SPRINGER CORRESPONDENCE FOR DISCONNECTED GROUPS

GUNTER MALLE AND KARINE SORLIN

ABSTRACT. The Springer correspondence is a map from the set of unipotent conjugacy classes of a reductive algebraic group to the set of irreducible complex characters of the Weyl group. Here, we determine this map explicitly in the case of disconnected classical algebraic groups.

CONTENTS

1. INTRODUCTION

We determine the Springer correspondence for disconnected reductive groups of classical types. The Springer correspondence was originally introduced by Tonny Springer as an injective map from the set of unipotent conjugacy classes of a connected reductive algebraic group G over an algebraic closed field to the set of irreducible characters of the Weyl group $W(G)$ of G. This correspondence and its later generalization by Lusztig (see [10] for references) turned out to be very important for the computation of character values of reductive groups over finite fields via Shoji's and Lusztig's algorithm. In the connected case, the Springer correspondence is explicitly known in all cases.

The second author showed in [12] that the definition of the Springer correspondence can be extended to the case of disconnected groups $\tilde{G}$ obtained from a connected reductive algebraic group G by adjoining an automorphism σ of order a power of the characteristic. Here, the correspondence is an injection from the set of unipotent conjugacy classes in the coset $G\sigma$ to the irreducible characters of the fixed points $W(G)^\sigma$ under σ of the Weyl group of G (see Section 2.2 for precise definitions).

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It is not yet known whether the correspondence plays a role in the computation of character values of such disconnected groups, but experiments on small examples show that a straightforward generalization of Lusztig's algorithm, with this new correspondence as input, yield the values of the Green functions computed by the first author [8] by different methods.

It turns out that two properties proved by the second author in [12], which we recall in Theorems 2.8 and 2.9, together with three reduction results, Theorems 3.1, 3.3 and 3.9, suffice to determine the Springer correspondence for disconnected groups of classical type. For the exceptional types, the Green functions were already obtained by the first author in [9].

Our proof makes essential use of results of Spaltenstein [13] on unipotent classes in disconnected groups, in particular the description of induction of classes (which we are able to make more explicit in Theorems 5.5 and 6.5) and a formula for the dimension of the variety of Borel subgroups containing a given unipotent element (see Propositions 5.2(b) and 6.2(b)).

2. Disconnected groups and Springer correspondence

2.1. Disconnected groups. In this article we will be interested in disconnected reductive groups of the form a semidirect product $G.\langle\sigma\rangle$, where G is a connected reductive group and σ an automorphism of G of finite order.

Let us recall a few results about the automorphisms of connected reductive groups. Let G be a connected reductive group defined over an algebraically closed field K of characteristic p. An automorphism σ of G is called *quasi-semisimple* if there exists a σ-stable pair (T, B) where T is a maximal torus included in a Borel subgroup B of G. According to [3, Thm. 1.8], when σ is quasi-semisimple, the group G^σ of fixed points of G under σ is a reductive group, and there is a bijection between σ-stable maximal tori T of G contained in a σ-stable Borel subgroup of G and maximal tori of G^σ defined by $T \mapsto T \cap (G^\sigma)^0$. Moreover, also according to [3, Thm. 1.33], when σ is quasi-semisimple unipotent, then G^σ is connected and the following equivalent properties hold true (σ is then said to be *quasi-central*):

(i) $\dim(G^\sigma) \geq \dim(G^{\sigma'})$ for any quasi-semisimple automorphism $\sigma' = \sigma \circ \mathrm{ad}(g)$, where $\mathrm{ad}(g)$ is the conjugation by g for all $g \in G$.

(ii) There is a bijection between σ-stable Borel subgroups B of G and Borel subgroups of G^σ defined by $B \mapsto B \cap (G^\sigma)^0$.

(iii) The Weyl group $W(G^\sigma)$ of G^σ is isomorphic to the group $W(G)^\sigma$ of fixed points under σ of the Weyl group of G.

Till the end of this section, we consider a disconnected reductive group $\hat{G} = G.\langle\sigma\rangle$, where G is a connected reductive group and σ is a unipotent quasi-semisimple automorphism of G. We say that an element of $\hat{G}$ is quasi-semisimple if it induces a quasi-semisimple automorphism of G.

Regular quasi-semisimple elements of $G\sigma$ play an important rôle in the Springer correspondence for $G\sigma$. An element x of $\hat{G}$ is said to be *regular* if it is regular in the connected component Gx of $\hat{G}$ which it belongs to, that is to say, if the dimension of its centralizer $\mathrm{C}_G(x)$ in G is minimal in Gx. By [12, Thm. 2.2] regular quasi-semisimple elements are open dense in $G\sigma$.

2.2. Springer correspondence. In this section, we recall the construction of the Springer correspondence for $G\sigma$.

Let us consider the following variety:

$$(\tilde{G\sigma}) := \{(g\sigma, B) \in G\sigma \times \mathcal{B}^G | B \in \mathcal{B}^G_{g\sigma}\},$$

where $\mathcal{B}^G$ is the variety of Borel subgroups of G and $\mathcal{B}^G_{g\sigma}$ the variety of Borel subgroups of G which are stable under conjugation by $g\sigma$. Let $\pi : (\tilde{G\sigma}) \to G\sigma$ be the projection on the first factor. Let $(G\sigma)_{\mathrm{reg}}$ be the set of regular quasi-semisimple elements of $G\sigma$ and $\pi_0 : \pi^{-1}((G\sigma)_{\mathrm{reg}}) \to (G\sigma)_{\mathrm{reg}}$ the restriction of π to $\pi^{-1}((G\sigma)_{\mathrm{reg}})$.

The group $W(G)^\sigma$ of σ-stable elements of the Weyl group $W(G)$ of G acts on $\pi^{-1}((G\sigma)_{\mathrm{reg}})$ from the right and $\pi_0 : \pi^{-1}((G\sigma)_{\mathrm{reg}}) \to (G\sigma)_{\mathrm{reg}}$ is $W(G)^\sigma$-equivariant with respect to the trivial action of $W(G)^\sigma$ on $(G\sigma)_{\mathrm{reg}}$. It is a Galois covering with group $W(G)^\sigma$. Thus $W(G)^\sigma$ acts on $\mathcal{L} := (\pi_0)_*\overline{\mathbb{Q}}_l$, the direct image of the constant sheaf $\overline{\mathbb{Q}}_l$ on $\pi^{-1}((G\sigma)_{\mathrm{reg}})$ under the map π_0, where l denotes a prime different from p.

Let us now consider the intersection cohomology complex $\mathrm{IC}(G\sigma, \mathcal{L})$ on $\overline{(G\sigma)}_{\mathrm{reg}} = G\sigma$ associated with $\mathcal{L}$. As IC is a faithful functor from the category of local systems on $(G\sigma)_{\mathrm{reg}}$ to the category $\mathcal{M}(G\sigma)$ of perverse sheaves on $G\sigma$, the action of $W(G)^\sigma$ on $\mathcal{L}$ induces an action of $W(G)^\sigma$ on $\mathrm{IC}(G\sigma, \mathcal{L})[\dim(G)]$.

Finally, $\mathrm{IC}(G\sigma, \mathcal{L})[\dim(G)]$ is isomorphic to $\pi_*\overline{\mathbb{Q}}_l[\dim(G)]$ in $\mathcal{M}(G\sigma)$, where $\pi_*\overline{\mathbb{Q}}_l$ is the direct image of the constant sheaf $\overline{\mathbb{Q}}_l$ on $(\tilde{G\sigma})$ under the map π. Thus, $W(G)^\sigma$ acts on $\pi_*\overline{\mathbb{Q}}_l$.

Let us now consider the restriction S of $\pi_*\overline{\mathbb{Q}}_l$ to the variety $\mathcal{U}(G\sigma)$ of unipotent elements of $G\sigma$. Then, $S[\dim(\mathcal{U}(G\sigma))]$ is a semisimple perverse sheaf on $\mathcal{U}(G\sigma)$ and by [12, Thm. 6.2] it decomposes as

$$S[\dim(\mathcal{U}(G\sigma))] \simeq \bigoplus_{(C,\mathcal{E})} V_{C,\mathcal{E}} \otimes \mathrm{IC}(\overline{C}, \mathcal{E})[\dim(C)],$$

where C runs over the unipotent conjugacy classes of $G\sigma$, $\mathcal{E}$ runs over the G-equivariant irreducible local systems on C, and $V_{C,\mathcal{E}}$ is a $\overline{\mathbb{Q}}_l$-vector space whose dimension is equal to the multiplicity of the simple object corresponding to $(C, \mathcal{E})$. Moreover, the action of $W(G)^\sigma$ on S induces an action of $W(G)^\sigma$ on all $V_{C,\mathcal{E}}$. By [12, Thm. 6.3(ii)] all non-zero $W(G)^\sigma$-modules $V_{C,\mathcal{E}}$ are irreducible, and the map $(C, \mathcal{E}) \mapsto V_{C,\mathcal{E}}$ induces a bijection from $\{(C, \mathcal{E}) | V_{C,\mathcal{E}} \neq 0\}$ to $\mathrm{Irr}(W(G)^\sigma)$ which is the Springer correspondence for $G\sigma$.

For any unipotent class C in $G\sigma$, the $\overline{\mathbb{Q}}_l$-vector space $V_{C,\overline{\mathbb{Q}}_l}$ is different from zero [12, Cor. 6.5]. The Springer correspondence thus induces an injection from the set of unipotent conjugacy classes of $G\sigma$ to the set of irreducible representations of W^σ defined by $C \mapsto V_{C,\overline{\mathbb{Q}}_l}$, which we call the *restricted Springer correspondence*. It is this map which we shall determine in this paper for groups of classical type.

2.3. Induction of unipotent classes. Let L be a σ-stable Levi subgroup of a σ-stable parabolic subgroup P of G. Let C be a unipotent L-conjugacy class of $L\sigma$. Then, there exists a unique G-conjugacy class $\tilde{C}$ in the coset $G\sigma$ such that

$\tilde{C} \cap CU_P$ is dense in CU_P, where U_P is the unipotent radical of P. The class $\tilde{C}$ is then called the unipotent class obtained by *inducing the class C from $L\sigma$ to $G\sigma$*. In this situation we write $\tilde{C} = \mathrm{Ind}_L^G(C)$.

Proposition 2.4. *Let C be a unipotent L-conjugacy class of $L\sigma$ and let $\tilde{C} = \mathrm{Ind}_L^G(C)$ be the unipotent G-conjugacy class of $G\sigma$ obtained by inducing the class C from $L\sigma$ to $G\sigma$. Let u be in C and let $\tilde{u}$ be in $\tilde{C}$. Then*

$$\dim(\mathcal{B}_u^L) = \dim(\mathcal{B}_{\tilde{u}}^G).$$

Proof. This is Proposition II.3.2 of [13]. □

If C is the quasi-semisimple unipotent class of $L\sigma$, then $\tilde{C}$ is called the *Richardson class* associated to P and contained in $G\sigma$. We have the following transitivity:

Proposition 2.5. *Let L be a σ-stable Levi subgroup of a σ-stable parabolic subgroup P of G and let L' be a σ-stable Levi subgroup of a σ-stable parabolic subgroup P' of L. Then for every unipotent L'-conjugacy class C' of $L'\sigma$ we have*

$$\mathrm{Ind}_L^G\left(\mathrm{Ind}_{L'}^L(C')\right) = \mathrm{Ind}_{L'}^G(C').$$

Proof. This comes from [13, II.3.6]. □

2.6. j-induction. We recall the Macdonald-Lusztig-Spaltenstein induction, also known as j-induction, in finite Coxeter groups. Let W be a finite Coxeter group and V a real vector space on which W acts as a reflection group. Then for every $d \geq 0$ the space $S^d(V)$ of homogeneous symmetric polynomials on V of degree d is naturally a W module. For an irreducible complex character χ of W let $b(\chi)$ denote the minimal d such that χ occurs in $S^d(V)$. It is known that such a d always exists (see [5, 5.2]).

Let W' be a reflection subgroup of W and $\psi \in \mathrm{Irr}(W')$. Then the b-invariant of any constituent χ of $\mathrm{Ind}_{W'}^W(\psi)$ satisfies $b(\chi) \geqslant b(\psi)$ [5, Lemma 5.2.5]. The Macdonald-Lusztig-Spaltenstein induction of ψ is defined as

$$j_{W'}^W(\psi) := \sum_{\chi \in \mathrm{Irr}(W): b(\chi) = b(\psi)} \langle \mathrm{Ind}_{W'}^W(\psi), \chi \rangle \, \chi.$$

That is, $j_{W'}^W(\psi)$ consists of those constituents χ of $\mathrm{Ind}_{W'}^W(\psi)$ with b-invariant equal to $b(\psi)$. Since any constituent of $\mathrm{Ind}_{W'}^W(\psi)$ has b-invariant at least $b(\psi)$, we see that j-induction is transitive. It is known moreover that, in Coxeter groups of type B_m, the j-induction sends irreducibles to irreducibles (see Proposition 4.6(a)).

2.7. Some properties of the Springer correspondence. We recall here two properties of the Springer correspondence for $G\sigma$ which were proved by the second author [12]. For all unipotent element $u \in G\sigma$, let d_u denote the dimension of $\mathcal{B}_u^G$.

Theorem 2.8. *Let C be the unipotent conjugacy class of $u \in G\sigma$, let χ_u be the irreducible representation of $W(G)^\sigma$ corresponding to the pair $(C, \overline{\mathbb{Q}}_l)$ by the Springer correspondence for $G\sigma$. Then*

$$d_u = b(\chi_u).$$

Proof. This is Proposition 6.7(ii) in [12]. □

Theorem 2.9. *Let L be a σ-stable Levi subgroup of a σ-stable parabolic subgroup P of G. Let C be the unipotent class of $u \in L\sigma$ and let $\tilde{u} \in \tilde{C} = \mathrm{Ind}_L^G(C)$.*

Let χ_u be the irreducible representation of $W(L)^\sigma$ which corresponds to $(C, \overline{\mathbb{Q}}_l)$ by the Springer correspondence for $L\sigma$.

