CONTINUITY OF SOME RANDOMLY SAMPLED SERIES OF FUNCTIONS

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Abstract. In this article we study the continuity properties of trajectories for some randomly sampled series of functions, $\sum a_k f(\alpha X_k(\omega))$ where $(a_k)_{k\geqslant 0}$ is a sequence of complex numbers, $(X_k)_{k\geqslant 0}$ is a sequence of real independent random variables, f is a real valued function with period one and summable Fourier coefficients. We obtain almost sure continuity results for these periodic or almost periodic series for a large class of functions f, where the "almost sure" does not depend on the function. We show optimality of the results in some cases.

Keywords : sampled fourier series, almost sure continuity of trajectories, almost periodic functions, random trigonometric polynomial

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1. Introduction

The almost sure convergence of series defined by

$$\sum a_k f(\alpha\, k)$$

as well as lacunary series of the type

$$\sum a_k f(\alpha\, n_k)$$

where f verifies

$$f(x+1) = f(x) \qquad \int_0^1 f(x)dx = 0 \qquad \int_0^1 f^2(x)dx = 1 \tag{1}$$

has been studied especially when the sequence $(n_k)_{k\geqslant 1}$ grows rapidly, meaning that $(n_k)_{k\geqslant 1}$ satisfies the Hadamard gap condition

$$\frac{n_{k+1}}{n_k} \geqslant q > 1.$$

In this case, in [4], Kac proves that $\sum_k a_k f(\alpha n_k)$ converges a.e. if $\sum_k |a_k|^2 < +\infty$ and $f \in \mathrm{Lip}(\gamma)$ with $\gamma > 0$. In another direction, in [1], Berkes proves that there exists a function $f \in \mathrm{Lip}(\frac{1}{2})$ satisfying (1) and for any sequence $(\varepsilon_k)_{k\geqslant 1}$ such that $\varepsilon_k > 0$, there exists a sequence of integers $(n_k)_{k\geqslant 1}$ with

$$\frac{n_{k+1}}{n_k} \geqslant 1 + \varepsilon_k$$

and a sequence $(a_k)_{k\geqslant 1}$ with $\sum_k |a_k|^2 < +\infty$ such that the series $\sum_k a_k f(\alpha\, n_k)$ is a.e. divergent.
We can naturally adress the question whether the convergence still holds when the sequence $(n_k)_{k\geqslant 1}$ grows polynomially ($\mathcal{O}(k^d)$ with $d > 0$) or subexponentially ($\mathcal{O}(2^{k^\gamma})$ with $\gamma \in]0, 1[$), and for which class of functions. We are going to answer the question when the sequence $(n_k)_{k\geqslant 1}$ is randomly generated, that is when $n_k = X_k(\omega)$ where $(X_k)_{k\geqslant 0}$ is a sequence of independent real random variables defined on the probability space $(\Omega, \mathcal{A}, \mathbb{P})$.
More precisely, consider the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and define $A(\mathbb{T})$ as the set of complex valued functions whose Fourier coefficients are absolutely summable:

$$A(\mathbb{T}) = \{f : \mathbb{T} \to \mathbb{C}, f(\alpha) = \sum_{j\in\mathbb{Z}} \hat{f}(j) \exp(2i\pi\alpha j), \sum_{j\in\mathbb{Z}} |\hat{f}(j)| < +\infty\},$$

$(a_k)_{k\geqslant 0}$ will denote a sequence of complex numbers.
Our aim is to study the convergence, when $\omega \in \Omega$ is fixed, of the series of functions

$$\begin{array}{ccl} \mathbb{R} & \to & \mathbb{C} \\ \alpha & \mapsto & F(\alpha,\omega) = \sum_{k=0}^{\infty} a_k f(\alpha X_k(\omega)). \end{array}$$

Our strategy is to study on one hand the convergence of the random part $F(\alpha,\omega) - \mathbb{E}(F(\alpha,.))$, which is the main interest of this work, and on the other hand to give some examples where we get the convergence of the deterministic part $\mathbb{E}(F(\alpha,.))$.

Let us precise the questions raised by the study of the random part $F(\alpha,\omega) - \mathbb{E}(F(\alpha,.))$ by considering convergence in $L^2(\lambda_I \times \mathbb{P})$ where λ_I denotes Lebesgue measure restricted on an interval $I \subset \mathbb{R}$. Fix $f \in A(\mathbb{T})$.

$$\| \sum_{n}^{m} a_k(f(\alpha X_k(\omega)) - \mathbb{E}(f(\alpha X_k(.)))\|^2_{L^2(\lambda_I \times \mathbb{P})} = \sum_{k=n}^{m} |a_k^2| \int_I \mathrm{var}(f(\alpha X_k(.)))\, d\lambda$$

As f is bounded, the integral is finite. Therefore, if $\sum |a_k^2|$ converges then, for any sequence $(X_k)_{k\geqslant 0}$ of independent real random variables, the series defining $F(\alpha,\omega) - \mathbb{E}(F(\alpha,.))$ converges in $L^2(\lambda_I \times \mathbb{P})$ and there exists a subsequence (n_m) such that $\lim_{m\to\infty} \sum_{k=0}^{n_m} a_k(f(\alpha X_k(\omega)) - \mathbb{E}(f(\alpha X_k(.)))$ exists $\lambda_I \times \mathbb{P}$-almost surely. Hence there exists Ω_0 with $\mathbb{P}(\Omega_0) = 1$ such that for all $\omega \in \Omega_0$, $\sum a_k(f(\alpha X_k(\omega)) - \mathbb{E}(f(\alpha X_k(.)))$ converges λ_I-almost surely along a subsequence.

