Statistics of return times for weighted maps of the interval

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April 1999

Abstract

For non markovian, piecewise monotonic maps of the interval associated to a potential, we prove that the law of the entrance time in a cylinder, when renormalized by the measure of the cylinder, converges to an exponential law for almost all cylinders. Thanks to this result, we prove that the fluctuations of R_n, first return time in a cylinder, are lognormal.

1 Introduction

In this article, we study the asymptotic law of R_n, which is, for a stationary stochastic process, the first time when the process repeats its n first symbols. In the same way, for a piecewise monotonic map T of the interval, R_n is the first return time in an interval of continuity of T^n. When the dynamical system is ergodic, Ornstein and Weiss [8] have proved that $\lim_{n\to\infty} \frac{1}{n} log\ R_n = h$, where the convergence is almost sure and h is the entropy of the system. Results about fluctuations of log R_n around nh are obtained for systems with the Gibbs property by Collet, Galves and Schmitt [3]. Showing that the non-markov part of the system can be disregarded, and proving something similar to the Gibbs property defined in [1], (third part), we give the same results for piecewise monotonic maps of the interval associated to a bounded variation weight, that is to say: the law of R_n, correctly renormalized, converges to a lognormal distribution. This convergence strongly uses the fact that we can approximate the law of the entrance time in a cylinder by an exponential law, which is proved in the fifth part.
Consider the following setting: T is a piecewise monotonic transformation (with b branches). T is piecewise $\mathcal{C}^2$, which means that there is a subdivision $(a_i)_{i=0}^{i=b}$ of [0,1] such that T is monotonic and extends to a $\mathcal{C}^2$ map on each $]a_i, a_{i+1}[$. Denote by $sing(T)$ the set $\{a_i, i = 0, \ldots, b\}$ of the points where T is not continuous and let $A_i =]a_i, a_{i+1}[$. We call n-cylinder a set as follows: $A_{i_1}^{i_n} = A_{i_1} \cap T^{-1}A_{i_2} \cap \ldots \cap T^{-n+1}A_{i_n}$. Denote by $\mathcal{P}^n$ the set of n-cylinders. For all x in $[0,1] \setminus \cup_0^\infty T^{-n}(sing(T))$ and all n, there is a unique n-cylinder containing x, called $\mathcal{P}^n(x)$.
We assume that the borelian σ-field $\mathcal{B}$ is generated by the finite partition $]a_i, a_{i+1}[$.
We are going to study the asymptotic law of R_n for a measure μ_φ invariant by T, where φ is a

AMS classification 2000 : 37A50, 37E05, 60F99 measurable potential. The study of dynamical systems associated to a potential (different from the inverse of the jacobian of T) arise from statistical mechanics, where the potential figures the interaction between the particles (see [1]). Another motivation is when the potential is equal to zero, the equilibrium states are then measures which maximize the entropy.

Given a measurable potential φ, define the associated transfer operator (for f measurable) by:

$$P_\varphi f(x) = \sum_{T(y)=x} e^{\varphi(y)} f(y)$$

We define the topological pressure of the system as follows:

$$p(\varphi) = \lim_{n\to\infty} \frac{1}{n} \log \sup_{x\in[0,1]} P_\varphi^n \mathbf{1}(x)$$

($p(\varphi)$ is well defined because the sequence $(\sup_{x\in[0,1]} P_\varphi^n \mathbf{1}(x))_{n\in\mathbb{N}}$ is submultiplicative)

Definition 1.1 *A measurable function f on [0,1] has bounded variation ($f \in BV([0,1])$) if $var_{[0,1]} f = var\ f < \infty$, where we define the variation on a set A by:*

$$var_A\ f = \sup \sum_{i=1}^{n} |f(x_{i-1}) - f(x_i)|$$

the supremum is taken over all finite partitions of A: $0 = x_0 < \ldots < x_n = 1, \quad n \geq 1$.

Recall that a measure is $e^{p(\varphi)-\varphi}$-conformal (in the sense of Denker et Urbanski [4]) if for all measurable sets A such that $T : A \to T(A)$ is invertible:

$$\nu(T(A)) = \int_A e^{p(\varphi)-\varphi}\ d\nu$$

Assuming certain hypothesis on the potential (see the next section), Liverani, Saussol and Vaienti [7] prove the existence of a conformal measure ν and the existence of a unique measure invariant by T, μ_φ, absolutely continuous with respect to ν and satisfying exponential decay of correlations. Under the same hypothesis on the weight, we can state our main result:
Define the entrance time in a cylinder A by:

$$\tau_A(x) = \inf\{k \geq 0, T^k(x) \in A\}$$

In the same way, we define the return time in a cylinder:

$$R_n(x) = \inf\{k > 0, T^k(x) \in \mathcal{P}^n(x)\}$$

Define, for f with bounded variation, the quantity that usually appears in the central limit theorem, i.e the asymptotic variance $\sigma(f)$ (see [6]):

$$C_n(f) = \int f \circ T^n f d\mu_\varphi - (\int f d\mu_\varphi)^2$$

$$\sigma^2(f) = C_0(f) + 2\sum_{n=1}^{\infty} C_n(f)$$

$\sigma^2(f)$ is well defined because $C_n(f)$ is the autocorrelation of f and so, it decays exponentially fast.
Let $h = h_{\mu_\varphi}$ be the entropy associated to the measure μ_φ i.e:

$$h = \lim_{n\to\infty} \frac{1}{n} \log \#\{A \in \mathcal{P}^n, \mu_\varphi(A) > 0\}$$

Theorem 1.1 *Assume* $\sigma(\varphi) \neq 0$, *then* $\left(\frac{\log R_n - nh}{\sigma(\varphi)\sqrt{n}}\right)_{n\in\mathbb{N}}$ *is a sequence of well defined random variables on the probability space* $([0,1], \mathcal{B}, \mu_\varphi)$ *and:*

$$\frac{\log R_n - nh}{\sigma(\varphi)\sqrt{n}} \Rightarrow \mathcal{N}(0,1)$$

where $\Rightarrow$ *is a convergence in law.*
(and $\sigma(\varphi) = 0$ *if and only if there exists a measurable* g *such that* $\varphi = g - g \circ T$*).*

2 Piecewise monotonic maps of the interval

Recall that T is a piecewise monotonic map of the interval. For $x \in [0,1]$, let:

$$S_n(x) = \exp(\sum_{i=0}^{n-1} \varphi \circ T^i(x))$$

Let us make the following hypothesis on the potential and the system:

(H1) $\exp(\varphi)$ has bounded variation.

(H2) (distortion) $\sum_{n=1}^{+\infty} \sup_{C\in\mathcal{P}^n} var_C \varphi < \infty$.

(H3) (dilatation) $\sup \varphi < p(\varphi)$.

(H4) (covering) $\forall I$ *interval* $, \exists N(I) \in \mathbb{N}^*, \quad C(I) > 0, \quad \inf P_\varphi^{N(I)} \mathbf{1}_I \geq C(I)$.

(H2) is called a distortion hypothesis because it allows us to show the distortion property (see lemma 2.5).
(H3) is called a dilatation hypothesis because it really plays the same role as the hypothesis $\inf |T'| \geq \rho > 1$ when the potential is the logarithm of the inverse of the derivative of T.
(H4) is equivalent, when φ is bounded from below (for example when φ is the logarithm of the inverse of the derivative of T and T is strictly expanding), to the following:

$$\forall I \ \ interval \ \ , \exists N(I) \in \mathbb{N}^*, \quad T^{N(I)} I \supset [0,1]$$

Lasota-Yorke inequality:

Theorem 2.1 *Under the hypothesis (H1), (H2), (H3), there exist* $\alpha < 1$ *and* $\xi > 0$ *such that for all* $f \in BV([0,1])$, $f \geq 0$*:*

$$\frac{1}{\lambda} var(P_\varphi(f)) \leq \alpha \ var(f) + \xi \nu(f)$$

Proof : The proof is deeply based on the sub-lemma 4.1.1 of [7]:

Sub-lemma 4.1.1:
For all integer m, there exists $B_m < \infty$ such that, for all positive function f with bounded variation:

$$var(P_\varphi^m f) \leq 9 \sup S_m \; var(f) + B_m \int f d\nu$$

By hypothesis: $\sup S_m \leq e^{m \sup \varphi} < \lambda^m$; let m such that $e^{m(\sup \varphi - p(\varphi))} < \frac{1}{9}$ (recall that $\lambda = e^{p(\varphi)}$):

$$\frac{1}{\lambda^m} var(P_\varphi^m f) \leq \alpha_m var(f) + B_m \nu(f)$$

with $\alpha_m < 1$ and $B_m < \infty$. It is then sufficient to consider the iterate P_φ^m to get the desired inequality. □

Existence of conformal and invariant measures:

Theorem 2.2 *(Liverani, Saussol, Vaienti [7])*
Under the hypothesis (H1)...(H4), there exists a non atomic $e^{p(\varphi)-\varphi}$-conformal measure ν and there exists a unique invariant probability measure μ_φ absolutely continuous with respect to ν.
ν and μ_φ are obtained in the following way:
there exist $\lambda > 0$ and h_φ such that:

$$P_\varphi h_\varphi = \lambda h_\varphi \quad , \quad \nu(h_\varphi) = 1 \quad , \quad P_\varphi^*(\nu) = \lambda \nu$$

$\mu_\varphi = h_\varphi \nu$, the density h_φ is positive, has bounded variation and $\lambda = e^{p(\varphi)}$. Moreover, $\inf(h_\varphi) > 0$.