Let $\chi_{\tilde{u}}$ be the irreducible representation of $W(G)^\sigma$ which corresponds to $(\tilde{C}, \overline{\mathbb{Q}}_l)$ by the Springer correspondence for $G\sigma$. Then,

$$\langle \mathrm{Res}_{W(L)^\sigma}^{W(G)^\sigma}(\chi_{\tilde{u}}), \chi_u \rangle \neq 0\,.$$

Proof. This is Proposition 7.13(vi) in [12]. □

The preceding results can be conveniently reformulated in terms of j-induction.

Corollary 2.10. *Let L be a σ-stable Levi subgroup of a σ-stable parabolic subgroup of G. Let C be the unipotent class of $u \in L\sigma$ and let $\tilde{u} \in \tilde{C} = \mathrm{Ind}_L^G(C)$.*

Let χ_u, $\chi_{\tilde{u}}$ be the Springer correspondents of $(C, \overline{\mathbb{Q}}_l)$, $(\tilde{C}, \overline{\mathbb{Q}}_l)$ respectively. Assume that $j_{W(L)^\sigma}^{W(G)^\sigma}(\chi_u)$ is irreducible. Then

$$\chi_{\tilde{u}} = j_{W(L)^\sigma}^{W(G)^\sigma}(\chi_u).$$

Proof. According to Theorem 2.9, $\chi_{\tilde{u}}$ is an irreducible component of $\mathrm{Ind}_{W(L)^\sigma}^{W(G)^\sigma}(\chi_u)$. Moreover, as $\tilde{C}$ is the unipotent class of $G\sigma$ obtained by inducing C from $L\sigma$ to $G\sigma$, we have $d_{\tilde{u}} = d_u$ by Proposition 2.4. According to Theorem 2.8, we have $d_u = b(\chi_u)$ and $d_{\tilde{u}} = b(\chi_{\tilde{u}})$, so $b(\chi_u) = b(\chi_{\tilde{u}})$. Thus indeed by definition, $\chi_{\tilde{u}}$ occurs with non-zero multiplicity in $j_{W(L)^\sigma}^{W(G)^\sigma}(\chi_u)$. □

3. Reduction results

In this section we show that in order to determine the Springer correspondence for disconnected groups, it is sufficient to consider the case where the connected component is simple.

Theorem 3.1. *Assume that $G = G_1 \times G_2$ is a direct product and σ stabilizes both factors. Let $\mathcal{CU}(G\sigma)$ be the set of unipotent classes of $G\sigma$ and for $i = 1, 2$, let $\mathcal{CU}(G_i\sigma)$ be the set of unipotent classes of $G_i\sigma$. Then there is a natural bijection between $\mathcal{CU}(G_1\sigma) \times \mathcal{CU}(G_2\sigma)$ and $\mathcal{CU}(G\sigma)$. Moreover, the Springer correspondence commutes with this map.*

Proof. The map $\mathcal{CU}(G_1\sigma) \times \mathcal{CU}(G_2\sigma) \to \mathcal{CU}(G\sigma)$ defined by $(C_1, C_2) \mapsto C_1 \times C_2$ is clearly a bijection. Similarly, there is a bijection between $\mathcal{B}^{G_1} \times \mathcal{B}^{G_2}$ and $\mathcal{B}^G$ defined by $(B_1, B_2) \mapsto B_1 \times B_2$.

The variety $(\tilde{G\sigma})$, defined in 2.2, is the direct product $(\tilde{G_1\sigma}) \times (\tilde{G_2\sigma})$, where, for $i = 1, 2$, $(\tilde{G_i\sigma})$ is defined by:

$$(\tilde{G_i\sigma}) := \{(g\sigma, B) \in G_i\sigma \times \mathcal{B}^{G_i} | B \in \mathcal{B}_{g\sigma}^{G_i}\},$$

where $\mathcal{B}^{G_i}$ is the variety of Borel subgroups of G_i and $\mathcal{B}_{g\sigma}^{G_i}$ the variety of Borel subgroups of G_i which are stable under conjugation by $g\sigma$ and the projection π : $(\tilde{G\sigma}) \to G\sigma$ is the product morphism $\pi_1 \times \pi_2$. Then, $\pi_* \overline{\mathbb{Q}}_l$ is the external product

$\pi_{1*}\overline{\mathbb{Q}}_l \boxtimes \pi_{2*}\overline{\mathbb{Q}}_l$ of the direct images $\pi_{i*}\overline{\mathbb{Q}}_l$ of the constant sheaves $\overline{\mathbb{Q}}_l$ on $(\tilde{G_i\sigma})$ under the maps π_i (for the external product commutes with the direct image functor [1, 4.2.7]) and the constant sheaf $\overline{\mathbb{Q}}_l$ on $\pi^{-1}(G\sigma) = \pi_1^{-1}(G_1\sigma) \times \pi_2^{-1}(G_2\sigma)$ is the external product $\overline{\mathbb{Q}}_l \boxtimes \overline{\mathbb{Q}}_l$.

As $W(G) = W(G_1) \times W(G_2)$ and as each $W(G_i)$ is σ-stable, $W(G)^\sigma = W(G_1)^\sigma \times W(G_2)^\sigma$. So, the Springer action of $W(G)^\sigma$ on $\pi_*\overline{\mathbb{Q}}_l$ is the product of the Springer action of $W(G_1)^\sigma$ on $\pi_{1*}\overline{\mathbb{Q}}_l$ and of the Springer action of $W(G_2)^\sigma$ on $\pi_{2*}\overline{\mathbb{Q}}_l$.

The variety $\mathcal{U}(G\sigma)$ is the direct product $\mathcal{U}(G_1\sigma) \times \mathcal{U}(G_2\sigma)$, where $\mathcal{U}(G_i\sigma)$ is the variety of the unipotent elements of $G_i\sigma$. Let S_i be the restriction of $\pi_{i*}\overline{\mathbb{Q}}_l$ to $\mathcal{U}(G_i\sigma)$. Then, $S_i[\dim(\mathcal{U}(G_i\sigma))]$ is a semisimple perverse sheaf on $\mathcal{U}(G_i\sigma)$ and it decomposes as

$$S_i[\dim(\mathcal{U}(G_i\sigma))] \simeq \bigoplus_{(C_i,\mathcal{E}_i)} V_{C_i,\mathcal{E}_i} \otimes \mathrm{IC}(\overline{C_i}, \mathcal{E}_i)[\dim(C_i)]$$

where C_i runs over the unipotent conjugacy classes of $G_i\sigma$, $\mathcal{E}_i$ runs over the G_i-equivariant irreducible local systems on C_i, the $\overline{\mathbb{Q}}_l$-vector spaces $V_{C_i,\mathcal{E}_i}$ are irreducible representations of $W(G_i)^\sigma$ (when $V_{C_i,\mathcal{E}_i} \neq 0$) and the map from $\{(C_i,\mathcal{E}_i) | V_{C_i,\mathcal{E}_i} \neq 0\}$ to $\mathrm{Irr}(W(G_i)^\sigma)$ defined by $(C_i,\mathcal{E}_i) \mapsto V_{C_i,\mathcal{E}_i}$ is the Springer correspondence for $G_i\sigma$.

But now denote by S the restriction of $\pi_*\overline{\mathbb{Q}}_l$ to $\mathcal{U}(G\sigma)$, like in 2.2. Then, S is the external product $S_1 \boxtimes S_2$ (for the external product commutes with the reciprocal image functor, [1, 4.2.7]) and $W(G)^\sigma$ acts on S via the actions of $W(G_1)^\sigma$ on S_1 and the action of $W(G_2)^\sigma$ on S_2.

We set $\dim(\mathcal{U}(G\sigma)) = n$ and $\dim(\mathcal{U}(G_i\sigma)) = n_i$ for $i = 1, 2$. We have $S[n] = (S_1 \boxtimes S_2)[n]$ in $\mathcal{M}(\mathcal{U}(G\sigma))$. So we get two decompositions of $S[n]$ as a semisimple perverse sheaf. One, given by 2.2, induces the Springer correspondence for $G\sigma$:

$$S[n] \simeq \bigoplus_{(C,\mathcal{E})} V_{C,\mathcal{E}} \otimes \mathrm{IC}(\overline{C}, \mathcal{E})[\dim(C)],$$

where C runs over the unipotent conjugacy classes of $G\sigma$, $\mathcal{E}$ runs over the G-equivariant irreducible local systems on C, and the other given by the decomposition of the perverse sheaves $S_i[n_i]$:

$$S[n] \simeq \bigoplus_{(C_1,\mathcal{E}_1)} \bigoplus_{(C_2,\mathcal{E}_2)} V_{C_1,\mathcal{E}_1} \otimes V_{C_2,\mathcal{E}_2} \otimes (\mathrm{IC}(\overline{C_1}, \mathcal{E}_1)[\dim(C_1)] \boxtimes \mathrm{IC}(\overline{C_2}, \mathcal{E}_2)[\dim(C_2)]).$$

So the theorem will be proved if we prove the following lemma:

Lemma 3.2. *Let C be a unipotent class of $G\sigma$ and let $\mathcal{E}$ be an irreducible G-equivariant local system on C, then, for $i = 1, 2$, there is a unipotent class C_i of $G_i\sigma$ and an irreducible G_i-equivariant local system on C_i such that $C = C_1 \times C_2$ and $\mathcal{E} = \mathcal{E}_1 \boxtimes \mathcal{E}_2$. Moreover, we have*

$$\mathrm{IC}(\overline{C}, \mathcal{E})[\dim(C)] = \mathrm{IC}(\overline{C_1}, \mathcal{E}_1)[\dim(C_1)] \boxtimes \mathrm{IC}(\overline{C_2}, \mathcal{E}_2)[\dim(C_2)]$$

Proof. The first part of the lemma is clear, let us prove the last equality.

Let us set $d = \dim(C)$ and $d_i = \dim(C_i)$ for $i = 1, 2$. We denote by j_i the inclusion $C_i \hookrightarrow \overline{C_i}$ for $i = 1, 2$ and we set $j = j_1 \times j_2$. Then, $\mathrm{IC}(\overline{C_i}, \mathcal{E}_i)[d_i]$ is the perverse extension $j_{i!*}\mathcal{E}_i[d_i]$ of $\mathcal{E}_i[d_i]$ and $\mathrm{IC}(\overline{C}, \mathcal{E})[d] = j_{!*}\mathcal{E}[d]$.

According to [1, 4.2.8], $\boxtimes$ is a t-exact bifunctor. Then, it induces an exact bifunctor of perverse sheaves and

$$j_{1!*}\mathcal{E}_1[d_1] \boxtimes j_{2!*}\mathcal{E}_2[d_2] = \mathrm{im}({}^p j_{1!}\mathcal{E}_1[d_1] \boxtimes {}^p j_{2!}\mathcal{E}_2[d_2] \to {}^p j_{1*}\mathcal{E}_1[d_1] \boxtimes {}^p j_{2*}\mathcal{E}_2[d_2]).$$

But the bifunctor $\boxtimes$ commutes with the usual operations by [1, 4.2.7]. Hence,

$${}^p j_{1!}\mathcal{E}_1[d_1] \boxtimes {}^p j_{2!}\mathcal{E}_2[d_2] = {}^p j_!\mathcal{E}[d] \qquad \text{and} \qquad {}^p j_{1*}\mathcal{E}_1[d_1] \boxtimes {}^p j_{2*}\mathcal{E}_2[d_2] = {}^p j_*\mathcal{E}[d].$$

So, we get $j_{1!*}\mathcal{E}_1[d_1] \boxtimes j_{2!*}\mathcal{E}_2[d_2] = j_{!*}\mathcal{E}[d]$. □

This completes the proof of Theorem 3.1. □

Theorem 3.3. *Let $G = G_1 \times \cdots \times G_r$ be a direct product of r isomorphic copies of a connected reductive group over an algebraically closed field of characteristic p, where r is a power of p. Let σ be an automorphism of G_1 such that σ^r is quasi-central unipotent. We also denote by σ the automorphism of G defined by $(g_1, g_2, \dots, g_r) \mapsto (\sigma(g_2), \dots, \sigma(g_r), \sigma(g_1))$. Then, the map*

$$\phi : G\sigma \to G_1\sigma^r, \qquad (u_1, \dots, u_r)\sigma \mapsto u_1 \cdots u_r\sigma^r\,,$$

induces a natural bijection between the unipotent classes of $G\sigma$ and the unipotent classes of $G_1\sigma^r$. Moreover, the Springer correspondence commutes with this bijection.

