Conformal measures for multidimensional piecewise invertible maps

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Abstract

Given a piecewise invertible map $T : X \to X$ and a weight $g : X \to]0, \infty[$, a conformal measure ν is a probability measure on X such that, for all measurable $A \subset X$ with $T : A \to TA$ invertible,

$$\nu(TA) = \lambda \int_A \frac{1}{g} \, d\nu$$

with $\lambda > 0$. Such a measure is an essential tool for the study of equilibrium states. Assuming that the topological pressure of the boundary is small, that $\log g$ has bounded distortion and an irreducibility condition, we build such a conformal measure.

1 Introduction

Consider a weighted dynamical system, i.e., a self-map T of some space X together with a positively valued function $g : X \to]0, \infty[$. The purpose of thermodynamical formalism is to study *equilibrium states* and their properties. An equilibrium state is a T-invariant probability measure on X, characterized by the following variational principle:

$$P_g(\mu_e, T) = \sup_{\mu} P_g(\mu, T) \text{ where } P_g(\mu, T) = h(\mu, T) - \int_X \log g \, d\mu.$$

$P_g(\mu, T)$ is called the *pressure* and $h(\mu, T)$ is the entropy — see [4] for background on thermodynamical formalism.

An indication of the importance of this notion is that if X is a manifold, T differentiable and g is the inverse of the Jacobian of T w.r.t. Lebesgue measure, then (in many settings) the equilibrium states are exactly the *a.c.i.m.*'s, i.e., the absolutely continuous (w.r.t. Lebesgue measure) invariant probability measures.

For general weight g, the equilibrium states are singular w.r.t. Lebesgue measure. But it turns out that (in many setting), they are again characterized by their absolute continuity w.r.t. to a special, not invariant, probability measure, *the conformal measure*, i.e., a probability measure with Jacobian $1/g$ w.r.t. T. Such results for the setting considered in this paper were proved in [2], but the existence of a conformal measure was proved only in the piecewise affine case (and assumed in the other cases).

In this paper we *build such conformal measures* for general, multi-dimensional and non-Markov piecewise invertible maps, completing the study of thermodynamical formalism of such systems done in [2], [3]. Our assumptions are natural generalizations of those encountered in the one-dimensional case (see, e.g., [7] and the references therein) and they are satified in many higher dimensional cases, see below.

To state our results we need some definitions.

Definition 1.1 *A piecewise invertible dynamical system is a triple* $(X, \mathcal{Z}, T)$ *with:*

- X *a compact subset of* $\mathbb{R}^d$.
- $\mathcal{Z}$ *a finite collection of pairwise disjoint, open subsets of* $\mathbb{R}^d$ *such that* $Y := \bigcup_{Z\in\mathcal{Z}} Z$ *is dense in* X.
- $T : Y \to X$ *such that for each* $Z \in \mathcal{Z}$, $T_{|Z}$ *is the restriction of a homeomorphism* T_Z *between neighborhoods of* $\overline{Z}$ *and* $\overline{TZ}$.

A weight for $(X, \mathcal{Z}, T)$ *is a continuous function* $g : Y \to]0, \infty[$. *We write* $g_n(x) = g(x) \cdot g(Tx) \dots g(T^{n-1}x)$ *where this is well-defined, that is to say, on* $\cap_{k=0}^{n-1} T^{-k}Y$.

Let $\mathcal{Z}_n$ be the set of n-cylinders associated to the collection $\mathcal{Z}$, i.e, for $n \geq 1$:

$$\mathcal{Z}_n = \{A_0 \cap T^{-1}A_1 \cap \dots \cap T^{-(n-1)}A_{n-1} \neq \emptyset : A_0 \dots A_{n-1} \in \mathcal{Z}\}.$$

The n-cylinder $A_0 \cap T^{-1}A_1 \cap \dots \cap T^{n-1}A_{n-1}$ will be written $[A_0, \dots, A_{n-1}]$. ∂Z denotes the topological boundary of Z as a subset of $\mathbb{R}^d$. We will use the following notation:

$$\partial\mathcal{Z}_n = \bigcup_{Z\in\mathcal{Z}_n} \partial Z.$$

The *topological pressure* of $S \subset X$ (w.r.t. the weight g) is

$$P_g(S, T) = \limsup_{n\to\infty} \frac{1}{n} \log \sum_{\substack{Z\in\mathcal{Z}_n \\ \bar{Z}\cap S\neq\emptyset}} \sup_Z g_n.$$

A *conformal measure* is a probability measure ν on X such that, for some number $\lambda > 0$,

$$\frac{d\nu \circ T}{d\nu} = \lambda \frac{1}{g} \ \nu\text{-a.e.}$$

Observe that this is equivalent to: $\nu(TA) = \lambda \int_A g^{-1}\, d\nu$ for all measurable A on which T is injective. It is also equivalent to requiring ν to be an eigenfunctional of $\mathcal{L}^*$ for a positive eigenvalue (see section 2 for definitions).

Main Theorem *Let* $(X, \mathcal{Z}, T)$ *be a piecewise invertible system with weight* g. *Assume that:*

(a) $P_g(\partial\mathcal{Z}, T) < P_g(X, T)$.

(b) g *has bounded distortion under* T *in the following sense:*

$$\lim_{m\to\infty} \limsup_{n\to\infty} \max_{Z\in\mathcal{Z}_{n+m}} \sup_{x,y\in Z} \log \frac{g_n(x)}{g_n(y)} = 0.$$

(c) *$\mathcal{Z}$ generates, i.e.,* $\lim_{n\to\infty}\sup_{Z\in\mathcal{Z}_n}\ \mathrm{diam}(Z)=0$.

(d) $T(Y)\supset Y$ *and* $\inf_Y g>0$.

(e) *each* $Z\in\mathcal{Z}_n$, $n\geq 1$, *has finitely many connected components.*

(f) *for every non-empty open set* $U\subset X$, *there is* $n<\infty$ *such that* $T^n(U\cap Y)\supset Y$.

Then there exists a conformal measure ν. *Moreover the associated eigenvalue is* $\lambda=e^{P_g(X,T)}$.