Theorem 2.3 *([7])*
Under the same hypothesis, μ_φ is the unique equilibrium state for φ, i.e:

$$p(\varphi) = h_{\mu_\varphi}(T) + \int \varphi d\mu_\varphi = \sup\{h_m(T) + \int \varphi dm\}$$

where $h_m(T)$ denotes the entropy of the measurable system (T, m) and the supremum is taken over all the T-invariant measures m.

The main ingredient to show these theorems is the Lasota-Yorke inequality. The covering hypothesis is needed to get a strictly positive density h_φ.

Decay of correlations:

Theorem 2.4 *([7])*
Assuming the same hypothesis as before, the decay of correlation is exponential: there is $\gamma > 0$ and a constant K such that, if f has bounded variation and g is integrable:

$$\left| \int f g \circ T^n d\mu_\varphi - \int f d\mu_\varphi \int g d\mu_\varphi \right| \leq K e^{-\gamma n} \left(\int |f| d\mu_\varphi + var\, f \right) \int |g| d\mu_\varphi$$

in particular, if $f = \mathbf{1}_A$ and $g = \mathbf{1}_B$ with A interval and B measurable, then $var\, f$=2 and for all n:

$$|\mu_\varphi(A \cap T^{-n}B) - \mu_\varphi(A)\mu_\varphi(B)| \leq K e^{-\gamma n}(2 + \mu_\varphi(A))\mu_\varphi(B)$$

This kind of mixing, which is weaker than Φ-mixing, is a key tool in the following.

Central limit theorem: For functions with summable decay of correlation (which is the case for $\varphi_0 = \mu_\varphi(\varphi) - \varphi$ since it has bounded variation and then decays exponentially fast), the central limit theorem is true(see [6]), i.e, recall that:

$$\sigma^2(f) = C_0(f) + 2\sum_{n=1}^{\infty} C_n(f)$$

and assume that $\sigma(\varphi) \neq 0$, then we have:

$$\frac{\sum_{i=0}^{n-1} \varphi_0 \circ T^i}{\sigma(\varphi_0)\sqrt{n}} \Rightarrow \mathcal{N}(0,1)$$

which is equivalent to:

$$\frac{-\log\ S_n + n\mu_\varphi(\varphi)}{\sigma(\varphi)\sqrt{n}} \Rightarrow \mathcal{N}(0,1)$$

(and $\sigma(\varphi) = 0$ if and only if there exists a measurable function g such that $\varphi = g - g \circ T$)

Distortion property:

Lemma 2.5 *Assume (H2), then there is a constant c>1 such that, for all n, all A∈ $\mathcal{P}^n$, all x and y in A:*

$$\frac{1}{c} \leq \frac{S_n(y)}{S_n(x)} \leq c$$

Proof :

$$\frac{S_n(y)}{S_n(x)} = e^{(\varphi(y)-\varphi(x))+\ldots+(\varphi \circ T^{n-1}(y) - \varphi \circ T^{n-1}(x))}$$

x and y are in the same n-cylinder, therefore, for all k, $T^{n-k}(x)$ and $T^{n-k}(y)$ are in the same k-cylinder and

$$\frac{S_n(y)}{S_n(x)} \leq \exp\ (\sum_{k=1}^{n} var_{C_k(T^{n-k}(x))}\varphi) \leq \exp\ (\sum_{n=1}^{+\infty} \sup_{C \in \mathcal{P}^n} var_C \varphi)$$

We get the other inequality by changing x and y. □

Remark 2.1 *In case when e^φ is the inverse of the derivative of the transformation, the bounded distortion property comes from the fact that T is $\mathcal{C}^2$ and from the uniform dilatation hypothesis made for T (see [2]).*

3 Estimates of the measure of a cylinder

In the following, K and β are generic positive constants independant from n and A. It is proven in this section first that the measure of a n-cylinder decays exponentially fast to zero, then that, for most n-cylinders, we can give an equivalent for this measure.

Lemma 3.1 *There exists $\theta > 0$ and a constant C such that, for all n and all n-cylinder A:*

$$\mu_\varphi(A) \leq Ce^{-\theta n}$$

Proof : Let A=$A_{i_1}^{i_n}$ be a n-cylinder. For all n_0 <n we get:

$$\mu_\varphi(A) \leq \mu_\varphi(A_{i_1} \cap T^{-n_0} A_{i_{n_0}} \cap \ldots \cap T^{-\left[\frac{n}{n_0}\right]n_0} A_{i_{\left[\frac{n}{n_0}\right]n_0}})$$

Let us use the mixing inequality with the interval A_{i_1} and the measurable set $A_{i_{n_0}} \cap \ldots \cap T^{-(\left[\frac{n}{n_0}\right]-1)n_0} A_{i_{\left[\frac{n}{n_0}\right]n_0}}$:

$$\begin{aligned}
\mu_\varphi(A_{i_1} \cap \ldots \quad \cap \quad & T^{-\left[\frac{n}{n_0}\right]n_0} A_{i_{\left[\frac{n}{n_0}\right]n_0}}) - \mu_\varphi(A_{i_1})\mu_\varphi(A_{i_{n_0}} \cap \ldots \cap T^{-(\left[\frac{n}{n_0}\right]-1)n_0} A_{i_{\left[\frac{n}{n_0}\right]n_0}}) \\
\leq \quad & Ke^{-\gamma n_0}(2 + \mu_\varphi(A_{i_1}))\mu_\varphi(A_{i_{n_0}} \cap \ldots \cap T^{-(\left[\frac{n}{n_0}\right]-1)n_0} A_{i_{\left[\frac{n}{n_0}\right]n_0}}) \\
\leq \quad & 3Ke^{-\gamma n_0}\mu_\varphi(A_{i_{n_0}} \cap \ldots \cap T^{-(\left[\frac{n}{n_0}\right]-1)n_0} A_{i_{\left[\frac{n}{n_0}\right]n_0}})
\end{aligned}$$

if we call $s = \sup\{\mu_\varphi(A_i), i = 0, \ldots, b-1\}$ we have:

$$\mu_\varphi(A) \leq (s + 3Ke^{-\gamma n_0})\mu_\varphi(A_{i_{n_0}} \cap \ldots \cap T^{-(\left[\frac{n}{n_0}\right]-1)n_0} A_{i_{\left[\frac{n}{n_0}\right]n_0}})$$

and, by induction:

$$\mu_\varphi(A) \leq (s + 3Ke^{-\gamma n_0})^{\left[\frac{n}{n_0}\right]+1}$$

Now, there is n_0 such that $s + 3Ke^{-\gamma n_0} < 1$ which ends the proof. □

The following lemma gives an equivalent of the measure of almost all n-cylinders (which are intervals). We cannot get the equivalent for all cylinders because of the following remark:

Remark 3.1 *Let A be a n-cylinder whose boundary does not contain any singularity of T, then T(A) is a (n-1)-cylinder. (When the system is markovian, the image of a n-cylinder is always a (n-1)-cylinder, that is why we get the equivalent for all cylinders). Conversely, if the boundary of A contains a singularity of T, T(A) can be much smaller than the (n-1)-cylinder it is included in.*

Proof of the remark:
If A is a n-cylinder, its boundary in contained in $\cup_{i=0}^{n-1} T^{-i}(sing\ T)$. If its boundary does not contain any singularity of T then it is included in $\cup_{i=1}^{n-1} T^{-i}(sing\ T)$. The boundary of $T(A)$ is then included in $\cup_{i=0}^{n-2} T^{-i}(sing\ T)$ and $T(A)$ is a union of (n-1)-cylinders. By an argument of connexity, as $T_{|A}$ is continuous, $T(A)$ is one n-cylinder.

Example:

In this example, A is a 2-cylinder, the boundary of A contains a singularity of T and $T(A)$ is strictly included in the 1-cylinder B.

