SPRINGER CORRESPONDENCE FOR DISCONNECTED REDUCTIVE GROUPS

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Abstract. The Springer correspondence is a map from the set of unipotent conjugacy classes of a reductive algebraic group to the set of irreducible complex characters of the Weyl group. This paper gives a résumé of the construction of the Springer correspondence for disconnected reductive groups of the form $G.\langle\sigma\rangle$ when σ is a quasi-semi-simple unipotent automorphism of G and more precisely for the connected component $G\sigma$. This correspondence is then described for $GL(n)\sigma$ in characteristic 2 when σ is an automorphism of order 2.

Introduction

Disconnected algebraic reductive groups naturally appear as extensions of connected reductive groups by an automorphism. Such groups were used by G. Malle to solve the Inverse Galois Problem in [10]. In this paper, we consider a disconnected reductive algebraic group of the form $G.\langle\sigma\rangle$, where G is a connected reductive group over an algebraic closed field of characteristic $p > 0$ and σ is a unipotent quasi-semi-simple automorphism of G. An automorphism σ of G is said to be quasi-semi-simple if there is a σ-stable pair (T, B), where T is a maximal torus included in a Borel subgroup B of G. If $G = GL(n)$, the subgroup B of upper triangular matrices is a Borel subgroup of G and the subgroup of diagonal matrices is a maximal torus of G included in B. We then consider the automorphism σ of G defined by $A \mapsto J\,{}^tA^{-1}J^{-1}$, where J is the matrix

$$J = \begin{pmatrix} & & 1 \\ & \dots & \\ 1 & & \end{pmatrix}$$

Then σ stabilises both T and B, so it is quasi-semi-simple.

The subject of this paper is the Springer correspondence for the disconnected group $G.\langle\sigma\rangle$ and more precisely for the connected component $G\sigma$.

In the connected case, the Springer correspondence has been studied from many different points of view. It is easely described in the case of $GL(n)$. The Weyl group of $GL(n)$ is isomorphic to the symmetric group S_n and irreducible representations of S_n are parametrized by partitions of n. But unipotent conjugacy classes of $GL(n)$ are also parametrized by partitions of n thanks to Jordan's normal form of matrices. The Springer correspondence is then the induced bijection from the set of unipotent conjugacy classes of $GL(n)$ to the set of irreducible representations of S_n.

The Springer correspondence for a connected reductive group G is not a bijection in general. It is an injection from the set of irreducible representations of the Weyl group of G to the set of pairs (u, ρ_u), where u is a representative of a unipotent G-conjugacy class in G and ρ is an irreducible representation of $\mathrm{A}_G(u) = \mathrm{Z}_G(u)/\mathrm{Z}_G(u)^0$, where $\mathrm{Z}_G(u)$ is the centralizer of u in G.

Let us now consider the case of $G\sigma$ when σ is a unipotent quasi-semi-simple automorphism of G. The Springer correspondence is then a map from the set of irreducible representations of the subgroup $W(G)^\sigma$ of σ-fixed elements of the Weyl group $W(G)$ of G to the set of pairs $(u\sigma, \rho_{u\sigma})$, where $u\sigma$ is a representative of a unipotent G-conjugacy class in $G\sigma$ and $\rho_{u\sigma}$ is an irreducible representation of $\mathrm{A}_G(u\sigma) = \mathrm{Z}_G(u\sigma)/\mathrm{Z}_G(u\sigma)^0$, where $\mathrm{Z}_G(u\sigma)$ is the centralizer of $u\sigma$ in G. This correspondence induces an injective map from the set of unipotent conjugacy classes in $G\sigma$ to the set of irreducible representations of $W(G)^\sigma$. We will call this injection the restricted Springer correspondence.

In the first section of this paper, we give some main results from [17] and [3] about quasi-semi-simple automorphisms of a connected reductive group. The second section is a summary of the construction of the Springer correspondence for $G\sigma$ with σ unipotent quasi-semi-simple, which was done in [14] and [15]. In the third section, we state two properties of the correspondence which were proved in [15] and which were used in [12] to calculate the restricted Springer correspondence for any reductive group $G.\langle\sigma\rangle$ when σ is unipotent quasi-semi-simple and when G is a connected reductive group of classical type. Following [12], this restrictive Springer correspondence is described in section 4 for $GL(n)\sigma$ in characteristic 2, when σ is a quasi-semi-simple automorphism of order 2 of $GL(n)$.