Proof. Let us first show that ϕ induces a bijection between the conjugacy classes of $G\sigma$ and those of $G_1\sigma^r$. Let $g = (g_1, \dots, g_r)$ and $x = (x_1, \dots, x_r)$ be in G, then

$$xg\sigma x^{-1} = (x_1 g_1 \sigma(x_2)^{-1}, \dots, x_{r-1}g_{r-1}\sigma(x_r)^{-1}, x_r g_r \sigma(x_1)^{-1}),$$

and

$$\phi(xg\sigma x^{-1}) = x_1 g_1 \sigma(g_2) \dots \sigma^{r-1}(g_r)\sigma^r(x_1)^{-1}\sigma^r = x_1\phi(g\sigma)x_1^{-1}.$$

So, ϕ induces a map $\Phi : C \mapsto \phi(C)$ from the conjugacy classes of $G\sigma$ to those of $G_1\sigma^r$. Let us prove that this is a bijection. Let $g = (g_1, \dots, g_r)$ be in G. If we set $x = (1, (g_2 \dots \sigma^{r-2}(g_r))^{-1}, (g_3 \dots \sigma^{r-3}(g_r))^{-1}, \dots, g_r^{-1})$ then we have $xg\sigma x^{-1} = (g_1\sigma(g_2)\dots\sigma^{r-1}(g_r), 1, \dots, 1)\sigma$. So, Φ is a bijection whose inverse associates to the conjugacy class of $g\sigma^r$ in $G_1\sigma^r$ the conjugacy class of $(g, 1, \dots, 1)\sigma$ in $G\sigma$.

The morphism ϕ may also be defined by $g\sigma \mapsto (g\sigma)^r$. Let g be in G. As r is a power of the characteristic p of K, $\phi(g\sigma)$ is unipotent if and only if $g\sigma$ is unipotent.

Thus, $\Phi : C \mapsto \phi(C)$ induces a bijection between unipotent conjugacy classes in $G\sigma$ and those in $G_1\sigma^r$.

Lemma 3.4. *Let C be a unipotent conjugacy class of $G\sigma$. Then we have:*

$$\dim\mathcal{U}(G\sigma) - \dim(C) = \dim\mathcal{U}(G_1\sigma^r) - \dim(\phi(C)).$$

Proof. We have $\dim\mathcal{U}(G\sigma) = \dim(G\sigma) - \mathrm{rank}(G^\sigma) = \dim(G\sigma) - \mathrm{rank}((G_1)^{\sigma^r})$ and $\dim\mathcal{U}(G_1\sigma^r) = \dim(G_1) - \mathrm{rank}((G_1)^{\sigma^r})$. Thus, the equality in the lemma is equivalent to $\dim(\mathrm{C}_G(g\sigma)) = \dim(\mathrm{C}_{G_1}(\phi(g\sigma)))$, where $g\sigma \in C$. But, for $g\sigma = (g_1, \dots, g_r)\sigma \in C$, the centralizer $\mathrm{C}_G(g\sigma) =$

$$\{(x, g_2 \dots \sigma^{r-2}(g_r)\sigma^{r-1}(x)(g_2 \dots \sigma^{r-2}(g_r))^{-1}, \dots, g_r\sigma(x)g_r^{-1}) | x \in \mathrm{C}_{G_1}(\phi(g\sigma))\}$$

is clearly isomorphic to $\mathrm{C}_{G_1}(\phi(g\sigma))$, whence the result. □

We will use the same notations as in 2.2 for $G\sigma$. As G^σ is isomorphic to $(G_1)^{\sigma^r}$, the Weyl group $W(G^\sigma)$ of G^σ is isomorphic to the Weyl group $W((G_1)^{\sigma^r})$ of $(G_1)^{\sigma^r}$.

Let us introduce the following notations for G_1:

$$\tilde{G_1} = \{(g\sigma^r, B) \in G_1\sigma^r \times \mathcal{B}^{G_1} | B \in \mathcal{B}^{G^1}_{g\sigma^r}\},$$

where $\mathcal{B}^{G_1}$ is the variety of Borel subgroups of G_1. Let $\pi' : \tilde{G_1} \to G_1\sigma^r$ be the projection on the first factor. Let $(G_1\sigma^r)_{\mathrm{reg}}$ be the set of regular quasi-semisimple elements of $G_1\sigma^r$ and $\pi'_0 : \pi'^{-1}((G_1\sigma^r)_{\mathrm{reg}}) \to (G_1\sigma^r)_{\mathrm{reg}}$ the restriction of π' to $\pi'^{-1}((G_1\sigma^r)_{\mathrm{reg}})$. Let us consider the following cartesian diagram:

$$\begin{array}{ccc} \pi'^{-1}((G_1\sigma^r)_{\mathrm{reg}}) & \xrightarrow{\tilde{i'}} & \tilde{G_1} \\ \downarrow{\scriptstyle \pi'_0} & & \downarrow{\scriptstyle \pi'} \\ (G_1\sigma^r)_{\mathrm{reg}} & \xrightarrow{i'} & G_1\sigma^r \end{array}$$

Let $\mathcal{L}'$ be the local system $(\pi'_0)_*\overline{\mathbb{Q}}_l$ on $(G_1\sigma^r)_{\mathrm{reg}}$. According to 2.2, we have an action of the Weyl group $W((G_1)^{\sigma^r})$ on $\mathcal{L}'$, and the fact that $\mathrm{IC}(G_1\sigma^r, \mathcal{L}')[n']$ is isomorphic to $\pi'_*\overline{\mathbb{Q}}_l[n']$ in $\mathcal{M}(G_1\sigma^r)$ (where $n' = \dim(G_1)$) induces a Springer action of $W((G_1)^{\sigma^r})$ on $\pi'_*\overline{\mathbb{Q}}_l$.

Let us consider the following commutative diagram

$$\begin{array}{ccc} (\tilde{G\sigma}) & \xrightarrow{\tilde{\phi}} & (\tilde{G_1}) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ G\sigma & \xrightarrow{\phi} & G_1\sigma^r \end{array}$$

where $\tilde{\phi}$ is defined by $(g\sigma, B) \mapsto (\phi(g\sigma), p_1(B))$ where $p_1 : G = G_1 \times \ldots \times G_r \to G_1$ is the projection on the first factor.

Lemma 3.5. *The map $\tilde{\phi}$ is well-defined and the diagram above is cartesian.*

Proof. Let $g\sigma = (g_1, \ldots, g_r)\sigma$ be in $G\sigma$ and let $B = B_1 \times \ldots \times B_r$ be a Borel subgroup of G which is stable under conjugation by $g\sigma$. Then, the Borel subgroup B_1 of G_1 is stable under conjugation by $\phi(g\sigma) = g_1\sigma(g_2)\ldots\sigma^{r-1}(g_r)$. Hence, $\tilde{\phi}$ is well-defined. Moreover, the map p_1 induces a morphism of varieties f from $\mathcal{B}^G_{g\sigma}$ to $\mathcal{B}^{G_1}_{\phi(g\sigma)}$ defined by $B \mapsto p_1(B)$. Let C be the conjugacy class of $g\sigma$. By Lemma 3.4, we have $\dim \mathcal{U}(G\sigma) - \dim(C) = \dim \mathcal{U}(G_1\sigma^r) - \dim(\phi(C))$. But, by [13, II.10.15], we have $2\dim \mathcal{B}^G_{g\sigma} = \dim(G) - \mathrm{rank}(G^\sigma) - \dim(C) = \dim \mathcal{U}(G\sigma) - \dim(C)$ and $2\dim \mathcal{B}^{G_1}_{\phi(g\sigma)} = \dim \mathcal{U}(G_1\sigma^r) - \dim(\phi(C))$. Then, we have $\dim \mathcal{B}^G_{g\sigma} = \dim \mathcal{B}^{G_1}_{\phi(g\sigma)}$ and f is an isomorphism. Hence, $(\tilde{G\sigma})$ is isomorphic to $(G\sigma) \times_{(G_1\sigma^r)} (\tilde{G_1})$ and the diagram above is cartesian. $\square$

By the proper base change theorem, we have $\phi^*\pi'_*\overline{\mathbb{Q}}_l \simeq \pi_*\tilde{\phi}^*\overline{\mathbb{Q}}_l = \pi_*\overline{\mathbb{Q}}_l$. Let us show that this isomorphism is $W((G_1)^{\sigma^r})$-equivariant for the Springer actions on $\pi'_*\overline{\mathbb{Q}}_l$ and on $\pi_*\overline{\mathbb{Q}}_l$.

Let us consider the following commutative diagram:

$$\begin{array}{ccc} (G\sigma)_{\mathrm{reg}} & \xrightarrow{\phi_{\mathrm{reg}}} & (G_1\sigma^r)_{\mathrm{reg}} \\ \Big\downarrow j_{\mathrm{reg}} & & \Big\downarrow j'_{\mathrm{reg}} \\ G\sigma & \xrightarrow{\phi} & G_1\sigma^r \end{array},$$

where ϕ_{reg} is the restriction of ϕ to $(G\sigma)_{\mathrm{reg}}$.

Lemma 3.6. *The image of $(G\sigma)_{\mathrm{reg}}$ by the morphism ϕ is $(G_1\sigma^r)_{\mathrm{reg}}$.*

Proof. Let $T := T_1 \times \ldots \times T_r$ be a σ-stable maximal torus of G included in a σ-stable Borel subgroup $B = B_1 \times \ldots \times B_r$ of G. Then, T_1 is a σ^r-stable maximal torus of G_1 included in the σ^r-stable Borel subgroup B_1 of G_1 and we have $T^\sigma = \{(t_1, \sigma^{-1}(t_1), \ldots, \sigma^{r-1}(t_1)) | t_1 \in (T_1)^{\sigma^r}\} \simeq (T_1)^{\sigma^r}$.

Let $g\sigma$ be a quasi-semisimple regular element of $G\sigma$. Let us prove that $\phi(g\sigma)$ is semisimple regular in $G_1\sigma^r$. There is $t = (t_1, \sigma^{-1}(t_1), \ldots, \sigma^{r-1}(t_1)) \in T^\sigma$, which is regular in G^σ, such that $g\sigma$ is conjugate to $t\sigma$ under G ([12, 2.4]). Moreover, $\phi(g\sigma)$ and $\phi(t\sigma)$ are conjugate under G_1. So, it is sufficient to prove that $\phi(t\sigma) = (t_1)^r\sigma^r$ is quasi-semisimple and regular in $G_1\sigma^r$. It is quasi-semisimple as it stabilizes the pair (T_1, B_1). Let us prove it is regular in $G_1\sigma^r$.

As $t \in T^\sigma$ is semisimple regular in G^σ, we have $\alpha(t_1) \neq 1$ for all roots α of $(G_1)^{\sigma^r}$ relative to $(T_1)^{\sigma^r}$ by [2, 14.4]. But, r is a power of the characteristic p of K. So we have $\alpha(t_1^r) = \alpha(t_1)^r \neq 1$, for all roots α of $(G_1)^{\sigma^r}$ relative to $(T_1)^{\sigma^r}$ and $(t_1)^r$ is regular in $(G_1)^{\sigma^r}$ by [2, 14.4]. Hence, by [12, 2.4], $\phi(t\sigma) = (t_1)^r\sigma^r$ is regular in $G_1\sigma^r$. $\square$

As $\pi'_*\overline{\mathbb{Q}}_l[n'] \simeq \mathrm{IC}(G_1\sigma^r, \mathcal{L}')[n']$ (where $n' = \dim(G_1)$) and $\pi_*\overline{\mathbb{Q}}_l[n] \simeq \mathrm{IC}(G\sigma, \mathcal{L})[n]$ (where $n = \dim(G\sigma)$), and as IC is a fully faithful functor from the category of local systems on $(G_1\sigma^r)_{\mathrm{reg}}$ (resp. on $(G\sigma)_{\mathrm{reg}}$) to the category of perverse sheaves on $G_1\sigma^r$ (resp. on $G\sigma$), it is sufficient to check that the isomorphism $\phi^*_{\mathrm{reg}}\pi'_{0*}\overline{\mathbb{Q}}_l \simeq \pi_{0*}\overline{\mathbb{Q}}_l$ (obtained by applying the functor j^*_{reg} to the previous isomorphism) is $W((G_1)^{\sigma^r})$-equivariant.

Let us consider the following cartesian diagram:

$$\begin{array}{ccc} \pi^{-1}((G\sigma)_{\mathrm{reg}}) & \xrightarrow{\tilde{\phi}_{\mathrm{reg}}} & \pi'^{-1}((G_1\sigma^r)_{\mathrm{reg}}) \\ \Big\downarrow \pi_0 & & \Big\downarrow \pi'_0 \\ (G\sigma)_{\mathrm{reg}} & \xrightarrow{\phi_{\mathrm{reg}}} & (G_1\sigma^r)_{\mathrm{reg}} \end{array},$$

where $\tilde{\phi}_{\mathrm{reg}}$ is the restriction of $\tilde{\phi}$ to $\pi^{-1}((G\sigma)_{\mathrm{reg}})$. By the proper base change theorem, we have $\phi^*_{\mathrm{reg}}\pi'_{0*}\overline{\mathbb{Q}}_l \simeq \pi_{0*}\tilde{\phi}^*_{\mathrm{reg}}\overline{\mathbb{Q}}_l$. Now recall that the action of $W((G_1)^{\sigma^r})$ on $\mathcal{L}$ comes from the action of $W((G_1)^{\sigma^r})$ on $\pi^{-1}((G\sigma)_{\mathrm{reg}})$ and the fact that π_0 is a Galois fibration with Galois group $W((G_1)^{\sigma^r})$. So the $W((G_1)^{\sigma^r})$-equivariance of the isomorphism $\phi^*\pi'_*\overline{\mathbb{Q}}_l \simeq \pi_*\overline{\mathbb{Q}}_l$ is proved with the following lemma.

Lemma 3.7. *$\tilde{\phi}_{\mathrm{reg}} : \pi^{-1}((G\sigma)_{\mathrm{reg}}) \to \pi'^{-1}((G_1\sigma^r)_{\mathrm{reg}})$ is $W((G_1)^{\sigma^r})$-equivariant.*

Proof. Let $T = T_1\times\ldots\times T_r$ be a σ-stable maximal torus of G and let $B = B_1\times\ldots\times B_r$ be a σ-stable Borel subgroup of G such that $T \subset B$.

