on Z and on X and that ϕ is H-equivariant. Then,*
(i) the perverse sheaf $\phi_*\overline{\mathbb{Q}}_l[\dim(X)]$ is semi-simple in $\mathcal{M}(X)$,
(ii) $\phi_*\overline{\mathbb{Q}}_l[\dim(X)]$ is decomposed as:

$$K \simeq \bigoplus_{(C,\mathcal{E})} \mathrm{IC}(\overline{C},\mathcal{E})[\dim(C)] \otimes V_{(C,\mathcal{E})},$$

where C runs over the set of H-stable smooth irreducible subvarieties of X and $\mathcal{E}$ the set of H-equivariant local systems over C and where $V_{(C,\mathcal{E})}$ is a $\overline{\mathbb{Q}}_l$-vector space whose dimension is equal to the multiplicity of the simple object corresponding to $(C,\mathcal{E})$.

Proof of theorem 6.2:
(i) As Z is smooth, $\overline{\mathbb{Q}}_l[n]$ is a simple perverse sheaf with finite monodromy. Moreover ϕ is semi-small, so, according to proposition 4.6, $\phi_*\overline{\mathbb{Q}}_l[n]$ is a perverse sheaf. We can then apply the decomposition theorem ([1], theorem 6.2.5). The result is that $\phi_*\overline{\mathbb{Q}}_l[n]$ is a semi-simple object of $\mathcal{M}(X)$.
(ii) By (i), $\phi_*\overline{\mathbb{Q}}_l[n]$ is a direct sum of simple objects of $\mathcal{M}(X)$. But, according to [1] (4.3.1), a simple object of $\mathcal{M}(X)$ is of the form $IC(Y,\mathcal{E})[\dim(Y)]$ where Y is a locally closed smooth irreducible subvariety of X and $\mathcal{E}$ an irreducible local system on Y.
Moreover, as $\overline{\mathbb{Q}}_l[n]$ is a H-equivariant perverse sheaf and as ϕ is a H-equivariant morphism, $\phi_*\overline{\mathbb{Q}}_l[n]$ is also H-equivariant ([12] 1.4.1). Hence, if $\mathrm{IC}(\bar{Y},\mathcal{E})[\dim(Y)]$ is a summand of $\phi_*\overline{\mathbb{Q}}_l[n]$, then $\mathrm{IC}(\bar{Y},\mathcal{E})[\dim(Y)]$ is G-equivariant, which means that Y is a H-stable subvariety and $\mathcal{E}$ a H-equivariant local system.
Thus, $\phi_*\overline{\mathbb{Q}}_l[n]$ is a direct sum of perverse sheaves of the form $\mathrm{IC}(\bar{Y},\mathcal{E})[\dim(Y)]$, where Y is a H-stable irreducible smooth subvariety of X and $\mathcal{E}$ is an irreducible H-equivariant local system on Y.

End of the proof of theorem 6.2.

Theorem 6.3. *Let n be the dimension of G and r' the rank of G^σ.*
(i) $\pi_*\overline{\mathbb{Q}}_l[n-r']|_{\mathcal{U}(G^1)}$ is a semi-simple object of $\mathcal{M}(\mathcal{U}(G^1))$ and decomposes as:

$$\pi_*\overline{\mathbb{Q}}_l[n-r']|_{\mathcal{U}(G^1)} \simeq \bigoplus_{(C,\mathcal{E})} V_{C,\mathcal{E}} \otimes \mathrm{IC}(\bar{C},\mathcal{E})[\dim(C)]$$

where $(C,\mathcal{E})$ runs over the set $\mathcal{N}$ and $V_{C,\mathcal{E}}$ is a $\overline{\mathbb{Q}}_l$-vector space whose dimension is equal to the multiplicity of the simple object corresponding to $(C,\mathcal{E})$.
(ii) The action of W^σ on $\pi_*\overline{\mathbb{Q}}_l|_{\mathcal{U}(G^1)}$ induces an action of W^σ on all $V_{C,\mathcal{E}}$, moreover, any W^σ-module $V_{C,\mathcal{E}}$ is irreducible if $V_{C,\mathcal{E}} \neq 0$ and the map $(C,\mathcal{E}) \mapsto V_{C,\mathcal{E}}$ induces a bijection from $\{(C,\mathcal{E}) | V_{C,\mathcal{E}} \neq 0\}$ to $\mathrm{Irr}(W^\sigma)$.

We call Springer correspondence for G^1 the bijection defined by (ii) in theorem 6.3.
Proof of theorem 6.3: The following proof is inspired by that for the connected case in [13] (proof of theorem 6.2).

(i) Let π_u be the restriction of $\pi : \tilde{G}^1 \to G^1$ to $\pi^{-1}(\mathcal{U}(G^1))$. Let us consider the following diagram:

$$\begin{array}{ccc} \pi^{-1}(\mathcal{U}(G^1)) & \xrightarrow{i} & \tilde{G}^1 \\ \downarrow \pi_u & & \downarrow \pi \\ \mathcal{U}(G^1) & \xrightarrow{i'} & G^1 \end{array}$$

Let us apply the proper base change theorem to π. We get $\pi_*\overline{\mathbb{Q}}_l|_{\mathcal{U}(G^1)} \simeq (\pi_u)_*\overline{\mathbb{Q}}_l$. We then apply theorem 6.2 to the morphism π_u. Let us check that the hypotheses of this theorem are satisfied in the situation of theorem 6.3.

First, $\pi_u : \pi^{-1}(\mathcal{U}(G^1)) \to \mathcal{U}(G^1)$ is a proper morphism (as proper morphisms are stable under base extension) between two varieties with the same dimension equal to $n - r'$ and $\mathcal{U}(G^1)$ is smooth. Moreover,

$$\dim\{g.\sigma \in \mathcal{U}(G^1) | 2\dim(\mathcal{B}^G_{g.\sigma}) \geq i\} \leq \dim(\mathcal{U}(G^1)) - i$$

(corollary 3.2) and $\pi_u^{-1}(g.\sigma) \simeq \mathcal{B}^G_{g.\sigma}$, so π_u is semi-small. Then, according to proposition 4.1, $\pi^{-1}(\mathcal{U}(G^1))$ is a smooth variety. Finally, π_u is a G-equivariant morphism.

Thus, $A = (\pi_u)_*\overline{\mathbb{Q}}_l[n - r']$ is a G-equivariant semi-simple object of $\mathcal{M}(\mathcal{U}(G^1))$ and A is a direct sum of $IC(\overline{Y}, \mathcal{E})[\dim(Y)]$, where Y is a G-stable smooth irreducible subvariety of $\mathcal{U}(G^1)$ and $\mathcal{E}$ is a G-equivariant irreducible local system on Y. The closure $\overline{Y}$ of Y is then an irreducible closed subvariety of $\mathcal{U}(G^1)$.

According to [14] (théorème I.4.1), there is a finite number of unipotent conjugacy classes in G^1. Any unipotent conjugacy class C of G^1 is a smooth subvariety of $\mathcal{U}(G^1)$, its closure $\overline{C}$ is an irreducible closed subvariety of $\mathcal{U}(G^1)$ and a union of unipotent conjugacy classes. Moreover, the G-stable irreducible closed subvarieties of $\mathcal{U}(G^1)$ are the closures of the unipotent conjugacy classes.

Thus, the G-equivariant simple objects of $\mathcal{M}(\mathcal{U}(G^1))$ are of the form $\mathrm{IC}(\overline{C}, \mathcal{E})[\dim(C)]$, where C is a unipotent conjugacy class of G^1 and $\mathcal{E}$ is a G-equivariant irreducible local system on C.

Thus, A is decomposed as:

$$A \simeq \bigoplus_{(C,\mathcal{E})} V_{C,\mathcal{E}} \otimes \mathrm{IC}(\bar{C}, \mathcal{E})[\dim(C)],$$

where, $(C, \mathcal{E})$ runs over all the pairs in $\mathcal{N}$ and we get 6.3 (i).

(ii) The action of W^σ on $\pi_*\overline{\mathbb{Q}}_l$ induces an action of W^σ on A and the action of W^σ on A induces a map $\alpha : \overline{\mathbb{Q}}_l[W^\sigma] \to \mathrm{End}(A)$. Let us prove that α is an isomorphism.

Let us first prove that α is injective. We consider the stalk $\mathcal{H}^i_\sigma(A)$ of A at $\sigma \in \mathcal{U}(G^1)$, We have $\mathcal{H}^i_\sigma(A) \simeq \mathrm{H}^{i+\gamma}(\mathcal{B}_\sigma, \overline{\mathbb{Q}}_l)$, where $\gamma = n - r'$. So, we have a natural morphism $\beta : \mathrm{End}(A) \to \mathrm{End}(\mathrm{H}^*(\mathcal{B}_\sigma, \overline{\mathbb{Q}}_l))$, and $\beta \circ \alpha$ is the action of W^σ on $\mathrm{H}^*(\mathcal{B}_\sigma, \overline{\mathbb{Q}}_l)$ that we have studied in the previous section. Thus, by corollary 5.4, $\beta \circ \alpha$ is injective which leads to the injectivity of α.

Let us now prove that α is surjective. As α is injective, we just have to check that $\dim(\overline{\mathbb{Q}}_l[W^\sigma]) \geq \dim(\mathrm{End}(A))$, which means, by (i), that

$$\sum_{(C,\mathcal{E})} \dim(V_{C,\mathcal{E}})^2 \quad \leq \quad |W^\sigma| \tag{1}$$

Let x be in $\mathcal{U}(G^1)$ and let C_x be the conjugacy class of x under G. We set $d_x = \dim(\mathcal{B}_x)$. As $\mathrm{H}^{2d_x}(\mathcal{B}_x, \overline{\mathbb{Q}}_l) \simeq \mathcal{H}^{2d_x}_x(A[-\gamma])$, we have:

$$\mathrm{H}^{2d_x}(\mathcal{B}_x, \overline{\mathbb{Q}}_l) \simeq \bigoplus_{(C,\mathcal{E})} V_{C,\mathcal{E}} \otimes \mathcal{H}^{2d_x - \gamma + \dim C}_x \mathrm{IC}(\bar{C}, \mathcal{E})$$

As A is G-equivariant, $\mathrm{C}_G(x)$ acts on the stalk $\mathcal{H}_x^{2d_x}(A[-\gamma])$. This induces an action of $A_G(x)$ which coincides with the action of $A_G(x)$ on $\mathrm{H}^{2d_x}(\mathcal{B}_x,\overline{\mathbb{Q}}_l)$ induced by the action of $C_G(x)$ on $\mathcal{B}_x$.

As G is a connected group which acts transitively on the variety C_x, there is a bijection between G-equivariant irreducible local systems on C_x and irreducible representations of $A_G(x)$ over $\overline{\mathbb{Q}}_l$ ([13] 3.5). Let ρ be an irreducible representation of $A_G(x)$ and let $\mathcal{E}$ an irreducible G-equivariant local system on C_x such that ρ is the image of $\mathcal{E}$ by the above bijection. We consider the direct summand in (6.1) corresponding to $(C_x,\mathcal{E})$. As, we have $2d_x-\gamma+\dim(C_x) = 2\dim(\mathcal{B}_x^G)+\dim(C_x)-(\dim(G)-\mathrm{rk}(G^\sigma)) = 0$ (theorem 3.1, (i)), this summand is equal to $V_{C_x,\mathcal{E}}\otimes\mathcal{H}_x^0(\mathrm{IC}(\bar{C}_x,\mathcal{E}_\rho))$, that is to say, to $V_{C_x,\mathcal{E}_\rho}\otimes(\mathcal{E})_x$ and $(\mathcal{E})_x$ is an irreducible $A_G(x)$-module isomorphic to ρ^γ ([13] 3.5).

Let us now consider $\mathrm{H}^{2d_x}(\mathcal{B}_x,\overline{\mathbb{Q}}_l)$ as a representation of $A_G(x)$. Let $m_{x,\rho}$ be the multiplicity of the irreducible representation ρ^γ in $\mathrm{H}^{2d_x}(\mathcal{B}_x,\overline{\mathbb{Q}}_l)$. Then, we have $\dim(V_{C_x,\mathcal{E}})\leq m_{x,\rho}$.

We have then proved that, for any unipotent conjugacy class C_x of G^1, and for any irreducible representation ρ of $A_G(x)$, we have $\dim(V_{C_x,\mathcal{E}_\rho})\leq m_{x,\rho}$.

However, we have

$$\dim(\mathrm{End}_{A_G(x)}(\mathrm{H}^{2d_x}(\mathcal{B}_x,\overline{\mathbb{Q}}_l))) = \sum_{\rho\in A_G(x)^\wedge} m_{x,\rho}^2$$

And

$$\dim(\mathrm{End}_{A_G(x)}(\mathrm{H}^{2d_x}(\mathcal{B}_x,\overline{\mathbb{Q}}_l))) = \dim(\mathrm{End}_{\overline{\mathbb{Q}}_l}(\mathrm{H}^{2d_x}(\mathcal{B}_x,\overline{\mathbb{Q}}_l))^{A_G(x)})$$

The space $\mathrm{H}^{2d_x}(\mathcal{B}_x,\overline{\mathbb{Q}}_l)$ has a basis parametrised by the set $I(\mathcal{B}_x)$ of irreducible components of $\mathcal{B}_x$, and the action of $A_G(x)$ on $\mathrm{H}^{2d_x}(\mathcal{B}_x,\overline{\mathbb{Q}}_l)$ is induced by its action on $I(\mathcal{B}_x)$. So, the basis of $\mathrm{End}_{\overline{\mathbb{Q}}_l}(\mathrm{H}^{2d_x}(\mathcal{B}_x,\overline{\mathbb{Q}}_l))$ is indexed by $I(\mathcal{B}_x)\times I(\mathcal{B}_x)$ and the action of $A_G(x)$ on $\mathrm{End}_{\overline{\mathbb{Q}}_l}(\mathrm{H}^{2d_x}(\mathcal{B}_x,\overline{\mathbb{Q}}_l))$ is induced by its diagonal action on $I(\mathcal{B}_x)\times I(\mathcal{B}_x)$. Thus, the dimension of $\mathrm{End}_{\overline{\mathbb{Q}}_l}(\mathrm{H}^{2d_x}(\mathcal{B}_x,\overline{\mathbb{Q}}_l))^{A_G(x)}$ is equal to the number of $A_G(x)$-orbits of $I(\mathcal{B}_x)\times I(\mathcal{B}_x)$.