The questions we are going to answer in this paper are the following :

- is it possible to have Ω_0 independant of $f \in A(\mathbb{T})$?
- can we have convergence for the whole sequence ?
- when we have convergence, does the function $F(.,\omega)$ have continuous sample paths ?

In general, these are difficult and interesting questions.

Remark 1.1. *If $\sum |a_k|^2$ diverges, then we can construct a sequence of independent random variables $(X_k)_{k\geqslant 1}$ and find $f \in A(\mathbb{T})$ such that the series defining $F(\alpha,\omega) - \mathbb{E}(F(\alpha,.))$ does not converge in $L^2(\lambda_I \times \mathbb{P})$. Take for example X_k with a uniform law on the $2k+1$ integers of $[k^2,(k+1)^2-1]$ for all $k \geqslant 1$ and choose f such that for all $\alpha \in \mathbb{T}$, $f(\alpha) = \exp(2i\pi\alpha)$.*

In the following section we will prove the following : depending on the growth of $\mathbb{E}|X_k|^\beta$ for $\beta > 0$ (polynomial or subexponential), we give conditions on the sequence $(a_k)_{k\geqslant 1}$ and a class of functions $\mathcal{C} \subset A(\mathbb{T})$ such that

$$\exists \Omega_0, \mathbb{P}(\Omega_0) = 1, \forall \omega \in \Omega_0, \forall f \in \mathcal{C}, \alpha \mapsto F(\alpha,\omega) - \mathbb{E}(F(\alpha,.)) \text{ is continuous on } \mathbb{R}.$$

The key ingredient will be estimates on trigonometric polynomials like

$$\sum_{k=\lambda}^{\Lambda} a_k \left(e^{2i\pi\alpha j X_k(\omega)} - \mathbb{E}(e^{2i\pi\alpha j X_k})\right).$$

Then , in section 3, we will give examples where we have continuity of $\alpha \mapsto F(\alpha,\omega)$. We will in particular explain how to deal with the deterministic part $\mathbb{E}(F(\alpha,.))$ in both cases when F is periodic or non periodic.

2. Main results

2.1. Notations.

For $f \in A(\mathbb{T})$, define

$$||f||_A := \sum_{j\in\mathbb{Z}} |\hat{f}(j)|.$$

Define also, for $f : \mathbb{T} \to \mathbb{C}$

$$B(\mathbb{T}) = \{f : \mathbb{T} \to \mathbb{C}, \sum_{j\in\mathbb{Z}} |\hat{f}(j)|\sqrt{\log(|j|+2)} < +\infty\},$$

$$||f||_B = \sum_{j\in\mathbb{Z}} |\hat{f}(j)|\sqrt{\log(|j|+2)},$$

Notice that $B(\mathbb{T}) \subset A(\mathbb{T})$.
We will denote by φ_X the characteristic function of the random variable X

$$\forall t \in \mathbb{R}, \varphi_X(t) = \mathbb{E}(e^{2i\pi tX}).$$

We will distinguish two cases depending wether $X(\Omega) \subset \mathbb{Z}$ or $G(X(\Omega)) = \mathbb{R}$, where $G(X(\Omega))$ is the additive group generated by the support of the random variable X. The more fruitful results will be obtained when $G(X(\Omega)) = \mathbb{R}$. Remark that in the first case, φ_X is periodic whereas in the second case, it is not. Note that there are also cases where φ_X is periodic and $X(\Omega) \not\subset \mathbb{Z}$. Still we will call (with a slight abuse of language):
periodic case : for all k, $X_k(\Omega) \subset \mathbb{Z}$,
non periodic case : $\exists K, \forall k \geqslant K$, $G(X_k(\Omega)) = \mathbb{R}$.
We will also distinguish two types of growth for $\mathbb{E}|X_k|$:
polynomial case: there exist $\beta > 0$ and $d > 0$ with $\mathbb{E}|X_k|^\beta = \mathcal{O}(k^d)$,
subexponential case: there exists $\beta > 0$ and $\gamma \in]0,1[$ with $\mathbb{E}|X_k|^\beta = \mathcal{O}(2^{k^\gamma})$.