1. Connected reductive groups with a quasi-semi-simple automorphism

Theorem 1.1 ([17] 9.4, [3] Theorem 1.8, (i)). *If G is a connected reductive group and σ a quasi-semi-simple automorphism of G, then the group G^σ of σ-fixed points of G is reductive.*

We consider a connected reductive group G endowed with a quasi-semi-simple automorphism σ. We fix a pair $T \subset B$, where T is a σ-stable maximal torus of G included in a σ-stable Borel subgroup B. We denote by U the unipotent radical of B, by Φ the set of roots of G relative to T and by Π the basis of simple roots relative to (T, B). For any $\alpha \in \Phi$ we fix an automorphism $x_\alpha : K^+ \to U_\alpha$ where U_α is the unipotent subgroup corresponding to α.

The automorphism σ induces an automorphism of Φ which stabilizes Π.

Definition 1.2. Let α be in Φ. If i is the order of the σ-orbit of α, then, σ^i induces an automorphism of U_α. We define $C_{\sigma,\alpha} \in k^\times$ by:

$$\sigma^i(x_\alpha(\lambda)) = x_\alpha(C_{\sigma,\alpha}\lambda)$$

Example 1.3. Let us consider the case of $G = GL(4)$, when σ is defined by $\sigma : A \in GL(4) \mapsto J^t A^{-1} J^{-1}$ with $J = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$.

We denote by T the set of diagonal matrices, and by B the set of upper triangular matrices. Then the simple roots of G relative to (T,B) are the $\alpha_i : t \mapsto t_{i+1}/t_i$ for $i \in \{1,2,3\}$, where $t = \mathrm{diag}(t_1,t_2,t_3,t_4)$. An easy matrices computation gives: $\sigma(x_{\alpha_1}(\lambda)) = x_{\alpha_3}(-\lambda)$, $\sigma(x_{\alpha_2}(\lambda)) = x_{\alpha_2}(-\lambda)$ and $\sigma(x_{\alpha_3}(\lambda)) = x_{\alpha_1}(-\lambda)$. Hence, we have two σ-orbits of simple roots: $\{\alpha_1,\alpha_3\}$ and $\{\alpha_2\}$.

Let us calculate the coefficients $C_{\sigma,\alpha}$ for both orbits.

The coefficient C_{σ,α_1} is defined by $\sigma^2(x_{\alpha_1}(\lambda)) = x_{\alpha_1}(C_{\sigma,\alpha_1}\lambda)$, so we get $C_{\sigma,\alpha_1} = 1$.

The coefficient C_{σ,α_2} is defined by $\sigma(x_{\alpha_2}(\lambda)) = x_{\alpha_2}(C_{\sigma,\alpha_2}\lambda)$, so we get $C_{\sigma,\alpha_2} = -1$.

Theorem 1.4 ([17] 8.2, [3] Theorem 1.8, (v)). *There is a natural surjection between the set of σ-orbits such that:*

$$(*)\begin{cases} C_{\sigma,\alpha} = \pm 1 \text{ where } -1 \text{ is allowed only if there are } \beta,\beta' \in \Phi \text{ in the } \sigma\text{-orbit of } \alpha \\ \text{such that } \beta+\beta' \in \Phi, \end{cases}$$

onto the set of roots of $(G^\sigma)^0$ relative to $(T^\sigma)^0$.

This is a bijection and any $C_{\sigma,\alpha}$ is equal to 1 except in the case when the Dynkin diagram of G has k components of type A_{2n} which are permuted by σ.

Theorem 1.5 ([3] Theorem 1.15, [4] Proposition 2.1). *A quasi-semi-simple automorphism σ of G is said to be quasi-central if one of the following equivalent properties is verified:*

1) If σ' is a quasi-semi-simple automorphism of G such that $\sigma' = \sigma \circ \mathrm{ad}(g)$, where $g \in G$, then $\dim(G^\sigma)^0 \geq \dim(G^{\sigma'})^0$.

2) The map $B \mapsto B \cap (G^\sigma)^0$ defines an isomorphism from the variety $\mathcal{B}^{(G^\sigma)^0}$ of Borel subgroups of $(G^\sigma)^0$ to the variety $\mathcal{B}^G_\sigma$ of σ-stable Borel subgroups of G.

3) Let $W(G)$ be the Weyl group of G. Then, the group $W(G)^\sigma$ of σ-fixed points of $W(G)$ is isomorphic to the Weyl group of $(G^\sigma)^0$.

4) There is a σ-stable pair $T \subset B$ where T is a maximal torus and B a Borel subgroup of G such that the condition $(*)$ of Theorem 1.4 is true for any simple root α of G relative to (T,B).

4') Property 4) is true for any σ-stable pair (T,B).

Example 1.6. With the same hypotheses as in Example 1.3 and in characteristic different from 2, σ is not quasi-central.