Remark 1.1

- *The conditions above are satisfied for instance, if* T *is piecewise* C^1, *expanding and generic, in the sense that (see [2, Prop. 5.2]):*

$$h_{\mathrm{mult}}(\mathcal{Z},T)=0$$

  and if g *has a summable modulus of continuity (a weaker condition than Hölder continuity) with*

$$\frac{\sup g}{\inf g}<\Lambda_-\left(\frac{\Lambda_-}{\Lambda_+}\right)^{d-1},$$

  Λ_+ *and* Λ_- *being the maximum and the minimum of the rate of expansion.*

- *Condition* (d) *could be replaced by the (very slightly) weaker condition:*

$$\inf_{y\in Y}\sum_{x\in T^{-1}y}g(y)>0.$$

- *Condition* (b) *is used only in Section 3 to prove that:*

  - $\prod_{k\geq 1}\frac{g(y_k)}{g(x_k)}$ *converges for every sequences* $(x_k),(y_k)$ *such that* $Tx_{k+1}=x_k$, $Ty_{k+1}=y_k$ *and* x_k,y_k *are in the same element of* $\mathcal{Z}$ *for every* k.

  - *the measurability of* $m(\cdot,x)$ *(see Section 3 for the definition).*

  Elsewhere only $\sup_Y g<\infty$ *is needed.*

As it was pointed in the introduction, the existence of conformal measures is motivated by the study of *equilibrium states*. Recall that these are the invariant probability measures μ_e such that:

$$h(\mu_e,T)+\int_X\log g\,d\mu_e=\sup_\mu h(\mu,T)+\int_X\log g\,d\mu=P_g(X,T).$$

Applying [2], we get:

Corollary 1.1 *Assume that* $(X,\mathcal{Z},T,g)$ *satifies the requirements of the theorem. Then there exists exactly one ergodic equilibrium state* μ_e.

Moreover it is ergodic and coincides with the unique invariant probability measure absolutely continuous with respect to the conformal measure ν.

[3] then gives an estimate of the rate of mixing depending on the smoothness of (T, g). For instance,

Corollary 1.2 *If T is piecewise uniformly expanding and g is piecewise Hölder continuous, then (T, μ_e) has exponential mixing w.r.t. Hölder continuous observables, i.e.,*

$$\int_X f \cdot h \circ T^n \, d\mu_e \to \int_X f d\mu_e \cdot \int_X h d\mu_e \qquad \textit{exponentially fast.}$$

for f Hölder continuous and h bounded and measurable.

(In non-Hölder situations, [3] gives sub-exponential speeds.)

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2 Construction of a positive eigenfunctional.

Let $(X, \mathcal{Z}, T, g)$ be as in the Main Theorem. A standard, key tool is the *transfer operator* associated to (T, g). It is the map $\mathcal{L}$ acting on functions according to:

$$(\mathcal{L}f)(x) = \sum_{y \in T^{-1}x} g(y) \cdot f(y)$$

which also can be written:

$$\mathcal{L}f = \sum_{Z \in \mathcal{Z}} (gf) \circ T_Z^{-1} \cdot \mathbf{1}_{TZ}.$$

Under condition (b) of the Main Theorem, $\sup_Y g < \infty$ so that $\sup_X \mathcal{L}\mathbf{1}_X < \infty$: $\mathcal{L}$ acts on the Banach space $\mathcal{B}$ of bounded functions. Finally let $\mathcal{B}'$ be the topological dual $\mathcal{B}$ and $\mathcal{L}^*$ be the dual operator of $\mathcal{L}$, i.e.

$$\mathcal{L}^*\alpha(f) = \alpha(\mathcal{L}f)$$

for all $\alpha \in \mathcal{B}'$ and $f \in \mathcal{B}$.

Observe that a probability measure belongs to $\mathcal{B}'$ and that being conformal is the same as being an eigenfunctional for $\mathcal{L}^*$ and some positive eigenvalue. Hence the first step of the proof of the Main Theorem is the

Proposition 2.1 *There exists a positive functional $\Lambda \in \mathcal{B}'$ with $\Lambda(\mathbf{1}_X) = \Lambda(\mathbf{1}_Y) = 1$ such that*

$$\mathcal{L}^*\Lambda = \lambda\Lambda$$

for some number $0 < \lambda < \infty$.

Remark 2.1 *Any such* Λ *has full support:* $\Lambda(\mathbf{1}_U) > 0$ *for all non-empty, open subsets* $U \subset X$*. This is an immediate consequence of conditions* (d) *and* (f)*. Namely, take* U *a non-emtpy open set and* n *such that* $T^n(U \cap Y) \supset Y$ *then:*

$$\begin{aligned}\Lambda(\mathbf{1}_U) &= \Lambda(\mathbf{1}_{U\cap Y}) \geq \frac{1}{\lambda^n}\mathcal{L}^n\mathbf{1}_{U\cap Y} \\ &\geq \frac{1}{\lambda^n}(\inf_Y \mathcal{L}^n\mathbf{1}_{U\cap Y})\Lambda(\mathbf{1}_Y) \geq \frac{(\inf g)^n}{\lambda^n} > 0.\end{aligned}$$

Remark 2.2 *Any such* Λ *doesn't give mass to the boundary because* $\mathcal{L}^n\mathbf{1}_{\partial\mathcal{Z}_n} = 0$ *and:*

$$\Lambda(\mathbf{1}_{\partial\mathcal{Z}_n}) = \frac{1}{\lambda^n}\Lambda(\mathcal{L}^n\mathbf{1}_{\partial\mathcal{Z}_n}) = 0.$$

Since we are looking for a probability measure, we will restrict the action of $\mathcal{L}^*$ to the following subset $\mathcal{C}$ of $\mathcal{B}'$:

$$\mathcal{C} = \{\alpha \in \mathcal{B}',\ \alpha(\mathbf{1}_X) = \alpha(\mathbf{1}_Y) = 1, \text{ and } \alpha(f) \geq 0 \text{ for all } f \geq 0\}.$$

We shall consider the *weak star topology* on $\mathcal{B}'$ and its subset $\mathcal{C}$ without further notice.