Remark 2.1.
Concerning the **subexponential** *case, if* $\gamma \geqslant 1$ *(*$\mathbb{E}|X_k|^\beta$ *growths exponentially) and if the sequence* $|a_k|$ *is decreasing, then condition (2) implies the convergence of the series* $\sum |a_k|$. *In this case, the function* F *is obviously well defined.*

Define lastly the following sequence:

$$c_n = \begin{cases} 1 + \sqrt{\log n} & \text{in the } \textbf{polynomial} \text{ case} \\ n^{-\frac{\gamma}{2}} & \text{in the } \textbf{subexponential} \text{ case} \end{cases}$$

2.2. **Continuity results.** In all the results stated here, the hypothesis made on the sequence $(a_k)_{k\geqslant 1}$ is the following

$$\sum_{n\geqslant 1} \frac{\sqrt{\sum_{k\geqslant n} |a_k|^2}}{n c_n} < +\infty \tag{2}$$

For example, in the **polynomial** case, if there exists $\varepsilon > 0$ such that $\sum_{k\geqslant n} |a_k|^2 = \mathcal{O}((\log n)^{-(1+\varepsilon)})$, then condition (2) holds. In the **subexponential** case, if $|a_k| = \mathcal{O}(k^{-\delta})$ with $\delta > \frac{\gamma+1}{2}$, then condition (2) holds.

Theorem 2.1. *Let* $(X_k)_{k\geqslant 0}$ *be a sequence of independent real valued random variables. Assume we are in the non periodic case. Let* $(a_k)_{k\geqslant 1}$ *be a sequence of complex numbers enjoying (2), then there exists a measurable set* Ω_0 *with* $\mathbb{P}(\Omega_0) = 1$ *such that for all* $\omega \in \Omega_0$, *for any* $f \in B(\mathbb{T})$, *for* $\alpha \in \mathbb{R}$, $F(\alpha, \omega) -$

$\mathbb{E}(F(\alpha,.))$ is well defined, $\alpha \mapsto F(\alpha,\omega)-\mathbb{E}(F(\alpha,.))$ is continuous, the series defining $F-\mathbb{E}(F)$ converges uniformly on every compact set and there exists $C_\omega > 0$ random variable with finite expectation such that for all $\alpha \in \mathbb{R}$:

$$|F(\alpha,\omega) - \mathbb{E}(F(\alpha,.))| \leqslant C_\omega ||f||_B \sqrt{\log(|\alpha|+2)}$$

Notice that the random variable C neither depends on α, nor on f.

Remark 2.2. *In the* **periodic case**, *using remark 2.3, we may take f in the larger set of functions $A(\mathbb{T})$ and in the previous inequality, $\sqrt{\log(|\alpha|+2)}$ is replaced by a constant, we thus get*

$$|F(\alpha,\omega) - \mathbb{E}(F(\alpha,.))| \leqslant C_\omega ||f||_A$$

This result relies on uniform estimations of the size of some trigonometric polynomials, more precisely on the following:

Theorem 2.2. *Let λ and Λ be two integers with $\lambda \leqslant \Lambda$, $(X_k)_{k\geqslant 0}$ be a sequence of independent real valued random variables for which there exists $\beta > 0$ such that, $\forall N \geqslant 0$, $\mathbb{E}|X_N|^\beta < \infty$. Define for every integer N*

$$\Phi_\beta(N) = 2 + \max(N, \mathbb{E}|X_N|^\beta).$$

Let $M \geqslant 1$ and $I_M = [-M, M]$. Let $(a_k)_{k\geqslant 1}$ be a sequence of real or complex numbers.

Define

$$A_{\lambda,\Lambda,M} = \sqrt{\log\left(M\Phi_\beta(\Lambda)\right)\sum_{k=\lambda}^{\Lambda}|a_k|^2},$$

then

$$\mathbb{E} \sup_{M\geqslant 1}\sup_{j\in\mathbb{Z}}\sup_{\lambda\geqslant 1}\sup_{\Lambda\geqslant\lambda}\sup_{\alpha\in I_M}\left|\frac{\sum_{k=\lambda}^{\Lambda} a_k\left[e^{2i\pi\alpha j X_k(\omega)} - \mathbb{E}e^{2i\pi\alpha j X_k}\right]}{\sqrt{A_{\lambda,\Lambda,M}^2\log\left(|j|+2\right)}}\right| < \infty. \tag{3}$$

Remark 2.3. *In the periodic case, the proof of Theorem 2.2 is easier. Namely, using the fact that $\alpha \mapsto j\alpha$ (mod 1) is onto for $j \neq 0$, we get*

$$\sup_{j\in\mathbb{Z}}\sup_{\alpha\in\mathbb{T}}\left|\sum_{k=\lambda}^{\Lambda} a_k\left[e^{2i\pi\alpha j X_k(\omega)} - \mathbb{E}e^{2i\pi\alpha j X_k}\right]\right|$$
$$= \sup_{\alpha\in\mathbb{T}}\left|\sum_{k=\lambda}^{\Lambda} a_k\left[e^{2i\pi\alpha X_k(\omega)} - \mathbb{E}e^{2i\pi\alpha X_k}\right]\right|.$$

The conclusion of Theorem 2.2 becomes then :

$$\mathbb{E}\sup_{j\in\mathbb{Z}}\sup_{\lambda\geqslant 1}\sup_{\Lambda\geqslant\lambda}\sup_{\alpha\in\mathbb{T}}\left|\frac{\sum_{k=\lambda}^{\Lambda} a_k\left[e^{2i\pi\alpha j X_k(\omega)} - \mathbb{E}e^{2i\pi\alpha j X_k}\right]}{\sqrt{A_{\lambda,\Lambda,1}^2}}\right| < \infty.$$