We then have the following equality, if we associate, to each pair $(C,\mathcal{E})\in\mathcal{N}$ a pair (x,ρ) with $x\in C$ and $\rho\in A_G(x)^\wedge$ is the image of C by the above bijection:

$$\begin{aligned}\sum_{(x,\rho)}\dim(V_{C_x,\mathcal{E}_\rho})^2 &\leq \sum_{(x,\rho)} m_{(x,\rho)}^2\\ &= \sum_{x\in\mathcal{U}(G^1)/\sim} |I(\mathcal{B}_x)\times I(\mathcal{B}_x)/A_G(x)|\\ &= |W^\sigma|\end{aligned}$$

The last equality is a result from [14] (II.2.11), hence we have equality in (1) and α is surjective.

End of the proof of theorem 6.3.

Corollary 6.4. *For any unipotent conjugacy class $C_{u.\sigma}$ in G^1, we have*

$$\mathrm{H}^{2d_{u.\sigma}}(\mathcal{B}_{u.\sigma},\overline{\mathbb{Q}}_l)\simeq\oplus_{\rho\in A_G(u.\sigma)^\wedge}V_{(C_{u.\sigma},\mathcal{E}_\rho)}\otimes\rho^\vee$$

as $W^\sigma\times A_G(u.\sigma)$-modules.

Corollary 6.5. *For any unipotent conjugacy class $C_{u.\sigma}$ in G^1, the $\overline{\mathbb{Q}}_l$-vector space $V_{(C_{u.\sigma},\overline{\mathbb{Q}}_l)}$ is different from zero. The Springer correspondence induces then an injection from the set of unipotent conjugacy classes of G^1 to the set of irreducible representations of W^σ defined by $C_{u.\sigma}\mapsto V_{(C_{u.\sigma},\overline{\mathbb{Q}}_l)}$.*

Proof: The constant sheaf $\overline{\mathbb{Q}}_l$ on the unipotent conjugacy class $C_{u.\sigma}$ corresponds to the trivial representation of $A_G(u.\sigma)$. Thus, we have the following isomorphism of W^σ-modules: $V_{(C_{u.\sigma},\overline{\mathbb{Q}}_l)}\simeq\mathrm{H}^{2d_{u.\sigma}}(\mathcal{B}_{u.\sigma},\overline{\mathbb{Q}}_l)^{A_G(u.\sigma)}$, whence the corollary as we have $\mathrm{H}^{2d_{u.\sigma}}(\mathcal{B}_{u.\sigma},\overline{\mathbb{Q}}_l)^{A_G(u.\sigma)}\neq 0$.

End of the proof.

According to corollary 5.4, the Springer action of W^σ from G^1 on $\mathrm{H}^*(\mathcal{B}^{G^\sigma},\overline{\mathbb{Q}}_l)$ is the regular representation of W^σ. Thus, for all i odd, we have $\mathrm{H}^i(\mathcal{B}^{G^\sigma},\overline{\mathbb{Q}}_l) = 0$, and for any $i = 2k$, $\mathrm{H}^i(\mathcal{B}^{G^\sigma},\overline{\mathbb{Q}}_l)$ is W^σ-isomorphic to a space of homogeneous polynomial functions of degree k. Let $n_k(\rho)$ be the multiplicity of an irreducible representation ρ in this space. Then, the polynomial function defined by $P_\rho(t) = \sum_k n_k(\rho)t^k$ is the fake degree of ρ ([5] section 11.1). The following corollary can be deduced from theorem 6.3.

Corollary 6.6.

$$n_k(\rho_{(C_{u.\sigma},\mathcal{E})}) = \dim(\mathcal{H}_\sigma^{2k-2d_{u.\sigma}}(IC(\overline{C_{u.\sigma}},\mathcal{E}))).$$

Let us denote by a_ρ the valuation of P_ρ for any irreducible representation ρ of W^σ.

Proposition 6.7.

(i) If $k < d_{u.\sigma}$, then $n_k(\rho_{(C_{u.\sigma},\mathcal{E})}) = 0$.

(ii) If $k = d_{u.\sigma}$ and if $\mathcal{E} = \overline{\mathbb{Q}}_l$, then $n_k(\rho_{(C_{u.\sigma},\mathcal{E})}) \neq 0$, which means that $d_{u.\sigma} = a_\rho$, where $\rho = \rho_{(C_{u.\sigma},\overline{\mathbb{Q}}_l)}$.

Proof: (i) is an immediate consequence of the previous corollary, for if $j < 0$, we have $\mathcal{H}_\sigma^{2j-2d_{u.\sigma}}(\mathrm{IC}(\overline{C_{u.\sigma}},\mathcal{E})) = 0$.

In order to prove (ii), we have to check that for all unipotent conjugacy class, we have $\mathcal{H}_\sigma^0(\mathrm{IC}(\overline{C},\overline{\mathbb{Q}}_l)) \neq 0$.

Let C be a unipotent conjugacy class. Then, its closure $\overline{C}$ is stratified by the unipotent conjugacy classes which are included in $\overline{C}$. Moreover, $\mathrm{Cl}_G(\sigma)$ is its smallest stratum and is closed. We set $V = \overline{C} - \mathrm{Cl}_G(\sigma)$ and we denote by j the immersion $V \hookrightarrow \overline{C}$. Then, $\dim(\mathrm{H}^0(j_!j^*(\mathrm{IC}(\overline{C},\overline{\mathbb{Q}}_l))))$ is the number of connected components of C whose closure in V is closed in $\overline{C}$ and $\dim(\mathrm{H}^0(\mathrm{IC}(\overline{C},\overline{\mathbb{Q}}_l)))$ is the number of connected components of C and is then equal to 1, as C is connected. As C is dense in $\overline{C}$ and as $\mathrm{Cl}_G(\sigma)$ is not empty, we have $\dim(\mathrm{H}^0(j_!j^*(\mathrm{IC}(\overline{C},\overline{\mathbb{Q}}_l)))) = 0$. But there is an injection from the cokernel of the injective morphism

$$0 \to \mathrm{H}^0(j_!j^*(\mathrm{IC}(\overline{C},\overline{\mathbb{Q}}_l))) \to \mathrm{H}^0(\mathrm{IC}(\overline{C},\overline{\mathbb{Q}}_l))$$

to $\mathrm{H}^0(\mathrm{IC}(\overline{C},\overline{\mathbb{Q}}_l)|_{\mathrm{Cl}_G(\sigma)})$, which is then different from zero. Moreover, $IC(\overline{C},\overline{\mathbb{Q}}_l)$ is G-equivariant, so we have $\mathcal{H}_\sigma^0(\mathrm{IC}(\overline{C},\overline{\mathbb{Q}}_l)) \neq 0$.

End of the proof.

7. Restriction of the Springer correspondence to a σ-stable Levi subgroup included in a σ-stable parabolic subgroup

Let L be a σ-stable Levi subgroup included in a σ-stable parabolic subgroup P. We denote by $\mathcal{P}$ the variety of parabolic subgroups which are conjugate to P. Let $W(L)$ be the Weyl group of L. Let C^L be a unipotent class of L^1 and C the unipotent class of G^1 obtained by inducing C^L from L^1 to G^1 (definition 7.1). The purpose of this section is to prove that the irreducible representation $\rho_{(C^L,\overline{\mathbb{Q}}_l)}$ of $W(L)^\sigma$, image of $(C^L,\overline{\mathbb{Q}}_l)$ by the Springer correspondence for L^1 is an irreducible component of the restriction to $W(L)^\sigma$ of the irreducible representation $\rho_{(C,\overline{\mathbb{Q}}_l)}$ of W^σ, image of $(C,\overline{\mathbb{Q}}_l)$ by the Springer correspondence from G^1 (proposition 7.13).

7.1. **Notations.** The variety $\tilde{G}^1$ is defined, like in section 4, by:

$$\tilde{G}^1 = \{(g.\sigma, B) \in G^1 \times \mathcal{B}^G | B \in \mathcal{B}^G_{g.\sigma}\},$$

where, $\mathcal{B}^G_{g.\sigma}$ is the variety of Borel subgroups of G which are stable under conjugation by $g.\sigma$. We still denote by π the projection $\tilde{G}^1 \to G^1$ on the first factor, $\tilde{\mathcal{U}}(G^1) = \pi^{-1}(\mathcal{U}(G^1))$ and π_u the restriction of π to $\tilde{\mathcal{U}}(G^1)$.

Notation 7.1. We put $A = \pi_{u*}\overline{\mathbb{Q}}_l$.

We denote by $\tilde{L}^1$, $\mathcal{B}^L_{l.\sigma}$, $\pi(L)$, $\tilde{\mathcal{U}}(L^1)$, $\pi(L)_u$, the analogous objects relative to L.

Notation 7.2. We put $A(L) = \pi(L)_{u*}\overline{\mathbb{Q}}_l$.

According to theorem 6.3, we have in $\mathcal{M}(\mathcal{U}(G^1))$:

$$A[\dim(\mathcal{U}(G^1))] \simeq \bigoplus_{(C,\mathcal{E})} \mathrm{IC}(\overline{C},\mathcal{E})[\dim(C)] \otimes V_{(C,\mathcal{E})} \tag{2}$$

where C runs over the set of unipotent classes of G^1 and $\mathcal{E}$ the set of irreducible G-equivariant local systems on C.

And by 6.3 in L, we have in $\mathcal{M}(\mathcal{U}(L^1))$:

$$A(L)[\dim(\mathcal{U}(L^1))] \simeq \bigoplus_{(C^L,\mathcal{F})} \mathrm{IC}(\overline{C^L},\mathcal{F})[\dim(C^L)] \otimes V_{(C^L,\mathcal{F})} \tag{3}$$

where C^L runs over the set of unipotent classes of L^1 and $\mathcal{F}$ the set of irreducible L-equivariant local systems on C^L.

The variety $\tilde{G}^{1P}$ is defined by

$$\tilde{G}^{1P} = \{(g.\sigma, P') \in G^1 \times \mathcal{P} | P' \in \mathcal{P}_{g.\sigma}\},$$

where $\mathcal{P}_{g.\sigma}$ is the variety of parabolic subgroups conjugate to P and stable under conjugation by $g.\sigma$. The variety $\tilde{G}^{1P}$ can be identified with $G \times^P P^1$ by the map $(g.\sigma, {}^xP) \mapsto (x, {}^{x^{-1}}(g.\sigma))$. Let $\xi : \tilde{G}^{1P} \to G^1$ be the projection on the first factor and $\eta : \tilde{G}^1 \to \tilde{G}^{1P}$ the morphism defined by $(g.\sigma, B') \mapsto (g.\sigma, P')$, where P' is the only parabolic subgroup in $\mathcal{P}$ such that $B' \subset P'$. Then, we have $\pi = \xi \circ \eta$.

Let $\tilde{\mathcal{U}}(G^1)^P$ be the inverse image of $\mathcal{U}(G^1)$ under ξ. Then, the variety $\tilde{\mathcal{U}}(G^1)^P$ can be identified with $G \times^P \mathcal{U}(P^1)$ by the map $(g.\sigma, {}^xP) \mapsto (x, {}^{x^{-1}}(g.\sigma))$. We denote by ξ_u the restriction of ξ to $\tilde{\mathcal{U}}(G^1)^P$ and by η_u the restriction of η to $\tilde{\mathcal{U}}(G^1)$.

Notation 7.3. We denote by A^P the complex of sheaves $\eta_{u*}\overline{\mathbb{Q}}_l$ on $\tilde{\mathcal{U}}(G^1)^P$.

Finally, let $\mathcal{V}$ be the variety $G \times^P L^1$, where the action of P on the left on L^1 is defined by $p : l.\sigma \mapsto \pi_P(p(l.\sigma)p^{-1})$, where π_P is the projection $\mathrm{N}_{\tilde{G}}(P) \to \mathrm{N}_{\tilde{G}}(L, P)$ whose kernel is U_P. Let us identify a point of $\mathcal{V}$ with its image under the map $(x, l.\sigma) \mapsto ({}^x(l.\sigma U_P), {}^xP)$. Then, $\mathcal{V}$ is the set of the pairs $(g.\sigma, P')$ with $P' \in \mathcal{P}$ such that $g.\sigma \in \mathrm{N}_{\tilde{G}}(P')/U_{P'}$, where $U_{P'}$ is the unipotent radical of P'. Let $r : \mathcal{V} \to \mathcal{P}$ be the projection on the second factor defined by $(g.\sigma, P') \mapsto P'$. We denote by $\mathcal{V}_u$ the subvariety $G \times^P \mathcal{U}(L^1)$ of $\mathcal{V}$ and by r_u the restriction of r to $\mathcal{V}_u$.