We still consider $G = GL(4)$ but now σ is defined by $\sigma : A \in GL(4) \mapsto J^t A^{-1} J^{-1}$ with
$$J = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$
In this case, we still have two σ-orbits of simple roots $\{\alpha_1, \alpha_3\}$ and $\{\alpha_2\}$, but now we have $C_{\sigma,\alpha_1} = C_{\sigma,\alpha_2} = 1$. So, by Theorem 1.5, we can conclude that σ is quasi-central.

Proposition 1.7 ([3] Corollary 1.33). *If σ is unipotent and quasi-semi-simple then G^σ is connected and σ is quasi-central.*

2. Springer correspondence for $G.\sigma$

In this section, σ is supposed to be unipotent and quasi-semi-simple.

The Springer correspondence for $G\sigma$ is then a map from the set of irreducible representations of the subgroup $W(G)^\sigma$ of σ-fixed elements of the Weyl group $W(G)$ of G to the set of pairs $(u\sigma, \rho_{u\sigma})$, where $u\sigma$ is a representative of a unipotent G-conjugacy class in $G\sigma$ and $\rho_{u\sigma}$ is an irreducible representation of $\mathrm{A}_G(u\sigma) = \mathrm{Z}_G(u\sigma)/\mathrm{Z}_G(u\sigma)^0$, where we denote by $\mathrm{Z}_G(u\sigma)$ the centralizer of $u\sigma$ in G. The construction of this Springer correspondence is described in this section. The method used here was inspired by that used by G. Lusztig for the connected case in [7] and is based on the theory of perverse sheaves.

We fix a pair $T \subset B$ where T is a σ-stable maximal torus of G included in a σ-stable Borel subgroup B.

2.1. **Springer representation of $W(G)^\sigma$ from $G\sigma$.** Let $j : (G\sigma)_{reg} \hookrightarrow G\sigma$ be the open dense subset of quasi-semi-simple regular elements of $G\sigma$. A element of $G\sigma$ is said to be regular if the dimension of its centralizer in G is minimal in $G\sigma$. We denote by $(T\sigma)_{reg}$ the intersection of $(G\sigma)_{reg}$ with $T\sigma$.

Let us consider the following variety:
$$(\tilde{G\sigma})_{reg} = \{(g.\sigma, xT) \in G\sigma \times (G/T) | x^{-1}(g.\sigma)x \in (T\sigma)_{reg}\}.$$
Let $\pi_{reg} : (\tilde{G\sigma})_{reg} \to (G\sigma)_{reg}$ be defined by $(g\sigma, hT) \mapsto g\sigma$. Then, the morphism π_{reg} is an unramified covering with Galois group $W(G)^\sigma$ for the action of $W(G)^\sigma$ on $(\tilde{G\sigma})_{reg}$ on the right defined by $w : (g.\sigma, xT) \mapsto (g.\sigma, xwT)$ and for the trivial action on $(G\sigma)_{reg}$.

Hence we get an action of $W(G)^\sigma$ on the local system $\pi_{reg*}\overline{\mathbb{Q}}_l$ on $(G\sigma)_{reg}$ and, by functoriality of $j_{!*}[\dim(G)]$, an action of $W(G)^\sigma$ on the perverse sheaf $K := j_{!*}\pi_{reg*}\overline{\mathbb{Q}}_l[\dim(G)]$ on $G\sigma$ which is called **Springer representation of $W(G)^\sigma$** with reference to the connected case (for the properties and definition of perverse sheaves and the intermediate extension $j_{!*}B$ for a perverse sheaf B, see [1] or [6]).

Now there is a second definition of the perverse sheaf K. Let us consider the following variety:
$$\tilde{G\sigma} = \{(g.\sigma, xB) \in G\sigma \times (G/B) | x^{-1}(g.\sigma)x \in B\sigma\},$$

and let $\pi : \tilde{G}\sigma \to G\sigma$ be the projection on the first factor. We then get an isomorphism

$$K \simeq \pi_* \overline{\mathbb{Q}}_l[\dim(G)]$$

in the category of perverse sheaves on $G\sigma$.

2.2. **How the Springer representation induces the regular representation of $W(G)^\sigma$.** The action of $W(G)^\sigma$ on $K \simeq \pi_*\overline{\mathbb{Q}}_l[\dim(G)]$ constructed above induces an action of $W(G)^\sigma$ on $\mathrm{H}^i(\mathcal{B}^G_{g\sigma}, \overline{\mathbb{Q}}_l) \simeq \mathcal{H}^i_{g\sigma}(\pi_*\overline{\mathbb{Q}}_l)$ for any $g\sigma \in G\sigma$, where we denote by $\mathcal{B}^G_{g\sigma}$ the variety of Borel subgroups of G which are stable by conjugation under $g\sigma$. In particular, we get an action of $W(G)^\sigma$ on $\mathrm{H}^*(\mathcal{B}^G_\sigma, \overline{\mathbb{Q}}_l)$ which is the regular representation of $W(G)^\sigma$.