The proof is more tedious. It relies on a fine inequality of G.Cohen and C.Cuny [2] *(see the following theorem) generalizing a previous work by A-H.Fan and D.Schneider* [3]. *We can see here why, for integer-valued X_k, we can work with the functional space $A(\mathbb{T})$, whereas for real-valued X_k, we need to introduce the space $B(\mathbb{T})$.*

Theorem 2.2 easily deduces from the following theorem, proved by G.Cohen and C.Cuny in [2] :

Theorem 2.3. (Cohen, Cuny) *Let (X_n) be a sequence of independent random variables, defined on $(\Omega, \mathcal{A}, \mathbb{P})$, with values in $\mathbb{R}$. Let φ be some positive non-decreasing function on $\mathbb{R}^+$ such that there exists $\eta > 0$ for which $\varphi(x) \geqslant x^\eta$ for every $x \geqslant 0$. Assume that there exists $\delta > 0$ such that $\sum_{n=1}^{\infty} \mathbb{P}(|X_n| > \varphi(n)^\delta) < \infty$. Let (a_n) be a sequence of complex numbers. Then there exists universal constants $\varepsilon > 0$ and $C > 0$ such that*

$$\mathbb{E} \sup_{m>n\geqslant 1} \sup_{T\geqslant 1} \exp\left\{ \varepsilon \frac{\max_{|t|\leqslant T} |\sum_{k=n+1}^m a_k (e^{2i\pi t X_k} - \mathbb{E}(e^{2i\pi t X_k}))|^2}{\log(1+T)\log(1+\varphi(m))\sum_{k=n+1}^m |a_k|^2} \right\} \leqslant C$$

In ordre to get Theorem 2.2, we choose $\eta = 1$, $\varphi(N) = \Phi_\beta(N)$, $\delta > 1 + \frac{1}{\beta}$ and $T = M|j|$ (with $j \neq 0$) and we use the inequality, true for all $x \geqslant 0$

$$1 + x \leqslant e^x$$

In fact, we also have finite higher order moments.

Finally, note that the theorem 2.2 could also be proved directly using Gaussian randomization techniques, see [3].

Proof : of Theorem 2.1 Let $(N_k)_{k\geqslant 1}$ be a strictly increasing sequence of integers and define

$$\forall k \geqslant 1, \qquad P_k(\alpha) = \sum_{l=N_k+1}^{N_{k+1}} a_l \left[f(\alpha X_l(\omega)) - \mathbb{E} f(\alpha X_l) \right]$$

where $f \in B(\mathbb{T})$. We want to study the following series, for all $M \geqslant 1$

$$\sum_k \sup_{\alpha \in [-M,M]} |P_k(\alpha)|.$$

We have

$$|P_k(\alpha)| \leqslant \sum_{j\in\mathbb{Z}} |\hat{f}(j)| \left| \sum_{l=N_k+1}^{N_{k+1}} a_l [e^{(2\pi j \alpha X_l(\omega))} - \mathbb{E} e^{(2\pi j \alpha X_l)}] \right|.$$

Hence, using Theorem 2.2, there exists a positive integrable random variable ξ such that

$$\sup_{\alpha\in[-M,M]} |P_k(\alpha)| \leqslant \xi ||f||_B \sqrt{\log\left(M\Phi_\beta(N_{k+1})\right) \sum_{j=N_k+1}^{N_{k+1}} |a_j|^2}. \tag{4}$$

First, in the polynomial case, that is to say when there exists $d>0$ with $\Phi_\beta(N)=\mathcal{O}(N^d)$, we choose $N_k=2^{2^k}$ and we need to prove that

$$\sum_k 2^{k/2}\left(\sum_{l=2^{2^k}+1}^{2^{2^{k+1}}}|a_l|^2\right)^{1/2}<+\infty.$$

Now we use the following equivalent

$$\sum_{l=2^{2^{k-1}}+1}^{2^{2^k}}\frac{1}{l(\log(l))^{1/2}}\approx 2^{(k-1)/2}$$

which may be computed by comparing series and integral, hence

$$\begin{aligned} 2^{k/2}\left(\sum_{l=2^{2^k}+1}^{2^{2^{k+1}}}|a_l|^2\right)^{1/2} &\leqslant C\sum_{l=2^{2^{k-1}}+1}^{2^{2^k}}\frac{1}{l(\log(l))^{1/2}}\left(\sum_{j=2^{2^k}+1}^{\infty}|a_j|^2\right)^{1/2}\\ &\leqslant C\sum_{l=2^{2^{k-1}}+1}^{2^{2^k}}\frac{\left(\sum_{j=l}^{\infty}|a_j|^2\right)^{1/2}}{l(\log(l))^{1/2}} \end{aligned}$$

and, using condition (2),

$$\sum_k 2^{k/2}\left(\sum_{l=2^{2^k}+1}^{2^{2^{k+1}}}|a_l|^2\right)^{1/2}\leqslant C\sum_{n\geqslant 2}\frac{\sqrt{\sum_{k\geqslant n}|a_k|^2}}{n\sqrt{\log n}}<+\infty.$$