Lemma 3.2 *Let $k_0 > 0$ and $n > k_0$. Let $A \in \mathcal{P}^n$ such that, for all $k \leq n - k_0$, $T^k(A)$ has no singularity of T in its boundary. Then, there exists a constant $c(k_0) > 1$ such that, for all x in A:*

$$\frac{1}{c(k_0)} \leq \frac{\mu_\varphi(A)}{\lambda^{-n} S_n(x)} \leq c(k_0).$$

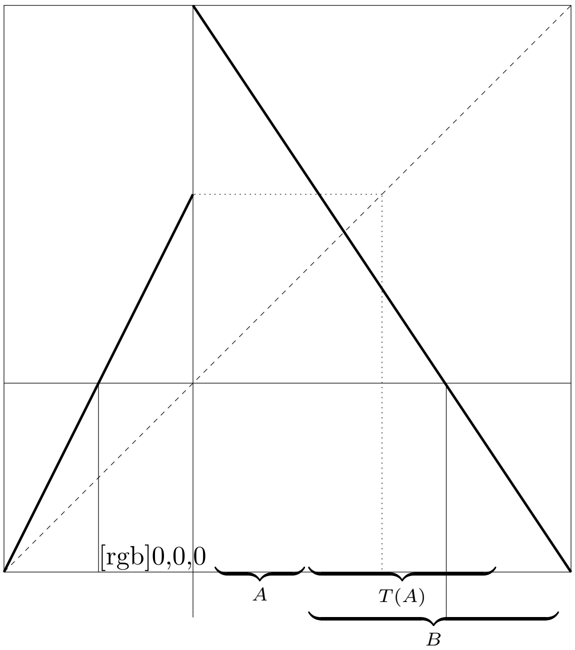

Figure 1: non markov map

Proof : Let $A \in \mathcal{P}^n$ such that, for all $k \leq n-k_0$, $T^k(A)$ has no singularity of T in its boundary and let $x \in A$:

$$\mu_\varphi(A) = \nu(h_\varphi \mathbf{1}_A) = \frac{1}{\lambda^{n-k_0}}\nu(P_\varphi^{n-k_0}(h_\varphi \mathbf{1}_A)).$$

$$P_\varphi^{n-k_0}(h_\varphi \mathbf{1}_A)(z) = \sum_{T^{n-k_0}(y)=z, y\in A} S_{n-k_0}(y) h_\varphi(y). \tag{1}$$

Let us take $z \in [0,1] \setminus \cup_{n\in\mathbb{N}} T^{-n}(singT)$ (we can restrict to such z without changing the integral because $\nu(\cup_{n\in\mathbb{N}} T^{-n}(singT)) = 0$), z is in a k_0-cylinder $C_{k_0}(z)$. $T^{-n+k_0}(C_{k_0}(z))$ is constituted at most by b^{n-k_0} n-cylinders and $T^{-n+k_0}(z)$ by at most b^{n-k_0} points. Each of them are in a different n-cylinder.
if A is one of these n-cylinders then $A \cap T^{-n+k_0}(z) = z_A$, if it's not the case then $A \cap T^{-n+k_0}(z) = \emptyset$. Therefore we get:

$$P_\varphi^{n-k_0}(h_\varphi \mathbf{1}_A)(z) \leq S_{n-k_0}(z_A) h_\varphi(z_A) \leq \sup(h_\varphi) S_{n-k_0}(z_A)$$

Let $x \in A$, we use the distorsion property (since x and z_A are in the same $n-k_0$-cylinder) in order to get:

$$P_\varphi^{n-k_0}(h_\varphi \mathbf{1}_A)(z) \leq K S_{n-k_0}(x)$$

$$\frac{\mu_\varphi(A)}{\lambda^{-n+k_0} S_{n-k_0}(x)} \leq K$$

Moreover, because of the previous remark, $T^{n-k_0}(A)$ is a k_0-cylinder and the sum (1) is not zero when $T^{-n+k_0}(z) \cap A \neq \emptyset$ which occurs when $T^{n-k_0}(A) = C_{k_0}(z)$ hence:

$$\begin{aligned} P_\varphi^{n-k_0}(h_\varphi \mathbf{1}_A)(z) &= \mathbf{1}_{T^{n-k_0}(A)}(z) \sum_{T^{n-k_0}(y)=z, y\in A} S_{n-k_0}(y) h_\varphi(y) \\ &\geq \mathbf{1}_{T^{n-k_0}(A)}(z) S_{n-k_0}(z_A) h_\varphi(z_A) \\ &\geq \frac{1}{c}\mathbf{1}_{T^{n-k_0}(A)}(z) S_{n-k_0}(x) \inf(h_\varphi) \end{aligned}$$

now we get:

$$\mu_\varphi(A) \geq \frac{\lambda^{-n+k_0}}{c}\nu(T^{n-k_0}(A))S_{n-k_0}(x)\inf(h_\varphi)$$

and $T^{n-k_0}(A)$ is a k_0-cylinder; now denoting $c(k_0) = (\frac{1}{c}\inf_{A\in C_{k_0}}\nu(A)\times\inf(h_\varphi))^{-1}$:

$$\frac{\mu_\varphi(A)}{\lambda^{-n+k_0}S_{n-k_0}(x)} \geq \frac{1}{c(k_0)}.$$

and $S_n(x) = S_{n-k_0}(x)S_{k_0}(T^{n-k_0}(x))$. But S_{k_0} is bounded and multiplying by λ^{k_0} we get the result. □

Lemma 3.3 *Let $B(n,C) = \{A \in \mathcal{P}^n, \forall x \in A : \frac{1}{C} \leq \frac{\mu_\varphi(A)}{\lambda^{-n}S_n(x)} \leq C\}$. There is K such that, for all $\epsilon > 0$, there exists $D(\epsilon)$ and N_ϵ such that, for $n > N_\epsilon$:*

$$\mu_\varphi(\bigcup_{A\in B(n,D(\epsilon))} A) \geq 1 - K\epsilon$$

Proof : Let $\rho = \lambda e^{-\sup\varphi}$. We use the hypothesis $\sup\varphi < p(\varphi)$ to state that $\rho > 1$.(recall that $\lambda = e^{p(\varphi)}$)
Let $\epsilon > 0$ and $k_0(\epsilon)$ such that:

$$k \geq k_0(\epsilon) \Rightarrow \frac{1}{\rho^k} < \frac{\epsilon}{k^2}$$

Let $n > k_0(\epsilon)$, according to the previous lemma, if A$\in \mathcal{P}^n$ and if, for all $k \leq n - k_0(\epsilon)$, $T^k(A)$ has no singularity of T in its boundary, then $A \in B(n, D(\epsilon))$; (with $D(\epsilon) = c(k_0(\epsilon))$). We show that the measure of this set is close to one by considering its complement:
Let $F(n,\epsilon) = \{A \in \mathcal{P}^n, \exists k \leq n - k_0, T^k(A)$ *has a singularity of* T *in its boundary* $\}$
Let $A \in F(n,\epsilon)$ and x in A: there exists $k \in [k_0, n]$ such that $T^{n-k}(A)$ has one singularity s of T in its boundary; we get then:

$$\nu([T^{n-k}(x), s]) \leq \nu(T^{n-k}(A))$$

But ν is a $\lambda e^{-\varphi}$ conformal measure so we get

$$1 \geq \nu(T^n(A)) = \int_{T^{n-1}(A)} \lambda e^{-\varphi} d\nu \geq \lambda e^{-\sup\varphi}\nu(T^{n-1}(A)) > \rho^k \nu(T^{n-k}(A))$$

hence $\nu(T^{n-k}(A)) \leq \frac{1}{\rho^k}$ and:

$$\nu([T^{n-k}(x), s]) < \frac{\epsilon}{k^2}$$

$\nu(\{s\}) = 0$ and the conformal measure ν is regular and has no atom, therefore, there exists a union of intervals V_k such that each singularity s is a bound of an interval and $\nu(V_k) = \frac{\epsilon}{k^2}$. Since the density h_φ is bounded, we obtain: $\mu_\varphi(V_k) \leq K\frac{\epsilon}{k^2}$ and, using the invariance by T of μ_φ:

$$\begin{aligned}\mu_\varphi(\bigcup_{A\in F(n,\epsilon)} A) &\leq \mu_\varphi(\bigcup_{k=k_0}^{n} T^{k-n}(V_k)) \\ &\leq \sum_{k=k_0}^{n} \mu_\varphi(V_k) \\ &\leq K\epsilon\end{aligned}$$

□

4 Return times and entrance times.

In this part, we show that, in some sense, the asymptotic law of R_n can be written as a sum of entrance times laws with fluctuating rates (these rates are the mass of the cylinders).

Definition 4.1 *A n-cylinder A is said k-recurrent (for $n > k$) if*

$$\forall\ l < k-1, A \cap T^{-l}(A) = \emptyset \text{ and } A \cap T^{-k+1}(A) \neq \emptyset$$

E_k is the set of the k-recurrent cylinders and $E_{<k}$ the set of the cylinders which recur before k.

Property 4.1 *If $k < n$:*

$$\#(E_k) \leq b^{k-1} \quad \text{and} \quad \#(E_{<k}) \leq b^k$$

Proof of the property:
If $A = A_{i_1}^{i_n} \in E_k$, there exists x in A such that $T^{k-1}(x)$ is in A.

$$x \in A \text{ and so } x \in A_{i_1}, T(x) \in A_{i_2}, \dots, T^{n-1}(x) \in A_{i_n}$$

$$T^{k-1}(x) \in A \text{ and so } T^{k-1}(x) \in A_{i_1}, \dots, T^{n+k-2}(x) \in A_{i_n}$$

Hence: $A_{i_k} = A_{i_1}, \dots, A_{i_n} = A_{i_{n-k+1}}$. For A we only have the choice for $A_{i_1}, \dots, A_{i_{k-1}}$ and $\#(E_k) \leq b^{k-1}$. Finally

$$\#(E_{<k}) \leq \sum_{i=1}^{k} \#(E_k) \leq b^k$$

Lemma 4.1 *Let (t_n) be a sequence such that $\lim_{n\to\infty} \frac{t_n}{n} = +\infty$, then:*

$$\lim_{n\to\infty} |\mu_\varphi\{R_n > t_n\} - \sum_{A\in\mathcal{P}^n} \mu_\varphi(A)\mu_\varphi\{\tau_A > t_n\}| = 0. \tag{2}$$