Let $\tilde{\mathcal{V}}$ be the variety $G \times^B (L \cap B)^1$ where the action of B on $(L \cap B)^1$ on the left is defined by $l : b.\sigma \mapsto \pi_P(l(b.\sigma)l^{-1})$ (the image of B^1 by π_P is $(L \cap B)^1$, where we have put $(L \cap B)^1 = (L \cap B).\sigma)$.

We identify $\tilde{\mathcal{V}}$ with the set of $(g.\sigma, B', P')$ where $P' \in \mathcal{P}$, B' is a Borel subgroup of P' and $g.\sigma \in \mathrm{N}_{\tilde{G}}(B')/U_{P'}$. Then, we can define a morphism $\zeta : \tilde{\mathcal{V}} \to \mathcal{V}$ by $(y, B', P') \mapsto (y, P')$. We set $\tilde{\mathcal{V}}_u = \zeta^{-1}(\mathcal{V}_u)$.

Notation 7.4. We denote by K the direct image of the constant sheaf $\overline{\mathbb{Q}}_l$ on $\tilde{\mathcal{V}}$ under ζ_u.

Let us consider the following diagram:

Diagram 7.1.

$$\begin{tikzcd}
\tilde{L}^1 = L \times^{L\cap B} (L\cap B)^1 \arrow[r, "\tilde{i}"] \arrow[d, "\pi(L)"] & \tilde{\mathcal{V}} = G \times^B (L\cap B)^1 \arrow[d, "\zeta"] & \tilde{G}^1 = G\times^B B^1 \arrow[l, "\tilde{q}"'] \arrow[d, "\eta"] \\
L^1 \arrow[r, "i"'] & \mathcal{V} = G\times^P L^1 & \tilde{G}^{1P} = G\times^P P^1 \arrow[l, "q"] \arrow[d, "\xi"] \\
 & & G^1
\end{tikzcd}$$

The morphism i is defined by $l.\sigma \mapsto (1, l.\sigma)$ (this is the inclusion of the fiber $r^{-1}(P)$); the morphism q is defined by $(x, p.\sigma) \mapsto (x, \pi_P(p.\sigma))$.

We denote by $\tilde{i}$ the morphism defined by $(l.\sigma, \tilde{B}') \mapsto (l.\sigma U_P, B', P)$ where B' is the Borel subgroup included in P such that $B' \cap L = \tilde{B}'$. The morphism $\tilde{q}$ is defined by $(g.\sigma, B') \mapsto (\overline{g.\sigma}, B', P')$, where P' is the only parabolic subgroup which is conjugate to P and such that $B' \subset P'$ and where we denote by $\overline{g.\sigma}$ the image of $g.\sigma$ in $\mathrm{N}_{\tilde{G}}(P')/U_{P'}$.

In the following diagram, the morphisms indexed by u are the restrictions to the unipotent varieties of the corresponding morphisms of diagram 7.1:

Diagram 7.2.

$$\begin{tikzcd}
\tilde{\mathcal{U}}(L^1) \arrow[r, "\tilde{i}_u"] \arrow[d, "\pi(L)_u"] & \tilde{\mathcal{V}}_u \arrow[d, "\zeta_u"] & \tilde{\mathcal{U}}(G^1) \arrow[l, "\tilde{q}_u"'] \arrow[d, "\eta_u"] \\
\mathcal{U}(L^1) \arrow[r, "i_u"'] & \mathcal{V}_u & \tilde{\mathcal{U}}(G^1)^P \arrow[l, "q_u"] \arrow[d, "\xi_u"] \\
 & & \mathcal{U}(G^1)
\end{tikzcd}$$

Using diagram 7.2 we will compare $A^P = \eta_{u*}\overline{\mathbb{Q}}_l$ and $A(L) = \pi(L)_{u*}\overline{\mathbb{Q}}_l$ with the help of $K = \zeta_{u*}\overline{\mathbb{Q}}_l$.

7.2. **Application of the decomposition theorem to A^P and to K.** The following proposition shows that the hypotheses of theorem 6.2 are verified for the morphisms $\eta_u : \tilde{\mathcal{U}}(G^1) \to \tilde{\mathcal{U}}(G^1)^P$ and $\zeta_u : \tilde{\mathcal{V}}_u \to \mathcal{V}_u$.

Proposition 7.1. *(i) The variety $\tilde{\mathcal{U}}(G^1)^P$ is smooth and*

$$\dim(\tilde{\mathcal{U}}(G^1)^P) = \dim(\mathcal{U}(G^1)) = \dim(G) - \mathrm{rk}(G^\sigma).$$

The varieties $\mathcal{V}_u$ and $\tilde{\mathcal{V}}_u$ are smooth and

$$\dim(\mathcal{V}_u) = \dim(\tilde{\mathcal{V}}_u) = \dim(\mathcal{P}) + \dim(\mathcal{U}(L^1)).$$

(ii) η and ζ are proper.
(ii') η_u and ζ_u are proper.
(iii) The morphisms η_u and ζ_u are semi-small.

Proof:

(i) Let us prove that the projection $p_2 : \tilde{\mathcal{U}}(G^1)^P \to \mathcal{P}$ on the second factor is a locally trivial fibration whose fiber is isomorphic to $\mathcal{U}(P^1) = \mathcal{U}(L^1)U_P$. We will be able to conclude that

$\tilde{\mathcal{U}}(G^1)^P$ is a smooth variety and that its dimension is equal to $\dim(G/P) + \dim(L) - \mathrm{rk}(L^\sigma) + \dim(U_P) = n - r'$, where $n = \dim(G)$ and $r' = \mathrm{rk}(G^\sigma)$ as $\mathrm{rk}(L^\sigma) = \mathrm{rk}(G^\sigma)$.

Let P' be in $\mathcal{P}$, then there is x in G such that $P' = {}^{xw_0}P$, where w_0 is the longest element of W and $V(P') = \{{}^{xuw_0}P; u \in U\}$ is an open neighbourhood of P' because Uw_0B a dense open set of G and Uw_0B/P is a dense open set of G/P. Moreover, $V(P') \simeq U_P$ as we have $w_0(u) \in U^-$ for all $u \in U$.

Let us prove that the morphism $\phi : U_P \times \mathcal{U}(L^1)U_P \to p_2^{-1}(V(P'))$ defined by $(u, p.\sigma) \mapsto ({}^{xuw_0}(p.\sigma), {}^{xuw_0}P)$ is an isomorphism. Let $(g.\sigma, P_1)$ be in $p_2^{-1}(V(P'))$, then there is only one $u \in U_P$ such that $P_1 = {}^{xuw_0}P$ and if we set $\psi((g.\sigma, P_1)) = (u, {}^{(xuw_0)^{-1}}(g.\sigma))$, then ψ and ϕ are inverse morphisms of varieties. So, $p_2 : \tilde{\mathcal{U}}(G^1)^P \to \mathcal{P}$ is a locally trivial fibration whose fiber is isomorphic to $\mathcal{U}(P^1)$.

Similarly, we have $\mathcal{V}_u = G \times^P \mathcal{U}(L^1)$ and $\tilde{\mathcal{V}}_u = G \times^B \mathcal{U}((L \cap B)^1)$. So, $r_u : \mathcal{V}_u \to \mathcal{P}$ is a locally trivial fibration whose fiber is isomorphic to $\mathcal{U}(L^1)$ and $\tilde{r}_u : \tilde{\mathcal{V}}_u \to \mathcal{B}$ is a locally trivial fibration whose fiber is isomorphic to $\mathcal{U}((L \cap B)^1)$.

Then, these varieties are smooth. Moreover, they have the same dimension, as $\dim(\mathcal{V}_u) = \dim(\mathcal{P}) + \dim(\mathcal{U}(L^1))$ and,

$$\begin{aligned}
\dim(\tilde{\mathcal{V}}_u) &= \dim(\mathcal{B}) + \dim(\mathcal{U}((L \cap B)^1)) \\
&= \dim(G) - \dim(B) + \dim(\mathcal{L}_\sigma(T)) + \dim(U) - \dim(U_P) \\
&= \dim(G) - \dim(B) + \dim(T) - \dim(T^\sigma) + \dim(U) - \dim(P) + \dim(L) \\
&= \dim(G) - \mathrm{rk}(L^\sigma) - \dim(P) + \dim(L) \\
&= \dim(\mathcal{P}) + \dim(\mathcal{U}(L^1))
\end{aligned}$$

(ii) $\tilde{G}^1$ is a closed subset of $G^1 \times \mathcal{B}$ and $\eta = (\mathrm{Id} \times f) \circ i$, where $i : \tilde{G}^1 \hookrightarrow G^1 \times G/B$ and where f associates to any $B' \in \mathcal{B}$ the only parabolic subgroup $P' \in \mathcal{P}$ such that $B' \subset P'$. The morphism i is proper as it is a closed immersion Moreover, the morphism $f : \mathcal{B} \to \mathcal{P}$ is proper as it is separated, of finite type and universally closed as $\mathcal{B}^G$ is complete. Thus, $\eta = (\mathrm{Id} \times f) \circ i$ is proper.

Let us now prove that ζ is proper. We consider the following cartesian diagram:

$$\begin{array}{ccc}
\mathcal{V} \times_{\mathcal{P}} \mathcal{B} & \to & \mathcal{B} \\
\downarrow f' & & \downarrow f, \\
\mathcal{V} & \xrightarrow{r} & \mathcal{P}
\end{array}$$

The variety $\mathcal{V} \times_{\mathcal{P}} \mathcal{B}$ is the set $\{(g.\sigma, P', B') | g.\sigma \in \mathrm{N}_{\tilde{G}}(P')/U_{P'} | f(B') = P'\}$. Recall that f is a proper, then, by base extension, f' is proper. Moreover, $\tilde{\mathcal{V}} = \{(g.\sigma, B', P') | g.\sigma \in \mathrm{N}_{\tilde{G}}(B')/U_{P'} | f(B') = P'\}$ is a closed subset of $\mathcal{V} \times_{\mathcal{P}} \mathcal{B}$ and the closed immersion $i : \tilde{\mathcal{V}} \hookrightarrow \mathcal{V} \times_{\mathcal{P}} \mathcal{B}$ is proper. So, $\zeta = f' \circ i$ is proper.

(ii') is a consequence of (ii) as $\tilde{\mathcal{V}}_u$ is a closed smooth subvariety of $\tilde{\mathcal{V}}$ and $\tilde{\mathcal{U}}(G^1)$ is a closed smooth subvariety of $\tilde{G}^1$.

In order to prove (iii), we need the following lemma:

Lemma 7.2.

a) The morphism $\tilde{\mathcal{U}}(G^1)^P \xrightarrow{q_u} \mathcal{V}_u$ is a fibration whose fiber at $v \in \mathcal{V}_u$ is isomorphic to U_P.

b) For any $y \in G \times^P (C^L U_P)$ (resp. $y \in G \times^P C^L$), we have $\eta_u^{-1}(y) \simeq \mathcal{B}_t^L$ (resp. $\zeta_u^{-1}(y) \simeq \mathcal{B}_t^L$), where t is in C^L.

c) For any unipotent L-conjugacy class C^L of L^1, the variety $G \times^P (C^L U_P)$ is a double fibration $G \times^P (C^L U_P) \xrightarrow{q} G \times^P C^L \xrightarrow{r} \mathcal{P}$, where the fiber of $q|_{G \times^P (C^L U_P)}$ is U_P and the fiber of $r|_{G \times^P C^L}$ is C^L.

d) the codimension of $G\times^P (C^L U_P)$ in $\tilde{\mathcal{U}}(G^1)^P$ is equal to that of C^L in $\mathcal{U}(L^1)$ and to that of $G\times^P C^L$ in $\mathcal{V}_u$.

Proof of lemma 7.2:

a) is clear using the identification $\tilde{\mathcal{U}}(G^1)^P = G\times^P \mathcal{U}(P^1)$, as we have $\mathcal{U}(P^1)\simeq \mathcal{U}(L^1)U_P$ and $\mathcal{V}_u = G\times^P \mathcal{U}(L^1)$.

b) Let C^L be a unipotent L-class in L^1. Then, if $l_1.\sigma$ and $l_2.\sigma$ are in C^L, the varieties $\mathcal{B}^L_{l_1.\sigma}$ are $\mathcal{B}^L_{l_2.\sigma}$ are isomorphic.

Let (y,P') be in $G\times^P C^L$. Then, there is $x\in G$ and $l.\sigma\in C^L$ such that $P' = {}^xP$ and $x^{-1}yx = l.\sigma U_P$ and the fiber of ζ_u at (y,P') is isomorphic to the variety $\mathcal{B}^P_{l.\sigma}$ of Borel subgroups of G which are contained in P and stable under conjugation by $l.\sigma$. Whence the result for ζ_u as $B'\mapsto B'\cap L$ is an isomorphism from $\mathcal{B}^P_{l.\sigma}$ onto $\mathcal{B}^L_{l.\sigma}$.