There is a similar result in the connected case but it is proved using the theory of Lie algebras. So this proof cannot be generalized to disconnected groups. The method which was used in [14] and [15] was to compare the connected component $G\sigma$ with the group G^σ of σ-fixed points of G which is, as σ is unipotent quasi-semi-simple, a connected reductive group (Proposition 1.7).

We can apply to G^σ the construction of the Springer representation for a connected reductive group described in [13] (section 4). Then we get an action of the Weyl group $W(G^\sigma)$ of G^σ on $\mathrm{H}^*(\mathcal{B}^{G^\sigma}, \overline{\mathbb{Q}}_l)$, where $\mathcal{B}^{G^\sigma}$ is the variety of Borel subgroups of G^σ. We will call this representation the Springer representation from G^σ.

Now, as σ is unipotent quasi-semi-simple, we have $W(G^\sigma) \simeq W(G)^\sigma$ and $\mathcal{B}^G_\sigma \simeq \mathcal{B}^{G^\sigma}$ (Theorem 1.5). Hence, we get two Springer representations of $W(G)^\sigma$ on $\mathrm{H}^*(\mathcal{B}^G_\sigma, \overline{\mathbb{Q}}_l)$, one from $G\sigma$ and one from G^σ. We are then reduced to proving the following proposition:

Proposition 2.1. *The Springer representation of $W(G)^\sigma$ on $\mathrm{H}^*(\mathcal{B}^G_\sigma, \overline{\mathbb{Q}}_l)$ from $G\sigma$ and that from G^σ coincide.*

This proposition was proved in [14] and [15] thanks to some arguments from the theory of perverse sheaves.

2.3. **Restriction to the variety of unipotent elements.** We consider the complex of sheaves $K' := K|_{\mathcal{U}(G\sigma)}[d]$ where we denote by $\mathcal{U}(G\sigma)$ the variety of unipotent elements of $G\sigma$ and we set $d = \dim(\mathcal{U}(G\sigma)) - \dim(G\sigma)$. Then, K' is a semi-simple perverse sheaf on $\mathcal{U}(G\sigma)$ and its decomposition $K' = \sum_{(u\sigma,\rho_{u\sigma})} K_{u\sigma,\rho_{u\sigma}} V_{u\sigma,\rho_{u\sigma}}$ as a sum of simple perverse sheaves $K_{u\sigma,\rho_{u\sigma}}$ is indexed by the set $\mathcal{N}$ of pairs $(u\sigma, \rho_{u\sigma})$ where $u\sigma$ is a representative of a unipotent G-conjugacy class of $G\sigma$ and $\rho_{u\sigma}$ an irreducible representation of $\mathrm{Z}_G(u\sigma)/\mathrm{Z}_G(u\sigma)^0$, where $\mathrm{Z}_G(u\sigma)$ is the centralizer of $u\sigma$ in G.

Moreover, the Springer action of $W(G)^\sigma$ on K induces an action of $W(G)^\sigma$ on the $\overline{\mathbb{Q}}_l$-vector spaces $V_{u\sigma,\rho_{u\sigma}}$ and we can show that if $V_{u\sigma,\rho_{u\sigma}} \neq 0$ then it is an irreducible $W(G)^\sigma$-module.

We call the **Springer correspondence for** $G\sigma$ the induced bijection from the set $\{(u\sigma, \rho_{u\sigma}) \in \mathcal{N} | V_{u\sigma,\rho_{u\sigma}} \neq 0\}$ to $\mathrm{Irr}(W(G)^\sigma)$ defined by $(u\sigma, \rho_{u\sigma}) \mapsto V_{u\sigma,\rho_{u\sigma}}$.

Then we have $V_{u\sigma,\rho_{u\sigma}} \neq 0$ if $(u\sigma, \rho_{u\sigma})$ is a pair of $\mathcal{N}$ such that $\rho_{u\sigma} = \mathrm{Id}_{u\sigma}$. Hence the Springer correspondence induces an injection from the set of conjugacy class of $G\sigma$ to the set of irreducible representations of $W(G)^\sigma$ which associates to the unipotent class C the representation $V_{u\sigma,\mathrm{Id}_{A_G(u\sigma)}}$ where $u\sigma \in C$. We will call this injection the **restricted Springer correspondence for** $G\sigma$.

2.4. **Green functions for** $G\sigma$**.** In [3], Digne and Michel define Deligne-Lusztig characters for $\tilde{G}^F$ where $\tilde{G}$ is a disconnected reductive groups endowed with a rational structure defined by the Frobenius endomorphism F. They show that the study of Deligne-Lusztig characters for disconnected reductive groups can be reduced to the study of these characters for $G^F.\langle\sigma\rangle$ where G is a connected reductive group endowed with a rational structure defined by F and σ is a quasi-semi-simple rational automorphism of G.