Let us consider the following varieties:

$$(\tilde{G\sigma})_{\mathrm{reg}} = \{(g\sigma, xT) \in G\sigma \times G/T | x^{-1}g\sigma x \in T\sigma \cap (G\sigma)_{\mathrm{reg}}\}.$$

and

$$(\tilde{G_1})_{\mathrm{reg}} = \{(g'\sigma^r, x_1T_1) \in G_1\sigma^r \times G_1/T_1 | x_1^{-1}g'\sigma^r x_1 \in T_1\sigma^r \cap (G_1\sigma^r)_{\mathrm{reg}}\}.$$

According to [12, Prop. 4.2], $\pi^{-1}((G\sigma)_{\mathrm{reg}})$ (resp. $\pi'^{-1}((G_1\sigma^r)_{\mathrm{reg}})$) is isomorphic to $(\tilde{G\sigma})_{\mathrm{reg}}$ (resp. $(\tilde{G_1})_{\mathrm{reg}}$) via the map $(g\sigma, xT) \mapsto (g\sigma, xB)$ (resp. $(g'\sigma^r, x_1T_1) \mapsto (g'\sigma^r, x_1B_1)$) and $W((G_1)^{\sigma^r})$ acts on $\pi^{-1}((G\sigma)_{\mathrm{reg}})$ (resp. on $\pi'^{-1}((G_1\sigma^r)_{\mathrm{reg}})$) via this isomorphism and the action of $W((G_1)^{\sigma^r})$ on $(\tilde{G\sigma})_{\mathrm{reg}}$ (resp. on $(\tilde{G_1})_{\mathrm{reg}}$) defined by

$$w : (g\sigma, (x_1T_1, \ldots, x_rT_r)) \mapsto (g\sigma, (x_1wT_1, \ldots, x_rwT_r))$$

(resp. by $w : (g'\sigma^r, x_1T_1) \mapsto (g'\sigma^r, x_1wT_1)$).

But the map $(\tilde{G\sigma})_{\mathrm{reg}} \to (\tilde{G_1})_{\mathrm{reg}}$ defined by

$$((g_1, \ldots, g_r)\sigma, (x_1T_1, \ldots, x_rT_r)) \mapsto (g_1\sigma(g_2)\cdots\sigma^{r-1}(g_r)\sigma^r, x_1T_1)$$

is $W((G_1)^{\sigma^r})$-equivariant for the actions described above. Whence, the lemma. □

The image under ϕ of the variety $\mathcal{U}(G\sigma)$ of unipotent elements of $G\sigma$ is the variety $\mathcal{U}(G_1\sigma^r)$ of unipotent elements of $G_1\sigma^r$. Let us consider the restriction of $\phi^*\pi'_*\overline{\mathbb{Q}}_l$ to $\mathcal{U}(G\sigma)$. By commutativity, we have $\phi^*\pi'_*\overline{\mathbb{Q}}_l|_{\mathcal{U}(G\sigma)} \simeq \phi_2^*(\pi'_*\overline{\mathbb{Q}}_l|_{\mathcal{U}(G_1\sigma^r)})$, where ϕ_2 is the restriction of ϕ to $\mathcal{U}(G\sigma)$. Hence, we have $\phi_2^*(\pi'_*\overline{\mathbb{Q}}_l|_{\mathcal{U}(G_1\sigma^r)}) \simeq \pi_*\overline{\mathbb{Q}}_l|_{\mathcal{U}(G\sigma)}$ and this isomorphism is $W((G_1)^{\sigma^r})$-equivariant for the Springer actions.

But, according to 2.2, we have the following decomposition formulas:

$$\pi'_*\overline{\mathbb{Q}}_l|_{\mathcal{U}(G_1\sigma^r)}[\dim(\mathcal{U}(G_1\sigma^r))] \simeq \bigoplus_{(C',\mathcal{E}')} V'_{C',\mathcal{E}'} \otimes \mathrm{IC}(\overline{C}', \mathcal{E}')[\dim(C')]$$

where C' runs over the unipotent conjugacy classes of $G_1\sigma^r$ and $\mathcal{E}'$ over the G_1-equivariant irreducible local systems on C' and $V'_{C',\mathcal{E}'}$ is a $\overline{\mathbb{Q}}_l$-vector space whose dimension is equal to the multiplicity of the simple object corresponding to $(C', \mathcal{E}')$, and

$$\pi_*\overline{\mathbb{Q}}_l|_{\mathcal{U}(G\sigma)}[\dim(\mathcal{U}(G\sigma))] \simeq \bigoplus_{(C,\mathcal{E})} V_{C,\mathcal{E}} \otimes \mathrm{IC}(\overline{C}, \mathcal{E})[\dim(C)]$$

where C runs over the unipotent conjugacy classes of $G\sigma$ and $\mathcal{E}$ over the G-equivariant irreducible local systems on C.

Moreover, the action of $W((G_1)^{\sigma^r})$ on $\pi_*\overline{\mathbb{Q}}_l|_{\mathcal{U}(G\sigma)}$ (resp. $\pi'_*\overline{\mathbb{Q}}_l|_{\mathcal{U}(G_1\sigma^r)}$) induces an action of $W((G_1)^{\sigma^r})$ on all $V_{C,\mathcal{E}}$ (resp. $V'_{C',\mathcal{E}'}$) and the map $(C, \mathcal{E}) \mapsto V_{C,\mathcal{E}}$ (resp. $(C', \mathcal{E}') \mapsto V'_{C',\mathcal{E}'}$) induces a bijection from $\{(C, \mathcal{E}) | V_{C,\mathcal{E}} \neq 0\}$ (resp. $\{(C', \mathcal{E}') | V'_{C',\mathcal{E}'} \neq 0\}$) to $\mathrm{Irr}(W((G_1)^{\sigma^r}))$ which is the Springer correspondence for $G\sigma$ (resp. for $G_1\sigma^r$).

Thus, according to Lemma 3.4, we have

$$\phi_2^*(\pi'_*\overline{\mathbb{Q}}_l|_{\mathcal{U}(G_1\sigma^r)})[\dim(\mathcal{U}(G\sigma))] \simeq \bigoplus_{(C',\mathcal{E}')} V'_{C',\mathcal{E}'} \otimes \phi_2^*(\mathrm{IC}(\overline{C}', \mathcal{E}'))[\dim(\Phi^{-1}(C'))].$$

So, the theorem will be proved with the following lemma:

Lemma 3.8. *For every unipotent class C' of $G_1\sigma^r$ and every irreducible G_1-equivariant local system $\mathcal{E}'$ on C', $(\phi_2|_C)^*\mathcal{E}'$ is an irreducible G-equivariant local system on $C = \Phi^{-1}(C')$ and all irreducible G-equivariant local systems on C can be obtained in that way. Moreover,*

$$\phi_2^*(\mathrm{IC}(\overline{C'}, \mathcal{E}'))[\dim(C)] \simeq \mathrm{IC}(\overline{C}, (\phi_2|_C)^*\mathcal{E}')[\dim(C)]$$

in $\mathcal{M}(\mathcal{U}(G\sigma))$.

Proof. As $\phi_2|_C$ is a smooth morphism with connected fibers, $(\phi_2|_C)^*\mathcal{E}'$ is an irreducible local system on C for all irreducible local systems on C'. Let us consider the following diagram,

$$\begin{tikzcd} G \times C \arrow[r, "p_1 \times \phi_2|_C"] \arrow[d, shift right, "p"'] \arrow[d, shift left, "m"] & G_1 \times C' \arrow[d, shift right, "m'"'] \arrow[d, shift left, "p'"] \\ C \arrow[r, "\phi_2|_C"'] & C' \end{tikzcd}$$

where p and p' are the projections on the second factor, m (resp. m') the action of G on C (resp. of G_1 on C') defined by

$$(g_1, g_2, ..., g_r) : (l_1, l_2, ..., l_r)\sigma \mapsto (g_1 l_1 \sigma(g_2)^{-1}, g_2 l_2 \sigma(g_3)^{-1}, ..., g_r l_r \sigma(g_1)^{-1})\sigma$$

(resp. by $g_1 : l\sigma^r \mapsto g_1 l \sigma^r g_1^{-1}$) and where $p_1 : G \to G_1$ is defined by $(g_1, \ldots, g_r) \mapsto g_1$. As this diagram is commutative, $(\phi_2|_C)^*\mathcal{E}'$ is a G-equivariant local system for all G_1-equivariant local systems $\mathcal{E}'$ on C'.

Moreover, as G (resp. G_1) acts transitively on C (resp. on C'), there is a $1-1$ correspondence between G-equivariant (resp. G_1-equivariant) irreducible local systems on C (resp. on C') and irreducible representations of $\mathrm{C}_G(g\sigma)/\mathrm{C}_G(g\sigma)^0$, where $g\sigma \in C$ (resp. of $\mathrm{C}_{G_1}(g'\sigma^r)/\mathrm{C}_{G_1}(g'\sigma^r)^0$, where $g'\sigma^r \in C'$).

But, according to the proof of Lemma 3.4, $\mathrm{C}_G(g\sigma) \simeq \mathrm{C}_{G_1}(g'\sigma^r)$. Thus, the functor $(\phi_2|_C)^*$ induces a bijection between G_1-equivariant irreducible local systems on C' and G-equivariant irreducible local systems on C.

The second part of the lemma comes from [1, 4.2.5]. Indeed, as $\phi_2|_C$ is a smooth morphism with connected fibers, the image of $\mathrm{IC}(\overline{C'}, \mathcal{E}')[\dim(C)]$ under $(\phi_2|_C)^*$ is $\mathrm{IC}(\overline{C}, (\phi_2|_C)^*\mathcal{E}')[\dim(C)]$. □

This completes the proof of Theorem 3.3. □

Finally, we deal with groups with non-trivial center. The following result is due to the referee; it replaces a weaker statement in a previous version:

Theorem 3.9. *Let S be a σ-stable subgroup of the center of G. We set $\overline{G} = G/S$ and let $\pi : G \to \overline{G}$ be the natural projection. Then, σ induces an automorphism of $\overline{G}$ that we still denote by σ, and the map*

$$\phi : G\sigma \to \overline{G}\sigma, \qquad g\sigma \mapsto (gS)\sigma,$$

induces a natural bijection between the unipotent classes of $G\sigma$ and the unipotent classes of $\overline{G}\sigma$. Moreover, the Springer correspondence for $G\sigma$ is completely described by the Springer correspondence for $\overline{G}\sigma$.

Proof. First, it is clear that ϕ induces a surjection Φ between the unipotent conjugacy classes of $G\sigma$ and those of $\overline{G}\sigma$ defined by $C \mapsto \phi(C)$. Moreover, by the following lemma, Φ is a bijection.

Lemma 3.10. *Let $h\sigma$ be a unipotent element of $G\sigma$ and let s be in S such that $hs\sigma$ is unipotent. Then $h\sigma$ and $hs\sigma$ are G-conjugate.*

Proof. We can suppose that h lies in a σ-stable Borel subgroup B. Then hs is also contained in B. Let T be a σ-stable torus of B and let U be the unipotent radical of B. Then, as $h\sigma$ and $hs\sigma$ are unipotent, there are $t, t' \in T$ and $v, v' \in U$ such that $h = t^{-1}\sigma(t)v$ and $hs = t'^{-1}\sigma(t')v'$. This implies that $s = (t^{-1}t')^{-1}\sigma(t^{-1}t')$ and that $h\sigma$ and $hs\sigma$ are conjugate. □

By [12, Cor. 6.4], the Springer correspondence for $G\sigma$ is given via the decomposition

$$\mathrm{H}^{2d_{u\sigma}}(\mathcal{B}^G_{u\sigma}, \overline{\mathbb{Q}}_l) \;=\; \bigoplus_{\rho\in\mathrm{Irr}(A_G(u\sigma))} V_{C_{u\sigma},\mathcal{E}_\rho} \otimes \rho\,\hat{}, \tag{1}$$

where $C_{u\sigma} \ni u\sigma$ is a unipotent conjugacy class in $G\sigma$, $A_G(u\sigma) = \mathrm{C}_G(u\sigma)/\mathrm{C}_G(u\sigma)^\circ$ and $\rho\,\hat{}$ is the dual representation of ρ.

Let us consider a unipotent element $u\sigma \in G\sigma$. We set $\overline{u}\sigma = \phi(u)\sigma$. Via the map $B \mapsto \pi(B)$, $\mathcal{B}^G_{u\sigma}$ can be naturally identified to $\mathcal{B}^{\overline{G}}_{\overline{u}\sigma}$. We then have $d_{u\sigma} = d_{\overline{u}\sigma}$.

Lemma 3.11. *The map π induces a surjection $\mathrm{C}_G(u\sigma) \to \mathrm{C}_{\overline{G}}(\overline{u}\sigma)$.*

Proof. Let us consider the subset S' of S defined by

$$S' = \{s \in S | s\sigma(s)\dots\sigma^{r-1}(s) = 1\}$$

where r is the order of σ. Then the map

$$\psi : S' \to S', s \mapsto s\sigma(s)^{-1}$$

is bijective. Indeed, if $\psi(s) = 1$, then s is a σ-stable element of S' which means that $s^r = 1$. But s is semisimple and r is the order of σ which is unipotent. Hence we get $s = 1$.

Let us now consider the set E of elements $x \in G$ such that $xu\sigma x^{-1} \in u\sigma S$. Clearly, we have $\pi(E) = \mathrm{C}_{\overline{G}}(\overline{u}\sigma)$. We will prove the lemma if we prove that $\pi(E) = \pi(\mathrm{C}_G(u\sigma))$.