This implies :

$$\sum_{k\geqslant 1}\sup_{\alpha\in[-M,M]}|P_k(\alpha)|<\infty$$

almost everywhere on the measurable set $\Omega_o=\{\omega\in\Omega,\xi(\omega)<\infty\}$. By construction, this set does not depend on the choice of f.
Secondly, in the subexponential case, that is when there exists $\gamma\in]0,1[$ with $\Phi_\beta(N)=\mathcal{O}(2^{N^\gamma})$, we choose $N_k=2^k$ and we need to prove that

$$\sum_k 2^{\gamma k/2}\left(\sum_{l=2^k+1}^{2^{k+1}}|a_l|^2\right)^{1/2}<+\infty.$$

Using the following equivalent

$$\sum_{l=2^{k-1}+1}^{2^k}\frac{1}{l^{1-\frac{\gamma}{2}}}\approx 2^{\frac{\gamma(k-1)}{2}}$$

and doing the same kind of computation as before, condition (2) implies

$$\sum_{k\geqslant 1}\sup_{\alpha\in[-M,M]}|P_k(\alpha)|<\infty.$$

We also get from (4), for all $\alpha \in \mathbb{R}$,

$$|P_k(\alpha)| \leqslant \xi ||f||_B \sqrt{\log(|\alpha|+2)} \sqrt{\log(\Phi_\beta(N_{k+1})) \sum_{j=N_k+1}^{N_{k+1}} |a_j|^2}.$$

By summing on k, we get the inequality

$$|F(\alpha,\omega) - \mathbb{E}(F(\alpha,.))| \leqslant C\xi(\omega)||f||_B \sqrt{\log(|\alpha|+2)}$$

where C only depends on the sequence $(a_k)_{k\geqslant 1}$ and $\mathbb{E}(\xi) < \infty$.
This ends the proof of Theorem 2.1. □

3. Examples

If the conditions of theorem 2.1 on (a_k), (X_k) and f are fulfilled, it remains to find conditions to ensure the convergence of the expectation part in order to get the convergence of $\sum a_k f(\alpha X_k)$. We have

$$\begin{aligned} |\sum_{k=n}^{m} a_k \mathbb{E}(f(\alpha X_k))| &= |\sum_{j\in\mathbb{Z}} \sum_{k=n}^{m} a_k \hat{f}(j) \mathbb{E}(\exp(2i\pi j\alpha X_k))| \\ &\leqslant (\sum_{j\in\mathbb{Z}} |\hat{f}(j)|) \sup_{j\in\mathbb{Z}} |\sum_{k=n}^{m} a_k \varphi_{X_k}(j\alpha)|. \end{aligned}$$

The term of this sum for $j=0$ is $\hat{f}(0)\sum_{k=n}^m a_k$, therefore, in order to have the convergence of the expectation part, we need either to assume that $\sum a_k$ converge (which is a too strong conditionand leads to a trivial case) or to suppose $\hat{f}(0) = \int_{\mathbb{T}} f(t)dt = 0$. This is what we will do in the following. For the same reason, we will always suppose $\alpha \neq 0$ in the sequel.

In the following examples, we will thus give conditions on (a_k), (X_k) and f in order to apply theorem 2.1 and also conditions to ensure the convergence of $\sum a_k \mathbb{E}(f(\alpha X_k))$

3.1. Examples in the non periodic case.

Let us begin by giving an example where the (X_k) are uniformly distributed :

Example 3.1. *Suppose that* $\mathcal{L}(X_k) = \mathcal{U}([\mu_k - \sigma_k/2, \mu_k + \sigma_k/2])$ *with* $\sigma_k > 0$ *and* $\mathbb{E}(X_k) = \mu_k$ *with* $\mu_k = \mathcal{O}(k^d)$ *for some* $d > 0$ *(polynomial case). The characteristic function of* X_k *can easily be computed :*

$$\varphi_{X_k}(t) = \frac{e^{2i\pi t\mu_k}}{\pi t\sigma_k} \sin(\pi t\sigma_k).$$

Using Cauchy-Schwarz inequality, the conditions

$$\sum_{n\geqslant 1} \frac{1}{\sigma_n^2} < +\infty.$$

and

$$\sum_{n\geqslant 2}\frac{\sqrt{\sum_{k\geqslant n}|a_k|^2}}{n\sqrt{\log n}} < +\infty$$

are sufficient to prove that there exists a measurable set Ω_o *with full measure* ($\mathbb{P}(\Omega_o) = 1$) *such that for any* $\omega \in \Omega_o$ *for all* $f \in B(\mathbb{T})$ *such that* $\int_{\mathbb{T}} f(t)dt = 0$ *: for any compact set* K *which does not contain* 0*, the application* $t \in K \mapsto F(t) = \sum_{k\geqslant 1} a_k f(tX_k(\omega))$ *is continuous and the series defining* F *converges uniformly on* K.

The subexponential case could be dealt with in the same way.

Here are other examples where the conditions of our theorems can be quite easily verified.