If we suppose in addition that σ is unipotent, we can define, as in the connected case, a Green function of $G^F.\langle\sigma\rangle$ as the restriction of a Deligne-Lusztig character to the set of unipotent elements. Then, the character formula proved in [3], reduces the determination of Deligne-Lusztig characters to the determination of Green functions.

It would be interesting to show that, similarly to which was proved in [8] for connected reductive groups, Green functions defined by Deligne-Lusztig characters as above coincide with the restriction to the set of unipotent elements of the characteristic function of some perverse sheaves as $K = \pi_*\overline{\mathbf{Q}}_l[\dim(G)]$ which is defined in section 2. It will certainly lead to the calculation of Green functions by an algorithm similar to Shoji's algorithm which is used in the connected case.

3. Properties of the Springer correspondence

In [12], the restricted Springer correspondence is calculated for any group $G.\langle\sigma\rangle$ where G is a connected reductive group of classical type. In fact, we are reduced to the calculation of two cases:

1) the case of the group $GL(n).\langle\sigma\rangle$ in characteristic 2 and where $\sigma : GL(n) \to GL(n)$ is a quasi-semi-simple automorphism of order 2 (cf. section 4).

2) the case of the general orhogonal group $GO(2n)$ which is the extension of $SO(2n)$ by a non-trivial automorphism of order 2, in characteristic 2.

As we have seen in section 2, the construction of the Springer correspondence is not descriptive. In order to calculate it explicitely we need to use its properties.

For groups of small rank (in particular for $G = GL(2)$ and $G = GL(3)$), the following property is sufficient:

Proposition 3.1 ([15] Proposition 6.7)**.** *If the pair $(u.\sigma, \mathrm{Id}_{\mathrm{A}_G(u.\sigma)}) \in \mathcal{N}$ is sent to $\chi \in \mathrm{Irr}(W(G)^\sigma)$ by the Springer correspondence for $G.\sigma$, then, $b(\chi) = \dim(\mathcal{B}^G_{u.\sigma})$, where $b(\chi)$ is the b-invariant of χ (definition of $b(\chi)$: see* [5] *5.2.2).*

But the calculation of [12] is especially based on the induction property of proposition 3.2. We consider a σ-stable Levi subgroup L included in a σ-stable parabolic subgroup P. This property links the Springer correspondence for $L\sigma$ with the Springer correspondence for $G\sigma$.

First, there is a map from unipotent L-conjugacy classes of $L\sigma$ to unipotent G-conjugacy classes of $G\sigma$ which is called induction of unipotent classes and which we will denote by $\mathrm{Ind}_{L\sigma}^{G\sigma}$ (definition and properties of induction of unipotent classes in disconnected reductive groups: see [16] II.3).

Let us consider the other side of the correspondence. We denote by $W(L)$ the Weyl group of L and by $W(L)^\sigma$ the subgroup of σ-fixed elements of $W(L)^\sigma$. Then $W(L)^\sigma$ is a Coxeter subgroup of $W(G)^\sigma$ and we can consider j-induction from $W(L)^\sigma$ to $W(G)^\sigma$ which we will denote by $j_{W(L)^\sigma}^{W(G)^\sigma}$ (definition and properties of j-induction: see [5] 5.2).

Proposition 3.2 ([15] Proposition 7.13). *Let $u\sigma$ be a unipotent element of $L\sigma$. We denote by C the unipotent L-conjugacy class of $u\sigma$. Let $\tilde{C}$ be the unipotent class of $G\sigma$ obtained by inducing C from $L\sigma$ to $G\sigma$ and let $v\sigma$ be in $\tilde{C}$.*

Let $\chi \in \mathrm{Irr}(W(L)^\sigma)$ be the image of the pair $(u, \mathrm{Id}_{\mathrm{A}_L(u\sigma)})$ by the Springer correspondence for $L\sigma$ and let $\tilde{\chi} \in \mathrm{Irr}(W(G)^\sigma)$ be the image of the pair $(v\sigma, \mathrm{Id}_{\mathrm{A}_G(v\sigma)})$ by the Springer correspondence for $G\sigma$.

Then, if $j_{W(L)^\sigma}^{W(G)^\sigma}(\chi)$ is irreducible, we have $\tilde{\chi} = j_{W(L)^\sigma}^{W(G)^\sigma}(\chi)$.