Proof : Recall the definition of R_n:

$$R_n(x) = \inf\{k > 0, T^k(x) \in \mathcal{P}^n(x)\}.$$

For all $t > 0$ We have:

$$\mu_\varphi\{R_n > t\} = \sum_{A\in\mathcal{P}^n} \mu_\varphi\{A \cap \tau_A > t\}$$

For all r with $n < r < t$ we get :

$$|\mu_\varphi\{A\cap\tau_A > t\} - \mu_\varphi(A)\mu_\varphi\{\tau_A > t\}| \leq |\mu_\varphi\{A\cap\tau_A > t\} - \mu_\varphi\{A\cap T^{-s+1}(A^c), r < s \leq t\}| +$$

$$|\mu_\varphi\{A\cap T^{-s+1}(A^c), r < s \leq t\} - \mu_\varphi(A)\mu_\varphi\{T^{-s+1}(A^c), r < s \leq t\}| +$$

$$\mu_\varphi(A)|\mu_\varphi\{T^{-s+1}(A^c), r < s \leq t\} - \mu_\varphi\{\tau_A > t\}|.$$

Bound for the third term:
Using the inclusion

$$\left(\bigcap_{r<s\leq t} T^{-s+1}A^c\right) \setminus \left(\bigcap_{1\leq s\leq t} T^{-s+1}A^c\right) \subset \left(\bigcup_{1\leq s\leq r} T^{-s+1}A\right)$$

it comes:

$$|\mu_\varphi\{T^{-s+1}(A^c), r < s \le t\} - \mu_\varphi\{T^{-s+1}(A^c), 1 \le s \le t\}| \le \mu_\varphi\{\cup_{1\le s\le r} T^{-s+1}(A)\} \le r\mu_\varphi(A)$$

so an upper bound for the third term is: $r\mu_\varphi(A)^2$. For the second one, the mixing inequality (see Th. 2.4) gives the following bound: $3Ke^{-\gamma r}$. As for the first one, we get the estimate:

$$\sum_{i=1}^{r} \mu_\varphi\{A \cap T^{-i+1}(A)\}$$

It remains to sum over all n-cylinders. For the third term, we get:

$$\sum_{A\in\mathcal{P}^n} r\mu_\varphi(A)^2 \le rCe^{-\theta n} \sum_{A\in\mathcal{P}^n} \mu_\varphi(A) \le rCe^{-\theta n}$$

For the second one, we get (since $\mathrm{card}(\mathcal{P}^n) \le b^n$):

$$\sum_{A\in\mathcal{P}^n} 3Ke^{-\gamma r} \le 3Ke^{nLog(b)-r\gamma}$$

A good choice of r will give the convergence to zero. For the first term, we must set apart the cylinders which recur too fast:
If $A \in E^c_{<k}$ then $\mu_\varphi\{A \cap T^{-i+1}(A)\} \le \mu_\varphi(A) \le Ce^{-\theta n}$ and

$$\sum_{A\in E^c_{<k}} \sum_{i=1}^{r} \mu_\varphi\{A \cap T^{-i+1}(A)\} \le \sum_{A\in E^c_{<k}} rCe^{-\theta n} \le rCe^{-\theta n + kLog(b)}$$

Besides, if $A \in E^c_{<k}$, $\forall i < k : \mu_\varphi\{A \cap T^{-i+1}(A)\} = 0$ and

$$\sum_{A\in E_{<k}} \sum_{i=1}^{r} \mu_\varphi\{A \cap T^{-i+1}(A)\} \le \sum_{A\in E_{<k}} \sum_{i=k}^{r} \mu_\varphi\{A \cap T^{-i+1}(A)\}$$

And if $i \ge k$, the mixing property yields to:

$$\mu_\varphi\{A \cap T^{-i+1}(A)\} \le (3Ke^{-\gamma k} + Ce^{-\theta n})\mu_\varphi(A)$$

$$\sum_{i=k}^{r} \mu_\varphi\{A \cap T^{-i+1}(A)\} \le r(3Ke^{-\gamma k} + Ce^{-\theta n})\mu_\varphi(A)$$

$$\sum_{A\in E^c_{<k}} \sum_{i=k}^{r} \mu_\varphi\{A \cap T^{-i+1}(A)\} \le r(3Ke^{-\gamma k} + Ce^{-\theta n})$$

Now we choose $r = min(n^2, \sqrt{nt_n})$ and $k = [\frac{\theta n}{\log(b)}]$ (we only have to change θ to ensure $k < n$) which gives us the convergence of all terms to zero. □

5 Approximation of the law of the entrance time in a cylinder by an exponential law.

This rather technical part is devoted to the control of the law of the entrance times in a cylinder. As it was pointed out in the previous part, this control is needed to estimate the asymptotic law of the return times.
Here the following theorem is proved:

Theorem 5.1 *For all $\epsilon > 0$, there exists N_ϵ such that, for all $n > N_\epsilon$ there exists $H_{n,\epsilon} \subset \mathcal{P}^n$ with:*

$$\mu_\varphi \left(\bigcup_{A \in H_{n,\epsilon}} A \right) > 1 - K\epsilon$$

There exists two strictly positive constants β and K such that, for all n-cylinder $A \in H_{n,\epsilon}$:

$$\sup_{t>0} \left| \mu_\varphi \left\{ \tau_A > \frac{t}{\mu_\varphi(A)} \right\} - e^{-t} \right| \leq K e^{-\beta n} \ .$$

In order to prove this theorem, we use the method of Galves and Schmitt ([5]).

Lemma 5.2 *For all $t > 0$, we have, if A is measurable:*

$$\mu_\varphi \left\{ \tau_A \leq \frac{t}{\mu_\varphi(A)} \right\} \leq t + \mu_\varphi(A)$$

The proof is in [5] (lemma 2). For all k and m positive real numbers, let:

$$X_k = \sum_{l=0}^{[k]} \chi_A \circ T^l \quad X_{[k,m]} = X_{[m]} - X_{[k]}$$

We have: $\{\tau_A \leq k\} = \{X_k \geq 1\}$.

Lemma 5.3 *There exists γ_0 such that, for all ϵ, there exists N_ϵ such that, for all $n > N_\epsilon$ there exists $I_{n,\epsilon} \subset \mathcal{P}^n$ such that, for all $A \in I_{n,\epsilon}$*

$$\mu_\varphi \left\{ \tau_A \leq \frac{t}{\mu_\varphi(A)} \right\} \geq \frac{t^2}{t^2 + \mu_\varphi(A)(1+t) + t(1+Ke^{-n\gamma_0})}$$

Moreover,

$$\mu_\varphi \left(\bigcup_{A \in I_{n,\epsilon}} A \right) > 1 - \epsilon$$

Proof : Let $X = X_{[\frac{t}{\mu_\varphi(A)}]}$. Using the Schwarz inequality, we get:

$$E(X)^2 \leq E(X^2) \mu_\varphi(X \geq 1)$$

and $E(X)^2 \geq t^2$. Moreover,

$$E(X^2) = \sum_{l=0}^{[\frac{t}{\mu_\varphi(A)}]} E(\chi_A \circ T^l) + 2 \sum_{l=1}^{[\frac{t}{\mu_\varphi(A)}]} \left([\frac{t}{\mu_\varphi(A)}] - l + 1 \right) \mu_\varphi\{A \cap T^{-l}(A)\}$$

The first term is $E(X) \leq t + \mu_\varphi(A)$. We bound the second for cylinders which don't recur too fast; For $A \in E^c_{<[ns]}$ (where s is positive) we get:

$$\sum_{l=1}^{[\frac{t}{\mu_\varphi(A)}]} \left([\frac{t}{\mu_\varphi(A)}] - l - 1\right) \mu_\varphi\{A \cap T^{-l}(A)\} = \sum_{l=[ns]}^{[\frac{t}{\mu_\varphi(A)}]} \left([\frac{t}{\mu_\varphi(A)}] - l - 1\right) \mu_\varphi\{A \cap T^{-l}(A)\}$$

the mixing property gives for this term:

$$\begin{aligned}
&\sum_{l=[ns]}^{[\frac{t}{\mu_\varphi(A)}]} \left([\frac{t}{\mu_\varphi(A)}] - l + 1\right) \left[K e^{-\gamma l}(2 + \mu_\varphi(A))\mu_\varphi(A) + \mu_\varphi(A)^2\right] \\
&\qquad \leq \mu_\varphi(A)^2 \sum_{l=[ns]}^{[\frac{t}{\mu_\varphi(A)}]} \left([\frac{t}{\mu_\varphi(A)}] - l + 1\right) + K\mu_\varphi(A) \sum_{l=[ns]}^{[\frac{t}{\mu_\varphi(A)}]} \left([\frac{t}{\mu_\varphi(A)}] - l + 1\right) e^{-\gamma l} \\
&\qquad \leq \mu_\varphi(A)^2 (\frac{t}{\mu_\varphi(A)})(\frac{t}{\mu_\varphi(A)} + 1) + K\mu_\varphi(A)(\frac{t}{\mu_\varphi(A)}) \sum_{l=[ns]}^{[\frac{t}{\mu_\varphi(A)}]} e^{-\gamma l} \\
&\qquad \leq t(t + \mu_\varphi(A)) + Kte^{-ns\gamma}
\end{aligned}$$

We choose now $s = \frac{\theta}{2logb}$ (where θ is given by lemma (3.1)) so that, for n big enough:

$$\mu_\varphi\left(\bigcup_{A \in E_{<[ns]}} A\right) \leq Cb^{ns}e^{-\theta n} \leq Ce^{-n\frac{\theta}{2}} < \epsilon$$

We take $I_{n,\epsilon} = E^c_{<[ns]}$. □

Let $g_A(t) = \mu_\varphi\left\{\tau_A > \frac{t}{\mu_\varphi(A)}\right\} = \mu_\varphi\{X = 0\}$.