Example 3.2. *Let* $(X_k)_{k\geqslant 1}$ *be a sequence of real independent random variables whose law can be written in the following way for all* $k \geqslant 1$: $\mathcal{L}(X_k) = \mathcal{L}(\sigma_k \cdot X + \mu_k)$ *where* X *verifies* $\mathbb{E}|X|^\beta < \infty$ *for some* $\beta > 0$ *and* $\sigma_k > 0$*. Moreover, we assume that there exist* $d > 0$ *and* $\delta > 0$ *such that* $|\sigma_k| = \mathcal{O}(k^d)$, $|\mu_k| = \mathcal{O}(k^d)$ *and the function* $t \mapsto t^\delta \mathbb{E}\exp(2i\pi tX)$ *is bounded on* $\mathbb{R}$*. Let* $(a_k)_{k\geqslant 1}$ *be a sequence of real or complex numbers and assume the two following conditions are fulfilled*

1. $|a_k| = \mathcal{O}(k^{-\gamma})$ *with* $\gamma > 1/2$
2. $\sum_{k=1}^{\infty} \frac{1}{|\sigma_k|^{2\delta}} < \infty$

Then there exists a measurable set Ω_o *with full measure* ($\mathbb{P}(\Omega_o) = 1$) *such that for any* $\omega \in \Omega_o$ *for all* $f \in B(\mathbb{T})$ *such that* $\int_{\mathbb{T}} f(t)dt = 0$ *: for any compact set* K *which does not contain* 0*, the application* $t \in K \mapsto F(t) = \sum_{k\geqslant 1} a_k f(tX_k(\omega))$ *is continuous and the series defining* F *converges uniformly on* K.

The random variable X *may have a Gaussian law with mean zero and variance one, a Cauchy law, the first Laplace law, an exponential law with parameter* $\lambda > 0$*. For example, if* $\mathcal{L}(X) = \mathcal{N}(0,1)$*, we have* $\mathbb{E}\exp 2i\pi tX = e^{-t^2/2}$ *and we use the fact that* $t \mapsto t^\delta \mathbb{E}\exp 2i\pi tX$ *is bounded on* $\mathbb{R}$ *for all* $\delta > 0$*. Namely, for simulation purpose, it would be interesting to have "localised" variables* X_k*, that is to say to choose the smallest* σ_k *(or the smallest* d*). In this case, it is possible as we only have to choose* δ *big enough so that the series* $\sum \frac{1}{|\sigma_k|^{2\delta}}$ *converges.*

We discuss now the case when the laws of X_k are generated by a convolution product of a given law μ :

Example 3.3. *Let* $(X_k)_{k\geqslant 1}$ *be an sequence of real valued independent random variables such that for every integer* $k \geqslant 1$, $\mathcal{L}(X_k) = \mu^{*k}$ *where* μ *is a probability measure on* $\mathbb{R}$ *with* $\mathbb{E}|X_1|^\beta < \infty$ *for some* β*. Assume the following*

(a) $$|\varphi_{X_1}(t)| = 1 \Longleftrightarrow t = 0 \qquad (X_1 \text{ aperiodic})$$

(b) $$\exists \delta > 0, \sup_{t\in\mathbb{R}} |t^{\delta}\varphi_{X_1}(t)| = q < \infty$$

Let $(a_k)_{k\geqslant 1}$ be a sequence of complex numbers satisfying (2). Then there exists a measurable set Ω_o with full measure ($\mathbb{P}(\Omega_o) = 1$) such that for any $\omega \in \Omega_o$, for all $f \in B(\mathbb{T})$ such that $\int_{\mathbb{T}} f(t)dt = 0$, for any compact set K which does not contain 0, the function $t \in K \mapsto F(t,\omega) = \sum_{k\geqslant 1} a_k f(t\, X_k(\omega))$ is continuous and the series defining F converges uniformly on K.

Here is how we prove the convergence of the expectation part. Let K be a compact set which does not contain 0. Using Cauchy-Schwarz inequality, it is sufficient to prove

$$\sup_{m>n\geqslant 0}\ \sup_{\alpha\in K}\ \sup_{j\in\mathbb{Z}-\{0\}} \sum_{k=n}^{m} |\varphi_{X_1}(j\alpha)|^{2k} < +\infty$$

Let us split the supremum on j respectively into the supremum on the indexes $J(q)$ and $\bar{J}(q)$ where $J(q) = \{j \in \mathbb{Z}^* : |j|^{\delta} \leqslant \left[\frac{2q}{\varepsilon^{\delta}}\right]\}$ and ε is the distance between 0 and the fixed compact K.
On one hand, using (a), one can prove that :

$$\forall\varepsilon > 0 \qquad \inf_{|t|>\varepsilon} |t|^{\delta}|1 - |\varphi_{X_1}(t)|^2| > 0$$

this implies

$$\begin{aligned}\sup_{m>n\geqslant 0}\ \sup_{\alpha\in K}\ \sup_{j\in J(q)} \sum_{k=n}^{m} |\varphi_{X_1}(j\alpha)|^{2k} &\leqslant \sup_{\alpha\in K}\ \sup_{j\in J(q)} \frac{2}{1-|\varphi_{X_1}(j\alpha)|^2}\\ &\leqslant \sup_{\alpha\in K}\ \sup_{j\in J(q)} C(\varepsilon)|j\alpha|^{\delta} \leqslant C(K)\left[\frac{2q}{\varepsilon^{\delta}}\right]\end{aligned}$$