The normalized operator $\mathcal{N} : \mathcal{C} \to \mathcal{C}$ is defined by:

$$\mathcal{N}\alpha = \frac{1}{\mathcal{L}^*\alpha(\mathbf{1}_Y)}\mathcal{L}^*\alpha \qquad \text{for all } \alpha \in \mathcal{C}.$$

Thus we are looking for fixed points of $\mathcal{N} : \mathcal{C} \to \mathcal{C}$. To find them, we are going to apply the Schauder-Tychonoff theorem [5, V.10.5]. To begin with, remark that $\mathcal{C}$ is obviously a convex subset of the topological vector space $\mathcal{B}'$, which is locally convex. Also $\mathcal{C}$ is non-empty as it contains for instance δ_x ($\delta_x(f) = f(x)$, $x \in X$ is given). Thus it remains to check the following hypothesis:

1. $\mathcal{N} : \mathcal{C} \to \mathcal{C}$ is well defined.
2. $\mathcal{C}$ is compact.
3. $\mathcal{N} : \mathcal{C} \to \mathcal{C}$ is continuous.

Condition 1: $\mathcal{N} : \mathcal{C} \to \mathcal{C}$ is well-defined.

To check this condition, take $\alpha \in \mathcal{C}$.

First we claim that $\mathcal{N}(\alpha) = (\mathcal{L}^*\alpha(1_Y))^{-1}\mathcal{L}^*\alpha$ is well defined as an element of $\mathcal{B}'$. Indeed, $\mathcal{L}^*\alpha \in \mathcal{B}'$ by construction and $\mathcal{L}^*\alpha(1_Y) > 0$: using that $\alpha(1_Y) = 1$ and the positivity of α

$$\alpha(\mathcal{L}\mathbf{1}_Y) \geq (\inf_Y \mathcal{L}\mathbf{1}_Y)\alpha(\mathbf{1}_Y) = \inf_Y \mathcal{L}\mathbf{1}_Y > 0$$

by condition (d) of the Main Theorem.

Second, $\mathcal{N}\alpha$ is positive, $(\mathcal{N}\alpha)(1_Y) = 1$ by construction and

$$\mathcal{N}\alpha(\mathbf{1}_X) = \frac{1}{\alpha(\mathcal{L}\mathbf{1}_Y)}\alpha(\mathcal{L}\mathbf{1}_X) = \frac{\alpha(\mathcal{L}\mathbf{1}_{X\setminus Y}) + \alpha(\mathcal{L}\mathbf{1}_Y)}{\alpha(\mathcal{L}\mathbf{1}_Y)} = \frac{\alpha(\mathcal{L}\mathbf{1}_Y)}{\alpha(\mathcal{L}\mathbf{1}_Y)} = 1$$

as $\mathcal{L}1_{X\setminus Y} = 0$ (only points in Y have an image).

Thus, condition 1 is satisfied.

Recall the

Theorem 2.2 *(Banach-Alaoglu [5, V.4.2]): The unit ball of the dual of any normed vector space is weakly* compact.*

Thus, $B_1 = \{\alpha \in \mathcal{B}' : \|\alpha\| \leq 1\}$, the unit ball of $\mathcal{B}'$, is compact and it is enough to check that $\mathcal{C}$ is a closed subset of $\mathcal{B}_1$ to establish

Condition 2: $\mathcal{C}$ is compact.

To check it, let $\alpha \in \mathcal{C}$ and $f \in \mathcal{B}$ with $\sup_X |f| < 1$:

$$0 \leq \sup_X |f| - f \leq 1 - f$$

so that $\alpha(1-f) \geq 0$ and, since $\alpha(1-f) = 1 - \alpha(f)$, $\alpha(f) \leq 1$. In the same way: $-\alpha(f) \leq 1$. Therefore $\alpha \in B_1$ and we have shown $\mathcal{C} \subset B_1$.

Recall that for any $f \in \mathcal{B}$, $\alpha \mapsto \alpha(f)$ is continuous by definition of the weak* topology. Hence $\mathcal{C}$, which is defined by a collection of inequalities of the type $\alpha(f) \leq \text{const}$, is closed and therefore compact.

Condition 3: $\mathcal{N} : \mathcal{C} \to \mathcal{C}$ is continuous.

Recall that a basis for the weak* topology of $\mathcal{B}'$ is given by the sets:

$$V(\alpha_0, \varepsilon, f_1, \ldots, f_n) := \{\alpha \in \mathcal{B}' : \forall i = 1, \ldots, n \ |\alpha(f_i) - \alpha_0(f_i)| < \varepsilon\}$$

with $\alpha_0 \in \mathcal{B}'$, $\varepsilon > 0$, $n = 1, 2, \ldots$ and $f_1, \ldots, f_n \in \mathcal{B}$. Thus to establish condition 3, i.e., continuity, we have to find, for every $V(\mathcal{N}\alpha_0, \varepsilon, f_1, \ldots, f_n)$, a number $\eta > 0$ and $g_1, \ldots, g_m \in \mathcal{B}$ such that:

$$\mathcal{N}V(\alpha_0, \eta, g_1, \ldots, g_m) \subset V(\mathcal{N}\alpha_0, \varepsilon, f_1, \ldots, f_n).$$

We claim that one can take $m = n + 1$ and:

$$\eta = \varepsilon \left(\frac{(\inf_Y \mathcal{L}1_Y)^2}{\inf_Y \mathcal{L}1_Y + (\sup_X \mathcal{L}1_Y) \max_{1 \leq i \leq n} \sup_X |f_i|} \right) \text{ and } g_1 = \mathcal{L}f_1, \ldots, g_n = \mathcal{L}f_m, g_{n+1} = \mathcal{L}1_Y.$$

Note that for all $j = 1, \ldots, n+1$, $g_j \in \mathcal{B}$.