Independence property We need to show that $g_A(t)$ is close to e^{-t}; for that, we show that this function satisfies some kind of independence property. We will first show that $g_A(t)$ is close to e^{-t} when t is equal to some power of $\mu_\varphi(A)$; then, given $t > 0$, we will divide it by this power of $\mu_\varphi(A)$.

Recall that we denote by K any constant independant of n and of the cylinders.

Lemma 5.4 *For n big enough and for all n-cylinder A :*

$$\sup_{s \geq \sqrt{\mu_\varphi(A)}} |g_A(\sqrt{\mu_\varphi(A)} + s) - g_A(\sqrt{\mu_\varphi(A)})g_A(s)| \leq K\mu_\varphi(A)^{\frac{3}{4}}$$

Proof : We must estimate $|g_A(t+s) - g_A(t)g_A(s)|$. To begin with, we dig a hole Δ between $[0, \frac{t}{\mu_\varphi(A)}]$ and $[\frac{t}{\mu_\varphi(A)}, \frac{t+s}{\mu_\varphi(A)}]$. This hole, thanks to the mixing inequality, will enable us to express the probability of not being in A during the time $[0, \frac{t}{\mu_\varphi(A)}] \cup [\frac{t+\Delta}{\mu_\varphi(A)}, \frac{t+s}{\mu_\varphi(A)}]$ in terms of the product of the probability of not being in A during each of the intervals $[0, \frac{t}{\mu_\varphi(A)}]$ and $[\frac{t+\Delta}{\mu_\varphi(A)}, \frac{t+s}{\mu_\varphi(A)}]$.

$$\begin{aligned}
|g_A(t+s) - g_A(t)g_A(s)| \;&\leq\; |g_A(t+s) - \mu_\varphi\{X_{[\frac{t}{\mu_\varphi(A)}]} + X_{[\frac{t}{\mu_\varphi(A)}+\Delta, \frac{t+s}{\mu_\varphi(A)}]} = 0\}| \\
&+\; |\mu_\varphi\{X_{[\frac{t}{\mu_\varphi(A)}]} + X_{[\frac{t}{\mu_\varphi(A)}+\Delta, \frac{t+s}{\mu_\varphi(A)}]} = 0\} - g_A(t)\mu_\varphi\{X_{[\Delta, \frac{s}{\mu_\varphi(A)}]} = 0\}| \\
&+\; |g_A(t)||\mu_\varphi\{X_{[\Delta, \frac{s}{\mu_\varphi(A)}]} = 0\} - g_A(s)|
\end{aligned}$$

Bounds for the first term:

$$\begin{aligned}|g_A(t+s) - \mu_\varphi\{X_{[\frac{t}{\mu_\varphi(A)}]} + X_{[\frac{t}{\mu_\varphi(A)}+\Delta, \frac{t+s}{\mu_\varphi(A)}]} = 0\}| &= \mu_\varphi\{X_{[\frac{t}{\mu_\varphi(A)}, \frac{t}{\mu_\varphi(A)}+\Delta]} > 0\} \\ &= \mu_\varphi\{X_{\Delta-1} > 0\} \leq \Delta\mu_\varphi(A) \qquad (3)\end{aligned}$$

because of the T-invariance. For the third term as well:

$$|\mu_\varphi\{X_{[\Delta, \frac{s}{\mu_\varphi(A)}]} = 0\} - g_A(s)| \leq \Delta\mu_\varphi(A) \qquad (4)$$

For the second term we use the mixing inequality and we denote:

$$P_A(f) = P_\varphi(f\mathbf{1}_A)$$

Let us renormalize P_φ with:

$$\mathcal{L}_\varphi = \frac{P_\varphi}{\lambda} \quad , \quad \mathcal{L}_A = \frac{P_A}{\lambda}$$

$$\begin{aligned}
&|\mu_\varphi\{X_{[\frac{t}{\mu_\varphi(A)}]} + X_{[\frac{t}{\mu_\varphi(A)}+\Delta, \frac{t+s}{\mu_\varphi(A)}]} = 0\} - g_A(t)\mu_\varphi\{X_{[\Delta, \frac{s}{\mu_\varphi(A)}]} = 0\}| \\
&= \left| \int \prod_{0}^{[\frac{t}{\mu_\varphi(A)}]} \mathbf{1}_{A^c} \circ T^i \prod_{[\frac{t}{\mu_\varphi(A)}]+\Delta+1}^{[\frac{t+s}{\mu_\varphi(A)}]} \mathbf{1}_{A^c} \circ T^i h_\varphi d\nu - \int \prod_{0}^{[\frac{t}{\mu_\varphi(A)}]} \mathbf{1}_{A^c} \circ T^i h_\varphi d\nu \int \prod_{\Delta+1}^{[\frac{s}{\mu_\varphi(A)}]} \mathbf{1}_{A^c} \circ T^i \, d\mu_\varphi \right| \\
&= \left| \int \mathbf{1}_{A^c} \mathcal{L}_{A^c}^{[\frac{t}{\mu_\varphi(A)}]}(h_\varphi) \prod_{\Delta+1}^{[\frac{s}{\mu_\varphi(A)}]} \mathbf{1}_{A^c} \circ T^i d\nu - \int \mathbf{1}_{A^c} \mathcal{L}_{A^c}^{[\frac{t}{\mu_\varphi(A)}]}(h_\varphi) d\nu \int \prod_{0}^{[\frac{s}{\mu_\varphi(A)}]-\Delta-1} \mathbf{1}_{A^c} \circ T^i \, d\mu_\varphi \right| \\
&= \left| \int \mathbf{1}_{A^c} \frac{\mathcal{L}_{A^c}^{[\frac{t}{\mu_\varphi(A)}]}(h_\varphi)}{h_\varphi} \left(\prod_{0}^{[\frac{s}{\mu_\varphi(A)}]-\Delta-1} \mathbf{1}_{A^c} \circ T^i \right) \circ T^{\Delta+1} \, d\mu_\varphi \right. \\
&\qquad \left. - \int \mathbf{1}_{A^c} \frac{\mathcal{L}_{A^c}^{[\frac{t}{\mu_\varphi(A)}]}(h_\varphi)}{h_\varphi} \, d\mu_\varphi \int \prod_{0}^{[\frac{s}{\mu_\varphi(A)}]-\Delta-1} \mathbf{1}_{A^c} \circ T^i \, d\mu_\varphi \right| \\
&\leq Ke^{-\gamma(\Delta+1)} \left[\int \mathbf{1}_{A^c} \mathcal{L}_{A^c}^{[\frac{t}{\mu_\varphi(A)}]}(h_\varphi) \, d\nu + var\left(\mathbf{1}_{A^c} \frac{\mathcal{L}_{A^c}^{[\frac{t}{\mu_\varphi(A)}]}(h_\varphi)}{h_\varphi} \right) \right] \int \prod_{0}^{[\frac{s}{\mu_\varphi(A)}]-\Delta-1} \mathbf{1}_{A^c} \circ T^i \, d\mu_\varphi \\
&\leq Ke^{-\gamma(\Delta+1)} \left[\int \mathbf{1}_{A^c} \mathcal{L}_{A^c}^{[\frac{t}{\mu_\varphi(A)}]}(h_\varphi) \, d\nu + var(\mathbf{1}_{A^c} \mathcal{L}_{A^c}^{[\frac{t}{\mu_\varphi(A)}]}(h_\varphi)) \| \frac{1}{h_\varphi} \|_\infty \right. \\
&\qquad \left. + var(\frac{1}{h_\varphi}) \| \mathbf{1}_{A^c} \mathcal{L}_{A^c}^{[\frac{t}{\mu_\varphi(A)}]}(h_\varphi) \|_\infty \right] \qquad (5)
\end{aligned}$$

In (5), we have used the following property of the variation:

$$var(fg) \leq \|f\|_\infty var(g) + \|g\|_\infty var(f).$$

Now, h_φ has bounded variation and $\inf(h_\varphi) > 0$. This implies: $\frac{1}{h_\varphi}$ has bounded variation. Moreover, $\|\mathbf{1}_{A^c} \mathcal{L}_{A^c}^{[\frac{t}{\mu_\varphi(A)}]}(h_\varphi)\|_\infty \leq K$ because $\mathcal{L}_{A^c}$ and $f \mapsto f\mathbf{1}_{A^c}$ are operators with norm less than one.