Let $(g.\sigma,P')$ be in $G\times^P (C^L U_P)$, then there is $x\in G$ and $p.\sigma\in C^L U_P$ such that $P'={}^xP$ and $x^{-1}g.\sigma x = p.\sigma$ and the fiber of η_u at $(g.\sigma,P')$ is isomorphic to the variety $\mathcal{B}^P_{p.\sigma}$. Moreover, if we set $l.\sigma = \pi_P(p.\sigma)$, then, $\mathcal{B}^P_{p.\sigma}\simeq \mathcal{B}^L_{l.\sigma}$. Whence the result for η_u.

c) is clear.

d) is a consequence of a) and c). Indeed, by a), we have

$$\dim(\tilde{\mathcal{U}}(G^1)^P) = \dim(\mathcal{U}(L^1)) + \dim(U_P) + \dim(\mathcal{P})$$

and

$$\dim(\mathcal{V}_u) = \dim(\tilde{\mathcal{U}}(G^1)^P) - \dim(U_P)$$

and according to c), we have

$$\dim(G\times^P (C^L U_P)) = \dim(U_P) + \dim(C^L) + \dim(\mathcal{P})$$

and

$$\dim(G\times^P C^L) = \dim(G\times^P (C^L U_P)) - \dim(U_P).$$

So,

$$\dim(\tilde{\mathcal{U}}(G^1)^P) - \dim(G\times^P (C^L U_P)) = \dim(\mathcal{U}(L^1)) - \dim(C^L) = \dim(\mathcal{V}_u) - \dim(G\times^P C^L).$$

End of the proof of lemma 7.2

Let us prove that η_u is semi-small. It is a proper morphism between two equidimensional varieties. Let us set

$$f(i) = \dim\{y\in \tilde{\mathcal{U}}(G^1)^P | \dim(\eta_u^{-1}(y)) \geq i/2\}.$$

By b) in lemma 7.2, the dimension of the fiber $\eta_u^{-1}(y)$ is constant for $y\in G\times^P (C^L U_P)$.

Let us set $d_{C^L} = \dim(\eta_u^{-1}(y))$, where y is in $G\times^P (C^L U_P)$.

Let C^L_i be the element whose dimension is maximal in the set of unipotent L-conjugacy classes C^L of L^1 such that $d_{C^L}\geq i/2$; then $f(i)$ is equal to the dimension of $G\times^P (C^L_i U_P)$.

Let us now put

$$g(i) = \dim\{t\in \mathcal{U}(L^1) | \dim(\mathcal{B}^L_t)\geq i/2\}.$$

The dimension of $\mathcal{B}^L_t$ is constant for $t\in C^L$ where C^L is a unipotent L-conjugacy class in L^1. So, $g(i)$ is equal to the dimension of the unipotent L-conjugacy classes of L^1 whose dimension is maximal in the set of classes C^L such that $\dim\mathcal{B}^L_t\geq i/2$ for $t\in C^L$. Moreover, by b) in lemma 7.2, we have $d_{C^L} = \dim\mathcal{B}^L_t$, for $t\in C^L$ and by d) in lemma 7.2, we have $\operatorname{codim}_{\tilde{\mathcal{U}}(G^1)^P}(G\times^P (C^L U_P)) = \operatorname{codim}_{\mathcal{U}(L^1)}(C^L)$. So $f(i) = g(i) + \dim(\tilde{\mathcal{U}}(G^1)^P) - \dim(\mathcal{U}(L^1))$. But, applying corollary 3.2, we have $g(i)\leq \dim(\mathcal{U}(L^1)) - i$, hence, $f(i)\leq \dim(\tilde{\mathcal{U}}(G^1)^P) - i$ and η_u is semi-small.

The proof for ζ_u is similar but $f(i)$ has to be replaced by:

$$h(i)=\dim\{y\in\mathcal{V}_u|\dim(\zeta_u^{-1}(y))\geq i/2\}.$$

End of the proof of proposition 7.1.

According to proposition 7.1, the hypotheses of theorem 6.2 are verified for η_u and ζ_u. Hence, we have the following decomposition formula in $\mathcal{M}(\mathcal{U}(G^1))$:

$$A^P[\dim(\mathcal{U}(G^1))]=\eta_{u*}\overline{\mathbb{Q}}_l[\dim(\mathcal{U}(G^1))]\ \simeq\ \bigoplus_{(C',\mathcal{E}')}\mathrm{IC}(\overline{C'},\mathcal{E}')[\dim(C')]\otimes V'_{(C',\mathcal{E}')}\tag{4}$$

where C' runs over the set of G-stable irreducible smooth varieties of $\tilde{\mathcal{U}}(G^1)^P$ and $\mathcal{E}'$ over the set of G-equivariant irreducible local systems on C'.

Moreover, $K[\dim(\mathcal{V}_u)]=\zeta_{u*}\overline{\mathbb{Q}}_l[\dim(\mathcal{V}_u)]$ is a G-equivariant semi-simple object of $\mathcal{M}(\mathcal{V}_u)$ and if $\mathrm{IC}(\overline{Y},\mathcal{F}')[\dim(Y)]$ is a simple object whose multiplicity in $K[\dim(\mathcal{V}_u)]$ is non zero, then, Y is a G-stable irreducible smooth subvariety of $\mathcal{V}_u$ and $\mathcal{F}'$ is a G-equivariant irreducible local system on Y. Then, $\overline{Y}$ is a G-stable irreducible closed subset of $\mathcal{V}_u=G\times^P\mathcal{U}(L^1)$. So, we have $\overline{Y}=G\times^P X$, where X is a L-stable irreducible closed subset of $\mathcal{U}(L^1)$, that is to say, the closure of a unipotent L-conjugacy class inL^1. We then get the following decomposition formula for $K[\dim(\mathcal{V}_u)]=\zeta_{u*}\overline{\mathbb{Q}}_l[\dim(\mathcal{V}_u)]$ in $\mathcal{M}(\mathcal{V}_u)$:

$$K[\dim(\mathcal{V}_u)]\ \simeq\ \bigoplus_{(C^L,\mathcal{F}')}\mathrm{IC}(G\times^P\overline{C^L},\mathcal{F}')[\dim(G\times^P C^L)]\otimes V''_{(C^L,\mathcal{F}')}\tag{5}$$

where C^L runs over the set of unipotent L-conjugacy classes in L^1, and $\mathcal{F}'$ over the set of G-equivariant irreducible local systems on $G\times^P C^L$.

7.3. **Identification of the decompositions of A^P and of $A(L)$.** The purpose of this subsection is to prove proposition 7.4 which shows how we can identify the decompositions (3) of $A(L)$ and (4) of A^P via the functors q_u^* et i_u^*. We need the following lemma:

Lemma 7.3.

a) For any unipotent L-conjugacy class C^L of L^1, the map $\mathcal{F}\mapsto(i|_{C^L})^*\mathcal{F}$ is a bijection from the set of G-equivariant irreducible local systems on $G\times^P C^L$ to the set of L-equivariant irreducible local systems on C^L.

b) Let C^L be a unipotent L-conjugacy class in L^1. The image of $G\times^P(C^LU_P)$ under q is $G\times^P C^L$ and if $\mathcal{F}$ is a G-equivariant irreducible local system on $G\times^P C^L$, then, $(q|_{G\times^P(C^LU_P)})^*\mathcal{F}$ is a G-equivariant irreducible local system on $G\times^P(C^LU_P)$, where q is defined in diagram 7.1.

Proof:

a) Let $C^L=\mathrm{Cl}_L(x.\sigma)$ be a unipotent L-conjugacy class in L^1. Then, $y=(1,x.\sigma)$ is in $G\times^P C^L$.

According to [13] (3.5), as L is a connected group which acts transitively on C^L, there is a bijection f_1 between L-equivariant irreducible local systems on C^L and irreducible finite dimensional $\overline{\mathbb{Q}}_l$-representations of $A_L(x.\sigma)=\mathrm{C}_L(x.\sigma)/\mathrm{C}_L(x.\sigma)^0$. If $\mathcal{L}$ is a L-equivariant irreducible local system on C^L, then, $A_L(x.\sigma)$ acts on $\mathcal{L}_{x.\sigma}$, this is an irreducible representation ρ of $A_L(x.\sigma)$ and the bijection f_1 is defined by $\mathcal{L}\mapsto\rho$.

Similarly, the action of G on $G\times^P C^L$, defined by $g:(h,l.\sigma)\mapsto(gh,l.\sigma)$ for all $g\in G$, is transitive. Indeed, let $(h,l.\sigma)$ be $G\times^P C^L$, then, there is $l'\in L$ such that $l.\sigma={}^{l'}(x.\sigma)$ and $(h,l.\sigma)=(hl',x.\sigma)=(hl').(1,x.\sigma)$. Hence, there is a bijection f_2 between G-equivariant irreducible local systems on $G\times^P C^L$ and irreducible finite dimensional $\overline{\mathbb{Q}}_l$-representations of $A_G(y)=\mathrm{Stab}_G(y)/\mathrm{Stab}_G(y)^0$. If $\mathcal{L}'$ is a G-equivariant irreducible local system on $G\times^P C^L$, then, $A_G(y)$ acts on $\mathcal{L}'_y$, this is an irreducible representation ρ' of $A_G(y)$ and the bijection f_2 is defined by $\mathcal{L}' \mapsto \rho'$.

But, we have $\mathrm{Stab}_G(y) = \mathrm{C}_L(x.\sigma)U_P$, whence $A_G(y) = A_L(x.\sigma)$, for U_P is connected, and there is a bijection $(f_1^{-1} \circ f_2)$ between G-equivariant irreducible local systems on $G \times^P C^L$ and L-equivariant irreducible local systems on C^L.

Let us prove that this bijection is induced by the functor $(i|_{C^L})^*$. Let $\mathcal{L}'$ be a G-equivariant irreducible local system on $G \times^P C^L$, it is sent by the bijection f_2 to the representation ρ of $A_G(y)$ on $\mathcal{L}'_y$ and $(i|_{C^L})^*\mathcal{L}'$ is a L-equivariant irreducible local system on C^L which is sent by the bijection f_1 to the same representation ρ of $A_G(y)$ $(= A_L(x.\sigma))$ on $\mathcal{L}'_y$ $(= (i|_{C^L})^*(\mathcal{L}')_{x.\sigma}$ for $i(x.\sigma) = y)$. Indeed, as $i|_{C^L}$ is L-equivariant, it is compatible with the actions of $A_L(x.\sigma)$.

Whence the result.

b) Let us consider the variety $G \times^P (C^L U_P)$, then $q|_{G\times^P(C^L U_P)} : G \times^P C^L U_P \to G \times^P C^L$ is a fibration whose fiber is isomorphic to U_P. Let us set $d = \dim(U_P)$. The morphism $q|_{G\times^P(C^L U_P)}$ is smooth with connected fibers whose dimension is equal to d, so the image of an irreducible local system by the functor $(q|_{G\times^P(C^L U_P)})^*$ is still an irreducible local system. Moreover, $q|_{G\times^P(C^L U_P)}$ is G-equivariant. So, for all G-equivariant irreducible local system $\mathcal{L}$ on $G\times^P C^L$, the local system $(q|_{G\times^P(C^L U_P)})^*(\mathcal{L})$ is still irreducible and G-equivariant.

End of the proof

Let C^L be a unipotent L-conjugacy class in L^1. We denote by $\Phi(C^L)$ the bijection obtained using lemma 7.3 a) and b) between L-equivariant irreducible local systems on C^L and G-equivariant irreducible local systems on $G \times^P C^L U_P$.

Proposition 7.4.

(i) We have the following isomorphisms in the derived category.

$$A^P \simeq q_u^* K,$$

$$A(L) \simeq i_u^* K.$$

(ii) $\mathrm{End}K \simeq \mathrm{End}A^P$ *via the functor* q_u^* *and* $\mathrm{End}K \simeq \mathrm{End}A(L)$ *via the functor* i_u^*.

(iii) The perverse sheaf $A^P[\dim(\mathcal{U}(G^1))]$ *is decomposed in* $\mathcal{M}(\tilde{\mathcal{U}}(G^1)^P)$ *as:*

$$A^P[\dim(\mathcal{U}(G^1))] \simeq \bigoplus_{(C^L,\mathcal{E}')} \mathrm{IC}(G \times^P (\overline{C^L}U_P), \mathcal{E}')[\dim(G \times^P (\overline{C^L}U_P))] \otimes V'_{(C^L,\mathcal{E}')} \tag{6}$$

where C^L *runs over the set of unipotent* L*-conjugacy classes of* L^1 *and* $\mathcal{E}'$ *is the image under* $\Phi(C^L)$ *of an* L*-equivariant local system* $\mathcal{E}$ *on* C^L.

Moreover, if $V_{(C^L,\mathcal{E})}$ *is the* $\overline{\mathbb{Q}}_l$*-vector space corresponding to the simple object* $\mathrm{IC}(\overline{C^L}, \mathcal{E})[\dim(C^L)]$ *in the decomposition (3) of* $A(L)[\dim(\mathcal{U}(L^1))]$*, then* $V_{(C^L,\mathcal{E})}$ *is isomorphic to* $V'_{(C^L,\mathcal{E}')}$ *as* $\overline{\mathbb{Q}}_l$*-vector spaces.*

Proof:

Let us consider the constant sheaf $\overline{\mathbb{Q}}_l$ on $\tilde{\mathcal{V}}_u$. Then, we have the following isomorphisms in the derived category: $A^P \simeq \eta_{u*}\tilde{q}_u^*\overline{\mathbb{Q}}_l$ and $A(L) \simeq \pi(L)_{u*}\tilde{i}_u^*\overline{\mathbb{Q}}_l$. Moreover, we have $K = \zeta_{u*}\overline{\mathbb{Q}}_l$. As $\pi(L)_u$ and η_u are proper, (i) is a consequence of the proper base change theorem.