To prove the claim assume $\alpha \in V(\alpha_0, \varepsilon, g_1, \ldots, g_{n+1})$. By definition, we have for all $j = 1, \ldots, n+1$:

$$|\alpha(g_j) - \alpha_0(g_j)| \leq \eta.$$

Thus

$$\begin{aligned}
|\mathcal{N}\alpha(f_i) - \mathcal{N}\alpha_0(f_i)| &\leq |\frac{1}{\mathcal{L}^*\alpha(\mathbf{1}_Y)}\mathcal{L}^*\alpha(f_i) - \frac{1}{\mathcal{L}^*\alpha(\mathbf{1}_Y)}\mathcal{L}^*\alpha_0(f_i)| \\
&\quad + \left| \frac{1}{\mathcal{L}^*\alpha(\mathbf{1}_Y)} - \frac{1}{\mathcal{L}^*\alpha_0(\mathbf{1}_Y)} \right| |\mathcal{L}^*\alpha_0(f_i)| \\
&\leq \frac{1}{\inf_Y \mathcal{L}\mathbf{1}_Y} |\alpha(\mathcal{L}f_i) - \alpha_0(\mathcal{L}f_i)| \\
&\quad + \frac{1}{(\inf_Y \mathcal{L}\mathbf{1}_Y)^2} |\alpha(\mathcal{L}\mathbf{1}_Y) - \alpha_0(\mathcal{L}\mathbf{1}_Y)| |\alpha_0(\mathcal{L}f_i)| \\
&\leq \frac{\eta}{\inf_Y \mathcal{L}\mathbf{1}_Y} + \frac{\eta}{(\inf_Y \mathcal{L}\mathbf{1}_Y)^2} |\alpha_0(\mathcal{L}f_i)|.
\end{aligned}$$

Moreover,

$$|\alpha_0(\mathcal{L}f_i)| \leq \sup_X |\mathcal{L}f_i| \leq \sup_X |\mathcal{L}\mathbf{1}_X| \sup_X |f_i|$$

which gives:

$$\begin{aligned} |\mathcal{N}\alpha(f_i) - \mathcal{N}\alpha_0(f_i)| &\leq \frac{\eta}{\inf_Y \mathcal{L}1_Y} + \frac{\eta \sup_x \mathcal{L}1_Y \max_{i\in I} \sup_X |f_i|}{(\inf_Y \mathcal{L}1_Y)^2} \\ &\leq \varepsilon. \end{aligned}$$

This proves condition 3.

We are now in position to apply the Schauder-Tychonoff theorem: there exists $\Lambda \in \mathcal{C}$ such that $\mathcal{N}(\Lambda) = \Lambda$, i.e.:

$$\mathcal{L}^*\Lambda = \lambda\Lambda$$

with $\lambda = \Lambda(\mathcal{L}\mathbf{1}_Y) \in]0, \infty[$. This ends the proof of the proposition (2.1).

3 Computing the positive eigenvalue.

A prerequisite for proving that Λ obtained above is indeed a measure is the following

Proposition 3.1 *Let Λ be an eigenfunctional for $\mathcal{L}^*$. Assume that Λ is positive, satisfies $\Lambda(1) = 1$ and $\Lambda(\mathbf{1}_U) > 0$ for every non-empty open subset U.*

Then the eigenvalue associated to Λ is $\lambda = \exp P_g(X, T)$.

The inequality $\lambda \leq \exp P_g(X, T)$ follows directly from the conformality of Λ and the definition of the pressure as

$$1 = \Lambda(\mathbf{1}_X) = \lambda^{-n}\Lambda(\mathcal{L}^n\mathbf{1}_X) \leq \lambda^{-n} \sum_{Z\in\mathcal{Z}_n} \Lambda(\mathbf{1}_{T^nZ}) \sup_Z g_n \leq \lambda^{-n} \sum_{Z\in\mathcal{Z}_n} \sup_Z g_n.$$

The proof of the converse inequality is more delicate and will make heavy use of the condition $P_g(\partial\mathcal{Z}, T) < P_g(X, T)$. We proceed pretty much like in the proof of Proposition E in [2], except that Λ not being a measure, we shall have to check a measurability property (Lemma (3.2) below) "by hand".

Recall that the *measure-theoretic pressure* of μ is:

$$P_g(\mu, T) = h(\mu, T) - \int_X \log g \, d\mu.$$

We have the *variational principle*, i.e., the topological pressure is the supremum of the measure-theoretic pressures: this follows from paragraph 3.2 of [2] using that $P_g(\partial\mathcal{Z}, T) < P_g(T)$ and that $\mathcal{Z}$ generates. This is easy to see as for ergodic measures with pressure strictly greater than $P_g(\partial\mathcal{Z}, T)$, the symbolic dynamics can be identified with (X, T), and the variational principle holds for the symbolic dynamics by a classical result (see for instance [4, chap.18].)

To conclude we will prove that the measure-theoretic pressures are bounded by $\log \lambda$, using the approach of [2] which combines a result of Ledrappier [6] and "shadowing" estimates.

Fix an ergodic invariant probability measure μ. We can assume that: $P_g(\mu, T) > P_g(\partial\mathcal{Z}, T)$ (otherwise there is nothing to prove, as $P_g(\partial\mathcal{Z}, T) < P_g(X, T)$).

Let $p : (\tilde{X}, \tilde{T}, \tilde{\mu}) \to (X, T, \mu)$ be the natural extension. We realize it as

$$\tilde{X} = \{\tilde{x} = (A, x) \in \mathcal{Z}^{\mathbb{Z}_-} \times X : x \in \bigcap_{n \geq 0} T^n[A_{-n} \dots A_0]\}$$

with the topology induced by that of $\mathcal{Z}^{\mathbb{Z}_-} \times X$.

The dynamics is given by the (partially-defined) map: $\tilde{T} : (\dots A_{-1}A_0, x) \mapsto (\dots A_{-1}A_0B, Tx)$ if B is the element of $\mathcal{Z}$ containing Tx.

Let ξ be the finite partition $p^{-1}\mathcal{Z}$. Let η be the measurable partition $\bigvee_{n\geq 0} \tilde{T}^n\xi$, i.e., two points are in the same element of η iff they have the same past by $\tilde{T}$. Observe that $\eta(\tilde{x}) = \{A\} \times W(\tilde{x})$ with

$$\tilde{x} = (A, x) \text{ and } W(\tilde{x}) := \bigcap_{n\geq 0} T^n[A_{-n} \dots A_0].$$

The assumption that each cylinder has finitely many connected components is used for the proof of the following lemma:

Lemma 3.2 *If $P_g(\mu, T) > P_g(\partial\mathcal{Z}, T)$ then there exists a countable measurable partition $\tilde{P}$ of $\tilde{X}$ modulo $\tilde{\mu}$ such that $W(\tilde{x})$ is constant and equal to a non-empty open subset on each element of $\tilde{P}$.*

The proof is based on the notion of the *shadowing* of a measure by a set introduced in [1]. We recall it for the convenience of the reader.