$$\begin{aligned}
(5) \quad \leq \quad & Ke^{-\gamma(\Delta+1)}\left[\int \mathbf{1}_{A^c}\mathcal{L}_{A^c}^{[\frac{t}{\mu_\varphi(A)}]}(h_\varphi)\ d\nu + K + K\ var(\mathcal{L}_{A^c}^{[\frac{t}{\mu_\varphi(A)}]}(h_\varphi))\right] \\
\leq \quad & Ke^{-\gamma(\Delta+1)}\left[g_A(t) + K + K\ var(\mathcal{L}_{A^c}^{[\frac{t}{\mu_\varphi(A)}]}(h_\varphi))\right] \\
\leq \quad & Ke^{-\gamma(\Delta+1)}\left[K + K\ var(\mathcal{L}_{A^c}^{[\frac{t}{\mu_\varphi(A)}]}(h_\varphi))\right] \qquad (6)
\end{aligned}$$

Where we have used again:

$$var(fg) \leq \|f\|_\infty var(g) + \|g\|_\infty var(f).$$

We must estimate $var(\mathcal{L}_{A^c}^{[\frac{t}{\mu_\varphi(A)}]}(h_\varphi))$, for that, we use the fact that $\mathcal{L}_\varphi = \mathcal{L}_A + \mathcal{L}_{A^c}$.

$$\mathcal{L}_{A^c}^N = (\mathcal{L}_\varphi - \mathcal{L}_A)^N = \mathcal{L}_\varphi^N - \sum_{r=0}^{N-1} \mathcal{L}_\varphi^r \mathcal{L}_A \mathcal{L}_\varphi^{N-r-1} + \sum_{0\leq i+j\leq N-2} \mathcal{L}_\varphi^i \mathcal{L}_A \mathcal{L}_{A^c}^{N-i-j-2} \mathcal{L}_A \mathcal{L}_\varphi^j$$

Since $\mathcal{L}_\varphi(h_\varphi) = h_\varphi$, we get:

$$\mathcal{L}_{A^c}^N(h_\varphi) = h_\varphi - \sum_{r=0}^{N-1} \mathcal{L}_\varphi^r \mathcal{L}_A(h_\varphi) + \sum_{0\leq i+j\leq N-2} \mathcal{L}_\varphi^i \mathcal{L}_A \mathcal{L}_{A^c}^{N-i-j-2} \mathcal{L}_A \mathcal{L}_\varphi^j(h_\varphi)$$

A computation gives:

$$\mathcal{L}_\varphi^i \mathcal{L}_A \mathcal{L}_{A^c}^{N-i-j-2} \mathcal{L}_A \mathcal{L}_\varphi^j = \mathcal{L}_\varphi^i \mathcal{L}_{B_{i,j}} \mathcal{L}_\varphi^{N-i-1}$$

with $B_{i,j} = A \cap T^{-1}(A^c) \cap T^{-2}(A^c) \cap .. \cap T^{-(N-i-j-2)}(A^c) \cap T^{-(N-i-j-1)}(A)$. Assume that A is a n-cylinder with $n > N$ and let $k \leq N$: A is completely included in an interval where T^n is monotone. Besides, $T^{-k}(A)$ is made with at most b^k intervals and each of them is included in an interval where T^k is monotone. As a consequence, $A \cap T^{-k}(A)$ is either empty, or an interval, or the union of two intervals (when two branches of T^k with opposite slope meet in a single point). Moreover, as $k \leq N$, $T^{-k}(A^c)$ either countains A or is disjoint from A. That is why $B_{i,j}$ is either an interval (empty or not) or the union of two intervals, therefore:

$$\mathcal{L}_{A^c}^N(h_\varphi) = h_\varphi - \sum_{r=0}^{N-1} \mathcal{L}_\varphi^r \mathcal{L}_A(h_\varphi) + \sum_{0\leq i+j\leq N-2} \mathcal{L}_\varphi^i \mathcal{L}_{B_{i,j}}(h_\varphi) \qquad (7)$$

We shall estimate the variation of each term.
One the one hand, if A is an interval or the union of two intervals, we apply the Lasota-Yorke inequality to the function $\mathbf{1}_A h_\varphi$ to get:

$$\begin{aligned}
var\mathcal{L}_A(h_\varphi) &= var\mathcal{L}_\varphi(\mathbf{1}_A h_\varphi) \leq \alpha\ var(\mathbf{1}_A h_\varphi) + \xi\nu(\mathbf{1}_A h_\varphi) \\
&\leq \alpha(var(h_\varphi) + var(\mathbf{1}_A)\|h_\varphi\|_\infty) + \xi\nu(A)\|h_\varphi\|_\infty \\
&\leq \alpha\ var(h_\varphi) + (4\alpha + \xi\nu(A))\|h_\varphi\|_\infty \qquad (8)
\end{aligned}$$

On the other hand, iterating the Lasota-Yorke inequality and using the conformality of ν gives: (for f with bounded variation)

$$var\mathcal{L}_\varphi^N(h_\varphi) \leq \alpha^N var(h_\varphi) + K\nu(h_\varphi) \qquad (9)$$

grouping (8) and (9), we have:

$$\begin{aligned} var(\mathcal{L}_\varphi^r \mathcal{L}_A(h_\varphi)) &\le \alpha^r var(\mathcal{L}_A(h_\varphi)) + K\nu(\mathcal{L}_A(h_\varphi)) \\ &\le \alpha^{r+1} var(h_\varphi) + \alpha^r(4\alpha + \xi\nu(A))\|h_\varphi\|_\infty + K\nu(A)\|h_\varphi\|_\infty \end{aligned} \tag{10}$$

Since $B_{i,j}$ is either an interval or the union of two intervals (and it is included in A), we can apply (10):

$$\begin{aligned} var(\mathcal{L}_\varphi^i \mathcal{L}_{B_{i,j}}(h_\varphi)) &\le \alpha^{i+1} var(h_\varphi) + \alpha^i(4\alpha + \xi\nu(B_{i,j}))\|h_\varphi\|_\infty + K\nu(B_{i,j})\|h_\varphi\|_\infty \\ &\le \alpha^{i+1} var(h_\varphi) + \alpha^i(4\alpha + \xi\nu(A))\|h_\varphi\|_\infty + K\nu(A)\|h_\varphi\|_\infty \end{aligned}$$

As $\alpha < 1$, we can write:

$$\begin{aligned} &var(\mathcal{L}_\varphi^r \mathcal{L}_A(h_\varphi)) \le K + K\nu(A) \le K \\ &var(\mathcal{L}_\varphi^i \mathcal{L}_{B_{i,j}}(h_\varphi)) \le K \end{aligned}$$

Let us now sum over r, i and j by using the relation: $\sum_{0\le i+j\le N-2} 1 = \frac{N(N-1)}{2} \le N^2$:

$$\sum_{r=0}^{N-1} var(\mathcal{L}_\varphi^r \mathcal{L}_A(h_\varphi)) \le KN$$

$$\sum_{0\le i+j\le N-2} var(\mathcal{L}_\varphi^i \mathcal{L}_{B_{i,j}}(h_\varphi)) \le KN^2$$

and according to the relation (7), we obtain, for N big enough:

$$var(\mathcal{L}_{A^c}^N(h_\varphi)) \le K + KN + KN^2 \le KN^2$$

Combining (3), (4) and (6), we get (with $N = [\frac{t}{\mu_\varphi(A)}]$):

$$|g_A(t+s) - g_A(t)g_A(s)| \le K\left(\frac{t}{\mu_\varphi(A)}\right)^2 e^{-\gamma(\Delta+1)} + 2\Delta\mu_\varphi(A)$$

if $\frac{t}{\mu_\varphi(A)}$ is big enough. Now we choose the size of the hole Δ: the only requierement is $\Delta < \frac{s}{\mu_\varphi(A)}$. Take $\Delta = \frac{1}{\mu_\varphi(A)^{1/4}}$, $s \ge \sqrt{\mu_\varphi(A)}$ and $t = \sqrt{\mu_\varphi(A)}$:

$$\sup_{s\ge\sqrt{\mu_\varphi(A)}} |g_A(\sqrt{\mu_\varphi(A)} + s) - g_A(\sqrt{\mu_\varphi(A)})g_A(s)| \le K\frac{1}{\mu_\varphi(A)} e^{-\gamma(\frac{1}{\mu_\varphi(A)^{1/4}}+1)} + 2\mu_\varphi(A)^{3/4}$$

if n is big enough: $\frac{1}{\mu_\varphi(A)} e^{-\gamma(\frac{1}{2\sqrt{\mu_\varphi(A)}}+1)} \le \mu_\varphi(A)^{3/4}$ therefore

$$\sup_{s\ge\sqrt{\mu_\varphi(A)}} |g_A(\sqrt{\mu_\varphi(A)} + s) - g_A(\sqrt{\mu_\varphi(A)})g_A(s)| \le K\mu_\varphi(A)^{3/4}$$

□

Define $r = r(A) = \sqrt{\mu_\varphi(A)}$ and $\theta = \theta(A) = -log\ g_A(r)$.