On the other hand, using (b),

$$\begin{aligned}\sup_{m>n\geqslant 0}\ \sup_{\alpha\in K}\ \sup_{j\in \bar{J}(q)} \sum_{k=n}^{m} |\varphi_{X_1}(j\alpha)|^{2k} &\leqslant \sup_{m>n\geqslant 0}\ \sup_{\alpha\in K}\ \sup_{j\in \bar{J}(q)} \sum_{k=n}^{m}\left(\frac{q}{|j\alpha|^{\delta}}\right)^{2k}\\ &\leqslant \sup_{m>n\geqslant 0} C\sum_{k=n}^{m}\frac{1}{4^k} < +\infty\end{aligned}$$

where C is a universal constant.

3.2. **Examples in the periodic case.** In the periodic situation, to obtain convergence properties for a large class of functions, we need to control a phenomenon of uniform distribution of the sequence $\{\alpha j\}$, as the modulus of the characteristic function does not go to zero anymore when $|j|$ goes to infinity. One way to do this is to link the regularity of the function f with the arithmetical properties of α. Thus, we only obtain pointwise convergence in this case.

Define for $\eta \geqslant 1$

$$C_\eta(\mathbb{T}) = \{f : \mathbb{T} \to \mathbb{C}, \sum_{j\in\mathbb{Z}} |\hat{f}(j)||j|^\eta < +\infty\}.$$

and $\|f\|_\eta = \sum_{j\in\mathbb{Z}} |\hat{f}(j)||j|^\eta$. Notice that $\forall\eta \geqslant 1, C_\eta(\mathbb{T}) \subset B(\mathbb{T})$. For these examples, we need here an extra definition.
We say that the irrationality measure of $\alpha \in (\mathbb{R} \setminus \mathbb{Q})/\mathbb{Z}$ is $\eta \geqslant 1$ if for all $\iota > 0$, there exists $C(\alpha, \iota) > 0$ such that

$$\forall j \in \mathbb{Z} \setminus \{0\}, |j\alpha|_\mathbb{Z} \geqslant \frac{C(\alpha, \iota)}{|j|^{\eta+\iota}}$$

where $|x|_\mathbb{Z}$ denotes the distance between x and the nearest integer.
It is well kown that Lebesgue almost every irrational number has an irrationality measure 1.
Moreover, by the theorem of Thue-Siegel-Roth [7], the irrationality measure of every irrational real algebraic number is 1.

Example 3.4. *X_k is a sequence of independent random variables with disjoint supports. For each $k \geqslant 1$, the support of X_k is the set of integers belonging to $[k^2, (k+1)^2 - 1]$. The law of X_k is uniform on this set of integers. A computation shows that*

$$|\varphi_{X_k}(j\alpha)| = \frac{1}{2k+1}\left|\frac{\sin(\pi j\alpha(2k+1))}{\sin(\pi j\alpha)}\right|$$

and using Cauchy-Schwarz inequality, we get

$$\left|\sum_{k=n}^{m} a_k\varphi_{X_k}(j\alpha)\right| \leqslant \frac{1}{|\sin(\pi j\alpha)|}\sqrt{\sum_{k=n}^{m}|a_k|^2}\sqrt{\sum_{k=n}^{m}\frac{1}{(2k+1)^2}}$$

As $|\sin|$ has period π, $|\sin(\pi\alpha j)| = |\sin(\pi|\alpha j|_\mathbb{Z})|$ and as $|\alpha j|_\mathbb{Z} \leqslant \frac{1}{2}$,

$$|\sin(\pi|\alpha j|_\mathbb{Z})| \geqslant |\alpha j|_\mathbb{Z}$$

therefore

$$\left|\sum_{k=n}^{m} a_k\varphi_{X_k}(j\alpha)\right| \leqslant \frac{1}{|\alpha j|_\mathbb{Z}}\sqrt{\sum_{k=n}^{m}|a_k|^2}\sqrt{\sum_{k=n}^{m}\frac{1}{(2k+1)^2}}$$

At this point (for the expectation part), note that we do not need the convergence of the series $\sum |a_k|^2$. However, we still need the conditions necessary for the convergence of the centered part (see Theorem 2.1). Anyway we have

$$\left|\sum_{k=n}^{m} a_k\varphi_{X_k}(j\alpha)\right| \leqslant \frac{\varepsilon}{|\alpha j|_\mathbb{Z}}$$

for large enough m and n.
Take α with an irrationality measure 1 and fix $\iota > 0$, there exists $C(\alpha, \iota) > 0$

such that

$$\forall j \in \mathbb{Z} \setminus \{0\},\ |j\alpha|_{\mathbb{Z}} \geqslant \frac{C(\alpha,\iota)}{|j|^{1+\iota}}$$

and

$$\forall j \in \mathbb{Z} \setminus \{0\},\ |j|^{-(1+\iota)} \left| \sum_{k=n}^{m} a_k \varphi_{X_k}(j\alpha) \right| \leqslant \frac{C}{C(\alpha,\iota)}\varepsilon$$

for large enough n and m. Now, if we take $f \in C_{1+\iota}$, we get

$$|\sum_{k=n}^{m} a_k \mathbb{E}(f(\alpha X_k))| \leqslant \|f\|_{1+\iota} \frac{C}{C(\alpha,\iota)}\varepsilon$$

for large enough n and m.