4. Description of the restricted Springer correspondence for $GL(n).\langle\sigma\rangle$ and $p = 2$

We describe in this section the restricted Springer correspondence for $GL(n)\sigma$ in characteristic 2 and where $\sigma : GL(n) \to GL(n)$ is defined by $\sigma : A \mapsto J^t A^{-1} J^{-1}$ with

$$J = \begin{pmatrix} & & 1 \\ & \dots & \\ 1 & & \end{pmatrix}.$$

The Springer correspondence for $GL(n)$ is easely described via the parametrizations of unipotent classes and irreducible representations of S_n by partitions of n. In the case of $GL(n)\sigma$, we need the more complex symbols which are defined in 4.1. These symbols were first defined in [9].

The Weyl group involved in the Springer correspondence for $GL(n)\sigma$ is the subgroup $W(GL(n))^\sigma$ of σ-fixed elements of the Weyl group $W(GL(n))$ of $GL(n)$. This is a Weyl group of type B_m if n is even and of type C_m if n if odd, where we have set $m = \lfloor n/2 \rfloor$. We will see respectively in 4.2 and in 4.3 how to parametrize, with the symbols defined in 4.1, the irreducible representations of the Weyl group $W(B_m)$ of type B_m and the unipotent conjugacy classes of $GL(n)\sigma$.

The restricted Springer correspondence for $GL(n)\sigma$ will be described in Theorem 4.3 via the parametrizations of 4.2 and 4.3.

Theorem 4.3 was proved in [12] using proposition 3.2 applied to $L\sigma$, where L is a σ-stable Levi subgroup $GL(r)\times GL(k)\times GL(r)$ of $GL(n)$ and where σ interchanges the two factors $GL(r)$ and acts on $GL(k)$ as on $GL(n)$.

4.1. **Symbols.** We denote by $\tilde{X}_d^{r,s}$ with $r,s\in\mathbb{Z}_{\geq 0}$ and $d\in\mathbb{Z}$, the set of pairs $S=(S_0,S_1)$ of finite sequences $S_i=(\lambda_{i,j}\mid 1\leq j\leq t_i)$ of non-negative integers, such that:

$$\begin{cases} \lambda_{i,j}\geq\lambda_{i,j-1}+r+s \text{ for } i\in\{0,1\}, \quad 2\leq j\leq t_i,\\ \lambda_{1,1}\geq s,\\ \text{and } t_0-t_1=d\end{cases}$$

For instance, $S=\begin{pmatrix} 0 & 6 & 10\\ 1 & 5 & \end{pmatrix}$ is in $\tilde{X}_1^{3,1}$ and $S'=\begin{pmatrix} 0 & 6 & 10\\ 1 & 5 & 9\end{pmatrix}$ is in $\tilde{X}_0^{3,1}$.

We define the rank of a symbol by the following formula:

$$\operatorname{rank}(S):=\sum_{\lambda\in S}\lambda-(r+s)\left\lfloor\frac{(|S|-1)^2}{4}\right\rfloor-s\left\lfloor\frac{|S|}{2}\right\rfloor,$$

where $|S|$ denotes the number of entries in S and $\lfloor x\rfloor:=\max\{n\in\mathbb{Z}\mid n\leq x\}$ is the Gauß brackets.

The rank is invariant under the shift operation on $\tilde{X}_d^{r,s}$ which associates to a symbol $S=(S_0,S_1)$ of $\tilde{X}_d^{r,s}$ the symbol $S'=\big(\{0\}\cup(S_0+r+s),\{s\}\cup(S_1+r+s)\big)$.

For instance, the image of $S=\begin{pmatrix} 0 & 6 & 10\\ 1 & 5 & \end{pmatrix}\in\tilde{X}_1^{3,1}$ by the shift operation is $S'=\begin{pmatrix} 0 & 4 & 10 & 14\\ 1 & 5 & 9 & \end{pmatrix}$.

We denote by $X_d^{r,s}$ the set of classes in $\tilde{X}_d^{r,s}$ under the shift operation and by $X_{d,n}^{r,s}$ the set of classes of symbols in $X_d^{r,s}$ of rank n.

A symbol $S\in X_{d,n}^{r,s}$ with $d\in\{0,1\}$ is called distinguished if $\lambda_{0,j}\leq\lambda_{1,j}$ and $\lambda_{1,j}\leq\lambda_{0,j+1}$ for all relevant j.

For instance, $S=\begin{pmatrix} 0 & 6 & 10\\ 1 & 5 & \end{pmatrix}$ is not distinguished in $X_{1,4}^{3,1}$ but $S''=\begin{pmatrix} 0 & 5 & 10\\ 1 & 6 & \end{pmatrix}$ is distinguished in $X_{1,4}^{3,1}$.

We denote by $D_{d,n}^{r,s}$ the subset of $X_{d,n}^{r,s}$ of distinguished symbols.

4.2. **Parametrization of** $\operatorname{Irr}(W(B_m))$**.** The irreducible representations of the Weyl group $W(B_m)$ of type B_m are naturally parametrized by pairs of partitions (μ_1,μ_2) of m (cf. [5] Theorem 5.5.6).