Lemma 5.5 *For n big enough and for all n-cylinder A :*

$$|g_A(kr(A)) - e^{k\theta(A)}| \leq \frac{K\mu_\varphi(A)^{3/4}}{1-e^{-\theta(A)}}$$

Proof : See [G.S](lemma 6). □

Lemma 5.6 *There exists γ_1 and γ_2 such that, for all $\epsilon > 0$, there exists N_ϵ such that, for all $n > N_\epsilon$, for all $A \in I_{n,\epsilon}$ (where $I_{n,\epsilon}$ is given by lemma (5.3)):*

$$1 - Ke^{-\gamma_1 n} \leq \frac{\theta(A)}{r(A)} \leq 1 + Ke^{-\gamma_2 n}$$

Proof : On the one hand, for $0 \leq u \leq \frac{1}{2}$, $-log(1-u) \leq u + u^2$. Now we get, by choosing n big enough and using lemma (5.2):

$$\begin{aligned}
\theta(A) &\leq \mu_\varphi \left\{ \tau_A \leq \frac{r(A)}{\mu_\varphi(A)} \right\} + \left(\mu_\varphi \left\{ \tau_A \leq \frac{r(A)}{\mu_\varphi(A)} \right\} \right)^2 \\
&\leq r(A) + \mu_\varphi(A) + (r(A) + \mu_\varphi(A))^2 \\
&\leq r(A)(1 + Ke^{-n\gamma_2})
\end{aligned}$$

since, by lemma (3.1), $\mu_\varphi(A) \leq Ce^{-n\theta}$. On the other hand, by lemma (5.3), if $A \in I_{n,\epsilon}$:

$$\theta(A) \geq 1 - e^{-\theta(A)} \geq \frac{r(A)^2}{r(A)^2 + \mu_\varphi(A)(1 + r(A)) + r(A)(1 + Ke^{-n\gamma_0})} \tag{11}$$

$$\frac{\theta(A)}{r(A)} \geq \frac{\sqrt{\mu_\varphi(A)}}{\mu_\varphi(A) + \mu_\varphi(A)(1 + \sqrt{\mu_\varphi(A)}) + \sqrt{\mu_\varphi(A)}(1 + Ke^{-n\gamma_0})} \geq \frac{1}{1 + Ke^{-n\gamma_1}} \geq 1 - Ke^{-n\gamma_1}$$

which concludes the proof. □

Proof of theorem (5.1):

Let $\epsilon > 0$. We only consider cylinders $A \in I_{n,\epsilon}$ and n big enough so as to use the previous lemmas. Let $t > 0$, $t = kr(A) + v$ with $k = [\frac{t}{r(A)}]$ and $0 \leq v < r(A)$:

$$|g_A(t) - e^{-t}| \leq |g_A(t) - g_A(kr(A))| + |g_A(kr(A)) - e^{-\theta(A)k}| + |e^{-\theta(A)k} - e^{-r(A)k}| + |e^{-r(A)k} - e^{-t}|$$

In the rest of the proof, we use the lemma (3.1) which says that the measure of the n-cylinders decrease exponentially fast. First term, by lemma (5.2) and (5.3):

$$|g_A(t) - g_A(kr(A))| = \mu_\varphi \left\{ \frac{kr(A)}{\mu_\varphi(A)} < \tau_A \leq \frac{t}{\mu_\varphi(A)} \right\} = \mu_\varphi \left\{ 0 < \tau_A \leq \frac{v}{\mu_\varphi(A)} \right\}$$

$$\leq \mu_\varphi(A)(1 + \frac{v}{\mu_\varphi(A)}) \leq 2r(A) \leq Ke^{-\beta n}$$

Second term: by lemma (5.5) : $|g_A(kr(A)) - e^{-\theta(A)k}| \leq K\frac{\mu_\varphi(A)^{\frac{3}{4}}}{1-e^{-\theta(A)}}$ and, taking the inverse in the inequality (11):

$$\frac{1}{1 - e^{-\theta(A)}} \leq 2 + \sqrt{\mu_\varphi(A)} + \frac{1}{\sqrt{\mu_\varphi(A)}}(1 + Ke^{-n\gamma_0})$$

$$K\frac{\mu_\varphi(A)^{\frac{3}{4}}}{1-e^{-\theta(A)}} \leq K\mu_\varphi(A)^{\frac{3}{4}} + K\mu_\varphi(A)^{\frac{1}{4}} \leq Ke^{-n\beta}$$

Fourth term:

$$|e^{-r(A)k} - e^{-t}| \leq v \leq \sqrt{\mu_\varphi(A)} \leq Ke^{-n\beta}$$

Third term: a computation shows that

$$|e^{-\theta(A)k} - e^{-r(A)k}| \leq 2k|\theta(A) - r(A)|(e^{-\theta(A)k} + e^{-r(A)k})$$

Lemma (5.6) ensures that

$$-Ke^{-n\gamma_1}r(A) \leq \theta(A) - r(A) \leq Ke^{-n\gamma_2}r(A)$$

$$|e^{-\theta(A)k} - e^{-r(A)k}| \leq Kr(A)ke^{-n\beta}(e^{-\theta(A)k} + e^{-r(A)k})$$

$$\leq Ke^{-n\beta}(r(A)ke^{-r(A)k} + \theta(A)ke^{-\theta(A)k}\frac{r(A)}{\theta(A)}) \leq Ke^{-n\beta}$$

because ue^{-u} and $\frac{r(A)}{\theta(A)}$ are bounded. This ends the proof.

6 Proof of the main theorem 1.1.

The mass of the cylinders, on the one hand, and the laws of the entrance times on the other hand, have a different influence on the sum (2). So, we have to determinate which of the two is the most important and will give the behaviour of the law of R_n.

We have to prove the convergence in law which means the following:

$$\lim_{n\to+\infty} \mu_\varphi\{R_n > e^{nh}e^{u\sigma(\varphi)\sqrt{n}}\} = \frac{1}{\sqrt{2\pi}}\int_u^\infty e^{-\frac{x^2}{2}}dx$$

Let $\varepsilon > 0$. Let us cut this quantity in several parts so as to use the lemma (5.1) and the approximation of the law of entrance times:

$$\mu_\varphi\{R_n > e^{nh}e^{u\sigma(\varphi)\sqrt{n}}\} = \mu_\varphi\{R_n > e^{nh}e^{u\sigma(\varphi)\sqrt{n}}\} - \sum_{A\in\mathcal{P}^n} \mu_\varphi(A)\mu_\varphi\left\{\tau_A > e^{nh}e^{u\sigma(\varphi)\sqrt{n}}\right\} \quad (12)$$

$$+ \sum_{A\in\mathcal{P}^n} \mu_\varphi(A)\mu_\varphi\left\{\tau_A > e^{nh}e^{u\sigma(\varphi)\sqrt{n}}\right\} - \sum_{A\in H_{n,\epsilon}\cap G_{n,\epsilon}} \mu_\varphi(A)\mu_\varphi\left\{\tau_A > e^{nh}e^{u\sigma(\varphi)\sqrt{n}}\right\} \quad (13)$$

$$+ \sum_{A\in H_{n,\epsilon}\cap G_{n,\epsilon}} \mu_\varphi(A)\mu_\varphi\left\{\tau_A > e^{nh}e^{u\sigma(\varphi)\sqrt{n}}\right\} - \sum_{A\in H_{n,\epsilon}\cap G_{n,\epsilon}} \mu_\varphi(A)e^{-\mu_\varphi(A)e^{nh}e^{u\sigma(\varphi)\sqrt{n}}} \quad (14)$$

$$+ \sum_{A\in H_{n,\epsilon}\cap G_{n,\epsilon}} \mu_\varphi(A)e^{-\mu_\varphi(A)e^{nh}e^{u\sigma(\varphi)\sqrt{n}}} \quad (15)$$

Thanks to the lemma (4.1), $\lim_{n\to+\infty}(12) = 0$.
By the lemma (3.3), there exist N_ϵ and $D(\varepsilon)$ such that for all $n > N_\epsilon$, for all $A \in B(n, D(\varepsilon))$ and all x in A : (we use the notation $B(n, D(\varepsilon)) = G_{n,\epsilon}$)

$$\frac{1}{D(\epsilon)} \leq \frac{\mu_\varphi(A)}{\lambda^{-n}S_n(x)} \leq D(\epsilon) \quad (16)$$

$$\mu_\varphi\left(\bigcup_{A\in G_{n,\epsilon}} A\right) \geq 1-K\epsilon$$

By the theorem (5.1),there exists N'_ϵ such that, for all $n > N'_\epsilon$, there exists $H_{n,\epsilon}\in\mathcal{P}^n$ such that, for all A in this set:

$$\sup_{t>0}\left|\mu_\varphi\left\{\tau_A > t\right\} - e^{-t\mu_\varphi(A)}\right| \leq Ke^{-\beta n}$$

$$\mu_\varphi\left(\bigcup_{A\in H_{n,\epsilon}} A\right) \geq 1-K\epsilon \tag{17}$$

If $n > max(N_\epsilon, N'_\epsilon)$:

$$\begin{aligned}|(13)| = \left|\sum_{A\in(H_{n,\epsilon}\cap G_{n,\epsilon})^c} \mu_\varphi(A)\mu_\varphi\left\{\tau_A > e^{nh}e^{u\sigma(\varphi)\sqrt{n}}\right\}\right| &\leq \left|\sum_{A\in(H_{n,\epsilon}\cap G_{n,\epsilon})^c}\mu_\varphi(A)\right| \\ &\leq \sum_{A\in G^c_{n,\epsilon}}\mu_\varphi(A) + \sum_{A\in H^c_{n,\epsilon}}\mu_\varphi(A) \\ &\leq K\epsilon\end{aligned}$$