Let $x \in E$ and $s \in S$ be such that $xu\sigma x^{-1} = u\sigma s$. Then $xu\sigma x^{-1}$ is unipotent and r is a power of the characteristic, so

$$(u\sigma s)^r = (u\sigma(u)\dots\sigma^{r-1}(u))(s\sigma(s)\dots\sigma^{r-1}(s))$$

is unipotent. As this is the Jordan decomposition of the unipotent element $(u\sigma s)^r$, the semisimple part $s\sigma(s)\dots\sigma^{r-1}(s)$ is trivial. Hence, s is in S'.

By the first part there is $s' \in S'$ such that $s = s'\sigma(s')^{-1}$, so $x' = x\sigma(s')^{-1} \in \mathrm{C}_G(u\sigma)$. Hence, for any $x \in E$, there is $x' \in \mathrm{C}_G(u\sigma)$ such that $\pi(x) = \pi(x')$. This proves the lemma.

□

Now the surjection $\mathrm{C}_G(u\sigma) \to \mathrm{C}_{\overline{G}}(\overline{u}\sigma)$ induces a surjection $\pi : A_G(u\sigma) \to A_{\overline{G}}(\overline{u}\sigma)$. But the action of $A_G(u\sigma)$ on $\mathrm{H}^{2d_{u\sigma}}(\mathcal{B}^G_{u\sigma}, \overline{\mathbb{Q}}_l)$ factors through the action of $A_{\overline{G}}(\overline{u}\sigma)$ on $\mathrm{H}^{2d_{u\sigma}}(\mathcal{B}^{\overline{G}}_{\overline{u}\sigma}, \overline{\mathbb{Q}}_l)$ via π. This implies that $V_{C_{u\sigma},\mathcal{E}_\rho}$ is different from zero only when $V_{C_{\overline{u}\sigma},\mathcal{E}_{\rho'}}$ is different from zero and $\rho = \rho' \circ \pi$. Since the Springer correspondence for $G\sigma$ is given via the decomposition formula (1), where $u\sigma$ runs over a set of representatives of unipotent classes of $G\sigma$, the Springer correspondence for $G\sigma$ is completely described by the Springer correspondence for $\overline{G}\sigma$.

This completes the proof of Theorem 3.9. □

With Theorems 3.1, 3.3 and 3.9 we have reduced the determination of the Springer correspondence for disconnected groups of the form $G = G^\circ.\langle\sigma\rangle$ to the case where G° is simply connected almost simple and σ induces a graph automorphism on the Dynkin diagram of G°.

Indeed, let G be connected reductive. Then $G = \mathrm{D}(G).\mathrm{Z}(G)^\circ$ with $D(G) \cap \mathrm{Z}(G)^\circ$ finite, where $\mathrm{D}(G)$ is the derived group of G and $\mathrm{Z}(G)^\circ$ is the identity component of the center of G (see for example [2, 0.40]). Then there exists an isogeny $D(G) \times \mathrm{Z}(G)^\circ \to G$ and for $G' = D(G)$ again there is an isogeny $\tilde{G}' \to G'$ where $\tilde{G}'$ is simply connected semisimple. All these maps can be chosen to be σ-equivariant for an automorphism σ of G (see Steinberg [14, 9.16]), hence by Theorem 3.9 the determination of the Springer correspondence is reduced to the case of simply connected groups. In turn, these are direct products of almost simple simply connected groups.

If G is a direct product of simply connected almost simple groups, permuted by σ, Theorem 3.1 gives a reduction to the case where σ acts transitively on the factors. Finally, Theorem 3.3 leads to the case where G is simply connected almost simple.

By Theorem 3.9 we may instead consider the case of simple groups of adjoint type. There are two families of such groups: $G^\circ = \mathrm{PGL}_n$ ($n \geq 3$), $G^\circ = \mathrm{PCO}^\circ_{2n}$ ($n \geq 4$), both extended by the non-trivial graph automorphism of order 2; and two exceptional cases: $G^\circ = E_6$ in characteristic 2 with σ the non-trivial graph automorphism of order 2, and $G^\circ = \mathrm{PCO}^\circ_8$ in characteristic 3 with the non-trivial graph automorphism of order 3. We consider the classical types in Sections 5–6. Note that by Theorem 3.9 we may replace PGL_n by GL_n. Furthermore, note that in groups with connected center all Levi subgroups again have connected center, so that induction may be applied.

4. Some combinatorics

The unipotent classes and the Springer correspondence are both best described by symbols, combinatorial objects introduced by Lusztig.

4.1. Symbols. The following definition of symbols is taken from Lusztig and Spaltenstein [7, 1.1]. For non-negative integers r, s and for $d \in \mathbb{Z}$, we consider the following set $\tilde{X}^{r,s}_d$ of *symbols*: The elements $S \in \tilde{X}^{r,s}_d$ are pairs $S = (S_0, S_1)$ of finite sequences $S_i = (\lambda_{i,j} \mid 1 \leq j \leq t_i)$ of non-negative integers, subject to the conditions

$$\begin{aligned} &\lambda_{i,j} \geq \lambda_{i,j-1} + r + s \qquad \text{for } i \in \{0,1\},\ 2 \leq j \leq t_i, \\ &\lambda_{1,1} \geq s, \end{aligned}$$

$$t_0 - t_1 = d.$$

Note that for $s = 0$ the second condition is empty. We will also write $|S_i| = t_i$ for the number of entries of S_i. We call $d = d(S) = |S_0| - |S_1|$ the *defect of* S.

For any symbol $S = (S_0, S_1)$ define the *contents* by

$$|S| := |S_0| + |S_1|$$

and the *rank* by

$$\mathrm{rank}(S) := \sum_{\lambda \in S} \lambda - (r+s)\left\lfloor \frac{(|S|-1)^2}{4} \right\rfloor - s\left\lfloor \frac{|S|}{2} \right\rfloor ,$$

where $\lfloor x \rfloor := \max\{n \in \mathbb{Z} \mid n \leq x\}$ denotes the Gauß brackets. Clearly, $\mathrm{rank}(S) \in \mathbb{N}_0$ for any $S \in \tilde{X}_d^{r,s}$.

On $\tilde{X}_d^{r,s}$ we have the *shift operation* which to $S = (S_0, S_1)$ associates

$$S' = \big(\{0\} \cup (S_0 + r + s), \{s\} \cup (S_1 + r + s)\big) \in \tilde{X}_d^{r,s}.$$

We denote by $X_d^{r,s}$ the classes in $\tilde{X}_d^{r,s}$ under the equivalence relation defined by the symmetric and transitive hull of the shift operation.

It is clear that the rank is invariant under shift and also the contents is invariant modulo 2 under shift. We can thus write $X_{d,n}^{r,s}$ for the set of classes of symbols in $X_d^{r,s}$ of rank n.

An element $S \in X_{d,n}^{r,s}$ with $d \in \{0,1\}$ is called *distinguished* if $\lambda_{0,j} \leq \lambda_{1,j}$ and $\lambda_{1,j} \leq \lambda_{0,j+1}$ for all relevant j. Note that this definition is independent of the chosen representative in its class. The subset of $X_{d,n}^{r,s}$ of distinguished symbols is denoted by $D_{d,n}^{r,s}$.

4.2. The genus. Let the *genus of a symbol* S be defined by

$$g(S) := \sum_{\{\lambda,\lambda'\}} \min(\lambda,\lambda') - \begin{cases} \left(r\binom{t}{2}\frac{4t+1}{3} + s\binom{t+1}{2}\frac{4t-1}{3}\right) & \text{if } |S| = 2t+1, \\ \left(r\binom{t}{2}\frac{4t-5}{3} + s\binom{t}{2}\frac{4t+1}{3}\right) & \text{if } |S| = 2t \end{cases}$$

(see [4, (2.22)]). Here the sum extends over all unordered pairs $\lambda = \lambda_{i,j}$, $\lambda' = \lambda_{k,l}$ of entries of S with $(i,j) \neq (k,l)$. Thus we have $g(S) \in \mathbb{N}_0$ for any $S \in \tilde{X}_d^{r,s}$. Furthermore, the following is easily checked:

Lemma 4.3. *The genus is invariant under shift, hence well-defined on* $X_d^{r,s}$.

4.4. Cuspidal symbols, addition. The *kernel* of a symbol $S = (S_0, S_1)$ is the symbol $\ker(S) = (S_0', S_1')$ with

$$S_0' = (0, r+s, \ldots, (t_0-1)(r+s)), \qquad S_1' = (s, r+2s, \ldots, (t_1-1)r + t_1 s).$$

Thus the defect $d(\ker(S)) = d(S)$ remains unchanged while $\mathrm{rank}(\ker(S)) \leq \mathrm{rank}(S)$. If equality holds, then $\ker(S) = S$. Such symbols S will be called *cuspidal*.

For any given defect there exists an addition

$$+ \ : X_{d,n}^{r,s} \times X_{d,n'}^{r',s'} \longrightarrow X_{d,n+n'}^{r+r',s+s'}$$

defined by component-wise addition of representatives of the same length. For fixed r, s, d, n let $X(S')$ be the subset of (classes of) symbols in $X^{r,s}_{d,n}$, with kernel $S' \in X^{r,s}_{d,k}$ of rank $\operatorname{rank}(S') = k$. Then addition of S' defines a natural bijection

$$X^{0,0}_{d,n-k} \xrightarrow{\sim} X(S') \subseteq X^{r,s}_{d,n}, \quad S \mapsto S + S'.$$

On the other hand, the set $X^{0,0}_{d,n-k}$ can be naturally identified with the set of pairs of partitions of $n-k$, so that we have a natural parametrization of $X(S')$ by pairs of partitions of $n-k$.

If $s = 0$ then the cyclic group of order 2 acts on $X^{r,0}_{d} \cup X^{r,0}_{-d}$ by interchanging the S_i, that is, $(S_0, S_1) \mapsto (S_1, S_0)$. We then denote by S^{tr} the symbol (S_1, S_0) obtained by interchanging the two rows of $S = (S_0, S_1) \in X^{r,0}_{d}$. Let Y^r_d denote the set of equivalence classes of $X^{r,0}_{\pm d}$ modulo this action. Again, the rank, the genus and the contents are well defined on elements of Y^r_d, and the kernels of equivalent symbols are equivalent. The defect is well-defined up to sign. As before, we denote by $Y^r_{d,n}$ the subsets of symbol-classes of rank n. We say that $S \in Y^r_{d,n}$ is distinguished if one of S, S^{tr} is distinguished, and write $D^r_{d,n}$ for the set of distinguished elements in $Y^r_{d,n}$.

4.5. j-induction in $W(B_m)$. We need an explicit description of j-induction in Weyl groups of type B_m. The irreducible characters of $W(B_m)$ are naturally indexed by pairs of partitions (μ_1, μ_2) of m (see [5, Thm. 5.5.6], for example). The b-invariant of the irreducible character $\chi \in \operatorname{Irr}(W(B_m))$ indexed by (μ_1, μ_2) is then given by

$$b(\chi) = 2 \sum_{\{\mu_{1i}, \mu_{1j}\}} \min(\mu_{1i}, \mu_{1j}) + 2 \sum_{\{\mu_{2i}, \mu_{2j}\}} \min(\mu_{2i}, \mu_{2j}) + |\mu_2|\,,$$

where the sums run over unordered pairs of entries of μ_1, μ_2 respectively (see [5, 5.5.3]).

Let $(\mu_1, \mu_2) \vdash m$ be a pair of partitions of m. By adding parts 0 to μ_1 or μ_2 we may assume that μ_1 and μ_2 have the same number of parts. Then (μ_1, μ_2) is an element of $\tilde{X}^{0,0}_{0,m}$, and defines a unique element of $X^{0,0}_{0,m}$. In particular, this defines a natural bijection

$$\operatorname{Irr}(W(B_m)) \longrightarrow X^{0,0}_{0,m}.$$

Similarly, by adding parts of length 0 we may assume that μ_1 contains precisely one more part than μ_2, and this defines a further natural bijection

$$\operatorname{Irr}(W(B_m)) \longrightarrow X^{0,0}_{1,m}.$$

We denote the sign-representation of the symmetric group $\mathfrak{S}_k$ by sgn. The following is well-known:

Proposition 4.6. *Let $\chi \in \operatorname{Irr}(W(B_k))$ with $k \leq m$. Then we have:*

(a) *The j-induction of $\chi \otimes \operatorname{sgn}$ from $W(B_k) \times \mathfrak{S}_{m-k}$ to $W(B_m)$ is irreducible.*

(b) *If χ is parametrized by $S = (S_0, S_1) \in X^{0,0}_{d,k}$, $d \in \{0, 1\}$, then the symbol of*

$$\tilde{\chi} := j^{W(B_m)}_{W(B_k) \times \mathfrak{S}_{m-k}}(\chi \otimes \operatorname{sgn})$$

is obtained from S by increasing each of the $\lfloor (m-k+1)/2 \rfloor$ largest entries of S_0 and each of the $\lfloor (m-k)/2 \rfloor$ largest entries of S_1 by 1 (that is, by increasing each of the rightmost $m-k$ entries of S by 1).