A measure $\tilde{\mu}$ is shadowed by $S \subset X$ if for every $\varepsilon > 0$ and $\tilde{\mu}$-a.e. $\tilde{x} \in \tilde{X}$ there exists $n > \varepsilon^{-1}$ and $s \in S$ such that:

$$d(T^k(p(\tilde{T}^{-n}\tilde{x})), T^k(s)) < \varepsilon \qquad \forall 0 \leq k \leq n.$$

i.e., the orbit is made of arbitrarily long beginnings of orbits starting in S.

The main consequence of shadowing is that it implies that $P_g(\tilde{\mu}, \tilde{T}) \leq P_g(S, T)$ [2, Theorem S].

Proof : (of lemma 3.2) The assumption on the pressure of μ ensures that it is not shadowed by $\partial\mathcal{Z}$ by the above. We claim that this implies that for $\tilde{\mu}$-a.e. $(A, x) \in \tilde{X}$, there exists $N < \infty$ such that, for all $n \geq 0$,

$$(\partial T^n[A_{-N-n} \dots A_{-N-1}]) \cap [A_{-N} \dots A_0] = \emptyset.$$

Let M be such that $\text{diam}(\mathcal{Z}^M) < \varepsilon$. Contradicting the above claim gives, for a.e. $\tilde{x} \in \tilde{X}$, an infinite sequence $n_1 < n_2 < \dots$ such that:

$$(\partial T^{n_{k+1}-n_k}[A_{-n_{k+1}} \dots A_{-n_k-1}]) \cap [A_{-n_k} \dots A_0] \neq \emptyset.$$

We may assume n_{k+1} to be minimal for the given n_k. Then, we must have that $(\partial TA_{-n_{k+1}}) \cap [A_{-n_{k+1}+1} \dots A_0] \neq \emptyset$. Thus the itinerary of $\tilde{T}^{-n_{k+1}}\tilde{x}$ is the same as that of some point of $\partial\mathcal{Z}$. Using that $\text{diam}(\mathcal{Z}^M) < \varepsilon$ and shifting by $\tilde{T}^{-M}$, we get that $\tilde{\mu}$ is shadowed by $\partial\mathcal{Z}$, a contradiction which proves the claim.

Now the claim implies that $T^{N+n}[A_{-N-n} \dots A_0]$, which by construction is a subset of $T^N[A_{-N} \dots A_0]$, is in fact a union of connected components of $T^N[A_{-N} \dots A_0]$. By assumption (e) of the Main Theorem, there are only finitely many such components. Hence, there exists $M < \infty$ such that

$$W(A, x) = T^M[A_{-M} \dots A_0].$$

In particular, $W(\cdot)$ is a.e. a non-empty, open subset of X, and takes only countably many distinct values. Finally, observe that the sets where the value of W is fixed are measurable, being of the form:

$$\{(B,x) \in \tilde{X} : B_{-M} \dots B_0 = A_{-M} \dots A_0\} \setminus \{(B,x) \in \tilde{X} : \forall m > M,\ B_{-m} \dots B_0 \in \mathcal{A}\}$$

where $\mathcal{A}$ is the set of words $B_{-m} \dots B_0$ defining a strictly smaller W-set than $B_{-M} \dots B_0$. □

We are going to compare $\tilde{\mu}$ with the function (tentatively) defined on measurable subsets S of X by:

$$m(S) := \int_{\tilde{X}} m(\tilde{x}, S)\, \tilde{\mu}(d\tilde{x})$$

where

$$m(\tilde{x}, S) := \frac{\Lambda(\mathbf{1}_{W(\tilde{x})} \cdot \rho(\tilde{x}, \cdot) \cdot \mathbf{1}_S)}{\Lambda(\mathbf{1}_{W(\tilde{x})} \cdot \rho(\tilde{x}, \cdot))}$$

and $\rho(\tilde{x}, y) := \prod_{k \geq 1} \frac{g(T^{-1}_{A_{-k}} \circ \dots T^{-1}_{A_{-1}} y)}{g(\tilde{T}^{-k} \tilde{x})}$ if $\tilde{x} \in \tilde{X}$, $\tilde{x} = (A, x)$ and $y \in W(\tilde{x})$. We have set $g(A, x) := g(x)$.

We first check that the form m is well-defined, i.e., that $\tilde{x} \mapsto m(\tilde{x}, S)$ is measurable. Observe that by assumption (b) of the Main Theorem, ρ is uniformly continuous on its domain of definition. Thus we can extend it to a continuous function defined on the whole of $\tilde{X} \times \tilde{X}$. It is then routine to check that, if W is any given measurable subset of X, the following is measurable:

$$\tilde{x} \mapsto \Lambda(\mathbf{1}_W \cdot \rho(\tilde{x}, \cdot) \cdot \mathbf{1}_S).$$

Now, use the previous Lemma to write: $\mathbf{1}_{W(\tilde{x})} = \sum_{E \in \tilde{P}} \mathbf{1}_E \mathbf{1}_{W(E)}$. The measurability of $m(\cdot, S)$ follows. In particular, $m(S)$ is well-defined.

We follow closely Ledrappier [6] for the rest of the proof of Proposition 3.1.

Let

$$Q(\tilde{x}) = \Lambda(\mathbf{1}_{W(\tilde{x})} \cdot \rho(\tilde{x}, \cdot))$$

(remark that the function to which we apply Λ is indeed a function over X, not $\tilde{X}$). A crucial observation is that the previous Lemma implies:

$$0 < Q(\tilde{x}) < \infty \text{ for } \tilde{\mu}\text{-a.e. } \tilde{x}.$$

Indeed, $\rho(\tilde{x}, \cdot) > 0$ on $W(\tilde{x})$ which is a non-empty open subset and which therefore satisfies $\Lambda(\mathbf{1}_{W(\tilde{x})}) > 0$.