Let us prove the other assertions. We have the following decomposition formulas (cf. 7.1, formula (3) and 7.2, formulas (5) and (4)).

$$A(L) \simeq \bigoplus_{(C^L,\mathcal{F})} \mathrm{IC}(\overline{C^L}, \mathcal{F})[2d_{C^L}] \otimes V_{(C^L,\mathcal{F})}$$

where C^L runs over the set of unipotent L-conjugacy classes of L^1 and $\mathcal{F}$ runs over the set of L-equivariant irreducible local systems on $G \times^P C^L$ and where we set $d_{C^L} = 1/2(\dim(C^L) - \dim(\mathcal{U}(L^1)))$.

$$K \simeq \bigoplus_{(C^L,\mathcal{F})} \mathrm{IC}(G \times^P \overline{C^L}, \mathcal{F})[d_{C^L}] \otimes V''_{(C^L,\mathcal{F})}$$

where C^L runs over the set of unipotent L-conjugacy classes of L^1, and $\mathcal{F}$ the set of G-equivariant irreducible local systems on $G \times^P C^L$ and where $d_{C^L} = 1/2(\dim(G \times^P C^L) - \dim(\mathcal{V}_u))$ (lemma 7.2, d)).

And,

$$A^P \simeq \bigoplus_{(C',\mathcal{E}')} \mathrm{IC}(\overline{C'}, \mathcal{E}')[2d_{C'}] \otimes V'_{(C',\mathcal{E}')}$$

where C' runs over the set of G-stable irreducible smooth subvarieties of $\tilde{\mathcal{U}}(G^1)^P$ and $\mathcal{E}'$ the set of G-equivariant irreducible local systems on C' and where we set $d_{C'} = 1/2(\dim(C') - \dim(\tilde{\mathcal{U}}(G^1)^P))$.

But, as $A(L)$ is isomorphic to $i_u^* C$, we get another decomposition of $A(L)$:

$$A(L) \simeq \bigoplus_{(C^L,\mathcal{F})} i_u^* \mathrm{IC}(G \times^P \overline{C^L}, \mathcal{F})[2d_{C^L}] \otimes V''_{(C^L,\mathcal{F})}$$

And as A^P is isomorphic to $q_u^* C$, we get another decomposition of A^P:

$$A^P \simeq \bigoplus_{(C^L,\mathcal{F})} q_u^* \mathrm{IC}(G \times^P \overline{C^L}, \mathcal{F})[2d_{C^L}] \otimes V''_{(C^L,\mathcal{F})}$$

where we have $d_{G\times^P(C^L U_P)} = d_{C^L}$ (lemma 7.2, d)).

Then, applying lemma 7.5, for any unipotent class C^L of L^1 and for any G-equivariant irreducible local system $\mathcal{F}'$ on $G \times^P C^L$, we can identify the $\overline{\mathbb{Q}}_l$-vector spaces $V_{(C^L,(i_{C^L})^*\mathcal{F})}$ and $V''_{(C^L,\mathcal{F})}$. Moreover, for all G-stable irreducible smooth variety C' of $\tilde{G}^{1P}$ and for all G-equivariant irreducible local system $\mathcal{E}'$, we have $V'_{(C',\mathcal{E}')} = 0$ except if there is a unipotent L-conjugacy class C^L of L^1 such that $C' = G\times^P(C^L U_P)$ and a G-equivariant irreducible local system $\mathcal{F}$ on $G\times^P C^L$ such that $\mathcal{E}' = (q_{G\times^P C^L})^*\mathcal{F}$ in which case we have $V'_{(C',\mathcal{E}')} = V''_{(C^L,\mathcal{F})}$. Thus, proposition 7.4 will be proved if we prove lemma 7.5.

Lemma 7.5.

(i) For any unipotent L-conjugacy class C^L of L^1 and for any local system $\mathcal{L}$ on $G \times^P C^L$, the functor q^* sends $\mathrm{IC}(G \times^P \overline{C^L}, \mathcal{L})$ over $\mathrm{IC}(G \times^P (\overline{C^L} U_P), (q|_{G\times^P(C^L U_P)})^*(\mathcal{L}))$.

(ii) For any unipotent L-conjugacy class C^L of L^1 and for all G-equivariant irreducible local systems $\mathcal{L}$ on $G \times^P C^L$, the functor i^* maps $\mathrm{IC}(G \times^P \overline{C^L}, \mathcal{L})$ to $\mathrm{IC}(\overline{C^L}, (i|_{C^L})^*\mathcal{L})$.

Proof of lemma 7.5:

Set $d = \dim(G\times^P(C^L U_P)) - \dim(G\times^P C^L)$. As $q|_{G\times^P(\overline{C^L}U_P)}$ is a smooth morphism of relative dimension d with connected fibers, the image of the simple object $\mathrm{IC}(G\times^P\overline{C^L}, \mathcal{L})[\dim(G\times^P C^L)]$ of $\mathcal{M}(G \times^P \overline{C^L})$ by $q^*[d]$ is the simple object $\mathrm{IC}(G \times^P (\overline{C^L} U_P), (q|_{G\times^P(C^L U_P)})^*\mathcal{L})[\dim(G \times^P (C^L U_P))]$ of $\mathcal{M}(G \times^P \overline{C^L} U_P)$ ([1], 4.2.6), whence the result.

Let us now prove (ii).

We denote by i' the closed immersion from $\overline{C^L}$ to $G\times\overline{C^L}$ and by p_2 the projection on the second factor from $G\times\overline{C^L}$ to $\overline{C^L}$. Let pr be the quotient morphism from $G\times\overline{C^L}$ to $G\times^P\overline{C^L}$. We have $p_2\circ i'=\mathrm{Id}_{\overline{C^L}}$ and $i|_{\overline{C^L}}=pr\circ i'$. We get the following diagram:

$$\begin{tikzcd} \overline{C^L} \arrow[r, shift left, "i'"] \arrow[dr, "i_{\overline{C^L}}"'] & G\times\overline{C^L} \arrow[l, shift left, "p_2"] \arrow[d, "pr"] \\ & G\times^P\overline{C^L} \end{tikzcd}$$

Let $\mathcal{L}'$ be a local system on C^L. As p_2 is a smooth morphism of relative dimension $\dim(G)$ with connected fibers, we have $p_2^*\mathrm{IC}(\overline{C^L},\mathcal{L}')=\mathrm{IC}(G\times\overline{C^L},(p_2|_{G\times C^L})^*\mathcal{L}')$. So we get

$$i'^*\mathrm{IC}(G\times\overline{C^L},(p_2|_{G\times C^L})^*\mathcal{L}')=\mathrm{IC}(\overline{C^L},\mathcal{L}').$$

Moreover, let $\mathcal{L}$ be a G-equivariant irreducible local system on $G\times^P C^L$ then, as pr is a smooth morphism of relative dimension $\dim(P)$ with connected fibers, we have $pr^*\mathrm{IC}(G\times^P\overline{C^L},\mathcal{L})=\mathrm{IC}(G\times\overline{C^L},(pr|_{C^L})^*\mathcal{L})$. Then we get

$$(i|_{\overline{C^L}})^*\mathrm{IC}(G\times^P\overline{C^L},\mathcal{L})=i'^*\mathrm{IC}(G\times\overline{C^L},(pr|_{G\times C^L})^*\mathcal{L}).$$

So (ii) will be proved if we get

$$(pr|_{G\times C^L})^*\mathcal{L}=(p_2|_{G\times C^L})^*(i|_{C^L})^*(\mathcal{L}).$$

Let us consider the following commutative diagram:

$$\begin{tikzcd} G\times L\times G\times C^L \arrow[r, "\tilde{pr}"] \arrow[d, shift right, "p"'] \arrow[d, shift left, "m"] & G\times G\times^P C^L \arrow[d, shift right, "m'"'] \arrow[d, shift left, "p'"] \\ G\times C^L \arrow[r, "pr|_{G\times C^L}"'] & G\times^P C^L \end{tikzcd}$$

where p and p' are the projections on the second factor, m (resp. m') the action of $G\times L$ on $G\times C^L$ (resp. of G on $G\times^P C^L$) defined by $(g,l):(x,h.\sigma)\mapsto(gx,{}^l(h.\sigma))$ (resp. by $g:(x,h.\sigma)\mapsto(gx,h.\sigma)$) and where $\tilde{pr}:G\times L\times G\times C^L\to G\times G\times^P C^L$ is defined by $(g,l,x,h.\sigma)\mapsto(l^{-1}g,x,h.\sigma)$.

As $\mathcal{L}$ is a G-equivariant local system on $G\times^P C^L$, we have $m'^*\mathcal{L}=p'^*\mathcal{L}$, and as the above diagram is commutative, we get $m^*(pr|_{G\times C^L})^*\mathcal{L}=p^*(pr|_{G\times C^L})^*\mathcal{L}$. So, $(pr|_{G\times C^L})^*\mathcal{L}$ is a $G\times L$-equivariant local system on $G\times C^L$ and is of the form $\mathcal{L}_1\boxtimes\mathcal{L}_2$ where $\mathcal{L}_1$ is a G-equivariant local system on G and $\mathcal{L}_2$ is a L-equivariant local system on C^L. But any G-equivariant local system on G is isomorphic to $\overline{\mathbb{Q}}_l$. Hence, we have

$$(pr|_{G\times C^L})^*\mathcal{L}=\overline{\mathbb{Q}}_l\boxtimes\mathcal{L}_2=(p_2|_{G\times C^L})^*\mathcal{L}_2.$$

We also have

$$(i|_{C^L})^*(\mathcal{L})=(i'|_{C^L})^*(pr|_{G\times C^L})^*\mathcal{L}=(i'|_{C^L})^*(p_2|_{G\times C^L})^*\mathcal{L}_2=\mathcal{L}_2.$$

Whence, the result.

End of the proof of lemma 7.5.

End of the proof of proposition 7.4.

7.4. **Action of $W(L)^\sigma$ on A^P.** In this subsection, we first construct an action of $W(L)^\sigma$ on the direct image $\eta_*\overline{\mathbb{Q}}_l$ of the constant sheaf $\overline{\mathbb{Q}}_l$ on $\tilde{G}^1$ and an action of $W(L)^\sigma$ on the direct image $\zeta_*\overline{\mathbb{Q}}_l$ of the constant sheaf $\overline{\mathbb{Q}}_l$ on $\tilde{\mathcal{V}}$ (propositions 7.6 and 7.7). The action of $W(L)^\sigma$ on $\eta_*\overline{\mathbb{Q}}_l$ induces an action on $A^P = \eta_{u*}\overline{\mathbb{Q}}_l$. In proposition 7.9, we compare the Springer action of $W(L)^\sigma$ from L^1 on $A(L) = \pi(L)_{u*}\overline{\mathbb{Q}}_l$ and the above action of $W(L)^\sigma$ on A^P.

For any σ-stable subgroup H of G, we denote by H^1_{reg} the set of quasi-semi-simple elements of H^1 which are regular in G^1. As L^σ and G^σ have same rank, $x \in L^1$ is quasi-semi-simple regular in G^1 if and only if it is quasi-simple regular in L^1. So, if H is a σ-stable subgroup of L, then, H^1_{reg} is the set of quasi-semi-simple elements of H^1 which are regular in L^1.

Let us consider the following diagram:

Diagram 7.3.

$$\begin{tikzcd}
\pi(L)^{-1}(L^1_{reg}) \arrow[r, "\tilde{i}_{reg}"] \arrow[d, "\pi(L)_{reg}"] & \zeta^{-1}(\mathcal{V}_{reg}) \arrow[d, "\zeta_{reg}"] & \eta^{-1}(\tilde{G}^{1P}_{reg}) \arrow[l, "\tilde{q}_{reg}"'] \arrow[d, "\eta_{reg}"] \\
L^1_{reg} \arrow[r, "i_{reg}"'] & \mathcal{V}_{reg} = G \times^P L^1_{reg} & \tilde{G}^{1P}_{reg} = G \times^P P^1_{reg} \arrow[l, "q_{reg}"] \arrow[d, "\xi_{reg}"] \\
 & & G^1_{reg}
\end{tikzcd}$$

where we have denoted by the index reg the restrictions of the morphisms of diagram 7.1 to the regular elements.

Proposition 7.6.
(i) The restriction η_{reg} of $\eta : \tilde{G}^1 \to \tilde{G}^{1P}$ to $\eta^{-1}(\tilde{G}^{1P}_{reg}) = \pi^{-1}(G^1_{reg})$ is a galoisian covering with Galois group $W(L)^\sigma$.
(ii) The restriction ζ_{reg} of $\zeta : \tilde{\mathcal{V}} \to \mathcal{V}$ to $\zeta^{-1}(\mathcal{V}_{reg})$ is a galoisian covering with Galois group $W(L)^\sigma$.

Proof:
(i) Let us consider the following cartesian diagram:

$$\begin{tikzcd}
\eta^{-1}(G^{1P}_{reg}) \arrow[r] \arrow[d, "\eta_{reg}"] & \tilde{G}^1 \arrow[d, "\eta"] \\
\tilde{G}^{1P}_{reg} \arrow[r] & \tilde{G}^{1P}
\end{tikzcd}$$

The morphism η_{reg} is defined by $(g.\sigma, B') \mapsto (g.\sigma, f(B'))$, where $f(B')$ is the only parabolic subgroup P' of $\mathcal{P}$ such that $B' \subset P'$.

Let $(g.\sigma, P')$ be in $\tilde{G}^{1P}_{reg}$ and let us consider the fiber at $(g.\sigma, P')$. Quasi-semi-simple regular elements of G^1 can be obtained under conjugation by G from elements of $T^\sigma_{reg}.\sigma$, where T^σ_{reg} is the set of elements of T^σ which are regular in G^σ. So, we can suppose that $g.\sigma \in T^\sigma.\sigma$ and that $P' = P$.