As for the term (14), by the theorem (5.1), for all $\varepsilon > 0$:

$$\begin{aligned}|(14)| &\leq \sum_{A\in H_{n,\epsilon}\cap G_{n,\epsilon}}\mu_\varphi(A)\left|\mu_\varphi\left\{\tau_A > e^{nh}e^{u\sigma(\varphi)\sqrt{n}}\right\} - e^{-\mu_\varphi(A)e^{nh}e^{u\sigma(\varphi)\sqrt{n}}}\right| \\ &\leq Ke^{-\beta n}\sum_{A\in\mathcal{P}^n}\mu_\varphi(A) \leq Ke^{-\beta n}\end{aligned}$$

We now turn to the term (15), which we can write $\mu_\varphi(Y_{n,\varepsilon})$ if we call $Y_{n,\varepsilon}$ the random variable:

$$Y_{n,\varepsilon} = \sum_{A\in H_{n,\epsilon}\cap G_{n,\epsilon}} \mathbf{1}_A e^{-\mu_\varphi(A)e^{nh}e^{u\sigma(\varphi)\sqrt{n}}}$$

Let $\eta > 0$, the Markov inequality will give us some information about $\liminf\mu_\varphi(Y_{n,\varepsilon})$:

$$\mu_\varphi(Y_{n,\varepsilon}) \geq e^{-e^{-\eta\sqrt{n}}}\mu_\varphi\{(\log\ Y_{n,\varepsilon} \geq -e^{-\eta\sqrt{n}}) \cap (\bigcup_{G_{n,\varepsilon}\cap H_{n,\varepsilon}} A)\}$$

and by the lemma (3.3), we have the two following inclusions:

$$\left(\left(D(\epsilon)\lambda^{-n}S_n \leq \frac{e^{-\eta\sqrt{n}}}{e^{nh}e^{u\sigma(\varphi)\sqrt{n}}}\right)\cap(\bigcup_{G_{n,\varepsilon}\cap H_{n,\varepsilon}} A)\right) \subset \left((\log Y_{n,\varepsilon} \geq -e^{-\eta\sqrt{n}})\cap(\bigcup_{G_{n,\varepsilon}\cap H_{n,\varepsilon}} A)\right)$$

$$\left(\lambda^{-n}S_n \leq \frac{e^{-2\eta\sqrt{n}}}{e^{nh}e^{u\sigma(\varphi)\sqrt{n}}}\right) \subset \left(D(\epsilon)\lambda^{-n}S_n \leq \frac{e^{-\eta\sqrt{n}}}{e^{nh}e^{u\sigma(\varphi)\sqrt{n}}}\right)$$

for n big enough. Consequently, we get the inequalities:

$$\begin{aligned}\mu_\varphi(Y_{n,\varepsilon}) &\geq e^{-e^{-\eta\sqrt{n}}}\mu_\varphi\left(\left(\lambda^{-n}S_n \leq \frac{e^{-2\eta\sqrt{n}}}{e^{nh}e^{u\sigma(\varphi)\sqrt{n}}}\right)\cap(\bigcup_{G_{n,\varepsilon}\cap H_{n,\varepsilon}} A)\right) \\ &\geq e^{-e^{-\eta\sqrt{n}}}\mu_\varphi\left(\left(\frac{-\log\ S_n + n\log\lambda - nh}{\sigma(\varphi)\sqrt{n}} \geq u + \frac{2\eta}{\sigma(\varphi)}\right)\cap(\bigcup_{G_{n,\varepsilon}\cap H_{n,\varepsilon}} A)\right)\end{aligned}$$

and $p(\varphi) = \log \lambda = h + \mu_\varphi(\varphi)$ so

$$\begin{aligned}
e^{e^{-\eta\sqrt{n}}}\mu_\varphi(Y_{n,\varepsilon}) &\geq \mu_\varphi\left(\left(\frac{-\log\ S_n + n\mu_\varphi(\varphi)}{\sigma(\varphi)\sqrt{n}} \geq u + \frac{2\eta}{\sigma(\varphi)}\right) \cap (\bigcup_{G_{n,\varepsilon}\cap H_{n,\varepsilon}} A)\right) \\
&\geq \mu_\varphi\left(\frac{-\log\ S_n + n\mu_\varphi(\varphi)}{\sigma(\varphi)\sqrt{n}} \geq u + \frac{2\eta}{\sigma(\varphi)}\right) - \mu_\varphi(\bigcup_{(G_{n,\varepsilon}\cap H_{n,\varepsilon})^c} A) \\
&\geq \mu_\varphi\left(\frac{-\log\ S_n + n\mu_\varphi(\varphi)}{\sigma(\varphi)\sqrt{n}} \geq u + \frac{2\eta}{\sigma(\varphi)}\right) - \mu_\varphi(\bigcup_{G^c_{n,\varepsilon}} A) - \mu_\varphi(\bigcup_{H^c_{n,\varepsilon}} A) \\
&\geq \mu_\varphi\left(\frac{-\log\ S_n + n\mu_\varphi(\varphi)}{\sigma(\varphi)\sqrt{n}} \geq u + \frac{2\eta}{\sigma(\varphi)}\right) - K\epsilon
\end{aligned}$$

By applying the central-limit theorem to the system $(T, \mu_\varphi, \varphi)$, we obtain, letting first n go to infinity, then η to zero:

$$\liminf_{n\to\infty}\ \mu_\varphi(Y_{n,\varepsilon}) \geq \frac{1}{\sqrt{2\pi}}\int_u^\infty e^{-\frac{x^2}{2}}dx - K\varepsilon$$

For the lim sup, we use the inequality, for $\eta > 0$ (notice that $Y_{n,\varepsilon} \leq 1$):

$$\mu_\varphi(Y_{n,\varepsilon}) \leq e^{-e^{\eta\sqrt{n}}}\mu_\varphi\{(log\, Y_{n,\varepsilon} < -e^{\eta\sqrt{n}}) \cap (\bigcup_{G_{n,\varepsilon}\cap H_{n,\varepsilon}} A)\} + \mu_\varphi\{(log\, Y_{n,\varepsilon} \geq -e^{\eta\sqrt{n}}) \cap (\bigcup_{G_{n,\varepsilon}\cap H_{n,\varepsilon}} A)\}$$

Using the other inequality in the lemma (3.3), we get the following inclusions:

$$\left(\left(\frac{\lambda^{-n}S_n}{D(\epsilon)} \leq \frac{e^{\eta\sqrt{n}}}{e^{nh}e^{u\sigma(\varphi)\sqrt{n}}}\right) \cap (\bigcup_{G_{n,\varepsilon}\cap H_{n,\varepsilon}} A)\right) \supset \left((\log Y_{n,\varepsilon} \geq -e^{\eta\sqrt{n}}) \cap (\bigcup_{G_{n,\varepsilon}\cap H_{n,\varepsilon}} A)\right)$$

$$\left(\lambda^{-n}S_n \leq \frac{e^{2\eta\sqrt{n}}}{e^{nh}e^{u\sigma(\varphi)\sqrt{n}}}\right) \supset \left(\frac{\lambda^{-n}S_n}{D(\epsilon)} \leq \frac{e^{-\eta\sqrt{n}}}{e^{nh}e^{u\sigma(\varphi)\sqrt{n}}}\right)$$

for n big enough. Consequently, we get the inequalities:

$$\begin{aligned}
\mu_\varphi(Y_{n,\varepsilon}) &\leq e^{-e^{\eta\sqrt{n}}}\mu_\varphi\{(log\ Y_{n,\varepsilon} < -e^{\eta\sqrt{n}}) \cap (\bigcup_{G_{n,\varepsilon}\cap H_{n,\varepsilon}} A)\} \\
&\quad + \mu_\varphi\left(\left(\frac{-\log\ S_n + n\mu_\varphi(\varphi)}{\sigma(\varphi)\sqrt{n}} \geq u - \frac{2\eta}{\sigma(\varphi)}\right) \cap (\bigcup_{G_{n,\varepsilon}\cap H_{n,\varepsilon}} A)\right) \\
&\leq e^{-e^{\eta\sqrt{n}}} + \mu_\varphi\left(\frac{-\log\ S_n + n\mu_\varphi(\varphi)}{\sigma(\varphi)\sqrt{n}} \geq u - \frac{2\eta}{\sigma(\varphi)}\right)
\end{aligned}$$

Letting first n go to infinity, then η to zero:

$$\limsup_{n\to\infty}\ \mu_\varphi(Y_{n,\varepsilon}) \leq \frac{1}{\sqrt{2\pi}}\int_u^\infty e^{-\frac{x^2}{2}}dx$$

Gathering all the results about the terms (12), (14), (15), (16):

$$\liminf_{n\to\infty} \mu_\varphi\{R_n > e^{nh}e^{u\sigma(\varphi)\sqrt{n}}\} \geq \frac{1}{\sqrt{2\pi}}\int_u^\infty e^{-\frac{x^2}{2}}dx - K\varepsilon$$

$$\limsup_{n\to\infty} \mu_\varphi\{R_n > e^{nh} e^{u\sigma(\varphi)\sqrt{n}}\} \le \frac{1}{\sqrt{2\pi}} \int_u^\infty e^{-\frac{x^2}{2}} dx + K\varepsilon$$

This concludes the proof.

Acknowledgements: It is a pleasure to thank Pierre Collet, whose ideas and suggestions were very helpful for this article.

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