Let $\mathcal{Z}' = T^{-1}\mathcal{Z}$. Compute:

$$\begin{aligned}
m(\tilde{x}, \mathcal{Z}'(x)) &= \frac{1}{Q(\tilde{x})} \cdot \Lambda(\mathbf{1}_{W(\tilde{x})} \cdot \rho(\tilde{x}, \cdot) \cdot \mathbf{1}_{\mathcal{Z}'(x)}) \\
&= \frac{1}{Q(\tilde{x})} \lambda^{-1} \Lambda(\mathcal{L}(\mathbf{1}_{W(\tilde{x})} \cdot \rho(\tilde{x}, \cdot) \cdot \mathbf{1}_{\mathcal{Z}'(x)})) \\
&= \frac{1}{Q(\tilde{x})} \lambda^{-1} \Lambda(\mathbf{1}_{\mathcal{Z}(Tx) \cap TW(\tilde{x})} \cdot \rho(\tilde{x}, T^{-1}_{\mathcal{Z}(x)}(\cdot)) \cdot g \circ T^{-1}_{\mathcal{Z}(x)}) \\
&= \frac{1}{Q(\tilde{x})} \lambda^{-1} \Lambda(\mathbf{1}_{W(\tilde{T}\tilde{x})} \cdot \rho(\tilde{T}\tilde{x}, \cdot) \cdot g(x)) \\
&= \lambda^{-1} \frac{Q(\tilde{T}\tilde{x})}{Q(\tilde{x})} g(x) \qquad (1)
\end{aligned}$$

using that $\mathcal{Z}(Tx)\cap TW(\tilde{x}) = W(\tilde{T}\tilde{x})$ and $\rho(\tilde{T}\tilde{x},z)=\rho(\tilde{x},y)\cdot\frac{g(y)}{g(x)}$ where $y=T^{-1}_{\mathcal{Z}(x)}(z)$.

We claim that:

$$\int_{\tilde{X}} \log m(\tilde{x},\mathcal{Z}'(x))\,d\tilde{\mu}(\tilde{x}) = \int_{\tilde{X}} \log g\,d\tilde{\mu} - \log\lambda.$$

Remark that if $\log Q$ was integrable, this equality would be an immediate consequence of the above computation. But we only know that $\log Q$ is a.e. finite, so we use the subtler argument of [6].

To begin with, remark that as $m(\tilde{x},\mathcal{Z}'(x))\le 1$, $\int_{\tilde{X}}\log m(\tilde{x},\mathcal{Z}'(x))\,d\tilde{\mu}(\tilde{x})$ is well-defined in $[-\infty,0]$ and the Birkhoff ergodic theorem gives:

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\log m(\tilde{T}^k\tilde{x},\mathcal{Z}'(T^kx)) \text{ exists a.e. and is } \int_{\tilde{X}}\log m(\tilde{x},\mathcal{Z}'(x))\,d\tilde{\mu}(\tilde{x}).$$

Convergence a.e. implies convergence in probability. Now the above computation shows that the Birkhoff average in the previous equation is equal to

$$\frac{1}{n}\Big(\log Q\circ\tilde{T}^n - \log Q\Big) + \int_{\tilde{X}}\log g\,d\tilde{\mu} - \log\lambda.$$

As $\log Q$ is almost everywhere finite, this average must converge in probability to $\int_{\tilde{X}}\log g\,d\tilde{\mu} - \log\lambda$. The claim follows.

Recalling that $\eta=\bigvee_{k\ge 0}\tilde{T}^{-k}\xi$ and that ξ generates, it is easy to see that η generates and that η is finer than $\tilde{T}\eta$, hence the entropy of μ can be computed as:

$$\begin{aligned} h(\mu,T) &= h(\tilde{\mu},\tilde{T}) = h(\tilde{\mu},\tilde{T}^{-1}) = H_{\tilde{\mu}}(\eta|\bigvee_{k\ge1}\tilde{T}^k\eta) = H_{\tilde{\mu}}(\tilde{T}^{-1}\eta|\bigvee_{k\ge0}\tilde{T}^k\eta)\\ &= H_{\tilde{\mu}}(\tilde{T}^{-1}\eta|\eta) = H_{\tilde{\mu}}(\tilde{T}^{-1}\xi|\eta) = -\int_{\tilde{X}}\sum_{Z'\in\tilde{T}^{-1}\xi}\mathbf{1}_{Z'}(\tilde{x})\log\mathbb{E}_{\tilde{\mu}}(\mathbf{1}_{Z'}|\eta)\,d\tilde{\mu}(\tilde{x}).\end{aligned}$$

Thus,

$$\begin{aligned} P_g(\mu,T)-\log\lambda &= h(\tilde{\mu},\tilde{T}) + \int_{\tilde{X}}\log g\,d\tilde{\mu} - \log\lambda\\ &= \int_{\tilde{X}}\sum_{Z'\in\tilde{T}^{-1}\xi}\mathbf{1}_{Z'}(\tilde{x})\log\frac{m(\tilde{x},pZ')}{\mathbb{E}_{\tilde{\mu}}(\mathbf{1}_{Z'}|\eta)}\,d\tilde{\mu}(\tilde{x})\\ &\le \log\int_{\tilde{X}}\sum_{Z'\in\tilde{T}^{-1}\xi}\mathbf{1}_{Z'}(\tilde{x})\frac{m(\tilde{x},pZ')}{\mathbb{E}_{\tilde{\mu}}(\mathbf{1}_{Z'}|\eta)}\,d\tilde{\mu}(\tilde{x})\\ &= \log\int_{\tilde{X}}\mathbb{E}_{\tilde{\mu}}\left(\sum_{Z'\in\tilde{T}^{-1}\xi}\mathbf{1}_{Z'}(\tilde{x})\frac{m(\tilde{x},pZ')}{\mathbb{E}_{\tilde{\mu}}(\mathbf{1}_{Z'}|\eta)}\,\middle|\,\eta\right)d\tilde{\mu}(\tilde{x})\\ &= \log\int_{\tilde{X}}\sum_{Z'\in\tilde{T}^{-1}\xi}\mathbb{E}_{\tilde{\mu}}(\mathbf{1}_{Z'}|\eta)\frac{m(\tilde{x},pZ')}{\mathbb{E}_{\tilde{\mu}}(\mathbf{1}_{Z'}|\eta)}\,d\tilde{\mu}(\tilde{x})\\ &= \log\int_{\tilde{X}}\sum_{Z'\in\mathcal{Z}'}m(\tilde{x},Z')\,d\tilde{\mu}(\tilde{x})\\ &= \log\int_{\tilde{X}}m(\tilde{x},1)\,d\tilde{\mu}(\tilde{x}) = 0\end{aligned}$$

using the concavity of the log and that $m(\cdot, pZ')$ is η-measurable: a simple computation shows that $m(\tilde{x}, S) = m(\tilde{y}, S)$ if $\eta(\tilde{x}) = \eta(\tilde{y})$.