Proof. The irreducibility is proved in [5, Thm. 5.2.6 and Rem. 5.6.7], for example. For the second statement note that the symbol described in the assertion is contained in the induction
$$\mathrm{Ind}_{W(B_k)\times\mathfrak{S}_{m-k}}^{W(B_m)}(\chi\otimes\mathrm{sgn})$$
of $\chi\otimes\mathrm{sgn}$ by [5, Prop. 6.4.7]. Moreover, our above formula shows that its b-invariant equals the one of $\chi\otimes\mathrm{sgn}$. Thus we are done by part (a). □

We can formally define a j-restriction from $W(B_m)$ to $W(B_k)\times\mathfrak{S}_{m-k}$ by
$$j\text{-}\mathrm{Res}_{W(B_k)\times\mathfrak{S}_{m-k}}^{W(B_m)}(\chi) := \sum_{\psi}\langle\chi, j_{W(B_k)\times\mathfrak{S}_{m-k}}^{W(B_m)}(\psi)\rangle\,\psi,$$
where the sum runs over all irreducible characters ψ of $W(B_k)\times\mathfrak{S}_{m-k}$. By the previous proposition, all coefficients in the sum are 0 or 1. But in general, the j-restriction is not irreducible if it is non-zero. The smallest example is given by the case $m=3$, $k=1$, χ parametrized by $(2,1)$. Then χ is j-induced from both $(-,1)\otimes 2$ and $(1,-)\otimes 1^2$ in $\mathrm{Irr}(W(B_1)\times\mathfrak{S}_2)$. Nevertheless, Proposition 4.6 shows the following:

Corollary 4.7. *The j-restriction $j\text{-}\mathrm{Res}_{W(B_k)\times\mathfrak{S}_{m-k}}^{W(B_m)}(\chi)$ of an irreducible character $\chi\in\mathrm{Irr}(W(B_m))$ contains at most one constituent of the form $\chi'\otimes\mathrm{sgn}$ with $\chi'\in\mathrm{Irr}(W(B_k))$.*

5. Restricted Springer correspondence for GL_n: 2

Let GL_n, $n\geq 3$, be the general linear group over an algebraically closed field of characteristic 2, $G(n)$ the extension of GL_n by the non-trivial graph automorphism σ of order 2. In this section we determine the restricted Springer correspondence for the unipotent conjugacy classes in the coset $G(n)\setminus\mathrm{GL}_n$.

Both the classes and the correspondence are best described in terms of symbols, which we introduce first.

5.1. Unipotent classes. We start by recalling the parametrization of the unipotent conjugacy classes in $G(n)\setminus\mathrm{GL}_n$. The classes are parametrized by pairs (λ,ϵ) where $\lambda=(\lambda_1\leq\lambda_2\leq\ldots)\vdash n$ is a partition of n into non-zero parts such that any even part occurs an even number of times, and $\epsilon:\{\lambda_i\}\to\{0,1,\omega\}$ is a map from the set of distinct parts of λ to $\{0,1,\omega\}$ which is equal to ω on parts of even length, different from ω on parts of odd length, and equal to 1 on parts of odd length which appear an odd number of times (see [13, I.2.7]).

This combinatorial parametrization seems not very well adapted to the description of induction of classes or to the Springer correspondence. Consider the following operation: Let (λ,ϵ) be the parameter of a unipotent class C in $G(n)\setminus\mathrm{GL}_n$. We subdivide the sequence $\lambda_1\leq\lambda_2\leq\ldots$ into blocks of length 1 or 2, such that λ_i lies in a block of length 1 if and only if λ_i is odd and $\epsilon(\lambda_i) \neq 0$. Then define

$$\begin{array}{lll} c_i & := & (\lambda_i-1)/2+2(i-1) \end{array} \quad \text{if } \{\lambda_i\} \text{ is a block,}$$
$$\left.\begin{array}{lll} c_i & := & \lambda_i/2+2(i-1), \\ c_{i+1} & := & (\lambda_i-2)/2+2i=c_i+1 \end{array}\right\} \quad \text{if } \{\lambda_i,\lambda_{i+1}\} \text{ is a block, } \epsilon(\lambda_i)=\omega,$$
$$\left.\begin{array}{lll} c_i & := & (\lambda_i+1)/2+2(i-1), \\ c_{i+1} & := & (\lambda_i-3)/2+2i=c_i \end{array}\right\} \quad \text{if } \{\lambda_i,\lambda_{i+1}\} \text{ is a block, } \epsilon(\lambda_i)=0.$$

We write $\rho(C)$ for the symbol $((c_1,c_3,\ldots),(c_2,c_4,\ldots))$. We set $m := \lfloor \frac{n}{2} \rfloor$ and define $d \in \{0,1\}$ by $n \equiv d \pmod 2$.

Proposition 5.2. (a) *The map $C \mapsto \rho(C)$ is a bijection from the set of unipotent classes in $G(n) \setminus \mathrm{GL}_n$ to $D_{d,m}^{3,1}$.*

(b) *For any unipotent class C in $G(n) \setminus \mathrm{GL}_n$ we have*

$$d_u = g\left(\rho\left(C\right)\right) \qquad \text{for } u \in C.$$

Proof. It follows easily from the definitions that $\rho(C)$ is a distinguished symbol in $X_d^{3,1}$. Since (on average over a block) the c_i are equal to $(\lambda_i-5)/2+2i$, and by definition the number of parts of λ is congruent to n modulo 2, the rank of $\rho(C)$ is equal to $\lfloor \frac{n}{2} \rfloor$ by (4.1). Conversely, given a distinguished symbol $S = ((c_1,c_3,\ldots),(c_2,c_4,\ldots)) \in D_{d,m}^{3,1}$, the entries $c_1 \leq c_2 \leq \ldots$ may be uniquely subdivided into blocks of size 1 or 2 such that the difference between two blocks is at least 2, and inside a block at most 1. Inspired by the procedure in [7, 3.1] for connected groups we define

$$\left.\begin{array}{lll} \lambda_i & := & 2c_i-4i+5, \\ \epsilon(\lambda_i) & := & 1 \end{array}\right\} \quad \text{if } \{c_i\} \text{ is a block,}$$
$$\left.\begin{array}{lll} \lambda_i=\lambda_{i+1} & := & 2c_i-4i+4, \\ \epsilon(\lambda_i) & := & \omega \end{array}\right\} \quad \text{if } \{c_i,c_{i+1}=c_i+1\} \text{ is a block,}$$
$$\left.\begin{array}{lll} \lambda_i=\lambda_{i+1} & := & 2c_i-4i+3, \\ \epsilon(\lambda_i) & := & 0 \end{array}\right\} \quad \text{if } \{c_i,c_{i+1}=c_i\} \text{ is a block.}$$

It is clear that this map is the inverse of ρ, hence ρ is a bijection as claimed.

For part (b) we check that Spaltenstein's formula for d_u in [13, Prop. II.6.3] translates to our formula

$$g(S) := \sum_{\lambda,\lambda'} \min(\lambda,\lambda') - \begin{cases} t(8t^2-3t-2)/3 & \text{if } d=1, \\ t(t-1)(8t-7)/3 & \text{if } d=0, \end{cases}$$

for the genus under the map ρ. Indeed, for the empty partition both formulas yield 0. Now assume that the assertion has been proved for all classes (λ,ϵ) with partitions λ with at most k parts. Let λ be a partition of n into k parts, where we first assume that both n and $k=2t+1$ are odd. We compare the behavior of the two formulas when parts are added to λ. Let $\tilde\lambda$ be the partition of $n+2l$ obtained from λ by adding two even parts $\lambda_{k+1}=\lambda_{k+2}=l \geq \lambda_k$. Then our formula gives

$$g\big(\rho\big((\tilde\lambda,\tilde\epsilon)\big)\big) = g\big(\rho\big((\lambda,\epsilon)\big)\big) + n + \frac{\lambda_{k+1}}{2},$$

(where $\tilde\epsilon$ is obtained from ϵ by adding $\tilde\epsilon(\lambda_{k+1}) = \omega$). On the other hand [13, Prop. II.6.3] shows that the value of d_u for the corresponding unipotent classes (λ, ϵ) and $(\tilde\lambda, \tilde\epsilon)$ increases by the same amount.

Exactly the same happens if $\tilde\lambda$ is the partition of $n+2l$ obtained from λ by adding two odd parts $\lambda_{k+1} = \lambda_{k+2} = l \geq \lambda_k$ with $\epsilon(\lambda_{k+1}) = 0$. Finally, if $\tilde\lambda$ is the partition of $n + l$ obtained from λ by adding an odd part $\lambda_{k+1} = l \geq \lambda_k$ with $\epsilon(\lambda_{k+1}) = 1$, then both formulas increase by $(n-1)/2$. This completes the induction step.
A similar calculation works when n or k or both are even. □

In the connected case, the analogue of Proposition 5.2(a) was shown in [7, Lemma 2.2], and the analogue of part (b) in [4, Prop. 2.23].

5.3. Induced classes. Spaltenstein [13, II.7] gave an algorithm for the determination of the induction of unipotent classes. We rephrase his description in terms of the associated symbols and obtain a closed expression for such an induction.

We first recall the induction of unipotent classes in GL_k. The unipotent classes in GL_k are parametrized by partitions of k. By [13, II.5.14 and II.7.1] the class with label $\mu \vdash k$ is induced from the trivial class of the Levi subgroup $\mathrm{GL}_{\mu_1^*} \times \ldots \times \mathrm{GL}_{\mu_s^*}$, where μ^* is the dual partition of μ. In particular, every unipotent class of GL_k is a Richardson class, that is, it is induced from the trivial class of a suitable Levi subgroup.

Now let L be a σ-stable Levi subgroup of GL_n. Then L has the shape

$$L = \mathrm{GL}_k \times (\mathrm{GL}_{n_1} \times \ldots \times \mathrm{GL}_{n_s})^2, \qquad n = k + 2(n_1 + \ldots + n_s),$$

where σ acts as the graph automorphism on the first factor and interchanges the two factors $\mathrm{GL}_{n_1} \times \ldots \times \mathrm{GL}_{n_s}$. Thus a unipotent class C in $L\sigma$ may be described by a tuple $(C_0, C_1, \ldots, C_s)$ of unipotent classes C_0 of $G(k) \setminus \mathrm{GL}_k$, respectively C_i of GL_{n_i}, $1 \leq i \leq s$. Since induction of classes is transitive and all classes of GL_{n_i} are Richardson classes, it is sufficient to describe induction in the case that $C_i = 1$ for $1 \leq i \leq s$. Moreover, again by transitivity and induction on s, we may assume that $s = 1$.

We need to reformulate a result of Spaltenstein in terms of symbols. Let (λ, ϵ) be the label of a unipotent class in $G(n) \setminus \mathrm{GL}_n$. Write c_i for the number of parts in λ equal to i. Consider the following two types of operations:

(a_1) if $c_i \geq 2$, then decrease λ_i^* by 2,

(a_2) if i is odd, $\epsilon(i) = 1$ and $\lambda_{i-1}^* = \lambda_i^*$, then decrease λ_i^* and λ_{i-1}^* by 1 each.

Denote a partition obtained by one of these two operations from λ by λ'. Moreover, define ϵ' by $\epsilon'(j) = \epsilon(j)$ if $\lambda_j^* = (\lambda')_j^*$, and $\epsilon'(j) \neq 1$ if $\epsilon(j) = 0$, $\epsilon'(j) \neq 0$ if $\epsilon(j) = 1$.

Lemma 5.4. *Assume that (λ', ϵ') is obtained from (λ, ϵ) by application of (a_1) or (a_2) above for some index i. Then the symbol $\rho((\lambda', \epsilon'))$ is obtained by decreasing exactly one of the entries of the symbol $\rho((\lambda, \epsilon))$ by 1.*

Proof. Subdivide the parts of λ into blocks as in (5.1). The operation (a_2) then corresponds to replacing a block $\{i\}$ (with $\epsilon(i) = 1$) by the block $\{i-2\}$, with $\epsilon(i-2) = 1$. On the associated symbol, this corresponds to decreasing the corresponding entry by 1.

For the operation (a_1) we have to distinguish several cases. If i is odd, $\epsilon(i)=1$, then there exist at least two blocks $\{i\}$, and (a_1) replaces them by a block $\{i-1,i-1\}$, with $\epsilon(i-1)=\omega$. If i is odd, but $\epsilon(i)=0$, then a block $\{i,i\}$ is replaced by a block $\{i-1,i-1\}$ with $\epsilon(i-1)=\omega$. Finally, if i is even, hence $\epsilon(i)=\omega$, then a block $\{i,i\}$ is replaced by a block $\{i-1,i-1\}$ with $\epsilon(i-1)=0$. In all three cases, this corresponds to decreasing one entry in the symbol by 1. □

The following is now a consequence of results of Spaltenstein:

Theorem 5.5. *Let C be a unipotent class in $G(k)\setminus \mathrm{GL}_k$. Then the symbol $\rho(\tilde{C})$ of the induced class*

$$\tilde{C} := \mathrm{Ind}_{G(k)\times \mathrm{GL}_{(n-k)/2}\wr 2}^{G(n)}(C\times\{1\})$$

is obtained from the symbol $\rho(C)$ by increasing each of the largest $(n-k)/2$ entries by 1.

Proof. The situation of the theorem is considered in [13, 7.3]: Let C be parametrized by (λ,ϵ) and $\tilde{C}$ by $(\tilde\lambda,\tilde\epsilon)$. Then there exists a sequence

$$(\lambda,\epsilon) = (\tilde\lambda^k,\tilde\epsilon^k), (\tilde\lambda^{k+1},\tilde\epsilon^{k+1}),\ldots,(\tilde\lambda^n,\tilde\epsilon^n) = (\tilde\lambda,\tilde\epsilon)$$

of unipotent classes, such that each class $(\tilde\lambda^{k+i},\tilde\epsilon^{k+i})$ is obtained by inducing $C\times\{1\}$ from $G(k)\times \mathrm{GL}_i\wr 2$ to $G(k+2i)$. The values $b_i := d_u(\tilde\lambda^{k+i},\tilde\epsilon^{k+i})$ then satisfy $b_{k+i}-b_{k+i-1} = i-1$, for $1\le i\le n-k$.