By adding parts 0 to μ_1 or μ_2 where $(\mu_1,\mu_2)\vdash m$, we may assume that:

i) μ_1 and μ_2 have the same number of parts, so $(\mu_1,\mu_2)\in\tilde{X}_{0,m}^{0,0}$. This defines a bijection from $\operatorname{Irr}(W(B_m))$ to the set of symbols $X_{0,m}^{0,0}$.

ii) μ_1 contains precisely one more part than μ_2. This defines then a bijection from $\operatorname{Irr}(W(B_m))$ to the set of symbols $X_{1,m}^{0,0}$.

For example, $\mathrm{Irr}(W(B_2))$ may be parametrized by pairs of partitions of 2, by symbols of $X_{0,2}^{0,0}$ thanks to 1) or by symbols of $X_{1,2}^{0,0}$ thanks to 2).

$\mathrm{Irr}(W(B_2))$	$(2,-)$	$(-,2)$	$(11,-)$	$(-,11)$	$(1,1)$
$X_{0,2}^{0,0}$	$\begin{pmatrix} 2 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 2 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$
$X_{1,2}^{0,0}$	$\begin{pmatrix} 2 \\ \ \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 2 & \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 0 & \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & \end{pmatrix}$

Using the classes of symbols of $X_{0,m}^{0,0}$ or $X_{1,m}^{0,0}$ to parametrize irreducible representations of $W(B_m)$ is a very convenient way to describe j-induction (cf. [12] proposition 4.6).

4.3. **Parametrization of unipotent classes.**

4.3.1. *Spaltenstein's parametrization.* The unipotent conjugacy classes of $GL(n)\sigma$ are parametrized in [16] by pairs (λ, ϵ), where $\lambda = (\lambda_1 \leq \lambda_2 \leq \ldots) \vdash n$ is a partition of n into non-zero parts such that any even part occurs an even number of times, and ϵ is a map which associates to the parts λ_i of λ an element of $\{0, 1, \omega\}$ such that:

$$\begin{aligned} \epsilon(\lambda_i) &= \omega \text{ if } \lambda_i \text{ is even,} \\ &\neq \omega \text{ if } \lambda_i \text{ is odd,} \\ &= 1 \text{ if } \lambda_i \text{ is odd and if } |\{j \in \mathbb{Z}_{\geq 0} | \lambda_j = \lambda_i\}| \text{ is odd} \end{aligned}$$

Hence the map ϵ is completely determined except for the odd parts of λ which occur an even number of times, for which ϵ can be equal to 0 or to 1.

Example 4.1. In $GL(2)\sigma$, there are two unipotent conjugacy classes and they are parametrized by 1_0^2 and 1^2. The partition 2 does not parametrize any unipotent class of $GL(2)\sigma$ as 2 is an even part which occurs an odd number of times. The partition 1^2 parametrized two unipotent classes. The map ϵ is denoted by the index 0 when it is equal to 0 and when ϵ is equal to 1, we omit the index.

4.3.2. *From parametrization to symbols.* Spaltenstein's parametrization is not convenient to describe induction of unipotent classes. That is the reason why we need to parametrize them by the symbols of 4.1. We proceed in the following way:

We consider a unipotent class C of $GL(n)\sigma$ parametrized as above by a pair (λ, ϵ) with $\lambda = (\lambda_1 \leq \lambda_2 \leq \ldots)$.

First we subdivide the sequence $\lambda_1 \leq \lambda_2 \leq \ldots$ into blocks of length 1 or 2, such that λ_i lies in a block of length 1 if and only if λ_i is odd and $\epsilon(\lambda_i) \neq 0$.

Then we apply the following formulas :

a) If $\{\lambda_i\}$ is a block, we set $c_i := (\lambda_i - 1)/2 + 2(i-1)$

b) If $\{\lambda_i, \lambda_{i+1}\}$ is a block and $\epsilon(\lambda_i) = \omega$, we set $\begin{cases} c_i & := & \lambda_i/2 + 2(i-1) \\ c_{i+1} & := & (\lambda_i - 2)/2 + 2i = c_i + 1 \end{cases}$

c) If $\{\lambda_i, \lambda_{i+1}\}$ is a block and $\epsilon(\lambda_i) = 0$, we set $\begin{cases} c_i & := & (\lambda_i + 1)/2 + 2(i-1) \\ c_{i+1} & := & (\lambda_i - 3)/2 + 2i = c_i \end{cases}$

We set $\rho(C) := ((c_1, c_3, \ldots), (c_2, c_4, \ldots))$.

We set $m := \lfloor \frac{n}{2} \rfloor$ and define $d \in \{0,1\}$ by $n \equiv d \pmod 2$.