Then, we have $\mathcal{B}^G_{g.\sigma} = \{{}^wB; w \in W^\sigma\}$. But, if $\mathcal{B}^P$ is the set of Borel subgroups of G which are included in P, we have $\mathcal{B}^P = \{{}^wB; w \in W(L)\}$. Hence, $\eta^{-1}_{reg}((g.\sigma, P')) \simeq \mathcal{B}^P_{g.\sigma} = \{{}^wB; w \in W(L)^\sigma\}$.

The morphism $\eta_{reg} : \tilde{G}^1_{reg} \to \tilde{G}^{1P}_{reg}$ is a finite morphism of varieties and the cardinality of its fibers is equal to $|W(L)^\sigma|$. It is then a flat morphism and as it is surjective, it is faithfully flat.

The action of $W(L)^\sigma$ on $\pi^{-1}(G^1_{reg})$ is obtained through the isomorphism $\tilde{G}^1_{reg} \to \pi^{-1}(G^1_{reg})$ of proposition 4.2 from the restriction to $W(L)^\sigma$ of the action of $W(G)^\sigma$ on $\tilde{G}^1_{reg}$. This is an action on the right on $\tilde{G}^{1P}_{reg}$. Indeed, η_{reg} is $W(L)^\sigma$-equivariant with respect to the action of $W(L)^\sigma$ on $\tilde{G}^1_{reg} \simeq \pi^{-1}(G^1_{reg})$ defined by $w : (g.\sigma, xT) \mapsto (g.\sigma, xwT)$ and with respect to the trivial action of $W(L)^\sigma$ on $\tilde{G}^{1P}_{reg}$, for we have $f({}^wB) = P$ for all $w \in W(L)^\sigma$. Moreover, if $((g.\sigma, xT), (g'.\sigma, x'T))$ is in $\tilde{G}^1_{reg} \times_{\tilde{G}^{1P}_{reg}} \tilde{G}^1_{reg}$, then, $\eta_{reg}((g.\sigma, xT)) = \eta_{reg}((g'.\sigma, x'T))$, which means that $g'.\sigma = g.\sigma$, that there is $w \in W(L)^\sigma$ such that $x'T = xwT$ and that ${}^xP = {}^{x'}P$. Hence,

$$\tilde{G}^1_{reg} \times_{\tilde{G}^{1P}_{reg}} \tilde{G}^1_{reg} = \{((g.\sigma, xT), (g.\sigma, xwT)) | (g.\sigma, xT) \in \tilde{G}^1_{reg} \ and \ w \in W(L)^\sigma\}$$

Thus, the map from $W(L)^\sigma \times \tilde{G}^1_{reg}$ to $\tilde{G}^1_{reg} \times_{G^{1P}_{reg}} \tilde{G}^1_{reg}$ defined by $(w, (g.\sigma, xT)) \mapsto ((g.\sigma, xT), (g.\sigma, xwT))$ is an isomorphism.

(ii) Let us now consider the following cartesian diagram:

$$\begin{array}{ccc} \zeta^{-1}(\mathcal{V}_{reg}) = G \times^B (L \cap B)^1_{reg} & \xleftarrow{\tilde{q}_{reg}} & \tilde{G}^1_{reg} = G \times^B B^1_{reg} \\ \big\downarrow \zeta_{reg} & & \big\downarrow \eta_{reg} \\ \mathcal{V}_{reg} = G \times^P L^1_{reg} & \xleftarrow{q_{reg}} & \tilde{G}^{1P}_{reg} = G \times^P P^1_{reg} \end{array}$$

We have $B^1_{reg} = (L \cap B)^1_{reg} U_P$ and $P^1_{reg} = L^1_{reg} U_P$ for $B^1_{reg} = T^1_{reg} U$ (theorem 2.2). Then, the morphism q_{reg} and $\tilde{q}_{reg}$ are fibration whose fibers are isomorphic to U_P. Moreover, the action of $W(L)^\sigma$ on the fibers is trivial. Hence, the action of $W(L)^\sigma$ on $\tilde{G}^1_{reg}$ induces an action of $W(L)^\sigma$ on $\zeta^{-1}(\mathcal{V}_{reg})$ which is trivial and the action of $W(L)^\sigma$ on $\tilde{G}^{1P}_{reg}$ induces an action of $W(L)^\sigma$ on $G \times^P L^1_{reg}$ such that ζ_{reg} is a galoisian covering with Galois group $W(L)^\sigma$.

End of the proof.

We can deduce from proposition 7.6 that $W(L)^\sigma$ acts on the local system $\eta_{reg*}\overline{\mathbb{Q}}_l$ on $\tilde{G}^{1P}_{reg}$ and that $W(L)^\sigma$ acts on the local system $\zeta_{reg*}\overline{\mathbb{Q}}_l$ on $\mathcal{V}_{reg} = G \times^P L^1_{reg}$.

As L^1_{reg} is a dense subset of L^1, the subvariety $\mathcal{V}_{reg} = G \times^P L^1_{reg}$ is dense in $\mathcal{V} = G \times^P L^1$. Moreover, $P^1_{reg} = L^1_{reg} U_P$ is dense in P^1, so the subvariety $\tilde{G}^{1P}_{reg} = G \times^P P^1_{reg}$ is dense in $\tilde{G}^{1P} = G \times^P P^1$. Let us consider the intersection cohomology complexes $\mathrm{IC}(\tilde{G}^{1P}, \eta_{reg*}\overline{\mathbb{Q}}_l)$ and $\mathrm{IC}(\mathcal{V}, \zeta_{reg*}\overline{\mathbb{Q}}_l)$. As IC is a fully faithful functor and as $W(L)^\sigma$ acts on $\eta_{reg*}\overline{\mathbb{Q}}_l$ and on $\zeta_{reg*}\overline{\mathbb{Q}}_l)$, we get an action of $W(L)^\sigma$ on these two intersection cohomology complexes.

Proposition 7.7. *We have the following isomorphisms:*
(i) $\eta_*\overline{\mathbb{Q}}_l[n] \simeq \mathrm{IC}(\tilde{G}^{1P}, \eta_{reg*}\overline{\mathbb{Q}}_l)[n]$, *in* $\mathcal{M}(\tilde{G}^{1P})$ *where* $n = \dim(\tilde{G}^{1P})$.
(ii) $\zeta_*\overline{\mathbb{Q}}_l[n'] \simeq \mathrm{IC}(\mathcal{V}, \zeta_{reg*}\overline{\mathbb{Q}}_l)[n']$, *in* $\mathcal{M}(\mathcal{V})$ *where* $n' = \dim(\mathcal{V})$.

Proof: This is a consequence of the following result and of proposition 4.6:

Lemma 7.8. *(i) The variety* $\tilde{G^1}^P$ *is smooth and* $\dim(\tilde{G^1}^P) = \dim(\tilde{G}^1)$.
The varieties $\mathcal{V}_u$ *and* $\tilde{\mathcal{V}}_u$ *are smooth and* $\dim(\mathcal{V}_u) = \dim(\tilde{\mathcal{V}}_u) = \dim(\mathcal{P}) + \dim(L)$.
(ii) The morphisms η *and* ζ *are small.*

Proof of lemma 7.8: The proof of (i) is similar to that of proposition 7.1 (i).
Let us prove (ii): η and ζ are proper morphisms between equidimensional varieties.

Let us consider the following cartesian diagram:

$$\begin{array}{ccc} G\times^B (L\cap B)^1 & \xleftarrow{\tilde{q}} & \tilde{G}^1 = G\times^B B^1 \\ \big\downarrow \zeta & & \big\downarrow \eta \\ G\times^P L^1 & \xleftarrow{q} & \tilde{G}^{1P} = G\times^P P^1 \end{array}$$

The morphims q and $\tilde{q}$ are fibrations whose fibers are isomorphic to U_P.

For any integer i, we set

$$f(i) = \dim\{(x, l.\sigma) \in G\times^P L^1 | \dim\zeta^{-1}((x, l.\sigma)) \geq i/2\}$$

and

$$g(i) = \dim\{(x, p.\sigma) \in \tilde{G}^{1P} | \dim\eta^{-1}((x, p.\sigma)) \geq i/2\}$$

For $(x, p.\sigma) \in \tilde{G}^{1P}$, we have $\eta^{-1}((x, p.\sigma)) \simeq \mathcal{B}^P_{p.\sigma}$. Moreover, if π_P is the projection $\mathrm{N}_{\tilde{G}}(P) \to \mathrm{N}_{\tilde{G}}(L, B)$, we have $\mathcal{B}^P_{p.\sigma} \simeq \mathcal{B}^L_{\pi_P(p.\sigma)}$. Then, for $(x, l.\sigma) \in G\times^P L^1$, we have $\zeta^{-1}((x, l.\sigma)) \simeq \mathcal{B}^P_{l.\sigma} \simeq \mathcal{B}^L_{l.\sigma}$. Hence,

$$\begin{aligned} g(i) &= \dim\{(x, p.\sigma) \in \tilde{G}^{1P} | \dim\eta^{-1}((x, p.\sigma)) \geq i/2\} \\ &= \dim\{(x, p.\sigma) \in \tilde{G}^{1P} | \dim\mathcal{B}^P_{p.\sigma} \geq i/2\} \\ &= \dim\{(x, p.\sigma) \in \tilde{G}^{1P} | \dim\mathcal{B}^L_{\pi_P(p.\sigma)} \geq i/2\} \\ &= \dim\{(x, l.\sigma) \in G\times^P L^1 | \dim\mathcal{B}^L_{l.\sigma} \geq i/2\} + \dim(U_P) \\ &= f(i) + \dim(U_P) \end{aligned}$$

Moreover,

$$\begin{aligned} f(i) &= \dim\{(x, l.\sigma) \in G\times L^1 | \dim\mathcal{B}^L_{l.\sigma} \geq i/2\} - \dim(P) \\ &= \dim\{l.\sigma \in L^1 | \dim\mathcal{B}^L_{l.\sigma} \geq i/2\} + \dim(G) - \dim(P) \\ &\leq \dim(L^1) - i + \dim\mathcal{P} \\ &= \dim(G\times^P L^1) - i \end{aligned}$$

So, ζ is a small morphism and we have $g(i) \leq \dim(\tilde{G}^{1P}) - i$, whence η is also small.

End of the proof of lemma 7.8.

By lemma 7.8 and by proposition 4.6, $\eta_*\overline{\mathbb{Q}}_l[n]$ (resp. $\zeta_*\overline{\mathbb{Q}}_l[n']$) verifies properties (i), (iii) and (iv) of definition 4.1 which characterize $\mathrm{IC}(\tilde{G}^{1P}, \eta_{reg*}\overline{\mathbb{Q}}_l)[n]$ (resp. $\mathrm{IC}(G\times^P L^1, \zeta_{reg*}\overline{\mathbb{Q}}_l)[n']$).

Then, we just have to check that $\mathcal{H}^0\eta_*\overline{\mathbb{Q}}_l|_{\tilde{G}^{1P}_{reg}} = \eta_{reg*}\overline{\mathbb{Q}}_l$ and that $\mathcal{H}^0\zeta_*\overline{\mathbb{Q}}_l|_{G\times^P L^1_{reg}} = \zeta_{reg*}\overline{\mathbb{Q}}_l$, which comes from the proper base change theorem.

End of the proof of proposition 7.7.

By proposition 7.7, we have an action of $W(L)^\sigma$ on $\eta_*\overline{\mathbb{Q}}_l$ and on $\zeta_*\overline{\mathbb{Q}}_l$. Applying the proper base change theorem to η, we get $\eta_*\overline{\mathbb{Q}}_l|_{\tilde{\mathcal{U}}(G^1)^P} \simeq \eta_{u*}\overline{\mathbb{Q}}_l$. The action of $W(L)^\sigma$ on $\eta_*\overline{\mathbb{Q}}_l$ induces then an action of $W(L)^\sigma$ on $A^P = \eta_{u*}\overline{\mathbb{Q}}_l$.

Proposition 7.9.

a) The action of $W(L)^\sigma$ on A^P induces an action of $W(L)^\sigma$ on the vector spaces $V'_{(C^L,\mathcal{E}')}$ of the decomposition (6) in proposition 7.4. We denote by $\rho'_{(C^L,\mathcal{E}')}$ the representation of $W(L)^\sigma$ on $V'_{(C^L,\mathcal{E}')}$.

b) The map $(C^L, \mathcal{E}') \mapsto \rho'_{(C^L,\mathcal{E}')}$ can be identified with the Springer correspondence for L^1: if the local system $\mathcal{E}'$ is the image under Φ of a L-equivariant irreducible local system $\mathcal{F}$ on C^L

and if $\rho_{(C^L,\mathcal{F})} \in \mathrm{Irr}(W(L)^\sigma)$ *is the image of the pair* $(C^L,\mathcal{F})$ *by the Springer correspondence for* L^1*, then we have* $\rho_{(C^L,\mathcal{F})} = \rho'_{(C^L,\mathcal{E}')}$.

c) The action of $W(L)^\sigma$ *described in a) induces an isomorphism* $\overline{\mathbb{Q}}_l[W(L)^\sigma] \simeq \mathrm{End}A^P$.

Proof: a) The group $W(L)^\sigma$ acts on A^P by automorphisms of A^P. So, it induces an action of $W(L)^\sigma$ on the vector spaces $V'_{(C^L,\mathcal{E}')}$.