This concludes the proof of Proposition (2.1).

4 Λ defines a conformal measure.

We are going to prove that Λ defines a conformal measure and this will conclude the proof of our Main Theorem.

Let $C_c(S)$ be the set of continuous functions with compact compact support contained in S. Of course $C_c(X) = C(X)$, as X is compact.

Since $\Lambda_{|C(X)}$ is a positive linear form on $C(X)$, the Riesz representation theorem implies that there exists ν, a positive Borel measure, such that:

$$\forall f \in C(X): \quad \Lambda(f) = \nu(f).$$

In particular, $\nu(\mathbf{1}_X) = \Lambda(\mathbf{1}_X) = 1$ so that ν is a probability measure.

For all $f \in C(X)$, $\Lambda(\mathcal{L}f) = \lambda\Lambda(f) = \lambda\nu(f)$. But $\mathcal{L}f$ is not necessarily continuous at points of $\partial T\mathcal{Z} := \bigcup_{Z\in\mathcal{Z}} \partial TZ$, so it is not clear that $\nu(\mathcal{L}f) = \Lambda(\mathcal{L}f)$.

We claim the following:

Lemma 4.1

$$\nu(\partial T\mathcal{Z}) = 0.$$

Let us see how this claim implies the conformality of ν. Fix $f \in C(X)$, non-negative. We prove that $\nu(\mathcal{L}f) = \Lambda(\mathcal{L}f)$:

Since the probability ν is a regular measure (as a Borel measure on a compact set), we have the following property: For each $\varepsilon > 0$, there exists U_ε open neighbourhood of $\partial T\mathcal{Z}$, such that $\nu(U_\varepsilon) < \varepsilon$.

As usual, we consider $f_\varepsilon \in C_c(X \setminus \partial T\mathcal{Z})$ such that:

$$\begin{cases} f_\varepsilon = f & in \quad X \setminus U_\varepsilon \\ f_\varepsilon \leq f & in \quad U_\varepsilon. \end{cases}$$

Then:

$$\begin{aligned} |\nu(\mathcal{L}f) - \lambda\nu(f)| &= |\nu(\mathcal{L}f_\varepsilon) + \nu(\mathcal{L}(f - f_\varepsilon)) - \lambda\nu(f)| \\ &\leq |\lambda\nu(f_\varepsilon) + 2\sup_X \mathcal{L}\mathbf{1}_X \sup_X f \ \nu(U_\varepsilon) - \lambda\nu(f)| \\ &\leq |2\sup_X f(\sup_X \mathcal{L}\mathbf{1}_X + \lambda)\nu(U_\varepsilon)| \\ &\leq (2\sup_X f(\sup_X \mathcal{L}\mathbf{1}_X + \lambda))\varepsilon. \end{aligned}$$

The conformality of ν follows using the following approximations: Let A be a Borel set such that $T: A \to TA$ is invertible and let $\varepsilon > 0$. Since ν is regular, there exist K_ε compact and O_ε open such that $K_\varepsilon \subset A \subset O_\epsilon$ and:

$$\nu(A) - \nu(K_\varepsilon) < \varepsilon$$

$$\nu(O_\varepsilon) - \nu(A) < \varepsilon$$

As usual, we consider g_ε continuous with compact support such that:

$$\begin{cases} g_\varepsilon = 1/g \;\; in \;\; K_\varepsilon \\ g_\varepsilon = 0 \;\; in \;\; O_\varepsilon^c \\ g_\varepsilon \le 1/g \;\; in \;\; O \setminus K_\varepsilon \end{cases}$$

$$\begin{aligned} |\nu(\mathcal{L}(\mathbf{1}_A \cdot 1/g)) - \lambda\nu(\mathbf{1}_A \cdot 1/g)| &\le |\nu(\mathcal{L}(\mathbf{1}_A \cdot 1/g)) - \nu(\mathcal{L}g_\varepsilon) + \lambda\nu(g_\varepsilon - \lambda\nu(\mathbf{1}_A \cdot 1/g)| \\ &\le |\nu(\mathcal{L}(\mathbf{1}_A \cdot 1/g - g_\varepsilon))| + \lambda|\nu(g_\varepsilon - \mathbf{1}_A \cdot 1/g)| \\ &\le 2\frac{\sup_X \mathcal{L}\mathbf{1}_X}{\inf_X g}\nu(O_\varepsilon \setminus K_\varepsilon) + 2\frac{\lambda}{\inf_X g}\nu(O_\varepsilon \setminus K_\varepsilon) \\ &\le 4\varepsilon\left(\frac{\sup_X \mathcal{L}\mathbf{1}_X + \lambda}{\inf_X g}\right) \end{aligned}$$

and:

$$\lambda \int_A \frac{1}{g}\, d\nu = \lambda\nu(\mathbf{1}_A \cdot 1/g) = \nu(\mathcal{L}(\mathbf{1}_A \cdot 1/g)) = \nu(TA).$$

We turn to the proof of the lemma (4.1) . We shall need the following two lemmas:

Lemma 4.2

$$P_g(\partial T\mathcal{Z}, T) \le P_g(\partial\mathcal{Z}, T).$$

Lemma 4.3 *For all Borel sets B:*

$$\nu(B) \le \inf\{\Lambda(O),\ O \text{ open},\ O \supset B\}.$$

Proof : (of Lemma 4.2). By definition:

$$P_g(\partial T\mathcal{Z}, T) = \limsup_{n\to\infty} \frac{1}{n} \log \sum_{\substack{B\in\mathcal{Z}_n \\ \bar{B}\cap\partial T\mathcal{Z}\neq\emptyset}} \sup_B g_n.$$