By [13, 6.28] the parameter $(\tilde\lambda^{k+i-1},\tilde\epsilon^{k+i-1})$ can be obtained from $(\tilde\lambda^{k+i},\tilde\epsilon^{k+i})$ by one of the operations in [13, 6.7], which are just the operations (a_1), (a_2) introduced above. Hence, by Lemma 5.4, $\rho((\tilde\lambda^{k+i-1},\tilde\epsilon^{k+i-1}))$ is obtained from $\rho((\tilde\lambda^{k+i},\tilde\epsilon^{k+i}))$ by decreasing one of the entries by 1.

Furthermore, the definition of the genus of a symbol together with Proposition 5.2(b) yields that decreasing by 1 the $(l-i+1)$th entry of a symbol with l entries decreases d_u exactly by $i-1$. Thus the symbol for $(\tilde\lambda^{k+i-1},\tilde\epsilon^{k+i-1})$ is obtained from the symbol of $(\tilde\lambda^{k+i},\tilde\epsilon^{k+i})$ by decreasing the $(l-i+1)$th entry by 1. In conclusion, $\rho((\tilde\lambda^n,\tilde\epsilon^n))$ is obtained from $\rho((\lambda,\epsilon))$ by increasing each of the largest $(n-k)/2$ entries by 1. □

Note the formal similarity of this result with the description of j-induction in Proposition 4.6. This will be the basis for the determination of the Springer correspondence in Theorem 5.9.

This allows to give an explicit description of *rigid unipotent classes* of $G(n)$, that is, unipotent classes which are not induced from a unipotent class of a proper σ-stable Levi subgroup (compare with [13, II.7.9]):

Corollary 5.6. *A unipotent class C in $G(n)\setminus \mathrm{GL}_n$ is rigid if and only if the associated symbol $\rho(C)\in D_d^{3,1}$ has the following property:*

- *for all $1\le j\le|\rho(C)|$, either $c_j=c_{j-1}$ or $c_j-c_{j-2}\le 4$ or $c_{j+1}-c_{j-1}\le 4$,*

where we have set $c_{-1}:=-4$, $c_0:=-3$ (that is, reducing all entries $c_j, c_{j+1},\ldots$ by 1 does not lead to a symbol in $D_d^{3,1}$).

Example 5.7. The symbols in $D^{3,1}_{1,3}$ satisfying the above criterion are

$$\begin{pmatrix} 0 & 4 \\ 4 & \end{pmatrix}, \begin{pmatrix} 0 & 4 & 8 \\ 2 & 7 & \end{pmatrix}, \begin{pmatrix} 1 & 5 & 9 \\ 1 & 5 & \end{pmatrix}, \begin{pmatrix} 0 & 4 & 8 & 12 \\ 2 & 6 & 10 & \end{pmatrix},$$

so the rigid unipotent classes for $G(7)$ are $3^2_0 1, 2^2 1^3, 31^4_0, 1^7$. Here, following the convention of Spaltenstein, we denote the unipotent classes by the corresponding partitions λ, where a part λ_i with an index 0 indicates that $\epsilon(\lambda_i) = 0$. The symbols in $D^{3,1}_{0,4}$ satisfying the above criterion are

$$\begin{pmatrix} 0 & 4 \\ 3 & 7 \end{pmatrix}, \begin{pmatrix} 0 & 4 & 9 \\ 2 & 6 & 10 \end{pmatrix}, \begin{pmatrix} 0 & 4 & 8 & 12 \\ 2 & 6 & 10 & 14 \end{pmatrix}, \begin{pmatrix} 1 & 5 & 9 & 13 \\ 1 & 5 & 9 & 13 \end{pmatrix},$$

so the rigid unipotent classes for $G(8)$ are $32^2 1, 2^2 1^4, 1^8, 1^8_0$.

Note that the induction of unipotent classes does not satisfy a Harish-Chandra theory with the rigid classes as cuspidal objects. Indeed, as we saw before, the unipotent class C with symbol $\begin{pmatrix} 0 & 4 \\ 3 & 7 \end{pmatrix}$ is rigid in $G(8)$. By Theorem 5.5 the induction of $C \times \{1\}$ to $G(12)$ is the class $\tilde{C}$ with symbol $\begin{pmatrix} 0 & 5 \\ 3 & 8 \end{pmatrix}$. But that class is also induced from the unipotent class with symbol $\begin{pmatrix} 0 & 4 \\ 2 & 7 \end{pmatrix}$ in $G(6)$, which in turn is induced from the rigid class with symbol $\begin{pmatrix} 0 & 4 \\ 2 & 6 \end{pmatrix}$ in $G(4)$. Thus $\tilde{C}$ is induced from two different rigid classes. This observation is the key to the explicit determination of the Springer correspondence in the next section.

5.8. The restricted Springer correspondence. The centralizer W^σ of the graph automorphism in the Weyl group W of GL_n is a Weyl group of type B_m, where $m = \lfloor n/2 \rfloor$, if n is odd, respectively of type C_m if n is even. Recall that the unipotent classes in $G(n) \setminus \mathrm{GL}_n$ are parametrized by $D^{3,1}_{d,m}$, where $d \equiv n \pmod 2$, $d \in \{0, 1\}$, while the irreducible characters of W^σ of type B_m are indexed by $X^{0,0}_{d,m}$. We denote by

$$S_d := \begin{pmatrix} 0 & 4 & 8 & \dots \\ 1 & 5 & 9 & \dots \end{pmatrix}$$

the unique (cuspidal) symbol in $X^{3,1}_{d,0}$ for $d \in \{0, 1\}$.

Theorem 5.9. (a) *The map*

$$D^{3,1}_{1,m} \longrightarrow X^{0,0}_{1,m}, \quad S \mapsto S - S_1,$$

represents the restricted Springer correspondence for the outer unipotent classes of $G(2m+1)$ in characteristic 2.

(b) *The map*

$$D^{3,1}_{0,m} \longrightarrow X^{0,0}_{0,m}, \quad S \mapsto [S - S_0]^{\mathrm{tr}},$$

represents the restricted Springer correspondence for the outer unipotent classes of $G(2m)$ in characteristic 2.

Proof. The proof is by induction on n. For $n = 2$ there exist just two outer unipotent classes with symbols $((0),(2)),((1),(1))$, with d_u equal to $0,1$ respectively, and $W(B_1)$ has two irreducible characters, $(1,-),(-,1)$ with b-values $0,1$, so by Theorem 2.8 the Springer correspondence is given by

$$\begin{pmatrix} 0 \\ 2 \end{pmatrix} \mapsto (1,-), \qquad \begin{pmatrix} 1 \\ 1 \end{pmatrix} \mapsto (-,1),$$

as claimed. Similarly for $n = 3$ we have two outer unipotent classes with symbols $((1),-),((0,4),(2))$ and $d_u = 0,1$, so again by Theorem 2.8 the Springer correspondence is given by

$$\begin{pmatrix} 1 \\ \ \end{pmatrix} \mapsto (1,-), \qquad \begin{pmatrix} 0 \quad 4 \\ 2 \end{pmatrix} \mapsto (-,1).$$

Now assume that the assertion is proved for $G(k)$ for all $k < n$. Let $d \in \{0,1\}$ be defined by $n \equiv d \pmod 2$. Let C be an outer unipotent class of $G(n)$, with symbol $S = \rho(C) = ((c_1,c_3,\ldots),(c_2,c_4,\ldots))$ with l entries. Define $c_{-1} := -4$, $c_0 := -3$. Since $S \in D_d^{3,1}$ we then have

$$c_{-1} \leq c_0 \leq \ldots \leq c_l, \qquad \text{and} \quad c_i - c_{i-2} \geq 4 \text{ for all } i \in \{1,\ldots,l\}.$$

Let $1 \leq j \leq l$ be the largest index i such that $c_i - c_{i-2} > 4$ and $c_i > c_{i-1}$. (Thus j is maximal subject to the condition that decreasing c_j by 1 still gives a distinguished symbol in $D_d^{3,1}$.) If no such entry exists then $S = S_d$ has rank 0, which is not the case. Denote by $\tilde{S}$ the symbol obtained from S by increasing the entries $c_{j+1},\ldots,c_l$ by 1 each. By Theorem 5.5 the symbol $\tilde{S}$ parametrizes the induced class

$$\tilde{C} = \mathrm{Ind}_{G(n)\times \mathrm{GL}_{l-j}\wr 2}^{G(n+2(l-j))}(C \times \{1\}) \ .$$

By the choice of j, decreasing the $l-j+1$ largest entries of $\tilde{S}$ by 1 leads to a distinguished symbol S'. (In fact, S' is obtained from S by decreasing the entry c_j by 1.) So again by Theorem 5.5 the class $\tilde{C}$ is also induced from the class $C' \times \{1\}$ of the Levi subgroup $G(n-2)\times \mathrm{GL}_{l-j+1}\wr 2$ of $G(n+2(l-j))$, where C' has symbol $\rho(C') = S'$.

$$\begin{tikzcd}[column sep=small] & G(n+2(l-j)) & \\ G(n)\times \mathrm{GL}_{l-j}\wr 2 \arrow[ur] & & G(n-2)\times \mathrm{GL}_{l-j+1}\wr 2 \arrow[ul] \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & \tilde{C} & \\ C\times\{1\} \arrow[ur] & & C'\times\{1\} \arrow[ul] \end{tikzcd}$$

By induction, the Springer correspondent χ' of C' is already known to be indexed by $S' - S_d$ if $d = 1$, respectively its transpose if $d = 0$, so by Corollary 2.10 the Springer correspondent of $\tilde{C}$ is the j-induced of χ'. Since increasing the s rightmost entries of a symbol commutes with addition of S_d, Proposition 4.6 shows that the Springer correspondent $\tilde{\chi}$ of $\tilde{C}$ has symbol $\tilde{S} - S_d$, respectively its transpose.

Now the Springer correspondent of the unipotent class $C \times \{1\}$ of the Levi subgroup $G(n)\times \mathrm{GL}_{l-j}\wr 2$ must be of the form $\chi \otimes \mathrm{sgn}$ for some $\chi \in \mathrm{Irr}(W(B_m))$, $m = \lfloor n/2 \rfloor$, by the first reduction theorem 3.1. Proposition 4.6 and Corollary 4.7 now show that χ is uniquely determined, and its symbol is obtained from the symbol for $\tilde{\chi}$ by decreasing the rightmost $l-j$ entries of $\tilde{S}-S_d$ by 1. Again by the commutation of this operation with addition of S_d, we deduce that the Springer correspondent of C must have symbol $S-S_d$ respectively its transpose, as claimed. □

Example 5.10. We give tables for the restricted Springer correspondence in $G(n) \setminus \mathrm{GL}_n$, $3 \leq n \leq 6$. Recall the convention for the notation for unipotent classes introduced in Example 5.7.

TABLE 1. Restricted Springer correspondence for $G(3)$

C	d_u	$\rho(C)$	χ_u
3	0	$\begin{pmatrix} 1 \end{pmatrix}$	$(1,-)$
1^3	1	$\begin{pmatrix} 0 & & 4 \\ & 2 & \end{pmatrix}$	$(-,1)$

TABLE 2. Restricted Springer correspondence for $G(4)$ and $G(5)$

C	d_u	$\rho(C)$	χ_u
31	0	$\begin{pmatrix} 0 \\ 3 \end{pmatrix}$	$(2,-)$
2^2	1	$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$	$(1,1)$
1^4	2	$\begin{pmatrix} 0 & 4 \\ 2 & 6 \end{pmatrix}$	$(1^2,-)$
1^4_0	4	$\begin{pmatrix} 1 & 5 \\ 1 & 5 \end{pmatrix}$	$(-,1^2)$

C	d_u	$\rho(C)$	χ_u
5	0	$\begin{pmatrix} 2 \end{pmatrix}$	$(2,-)$
31^2	1	$\begin{pmatrix} 0 & 5 \\ 2 & \end{pmatrix}$	$(1,1)$
31^2_0	2	$\begin{pmatrix} 1 & 5 \\ 1 & \end{pmatrix}$	$(1^2,-)$
2^21	2	$\begin{pmatrix} 0 & 4 \\ 3 & \end{pmatrix}$	$(-,2)$
1^5	4	$\begin{pmatrix} 0 & 4 & 8 \\ 2 & 6 & \end{pmatrix}$	$(-,1^2)$

Recall that an irreducible character $\chi \in \mathrm{Irr}(W)$ of a Weyl group W is called *special* if its a- and b-invariants coincide (see [5, 9.4.8 and Remark 9.4.10] for the definition of the a-invariant). In the case of a Weyl group of type B_m, the special characters are precisely those indexed by a distinguished symbol $S \in X^{1,0}_{1,m}$ (see [5, 6.4.4]). In the case of connected groups, all special characters occur as images under the Springer correspondence of pairs $(u,1)$, see [6, 13.1.1] for the case of good characteristic, and [7, Lemma 2.2 and 3.2] for bad characteristic, or also the remark after Thm. 2.1 in [4]. The analogous result remains true for disconnected groups of type $G(n)$:

Corollary 5.11. *All special characters of $W(B_m)$ occur as images under the Springer correspondence of an outer unipotent class of $G(2m+1)$ as well as of an outer unipotent class of $G(2m)$.*

TABLE 3. Restricted Springer correspondence for $G(6)$

C	d_u	$\rho(C)$	χ_u
51	0	$\begin{pmatrix} 0 \\ 4 \end{pmatrix}$	$(3,-)$
3^2	1	$\begin{pmatrix} 1 \\ 3 \end{pmatrix}$	$(2,1)$
31^3	2	$\begin{pmatrix} 0 & 4 \\ 2 & 7 \end{pmatrix}$	$(21,-)$
3_0^2	2	$\begin{pmatrix} 2 \\ 2 \end{pmatrix}$	$(1,2)$
2^21^2	3	$\begin{pmatrix} 0 & 5 \\ 2 & 6 \end{pmatrix}$	$(1^2,1)$
$2^21_0^2$	4	$\begin{pmatrix} 1 & 5 \\ 1 & 6 \end{pmatrix}$	$(1,1^2)$
1^6	6	$\begin{pmatrix} 0 & 4 & 8 \\ 2 & 6 & 10 \end{pmatrix}$	$(1^3,-)$
1_0^6	9	$\begin{pmatrix} 1 & 5 & 9 \\ 1 & 5 & 9 \end{pmatrix}$	$(-,1^3)$

Proof. Let $S \in X_{1,m}^{1,0}$ be distinguished. Then $S + ((0,3,6,\ldots),(1,4,7,\ldots))$ is a distinguished symbol in $D_{1,m}^{3,1}$, so by Proposition 5.2(a) it parametrizes an outer unipotent class of $G(2m+1)$. By Theorem 5.9(a) its Springer correspondent is the character indexed by

$$S + \begin{pmatrix} 0 & 3 & 6 & \ldots \\ 1 & 4 & 7 & \ldots \end{pmatrix} - \begin{pmatrix} 0 & 4 & 8 & \ldots \\ 1 & 5 & 9 & \ldots \end{pmatrix} \in X_{1,m}^{0,0},$$

so by $S \in X_{1,m}^{1,0}$.