Finally, here is again an example where the laws of X_k are generated by a product of convolution.

Example 3.5. *Let $(X_k)_{k\geqslant 1}$ be an sequence of independent random variables such that for all integer $k \geqslant 1$, $\mathcal{L}(X_k) = \mu^{*k}$ where μ is a probability measure on $\mathbb{R}$ with $\mathbb{E}|X_1|^\beta < \infty$ for some $\beta > 0$. Assume we are in the* **periodic** *case. Let $(a_k)_{k\geqslant 1}$ be a sequence of complex numbers such that $|a_k| = O(k^{-\gamma})$ with $\gamma > 1/2$.*
Suppose that X_1 gives a strictly positive mass to two consecutive integers.
Then there exists a measurable set Ω_o with full measure ($\mathbb{P}(\Omega_o) = 1$) such that for any $\omega \in \Omega_o$, for all $\iota > 0$ and for all $f \in C_{1+\iota}(\mathbb{T})$ such that $\int_{\mathbb{T}} f(t)dt = 0$, the function $\alpha \mapsto F(t,\omega) = \sum_{k\geqslant 1} a_k f(\alpha\, X_k(\omega))$ is Lebesgue-almost everywhere well defined.

Let us describe how to get the convergence of the expectation part.

First note that as X_1 gives a strictly positive mass to two consecutive integers, X_1 is aperiodic meaning that

$$|\varphi_{X_1}(t)| = 1 \Leftrightarrow t = 0.$$

We first use Cauchy-Schwarz inequality, the fact that the law of X_k is a convolution product and that $\sum |a_k|^2 < +\infty$

$$\begin{aligned} \left| \sum_{k=n}^{m} a_k \varphi_{X_k}(j\alpha) \right| &\leqslant \sqrt{\sum_{k=n}^{m} |a_k|^2} \sqrt{\sum_{k=n}^{m} |\varphi_{X_1}(j\alpha)|^{2k}} \\ &\leqslant \varepsilon \frac{1}{\sqrt{1 - |\varphi_{X_1}(j\alpha)|^2}} \end{aligned}$$

for large enough n and m.
Remark that these inequalities are true for every α non zero irrational number as X_1 is aperiodic.
Now let Y a random variable such that $Y(\Omega) \subset \mathbb{Z}$. Let $p_k = \mathbb{P}(Y = k)$, $\varphi_Y(t) = \mathbb{E}(e^{2i\pi tY}) = \sum_{k\in\mathbb{Z}} p_k e^{2i\pi tk}$ and $\Re(\varphi_Y(t)) = \sum_{k\in\mathbb{Z}} p_k \cos(2\pi tk)$

hence

$$1-\Re(\varphi_Y(t))=\sum_{k\in\mathbb{Z}}p_k(1-\cos(2\pi tk))=\sum_{k\in\mathbb{Z}}2p_k\sin^2(\pi tk)\geqslant 2p_1\sin^2(\pi t)$$

We take now $Y=X_1-X_1'$ *where* X_1' *is an independent copy of* X_1. $\varphi_Y(t)=|\varphi_{X_1}(t)|^2$ *and as* X_1 *gives a strictly positive mass to two consecutive integers,* $p_1=\mathbb{P}(X_1-X_1'=1)>0$. *We thus get*

$$1-|\varphi_{X_1}(\alpha j)|^2\geqslant 2p_1\sin^2(\pi\alpha j).$$

As $x\mapsto\sin^2 x$ *has period* π, $\sin^2(\pi\alpha j)=\sin^2(\pi|\alpha j|_{\mathbb{Z}})$ *and as* $|\alpha j|_{\mathbb{Z}}\leqslant\frac{1}{2}$,

$$\sin^2(\pi|\alpha j|_{\mathbb{Z}})\geqslant|\alpha j|_{\mathbb{Z}}^2.$$

Consequently:

$$\frac{1}{\sqrt{1-|\varphi_{X_1}(j\alpha)|^2}}\leqslant\frac{C}{|\alpha j|_{\mathbb{Z}}}$$

where C *is a constant depending only on* X_1.
Take α *with an irrationality measure 1 and fix* $\iota>0$, *there exists* $C(\alpha,\iota)>0$ *such that*

$$\forall j\in\mathbb{Z}\setminus\{0\},\ |j\alpha|_{\mathbb{Z}}\geqslant\frac{C(\alpha,\iota)}{|j|^{1+\iota}}$$

and

$$\forall j\in\mathbb{Z}\setminus\{0\},\ |j|^{-(1+\iota)}\left|\sum_{k=n}^{m}a_k\varphi_{X_k}(j\alpha)\right|\leqslant\frac{C}{C(\alpha,\iota)}\varepsilon$$

for large enough n *and* m. *We conclude in the same way as in the previous example for* $f\in C_{1+\iota}$.

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