Then, the map $C \mapsto \rho(C)$ is a bijection from the set of unipotent conjugacy classes of $GL(n)\sigma$ to the set of distinguished symbols $D^{3,1}_{d,m}$.

Example 4.2. Let us consider $GL(4)\sigma$. There are four unipotent conjugacy classes:

$$(31)\text{: we get 2 blocks with formula a), so } \rho(C) = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

$$(2^2)\text{: we get 1 block with formula b), so } \rho(C) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$(1^4)\text{: we get 4 blocks with formula a), so } \rho(C) = \begin{pmatrix} 0 & 4 \\ 2 & 6 \end{pmatrix}$$

$$(1^4_0)\text{: we get 2 blocks with formula c), so } \rho(C) = \begin{pmatrix} 1 & 5 \\ 1 & 5 \end{pmatrix}$$

4.4. **Restricted Springer correspondence for $GL(n).\sigma$.** We define $S_0 \in X^{3,1}_{0,0}$ and $S_1 \in X^{3,1}_{1,0}$ by

$$S_0 := \begin{pmatrix} 0 \\ \end{pmatrix} = \begin{pmatrix} 0 & 4 \\ 1 & \end{pmatrix} = \begin{pmatrix} 0 & 4 & 8 \\ 1 & 9 & \end{pmatrix} = \ldots$$

$$S_1 := \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 4 \\ 1 & 5 \end{pmatrix} = \begin{pmatrix} 0 & 4 & 8 \\ 1 & 5 & 9 \end{pmatrix} = \ldots$$

Theorem 4.3. *We set $m = \lfloor n/2 \rfloor$.*

(a) *The map*

$$D^{3,1}_{1,m} \longrightarrow X^{0,0}_{1,m}, \quad S \mapsto S - S_1,$$

is the restricted Springer correspondence for $GL(n)\sigma$ when n is odd.

(b) *The map*

$$D^{3,1}_{0,m} \longrightarrow X^{0,0}_{0,m}, \quad S \mapsto [S - S_0]^{\mathrm{tr}},$$

is the restricted Springer correspondence for $GL(n)\sigma$ when n is even, where we denote by tr *the transposition of a symbol.*

The following tables give the restricted Springer correspondence for $GL(n)\sigma$ with $n = 2, 3, 4, 5$. The unipotent conjugacy classes C are parametrized with Spaltenstein's method in the first column and their images χ by the restricted Springer correspondence are in the fourth column. In the second column, we give for any conjugacy class C the dimension d_C of the variety $\mathcal{B}^G_{u\sigma}$ of Borel subgroups which are stable by conjugation under $u\sigma$, for any $u\sigma \in C$. According to proposition 3.1, we have $d_C = b(\chi)$. The third column gives the symbol of $D^{3,1}_{d,m}$ corresponding to C.

Restricted Springer correspondence for $GL(2)\sigma$

C	d_C	$\rho(C)$	χ
1^2	0	$\begin{pmatrix} 0 \\ 2 \end{pmatrix}$	$(1, -)$
1^2_0	1	$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$	$(-, 1)$

Restricted Springer correspondence for $GL(3)\sigma$

C	d_C	$\rho(C)$	χ
3	0	$\begin{pmatrix} 1 \\ \ \end{pmatrix}$	$(1, -)$
1^3	1	$\begin{pmatrix} 0 & 4 \\ 2 & \end{pmatrix}$	$(-, 1)$

Restricted Springer correspondence for $GL(4)\sigma$

C	d_C	$\rho(C)$	χ
31	0	$\begin{pmatrix} 0 \\ 3 \end{pmatrix}$	$(2, -)$
2^2	1	$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$	$(1, 1)$
1^4	2	$\begin{pmatrix} 0 & 4 \\ 2 & 6 \end{pmatrix}$	$(1^2, -)$
1^4_0	4	$\begin{pmatrix} 1 & 5 \\ 1 & 5 \end{pmatrix}$	$(-, 1^2)$

Restricted Springer correspondence for $GL(5)\sigma$

C	d_C	$\rho(C)$	χ
5	0	$\begin{pmatrix} 2 \\ \end{pmatrix}$	$(2,-)$
31^2	1	$\begin{pmatrix} 0 & & 5 \\ & 2 & \end{pmatrix}$	$(1,1)$
31^2_0	2	$\begin{pmatrix} 1 & & 5 \\ & 1 & \end{pmatrix}$	$(1^2,-)$
2^21	2	$\begin{pmatrix} 0 & & 4 \\ & 3 & \end{pmatrix}$	$(-,2)$
1^5	4	$\begin{pmatrix} 0 & & 4 & & 8 \\ & 2 & & 6 & \end{pmatrix}$	$(-,1^2)$

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