For $G(2m)$, let again $S \in X_{1,m}^{1,0}$ be distinguished with entries $c_1 \leq c_2 \leq c_3 \leq \ldots$. Let

$$S' := \begin{pmatrix} 0 & c_2+1 & c_4+1 & \ldots \\ c_1 & c_3 & c_5 & \ldots \end{pmatrix} \in X_{0,m}^{1,0}.$$

Then $S' + ((0,3,6,\ldots),(1,4,7,\ldots))$ is a distinguished symbol in $D_{0,m}^{3,1}$, hence by Proposition 5.2(a) it parametrizes an outer unipotent class of $G(2m)$. By Theorem 5.9(b) its Springer correspondent is the character indexed by

$$\left[S' + \begin{pmatrix} 0 & 3 & 6 & \ldots \\ 1 & 4 & 7 & \ldots \end{pmatrix} - \begin{pmatrix} 0 & 4 & 8 & \ldots \\ 1 & 5 & 9 & \ldots \end{pmatrix}\right]^{\mathrm{tr}} \in X_{0,m}^{0,0},$$

with entries $((c_1, c_3-1, c_5-2,\ldots),(0,c_2,c_4-1,\ldots))$, so by $S \in X_{0,m}^{1,0}$. $\square$

6. Restricted Springer correspondence for GO_{2n}

Let GO_{2n}, $n \geq 3$, be the general orthogonal group over an algebraically closed field of characteristic 2, the extension of SO_{2n} by a non-trivial graph automorphism σ of order 2. In this section we determine the restricted Springer correspondence for the unipotent conjugacy classes in the coset $\mathrm{GO}_{2n} \setminus \mathrm{SO}_{2n}$.

Again, the classes and the correspondence are best described in terms of symbols.

6.1. Unipotent classes. The unipotent conjugacy classes of GO_{2n} not contained in SO_{2n} are parametrized by pairs (λ, ϵ) where $\lambda = (\lambda_1 \leq \lambda_2 \leq \ldots) \vdash 2n$ is a partition of $2n$ into an odd number of non-zero parts such that any odd part occurs an even number of times, and $\epsilon : \{\lambda_i\} \to \{0, 1, \omega\}$ is a map from the set of parts of λ to $\{0, 1, \omega\}$ which is equal to ω on parts of odd length, different from ω on parts of even length, and equal to 1 on parts of even length which appear an odd number of times (see [13, I.2.6]).

Let (λ, ϵ) be the parameter of a unipotent class C in $\mathrm{GO}_{2n} \setminus \mathrm{SO}_{2n}$. We partition the sequence $\lambda_1 \leq \lambda_2 \leq \ldots$ into blocks of length 1 or 2 such that λ_i lies in a block of length 1 if and only if $\epsilon_{\lambda_i} = 1$. We then set

$$\begin{array}{rcll} c_i & := & (\lambda_i - 2)/2 + 2(i-1) & \text{if } \{\lambda_i\} \text{ is a block,} \\ c_i & := & (\lambda_i - 1)/2 + 2(i-1), & \multirow{2}{*}{} \\ c_{i+1} & := & (\lambda_i - 3)/2 + 2i = c_i + 1 & \left.\right\} \text{ if } \{\lambda_i, \lambda_{i+1}\} \text{ is a block, } \epsilon(\lambda_i) = \omega, \\ c_i & := & \lambda_i/2 + 2(i-1), & \\ c_{i+1} & := & \lambda_i/2 + 2(i-1) = c_i & \left.\right\} \text{ if } \{\lambda_i, \lambda_{i+1}\} \text{ is a block, } \epsilon(\lambda_i) = 0. \end{array}$$

Let $\rho(C)$ denote the distinguished symbol $((c_1, c_3, \ldots), (c_2, c_4, \ldots))$ in $X^{4,0}_{1,n-1}$. Note that it also defines an element of $Y^4_{1,n-1}$.

Proposition 6.2. (a) *The map $C \mapsto \rho(C)$ is a bijection from the set of unipotent classes in $\mathrm{GO}_{2n} \setminus \mathrm{SO}_{2n}$ to $D^{4,0}_{1,n-1}$.*

(b) *For any unipotent element $u \in \mathrm{GO}_{2n} \setminus \mathrm{SO}_{2n}$ we have*

$$d_u = g\left(\rho\left([u]\right)\right).$$

The proof is entirely analogous to that of Proposition 5.2, using the results of Spaltenstein [13, Prop. II.6.3] on d_u for part (b).

6.3. Induced classes. We now describe Spaltenstein's results on the induction of unipotent classes [13, II.7] in terms of the associated symbols.

Let L be a σ-stable Levi subgroup of GO_{2n}. Then L has the shape

$$L = \mathrm{GO}_{2m} \times \mathrm{GL}_{n_1} \times \ldots \times \mathrm{GL}_{n_s}, \quad n = m + n_1 + \ldots + n_s,$$

where σ acts as the graph automorphism on the first factor and centralizes the factor $\mathrm{GL}_{n_1} \times \ldots \times \mathrm{GL}_{n_s}$. Thus a unipotent class C in $L\sigma$ may be described by a tuple $(C_0, C_1 \ldots, C_s)$ of unipotent classes C_0 of $\mathrm{GO}_{2m} \setminus \mathrm{SO}_{2m}$, respectively C_i of GL_{n_i}, $1 \leq i \leq s$. As for the case of $G(n)$ it is sufficient to describe induction in the case when $s = 1$, $C_i = \{1\}$.

Let (λ, ϵ) be the label of a unipotent class in $\mathrm{GO}_{2n} \setminus \mathrm{SO}_{2n}$. Write c_i for the number of parts in λ equal to i. Consider the following two types of operations:

(a_1)' if $c_i \geq 2$, then decrease λ_i^* by 2,

(a_2)' if i is even, $\epsilon(i) = 1$ and $\lambda_{i-1}^* = \lambda_i^*$, then decrease λ_i^* and λ_{i-1}^* by 1 each.

Denote a partition obtained by one of these two operations from λ by λ'. Moreover, define ϵ' by $\epsilon'(j) = \epsilon(j)$ if $\lambda_j^* = (\lambda')_j^*$, and $\epsilon'(j) \neq 1$ if $\epsilon(j) = 0$, $\epsilon'(j) \neq 0$ if $\epsilon(j) = 1$.

The following can be proved just as Lemma 5.4:

Lemma 6.4. *Assume that (λ', ϵ') is obtained from (λ, ϵ) by application of (a_1)' or (a_2)' above for some index i. Then the symbol $\rho((\lambda', \epsilon'))$ is obtained by decreasing exactly one of the entries of the symbol $\rho((\lambda, \epsilon))$ by 1.*

Similarly as in Theorem 5.5 we obtain, using [13, 6.7 and 6.28]:

Theorem 6.5. *Let C be a unipotent class in $\mathrm{GO}_{2n} \setminus \mathrm{SO}_{2n}$. Then the symbol $\rho(\tilde{C})$ of the induced class*

$$\tilde{C} := \mathrm{Ind}_{\mathrm{GO}_{2m} \times \mathrm{GL}_{n-m}\sigma}^{\mathrm{GO}_{2n}} (C \times \{1\})$$

is obtained from the symbol $\rho(C)$ by increasing each of the largest $n-m$ entries by 1.

This allows to give an explicit description of rigid unipotent classes of GO_{2n} (compare with [13, II.7.9]):

Corollary 6.6. *A unipotent class C in $\mathrm{GO}_{2n} \setminus \mathrm{SO}_{2n}$ is rigid if and only if the associated symbol $\rho(C) \in D_1^{4,0}$ has the following property:*

- *for all $1 \leq j \leq |\rho(C)|$, either $c_j = c_{j-1}$ or $c_j - c_{j-2} \leq 4$ or $c_{j+1} - c_{j-1} \leq 4$,*

where we have set $c_{-1} = c_0 := -4$ (that is, reducing all entries $c_j, c_{j+1}, \ldots$ by 1 does not lead to a symbol in $D_1^{4,0}$).

6.7. The restricted Springer correspondence. We now have all ingredients to prove the analogue of Theorem 5.9:

Theorem 6.8. *The map*

$$D_{1,n-1}^{4,0} \longrightarrow X_{1,n-1}^{0,0}, \quad S \mapsto S - \begin{pmatrix} 0 & 4 & \ldots \\ 0 & 4 & \ldots \end{pmatrix},$$

represents the restricted Springer correspondence for the outer unipotent classes of GO_{2n} in characteristic 2.

Proof. This is entirely analogous to the proof of Theorem 5.9, using the description of induced classes in Theorem 6.5. □

Example 6.9. We give tables for the restricted Springer correspondence in $\mathrm{GO}_{2n} \setminus \mathrm{SO}_{2n}$, $n = 6, 8$. Since $\mathrm{GO}_6 \cong G(4)$, the results for GO_6 reproduce those for $G(4)$ in Example 5.10. But note that the Weyl group of $G(4)^\sigma$ has to be interpreted as $W(C_2)$, while that of $\mathrm{GO}_6 \cong G(4)$ has type $W(B_2)$, so the two correspondences differ by the non-trivial automorphism of $W(B_2)$, which interchanges the characters indexed by $(-, 2)$ and $(1^2, -)$.

As for $G(n)$, all special characters lie in the image of the Springer correspondence (compare with Corollary 5.11):

Corollary 6.10. *All special characters of $W(B_{n-1})$ occur as images under the Springer correspondence of a unipotent class in $\mathrm{GO}_{2n} \setminus \mathrm{SO}_{2n}$.*

Table 4. Restricted Springer correspondence for GO_6

C	d_u	$\rho(C)$	χ_u
6	0	$\begin{pmatrix} 2 \\ \end{pmatrix}$	$(2,-)$
41^2	1	$\begin{pmatrix} 0 & 5 \\ 1 & \end{pmatrix}$	$(1,1)$
2^3	2	$\begin{pmatrix} 0 & 4 \\ 2 & \end{pmatrix}$	$(-,2)$
21^4	4	$\begin{pmatrix} 0 & 4 & 8 \\ 1 & 5 & \end{pmatrix}$	$(-,1^2)$

Table 5. Restricted Springer correspondence for GO_8

C	d_u	$\rho(C)$	χ_u
8	0	$\begin{pmatrix} 3 \\ \end{pmatrix}$	$(3,-)$
61^2	1	$\begin{pmatrix} 0 & 6 \\ 1 & \end{pmatrix}$	$(2,1)$
42^2	2	$\begin{pmatrix} 0 & 5 \\ 2 & \end{pmatrix}$	$(1,2)$
42_0^2	3	$\begin{pmatrix} 1 & 5 \\ 1 & \end{pmatrix}$	$(1^2,1)$
3^22	3	$\begin{pmatrix} 0 & 4 \\ 3 & \end{pmatrix}$	$(-,3)$
41^4	4	$\begin{pmatrix} 0 & 4 & 9 \\ 1 & 5 & \end{pmatrix}$	$(1,1^2)$
2^31^2	5	$\begin{pmatrix} 0 & 4 & 8 \\ 1 & 6 & \end{pmatrix}$	$(-,21)$
21^6	9	$\begin{pmatrix} 0 & 4 & 8 & 12 \\ 1 & 5 & 9 & \end{pmatrix}$	$(-,1^3)$

Proof. Let $S \in X_{1,n-1}^{1,0}$ be distinguished. Then $S + ((0,3,6,\ldots),(0,3,6,\ldots))$ is a distinguished symbol in $D_{1,n-1}^{4,0}$, so by Proposition 6.2(a) it parametrizes an outer unipotent class of GO_{2n}. By Theorem 6.8 its Springer correspondent is the character indexed by

$$S + \begin{pmatrix} 0 & 3 & 6 & \ldots \\ 0 & 3 & 6 & \ldots \end{pmatrix} - \begin{pmatrix} 0 & 4 & 8 & \ldots \\ 0 & 4 & 8 & \ldots \end{pmatrix} \in X_{1,n-1}^{0,0},$$

so by $S \in X_{1,n-1}^{1,0}$. □

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G.M.: Fachbereich Mathematik/Informatik, Universität Kassel, Heinrich-Plett-Str. 40, D–34132 Kassel, Germany.

E-mail address: malle@mathematik.uni-kassel.de

K.S.: LAMFA, Université de Picardie Jules Verne, 33 rue Saint-Leu, F–80000 Amiens, France.

E-mail address: karine.sorlin@wanadoo.fr