Let $B \in \mathcal{Z}_n$ such that $\bar{B}\cap\partial T\mathcal{Z} \neq \emptyset$, there exists $Z \in \mathcal{Z}$ such that $\overline{Z \cap T_Z^{-1}B}\cap\partial\mathcal{Z} \neq \emptyset$. Moreover, let $x \in B$:

$$g_n(x) \le \frac{g_{n+1}(T_Z^{-1}(x))}{g(T_Z^{-1}(x))} \le \frac{1}{\inf g} \sup_{Z\cap T_Z^{-1}B} g_{n+1}.$$

Hence $\sup_B g_n \le \frac{1}{\inf g} \sup_{B'} g_{n+1}$ with $B' = Z \cap T_Z^{-1}B \in \mathcal{Z}_{n+1}$. Recall that $\inf g > 0$ by assumption (b) of the Main Theorem. We have:

$$\sum_{\substack{B\in\mathcal{Z}_n \\ \bar{B}\cap\partial T\mathcal{Z}\neq\emptyset}} \sup_B g_n \le \frac{1}{\inf g} \sum_{\substack{B'\in\mathcal{Z}_{n+1} \\ \bar{B}'\cap\partial\mathcal{Z}\neq\emptyset}} \sup_{B'} g_{n+1}$$

and thus:

$$\limsup_{n\to\infty} \frac{1}{n} \log \sum_{\substack{B\in\mathcal{Z}_n \\ \bar{B}\cap\partial T\mathcal{Z}\neq\emptyset}} \sup_B g_n \le \lim_{n\to\infty} \frac{1}{n} \log \frac{1}{\inf g} + \limsup_{n\to\infty} \frac{1}{n+1} \log \sum_{\substack{B'\in\mathcal{Z}_{n+1} \\ \bar{B}'\cap\partial\mathcal{Z}\neq\emptyset}} \sup_{B'} g_{n+1}.$$

□

Proof : (of Lemma 4.3) Since ν is regular:

$$\nu(B) = \inf\{\nu(O),\ O \text{ open},\ O \supset B\}.$$

Let us fix an open set O and show that: $\nu(O) \leq \Lambda(O)$, this will prove the lemma. Take $\varepsilon > 0$. Using again the regularity of ν, there exists K_ε, a compact subset of O, such that:

$$\nu(O) < \nu(K_\epsilon) + \varepsilon.$$

Let $g_\varepsilon : X \to [0,1]$ be continuous and such that:

$$\begin{cases} g_\varepsilon = 1 & in \quad K_\varepsilon \\ g_\varepsilon = 0 & in \quad O^c \\ g_\varepsilon \leq 1 & in \quad O \setminus K_\varepsilon. \end{cases}$$

On the one hand, $g_\varepsilon \leq \mathbf{1}_O$ so that

$$\nu(g_\varepsilon) = \Lambda(g_\varepsilon) \leq \Lambda(O) \text{ and } \sup_{\varepsilon>0} \nu(g_\varepsilon) \leq \Lambda(O).$$

On the other hand, $\nu(g_\varepsilon) \geq \nu(K_\varepsilon) > \nu(O) - \varepsilon$ so that:

$$\nu(O) < \nu(K_\varepsilon) + \varepsilon \leq \nu(g_\varepsilon) + \varepsilon$$

and $\nu(O) \leq \sup_{\varepsilon>0} \nu(g_\varepsilon) \leq \Lambda(O)$. □

Proof : (of Lemma 4.1). The claim will follow from the above Lemma if we can find neighborhoods O of $\partial T\mathcal{Z}$ with $\Lambda(O)$ arbitrarily small.

Let $\mathcal{A}_n$ be $\{Z \in \mathcal{Z}_n, \bar{Z} \cap \partial T\mathcal{Z} \neq \emptyset\}$. Using the conformality, we get, for any $\delta > 0$, $N(\delta)$ such that, for all $n > N(\delta)$:

$$\Lambda\left(\bigcup \mathcal{A}_n\right) \leq \frac{1}{\lambda^n} \sum_{\substack{Z\in\mathcal{Z}_n \\ \bar{Z}\cap\partial T\mathcal{Z}\neq\emptyset}} \sup g_n \leq \left(\frac{e^{P_g(\partial T\mathcal{Z},T)+\delta}}{e^{P_g(X,T)}}\right)^n \leq \left(\frac{e^{P_g(\partial \mathcal{Z},T)+\delta}}{e^{P_g(X,T)}}\right)^n$$

by Lemma 4.2. Taking $\delta = (P_g(X,T) - P_g(\partial\mathcal{Z},T))/2$, which is positive by assumption (a) of the Main Theorem, we get:

$$\lim_{n\to\infty} \Lambda\left(\bigcup \mathcal{A}_n\right) = 0 \tag{2}$$

If $Z \in \mathcal{Z}_n \setminus \mathcal{A}_n$, then $\bar{Z} \cap \partial T\mathcal{Z} = \emptyset$, and since both sets are compact: $d(\bar{Z}, \partial T\mathcal{Z}) > 0$.

As the set $\mathcal{Z}_n \setminus \mathcal{A}_n$ is finite, $\inf_{Z\in\mathcal{Z}_n\setminus\mathcal{A}_n} d(\bar{Z}, \partial T\mathcal{Z}) > 0$. Thus the following set is a neighborhood of $\partial T\mathcal{Z}$:

$$O_n = \{x \in X, d(x, \partial T\mathcal{Z}) < \inf_{Z\in\mathcal{Z}_n\setminus\mathcal{A}_n} d(\bar{Z}, \partial T\mathcal{Z})\}.$$

and it is included in $\bigcup_{Z\in\mathcal{A}_n} \bar{Z}$. Because of the remark 2.2, $\Lambda(\bigcup_{Z\in\mathcal{A}_n} \bar{Z}) = \Lambda(\bigcup \mathcal{A}_n)$; we deduce $\lim_{n\to\infty} \Lambda(O_n) = 0$, concluding the proof of the lemma 4.1 and therefore the proof of the Main Theorem. □

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