Let us prove b). Applying theorem 6.3 to L^1, the Springer action of $W(L)^\sigma$ from L^1 on $\pi(L)_*\overline{\mathbb{Q}}_l[\dim(L)]$ induces an action on $A(L)$ by restriction of $\pi(L)_*\overline{\mathbb{Q}}_l$ to $\mathcal{U}(L^1)$. To compare the action of $W(L)^\sigma$ on A^P and that on $A(L)$ is equivalent to compare the action of $W(L)^\sigma$ on $\eta_*\overline{\mathbb{Q}}_l$ and that on $\pi(L)_*\overline{\mathbb{Q}}_l$.

Let us apply the proper base change theorem to the morphisms $\pi(L)$ and η. We get the following isomorphims in the derived category: $q^*\zeta_*\overline{\mathbb{Q}}_l \simeq \eta_*\overline{\mathbb{Q}}_l$ et $i^*\zeta_*\overline{\mathbb{Q}}_l \simeq \pi(L)_*\overline{\mathbb{Q}}_l$.

Let us check that these isomorphisms are compatible with the actions of $W(L)^\sigma$, the Springer action on $\pi(L)_*\overline{\mathbb{Q}}_l$ and the actions on $\eta_*\overline{\mathbb{Q}}_l$ and $\zeta_*\overline{\mathbb{Q}}_l$ described in propositions 7.6 and 7.7.

The functor IC is fully faithful and we have the following isomorphisms in the category of perverse sheaves, first $\eta_*\overline{\mathbb{Q}}_l[\dim(G^1)] \simeq \mathrm{IC}(\tilde{G}^{1P},\eta_{reg*}\overline{\mathbb{Q}}_l)[\dim(G^1)]$, then $\pi(L)_*\overline{\mathbb{Q}}_l[\dim(L^1)] \simeq \mathrm{IC}(L^1,\pi(L)_{reg*}\overline{\mathbb{Q}}_l)[\dim(L^1)]$, finally $\zeta_*\overline{\mathbb{Q}}_l[\dim(G\times^P L^1)] \simeq \mathrm{IC}(G\times^P L^1,\eta_{reg*}\overline{\mathbb{Q}}_l)[\dim(G\times^P L^1)]$. So, it is sufficient to check that $q^*_{reg}\zeta_{reg*}\overline{\mathbb{Q}}_l \simeq \eta_{reg*}\overline{\mathbb{Q}}_l$ et $i^*_{reg}\zeta_{reg*}\overline{\mathbb{Q}}_l \simeq \pi(L)_{reg*}\overline{\mathbb{Q}}_l$ are compatible with the actions of $W(L)^\sigma$.

But the actions of $W(L)^\sigma$ on $\eta_{reg*}\overline{\mathbb{Q}}_l$, $\zeta_{reg*}\overline{\mathbb{Q}}_l$ and $\pi(L)_{reg*}\overline{\mathbb{Q}}_l$ come from the fact that η_{reg*}, ζ_{reg*} and $\pi(L)_{reg*}$ are galoisian coverings with Galois group $W(L)^\sigma$ and from the action of $W(L)^\sigma$ on the varieties $\pi^{-1}(G^1_{reg})$, $\zeta^{-1}(\mathcal{V}_{reg})$ and $\pi(L)^{-1}(L^1_{reg})$. So, it is sufficient to prove that the morphisms $\tilde{i}_{reg}$ and $\tilde{q}_{reg}$ are $W(L)^\sigma$-equivariant. The morphism $\tilde{q}_{reg}$ is $W(L)^\sigma$-equivariant by definition of the action of $W(L)^\sigma$ on $G\times^B(L\cap B)^1_{reg}$. Moreover, $\tilde{L}^1_{reg} = \{(l.\sigma,xT) \in L^1\times L/T | x^{-1}l.\sigma x \in T^1_{reg}\}$ and the action of $W(L)^\sigma$ on $\tilde{L}^1_{reg}$ is defined by $w : (l.\sigma,xT) \mapsto (l.\sigma,xwT)$ and the action of $W(L)^\sigma$ on $G\times^B(L\cap B)^1_{reg}$ comes from that on $\tilde{G}^1_{reg} = \{(g.\sigma,xT) \in G^1\times G/T | x^{-1}g.\sigma x \in T^1_{reg}\}$ where the action of $W(L)^\sigma$ is defined by $w : (g.\sigma,xT) \mapsto (g.\sigma,xwT)$. So, $\tilde{i}_{reg}$ is also $W(L)^\sigma$-equivariant. Whence the result.

c) is a consequence of b) as, applying theorem 6.3 to L^1, we get $\overline{\mathbb{Q}}_l[W(L)^\sigma] \simeq \mathrm{End}A(L)$.

End of the proof.

7.5. **Action of $W(L)^\sigma$ on A induced from that on A^P.** The complexes A and A^P are related by the formula $A = \xi_{u*}A^P$ and the action of $W(L)^\sigma$ on A^P described in 7.4 induces an action of $W(L)^\sigma$ on A by functoriality. Let us compare this action with the action of $W(L)^\sigma$ by restriction of the Springer action of $W(G)^\sigma$ on A from G^1 (proposition 7.11).

Proposition 7.10. *For all pair* $(C^L,\mathcal{E}')$ *such that* $V'_{(C^L,\mathcal{E}')} \neq 0$*, the direct image* $\xi_{u*}\mathrm{IC}(G\times^P(\overline{C}^LU_P),\mathcal{E}')[\dim(G\times^P(C^LU_P))]$ *is a semi-simple object of* $\mathcal{M}(\mathcal{U}(G^1))$ *and is decomposed as:*

$$(7)\quad \xi_{u*}(\mathrm{IC}(G\times^P\overline{C}^LU_P,\mathcal{E}')[\dim(G\times^P(C^LU_P))]) \simeq \bigoplus_{(C,\mathcal{E})}\mathrm{IC}(\overline{C},\mathcal{E})[\dim(C)]\otimes V^{(C^L,\mathcal{E}')}_{(C,\mathcal{E})}$$

where C *runs over the set of unipotent* G*-conjugacy classes in* G^1*,* $\mathcal{E}$ *runs over the set of* G*-equivariant irreducible local systems on* C *and where* $V^{(C^L,\mathcal{E}')}_{(C,\mathcal{E})}$ *is a* $\overline{\mathbb{Q}}_l$*-vector space whose dimension is equal to the multiplicity of the simple object corresponding to* $(C,\mathcal{E})$.

Proof: The perverse sheaf $\mathrm{IC}(G\times^P\overline{C}^LU_P,\mathcal{E}')[\dim(G\times^P(C^LU_P))]$ is a direct summand of $A^P[\dim(\mathcal{U}(G^1))]$. So, $\xi_{u*}(\mathrm{IC}(G\times^P\overline{C}^LU_P,\mathcal{E}')[\dim(G\times^P(C^LU_P))])$ is a direct summand of

$\xi_{u*}A^P[\dim(\mathcal{U}(G^1))] = A[\dim(\mathcal{U}(G^1))]$. But, $A[\dim(\mathcal{U}(G^1))]$ is semi-simple in $\mathcal{M}(\mathcal{U}(G^1))$, so all its direct summands are direct sums of simple objects of $\mathcal{M}(\mathcal{U}(G^1))$ which are direct summands of $A[\dim(\mathcal{U}(G^1))]$.

End of the proof.

If we put together the equality $A = \xi_{u*}A^P$, the decomposition formula (6) of A^P and the decomposition formula (7) of the previous proposition, we get the following double decomposition of the perverse sheaf $A[\dim(\mathcal{U}(G^1)]$:

$$A[\dim(\mathcal{U}(G^1)] \;=\; \bigoplus_{(C,\mathcal{E})}\bigoplus_{(C^L,\mathcal{E}')} \mathrm{IC}(\overline{C},\mathcal{E})[\dim(C)] \otimes V_{(C,\mathcal{E})}^{(C^L,\mathcal{E}')} \otimes V'_{(C^L,\mathcal{E}')}, \tag{8}$$

where C runs over the set of unipotent G-conjugacy classes of G^1 and $\mathcal{E}$ runs over the set of G-equivariant irreducible local systems on C, where C^L runs over the set of unipotent L-conjugacy classes of L^1 and $\mathcal{E}'$ runs over the set of G-equivariant irreducible local systems on $G\times^P(C^LU_P)$ and where $V_{(C,\mathcal{E})}^{(C^L,\mathcal{E}')}$ is the $\overline{\mathbb{Q}}_l$-vector space defined in the previous proposition.

Hence, the action of $W(L)^\sigma$ on A induced by that on A^P is given by the action of $W(L)^\sigma$ on the vector spaces $V'_{(C^L,\mathcal{E}')}$ which are $W(L)^\sigma$-modules, by proposition 7.9.

Proposition 7.11. *The action of $W(L)^\sigma$ on $A = \xi_{u*}A^P$ induced by that on A^P by functoriality of ξ_{u*} is the restriction to $W(L)^\sigma$ of the Springer action of $W(G)^\sigma$ on A.*

Proof: Let us consider the following diagram:

$$\begin{tikzcd}
\tilde{\mathcal{U}}(G^1) \arrow[r, hook, "\tilde{j}_u"] \arrow[d, "\eta_u"] & \tilde{G}^1 \arrow[d, "\eta"] & \tilde{G}^1_{reg} \arrow[l, hook', "\tilde{j}_{reg}"'] \arrow[d, "\eta_{reg}"] \\
\tilde{\mathcal{U}}(G^1)^P \arrow[r, hook, "j_u^P"] \arrow[d, "\xi_u"] & (\tilde{G}^1)^P \arrow[d, "\xi"] & \tilde{G}^{1P}_{reg} \arrow[l, hook', "j^P_{reg}"'] \arrow[d, "\xi_{reg}"] \\
\mathcal{U}(G^1) \arrow[r, hook, "j_u"] & G^1 & G^1_{reg} \arrow[l, hook', "j_{reg}"']
\end{tikzcd}$$

Both actions of $W(L)^\sigma$ on A come from actions on $\pi_*\overline{\mathbb{Q}}_l$. Let us compare the action of $W(L)^\sigma$ on $\pi_*\overline{\mathbb{Q}}_l[n] \simeq \mathrm{IC}(G^1, \pi_{reg*}\overline{\mathbb{Q}}_l)[n]$ which is the restriction of the Springer action of $W(G)^\sigma$ from G^1 (which is induced by functoriality of IC from the restriction to $W(L)^\sigma$ of the action of $W(G)^\sigma$ on $\pi_{reg*}\overline{\mathbb{Q}}_l$) and the action of $W(L)^\sigma$ which comes from the isomorphism $\pi_*\overline{\mathbb{Q}}_l[n] \simeq \xi_*\mathrm{IC}(\tilde{G}^{1P}, \eta_{reg*}\overline{\mathbb{Q}}_l)[n]$.

As IC is a fully faithful functor, it is sufficient to compare the restriction to $W(L)^\sigma$ of the action of $W(G)^\sigma$ on $\pi_{reg*}\overline{\mathbb{Q}}_l$ and the action of $W(L)^\sigma$ induced by functoriality of ξ_{reg*} from that on $\eta_{reg*}\overline{\mathbb{Q}}_l$.

But both these actions comes from the same action of $W(L)^\sigma$ on $\tilde{G}^1_{reg}$, whence the result.

End of the proof.

Hence, the restriction to $W(L)^\sigma$ of the Springer action of $W(G)^\sigma$ from G^1 on A is given by an action on the vector spaces $V'_{(C^L,\mathcal{E}')}$ of the double decomposition formula (7) of A. Moreover, if C is a unipotent G-conjugacy class of G^1 and if $\mathcal{E}$ is a G-equivariant irreducible local system on C, such that $(C,\mathcal{E})$ is associated by the Springer correspondence from G^1 to an irreducible representation $V_{(C,\mathcal{E})}$ of $W(G)^\sigma$, then, $V_{(C,\mathcal{E})}$ is decomposed as a direct sum of $W(L)^\sigma$-modules as:

$$V_{(C,\mathcal{E})} = \bigoplus_{(C^L,\mathcal{E}')} V_{(C,\mathcal{E})}^{(C^L,\mathcal{E}')} \otimes V'_{(C^L,\mathcal{E}')},$$

where C^L runs over the set of unipotent L-conjugacy classes of L^1 and $\mathcal{E}'$ the set of G-equivariant irreducible local systems on $G\times^P C^L U_P$ and where the vector spaces $V_{(C,\mathcal{E})}^{(C^L,\mathcal{E}')}$ are those of proposition 7.10.

7.6. **Induced unipotent classes.** We come to the purpose of section 7 which is assertion (vi) in proposition 7.13. Let us first recall the definition of an induced unipotent class:

Definition 7.1. ([14] page 60)
Let C^L be a unipotent L-conjugacy class in $L.\sigma$, then, the unipotent G-conjugacy class C is said to be obtained by induction of C^L from $L.\sigma$ to $G.\sigma$ if $C\cap C^LU_P$ is dense in C^LU_P, where U_P is the unipotent radical of P. If $t.\sigma$ is in C, we can choose $x.\sigma\in t.\sigma U_P$ such that $C=\mathrm{Cl}_G(x.\sigma)$.

Let us then recall a few properties of induced unipotent classes.

