Research report of Barbara Schapira

My mathematical interests lie in the study of ergodic properties of geodesic and horocycle flows acting on the unit tangent bundle of negatively curved manifolds : existence of invariant (finite or σ-finite) measures, entropy, unique ergodicity, equidistribution properies, generic vectors, ... This subject is very classical and was intensively studied during the last century, in the case of geodesic and horocyclic flows on compact or finite volume surfaces of constant negative curvature.

I am particularly interested in situations where the classical powerful methods issued from harmonic analysis or lattices of Lie groups do not work : non compact manifolds, infinite volume, infinitely generated groups, infinite measure of maximal entropy, variable curvature (and even nonpositive curvature),

I describe below the classical and well known case, and some of my works on the subject.

Hyperbolic surfaces of finite volume

If S is a hyperbolic surface, its unit tangent bundle identifies with $PSL(2,\mathbb{R})/\Gamma$, where Γ is a discrete subgroup of $PSL(2,\mathbb{R})$. The geodesic flow $(g^t)_{t\in\mathbb{R}}$ acts on the left as the one parameter group $\{\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix},\ t \in \mathbb{R}\}$, whereas the (unstable) horocycle flow $(h^s)_{s\in\mathbb{R}}$ corresponds to the unipotent group $\{\begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix},\ s \in \mathbb{R}\}$.

The geodesic flow is the typical geometrical example of a hyperbolic flow, with positive entropy, exponential mixing, dense orbits, dense periodic orbits, infinitely many invariant measures, ...

When S is compact, the horocyclic flow is minimal (Hedlund), ergodic with respect to the Liouville measure, and even uniquely ergodic (Furstenberg). In particular, all orbits $(h^s u)_{0\leq s\leq S}$ are equidistributed towards the Liouville measure when $S \to +\infty$. On finite volume surfaces appear periodic horocyclic orbits due to the presence of cusps (thin ends). But except the Dirac measures supported on periodic orbits, the Liouville measure is the unique ergodic probability measure on T^1S. And all nonperiodic orbits are equidistributed towards the Liouville measure. However, due to the presence of periodic orbits, this result becomes highly non trivial, and needs to understand the time spent by horocycles inside the cusps of the surface (Dani, and Dani-Smillie).

Geometrically finite negatively curved surfaces

In the particular case of hyperbolic surfaces, the notion of geometrical finiteness coincides with the fact that the fundamental group $\Gamma = \pi_1(S)$ is finitely generated.

If the surface has infinite volume, it has *funnels*, that is big ends of infinite volume, topologically homeomorphic to cylinders, separated from the compact part of the surface by a closed geodesic. Of course, the surface can still have *cusps*, that is thin ends of finite volume (also topologically homeomorphioc to cylinder). A geodesic orbit can enter a cusp and come back, and can do it even infinitely many times, during unbounded times, ... By contrast, if it enters a funnel, it never comes back.

The study must therefore restricted to the *nonwandering set* Ω *of the geodesic flow*. On this set, $(g^t)_{t\in\mathbb{R}}$ has the same qualitative properties as on a finite volume surface.

However, Ω is not invariant by the horocyclic flow, so that we need to consider also the *nonwandering set* $\mathcal{E}$ *of the horocyclic flow*, which consists of all horocyclic orbits intersecting Ω. Of course, $\Omega \subset \mathcal{E} \subset T^1S$, and all inclusions are strict.

As consequences, the measure of maximal entropy of the geodesic flow, supported on Ω, does not coincide anymore with the Liouville measure ; the Liouville measure, supported on T^1S, is no more ergodic under any of the two flows ; there is no measure invariant and ergodic under both flows.

However, most results true on finite volume surfaces can be extended in this context. For example, all horocyclic orbits of $\mathcal{E}$ are periodic or dense in $\mathcal{E}$. The horocyclic flow has a unique invariant ergodic measure on $\mathcal{E}$ except the Dirac measures supported on periodic orbits (Burger, Roblin). And this measure is infinite.

In my article *Lemme de l'Ombre et non divergence des horosphères d'une variété géométriquement finie*, Ann. Inst. Fourier (2004), I proved that in a certain sense, horocycles of $\mathcal{E}$ do not spend to much time inside the cusps of the surface. (The article was written in the context of manifolds of any dimension and variable negative curvature).

As a consequence, in another article *Equidistribution of the horocycles of a geometrically finite surface*, IMRN (2005), I obtained the equidistribution of nonperiodic horocyclic orbits of $\mathcal{E}$ towards the unique nonperiodic invariant ergodic measure on $\mathcal{E}$, which is infinite.

This result was obtained through other intermediate equidistribution results.

Half-horocycles

Usually, in classical ergodic theory, if a dynamical system is invertible, properties of the system and of its inverse are the same. For equidistribution property, for example, the usual statement, on a finite volume surface, says that for all nonperiodic vectors $v \in T^1S$, and all continuous functions f defined on T^1S, the Birkhoff average of f along the orbit $(h^s v)_{0 \le s \le S}$ converges to the integral of f w.r.t. the Liouville measure when $S \to +\infty$.

It turns out that in the statement of my equidistribution result mentioned above, as well as in an analogous result previously obtained by M. Burger, I consider symmetric orbits $(h^s v)_{-S \le s \le S}$ and their behaviour when $S \to +\infty$.

In a recent work *Density and equidistribution of half horocycles on geometrically finite surfaces* to appear in the Journal of the London Math. Society, I clarify the cases where one needs to consider symmetric orbits. Once again, it is mainly due to the presence of funnels. In such situations, it can happen that a half-orbit will enter a funnel and never come back, whereas the other half orbit is recurrent and even dense in $\mathcal{E}$.

Geometrically infinite surfaces

In a work in collaboration with Omri Sarig, we studied the question of equidistribution of orbits in the simplest case of geometrically infinite surfaces, the case of abelian covers of compact hyperbolic surfaces. In this situation, the invariant ergodic measures under the horocyclic flow were described and classified (Babillot-Ledrappier, Sarig), using the notion of *asymptotic cycle* of a vector. Roughly speaking, if you see a $\mathbb{Z}^d$-cover of a compact surface from very far, you see no details on the local geometry, but only $\mathbb{Z}^d$, and the asymptotic cycle of a vector v describes the asymptotic average displacement of $(g^t v)_{t \le 0}$ in $\mathbb{Z}^d$. Babillot-Ledrappier and Sarig showed that (h^s)-invariant ergodic measures are classified by asymptotic cycles, and we also classify generic vectors using this notion of asymptotic cycle. This work, *The generic points for the horocycle flow on a class of hyperbolic surfaces with infinite genus*, is published in IMRN (2008).

Possible talks

If the other participants wish, I can give a talk on any of the above works. Here is a generic title : "Equidistribution properties of horocycles on infinite volume hyperbolic surfaces".

Other works

I am also interested in thermodynamical formalism and quasi-invariant measures. See

- *On quasi-invariant transverse measures for the horospherical foliation of a negatively curved manifold* ETDS (2004)
- *Mesures transverses quasi-invariantes et limites de moyennes longitudinales,* CRAS (2003)

I participated to the collective book

- *Théorèmes ergodiques pour des actions de groupes*, L'Enseignement mathématique, 2010.

I am also interested in generic properties of invariant measures under the geodesic flow. I work with Yves Coudene on these questions. See

- *Generic measures for hyperbolic flows on non compact spaces*, Israel Journal of maths (2010).
- *Counterexamples in nonpositive curvature*, submitted, 2010.