Proposition 7.12. *([14] page 60)*
Let $t.\sigma$ be a unipotent element of L^1, let $x.\sigma$ be in $t.\sigma U_P$ such that $C=\mathrm{Cl}_G(x.\sigma)$ is obtained by inducing $C^L=\mathrm{Cl}_L(t.\sigma)$ from $L.\sigma$ to $G.\sigma$. Then,
(i) $\dim(\mathrm{C}_G(x.\sigma))=\dim(\mathrm{C}_L(t.\sigma))$ and $\dim(\mathcal{B}^L_{t.\sigma})=\dim(\mathcal{B}^G_{x.\sigma})$.
(ii) $C\cap C^LU_P$ is a P-conjugacy class in $\mathrm{N}_{\tilde{G}}(P)$.
(iii) $\mathrm{C}_G(x.\sigma)^0\subset P$ and, if $g\in G$ is such that $g^{-1}(x.\sigma)g\in C^LU_P$, then, there is z in $\mathrm{C}_G(x.\sigma)$ such that ${}^zP={}^gP$.
(iv) $\overline{C}=\cup_{g\in G}{}^g(\overline{C}^LU_P)$.

Proposition 7.13. *Let $t.\sigma$ be a unipotent element of L^1, let $x.\sigma$ be in $t.\sigma U_P$ such that $C=\mathrm{Cl}_G(x.\sigma)$ is obtained by inducing $C^L=\mathrm{Cl}_L(t.\sigma)$ from $L.\sigma$ to $G.\sigma$. Then,*
(i) ξ_u sends $G\times^P(\overline{C}^LU_P)$ on $\overline{C}$.
(ii) $\dim(G\times^P(C^LU_P))=\dim(C)$.
(iii) $\xi_u^{-1}(C)\cap(G\times^P(\overline{C}^LU_P))$ is a G-orbit included and dense in $G\times^P(C^LU_P)$.
(iv) $\xi_u^{-1}(x.\sigma)\cap(G\times^P(\overline{C}^LU_P))$ is finite and its cardinal is equal to $[\mathrm{C}_G(x.\sigma):\mathrm{C}_P(x.\sigma)]$.
(v) Let $\rho_{(C^L,\overline{\mathbb{Q}}_l)}$ denote the image of $(C^L,\overline{\mathbb{Q}}_l)$ by the Springer correspondence from L^1; then the multiplicity of $\rho_{(C^L,\overline{\mathbb{Q}}_l)}$ in $\mathrm{H}^i(\mathcal{B}_{x.\sigma},\overline{\mathbb{Q}}_l)$ is equal to $[\mathrm{C}_G(x.\sigma):\mathrm{C}_P(x.\sigma)]$ if $i=2\dim(\mathcal{B}_{x.\sigma})$ and is equal to zero otherwise.
(vi) If we denote by $\rho_{(C,\overline{\mathbb{Q}}_l)}$ the image of $(C,\overline{\mathbb{Q}}_l)$ by the Springer correspondence from G^1, we have:

$$\langle\rho_{(C^L,\overline{\mathbb{Q}}_l)},\rho_{(C,\overline{\mathbb{Q}}_l)}|_{W(L)^\sigma}\rangle_{W(L)^\sigma}=1.$$

Proof:
(i) comes from the equality $\overline{C}=\cup_{g\in G}{}^g(\overline{C}^LU_P)$ (proposition 7.12, (iv)).
(ii) is clear as $G\times^P(C^LU_P)\to G\times^P C^L\to\mathcal{P}$ is a double fibration with fibers isomorphic respectively to U_P and C^L, and as we have $\dim(\mathrm{C}_G(x.\sigma))=\dim(\mathrm{C}_L(t.\sigma))$ (proposition 7.12, (i)).
(iii) As ξ_u is G-equivariant, any G-orbit is sent on a G-conjugacy class in $\mathcal{U}(G^1)$. The variety $G\times^P(\overline{C}^LU_P)$ is a union of G-orbits. Let $\mathcal{O}$ be a G-orbit in $G\times^P(\overline{C}^LU_P)$ which is sent on C, then, it is sent surjectively onto C and $\dim(\mathcal{O})\geq\dim(C)$. But, by (ii), $\dim(\mathcal{O})\leq\dim(G\times^P(C^LU_P))=\dim(C)$, so $\dim(\mathcal{O})=\dim(G\times^P(C^LU_P))$. Then, as $G\times^P(\overline{C}^LU_P)$ is irreducible (for $\overline{C}^L$ is irreducible), we can deduce that $\overline{\mathcal{O}}=G\times^P(\overline{C}^LU_P)$ and that $\xi_u^{-1}(C)\cap(G\times^P(\overline{C}^LU_P))=\mathcal{O}$. Hence, $\xi_u^{-1}(C)\cap(G\times^P(\overline{C}^LU_P))$ is only one G-orbit.

Let us suppose that $\xi_u^{-1}(C) \cap (G \times^P (\overline{C}^L U_P))$ is not included in $G \times^P (C^L U_P)$. Then, as ξ_u is equivariant, $\xi_u^{-1}(C) \cap (G \times^P (\overline{C}^L U_P))$ is included in $(G \times^P (\overline{C}^L U_P)) - (G \times^P (C^L U_P))$, which is impossible for dimension reasons.

(iv) $\xi_u^{-1}(x.\sigma) \cap (G \times^P (\overline{C}^L U_P))$ is the closure of $\xi_u^{-1}(x.\sigma) \cap (G \times^P (C^L U_P))$ in $\xi_u^{-1}(x.\sigma) \simeq \mathcal{P}_{x.\sigma}$. But, we have:

$$\xi_u^{-1}(x.\sigma) \cap (G \times^P (C^L U_P)) = \{(x.\sigma, {}^g P) | g^{-1}(x.\sigma) g \in C^L U_P\}$$

So, we can deduce from the (iii) of proposition 7.12 that:

$$\xi_u^{-1}(x.\sigma) \cap (G \times^P (C^L U_P)) = \{(x.\sigma, {}^g P) | g \in \mathrm{C}_G(x.\sigma)\}$$

Hence, $\xi_u^{-1}(x.\sigma) \cap (G \times^P (C^L U_P)) \simeq \mathrm{C}_G(x.\sigma)/(\mathrm{C}_G(x.\sigma) \cap P) = \mathrm{C}_G(x.\sigma)/\mathrm{C}_P(x.\sigma)$, where $\mathrm{C}_P(x.\sigma)$ is a subgroup of $\mathrm{C}_G(x.\sigma)$ of finite index as we have $\mathrm{C}_G(x.\sigma)^0 \subset \mathrm{C}_P(x.\sigma)$.

Thus, $\xi_u^{-1}(x.\sigma) \cap (G \times^P (C^L U_P))$ is finite and its cardinal is equal to $[\mathrm{C}_G(x.\sigma) : \mathrm{C}_P(x.\sigma)]$, so $\xi_u^{-1}(x.\sigma) \cap (G \times^P (C^L U_P))$ is equal to $\xi_u^{-1}(x.\sigma) \cap (G \times^P (\overline{C}^L U_P))$.

(v) Applying the double decomposition formula (8) of A, we get:

$$\mathrm{H}^i(\mathcal{B}_{x.\sigma}, \overline{\mathbb{Q}}_l) = \bigoplus_{(C,\mathcal{E})} \bigoplus_{(C^L,\mathcal{E}')} \mathcal{H}^{i-2d_C}_{x.\sigma}(\mathrm{IC}(\overline{C}, \mathcal{E})) \otimes V_{(C,\mathcal{E})}^{(C^L,\mathcal{E}')} \otimes V'_{(C^L,\mathcal{E}')},$$

where $\mathcal{E}$ runs over the set of G-equivariant irreducible local systems on C and $\mathcal{E}'$ the set of G-equivariant irreducible local systems on $G \times^P (C^L U_P)$, where $V'_{(C^L,\mathcal{E}')}$ is a $W(L)^\sigma$-module and where for all unipotent G-conjugacy class C, we set $d_C = 1/2(\dim(\mathcal{U}(G^1)) - \dim(C))$.

Whence,

$$\mathrm{H}^i(\mathcal{B}_{x.\sigma}, \overline{\mathbb{Q}}_l)^{\rho'_{(C^L,\overline{\mathbb{Q}}_l)}} = (\bigoplus_{(C,\mathcal{E})} \mathcal{H}^{i-2d_C}_{x.\sigma}(\mathrm{IC}(\overline{C}, \mathcal{E})) \otimes V_{(C,\mathcal{E})}^{(C^L,\overline{\mathbb{Q}}_l)}) \otimes V'_{(C^L,\overline{\mathbb{Q}}_l)}$$

Moreover, by proposition 7.10, we have:

$$\xi_{u*}(\mathrm{IC}(G \times^P (\overline{C}^L U_P), \overline{\mathbb{Q}}_l)[\dim(G \times^P (C^L U_P))]) \simeq \bigoplus_{(C,\mathcal{E})} \mathrm{IC}(\bar{C}, \mathcal{E})[\dim(C)] \otimes V_{(C,\mathcal{E})}^{(C^L,\overline{\mathbb{Q}}_l)}.$$

Hence,

$$\mathrm{H}^i(\mathcal{B}_{x.\sigma}, \overline{\mathbb{Q}}_l)^{\rho'_{(C^L,\overline{\mathbb{Q}}_l)}} = \mathrm{H}^{i-2d_{C^L}}(\xi_u^{-1}(x.\sigma) \cap (G \times^P (\overline{C}^L U_P)), \mathrm{IC}(G \times^P (\overline{C}^L U_P), \overline{\mathbb{Q}}_l)) \otimes V'_{(C^L,\overline{\mathbb{Q}}_l)}.$$

But, by (iv), $\xi_u^{-1}(x.\sigma) \cap (G \times^P (\overline{C}^L U_P))$ is a finite set of cardinality $[\mathrm{C}_G(x.\sigma) : \mathrm{C}_P(x.\sigma)]$. So, $\mathrm{H}^{i-2d_{C^L}}(\xi^{-1}(x.\sigma) \cap (G \times^P (\overline{C}^L U_P)), \mathrm{IC}(G \times^P (\overline{C}^L U_P), \overline{\mathbb{Q}}_l))$ is equal to zero except for $i = 2d_{C^L}$, in which case, it is a vector space of dimension $[\mathrm{C}_G(x.\sigma) : \mathrm{C}_P(x.\sigma)]$.

Moreover, according to (ii),

$$\begin{aligned} d_{C^L} &= 1/2(\dim(\mathcal{U}(G^1)) - \dim(G \times^P (C^L U_P))) \\ &= 1/2(\dim(\mathcal{U}(G^1)) - \dim(C)) \\ &= \dim(\mathcal{B}_{x.\sigma}) \end{aligned}$$

Consequently, the multiplicity of $\rho'_{(C^L,\overline{\mathbb{Q}}_l)}$ in $\mathrm{H}^i(\mathcal{B}_{x.\sigma}, \overline{\mathbb{Q}}_l)$ is equal to $[\mathrm{C}_G(x.\sigma) : \mathrm{C}_P(x.\sigma)]$ if $i = 2\dim(\mathcal{B}_{x.\sigma})$ and is equal to zero otherwise. Hence, we have $\rho'_{(C^L,\overline{\mathbb{Q}}_l)} = \rho_{(C^L,\overline{\mathbb{Q}}_l)}$ (proposition 7.9, b)).

Whence (v).

(vi) By the proof of (iv), we have $\xi_u^{-1}(x.\sigma) \cap (G \times^P (\overline{C}^L U_P)) = \{(x.\sigma, {}^x P) | g \in \mathrm{C}_G(x.\sigma)\}$. Moreover, we have $\mathrm{C}_G(x.\sigma)^0 \subset \mathrm{C}_P(x.\sigma)$ (proposition 7.12, (iii)). Hence, $A_G(x.\sigma)$

$= \mathrm{C}_G(x.\sigma)/\mathrm{C}_G(x.\sigma)^0$ acts transitively on $\xi_u^{-1}(x.\sigma) \cap (G \times^P (\overline{C}^L U_P))$. Thus, the dimension of $\mathrm{H}^0(\xi_u^{-1}(x.\sigma) \cap (G \times^P (\overline{C}^L U_P)), \overline{\mathbb{Q}}_l)^{A_G(x.\sigma)}$ is equal to 1.

Moreover, according to the proof of (v),

$$\mathrm{H}^{2d_{C^L}}(\mathcal{B}_{x.\sigma})^{\rho_{(C^L,\overline{\mathbb{Q}}_l)}} = \mathrm{H}^0(\xi_u^{-1}(x.\sigma) \cap (G \times^P (\overline{C}^L U_P)), \overline{\mathbb{Q}}_l) \otimes V_{(C^L,\overline{\mathbb{Q}}_l)} \tag{9}$$

Then, by proposition 7.11, the action of $W(L)^\sigma$ on $\mathrm{H}^{2d_{C^L}}(\mathcal{B}_{x.\sigma})$ induced from that on A^P by functoriality of ξ_{u*} can be identified with the restriction to $W(L)^\sigma$ of the Springer action of W^σ from G^1. Hence, according to corollary 6.4,

$$\mathrm{H}^{2d_{C^L}}(\mathcal{B}_{x.\sigma}, \overline{\mathbb{Q}}_l) = \bigoplus_{\rho \in A_G(x.\sigma)^\wedge} V_{(C,\mathcal{E}_\rho)} \otimes \rho^\vee \tag{10}$$

So, we can deduce from equalities (9) and (10) that:

$$(V_{(C,\overline{\mathbb{Q}}_l)})^{\rho_{(C^L,\overline{\mathbb{Q}}_l)}} = (\mathrm{H}^0(\xi_u^{-1}(x.\sigma) \cap (G \times^P (\overline{C}^L U_P)), \overline{\mathbb{Q}}_l) \otimes V_{(C^L,\overline{\mathbb{Q}}_l)})^{A_G(x.\sigma)},$$

and we get $\dim((V_{(C,\overline{\mathbb{Q}}_l)})^{\rho_{(C^L,\overline{\mathbb{Q}}_l)}}) = 1$.

End of the proof.

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