

A MAXIMAL CUBIC QUOTIENT OF THE BRAID ALGEBRA

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Abstract. We study a quotient of the group algebra of the braid group in which the Artin generators satisfy a cubic relation. This quotient is maximal among the ones satisfying such a cubic relation. It is finite-dimensional for at least $n \leq 5$ and we investigate its module structure in this range. We also investigate the proper quotients of it that appear in the realm of quantum groups, and describe another maximal quotient related to the usual Hecke algebras. Finally, we describe the connection between this algebra and a quotient of the algebra of horizontal chord diagrams introduced by Vogel. We prove that these two are isomorphic for $n \leq 5$.

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1. INTRODUCTION

Since its introduction 80 years ago by E. Artin, the braid group and its representations have been recognized as object of great importance in a number of mathematical problems, in topology and elsewhere. There is one finite-dimensional quotient of the group algebra of the braid group – braid algebra for short – with a simple description that is well-understood. It is the Hecke algebra, where the additional relation is a quadratic relation on any Artin generator on 2 strands. It is thus worthwhile to get similar quotients.

Replacing the quadratic relation by a generic cubic one, one gets another quotient which deserves to be called in this context a cubic Hecke algebra, but which is not finite-dimensional (for at least one specialization of the parameters). However, a number of already defined finite-dimensional quotients of the braid algebra (some of them originating from the quantum world and in the realm of Vassiliev invariants) factor through this cubic Hecke algebra. It is thus tempting to look for a new quotient, covering the usual ones, and whose defining relations would involve as few strands as possible.

Since there is no other possible relation on 2 strands than the basic cubic one, the next possibility is to look for relations on 3 strands. It turns out that the generic cubic Hecke algebras *on at most 5 strands* are finite-dimensional and semisimple, and therefore there is only a finite number of ideals by which it is possible to divide out. Therefore a ‘maximal cubic quotient’ defined on the fewest possible number of strands is uniquely defined by one of (the isotypic component attached to) irreducible representations of the cubic Hecke algebra on 3 strands. This explains the term *maximal* used in the title of the present paper.

It turns out that, up to some Galois symmetries, there are only 3 possibilities for such a maximal quotient, corresponding to representations of dimension 1, 2 and 3. We prove below, elaborating on previous joint work with M. Cabanes, that the one corresponding to the 3-dimensional representation is finite dimensional but is closely related to the usual, quadratic Hecke algebra. The one related to the 1-dimensional one is still mysterious, although we proved in [6] that the quotient related to the collection of all 1-dimensional representations collapses on 5 strands. In the present paper we study in detail the maximal quotient related to an arbitrary 2-dimensional irreducible representation.

This quotient is particularly interesting because all the quantum constructions we know of which provide quotients of the cubic Hecke algebras factorize through this quotient. Moreover, there are reasons to hope that this quotient is itself finite-dimensional. Finally, it has the advantage of being defined simply by

- the generic cubic relation on 2 strands
- a single, relatively simple, braid relation on 3 strands

Although we do not solve the question of its finite dimensionality in this paper for $n \geq 6$, we tried to provide a thorough algebraic study of this quotient on at most 5 strands. This includes

- the comparison with an infinitesimal algebra introduced by P. Vogel in [33]

- the determination of a convenient $\mathbb{Z}[a^{\pm 1}, b^{\pm 1}, c^{\pm 1}]$ -module structure for it, which is proved to be free on at most 4 strands, with an explicit basis.
- determination of generators (for at most 5 strands) as modules over the subalgebra on 1 strand less

The $\mathbb{Z}[a^{\pm 1}, b^{\pm 1}, c^{\pm 1}]$ -module structure we define is the quotient of the group algebra of the braid group on n strands by the cubic relation $(s_1 - a)(s_1 - b)(s_1 - c) = 0$, where s_1, \dots, s_{n-1} denote the Artin generators, together with the relation

$$s_1^{-1} s_2 s_1 = s_1 s_2 s_1^{-1} - a^{-1} s_1 s_2 + a s_1 s_2^{-1} + a s_1^{-1} s_2 - a^3 s_1^{-1} s_2^{-1} + a^{-1} s_2 s_1 - a s_2 s_1^{-1} - a s_2^{-1} s_1 + a^3 s_2^{-1} s_1^{-1} + a^2 s_1^{-1} s_2^{-1} s_1 - a^2 s_1 s_2^{-1} s_1^{-1}$$

that is

This relation, to the best of our knowledge, has first been exhibited by Ishii, in his study of the Links-Gould polynomial (see [14]).

We hope that this algebraic work will be useful, both for the further study of this quotient on a higher number of strands, and concerning the study of specializations of these algebras as well. In particular, it is important to have a well-defined algebra over a generic ring of the form $\mathbb{Z}[a^{\pm 1}, b^{\pm 1}, c^{\pm 1}]$ in order to be able to deal with specialization, and it is known in particular that some of the Markov traces occurring on some of the cubic quotients cannot be defined over $\mathbb{Q}(a, b, c)$ – notably the one producing the Kauffman polynomial.

The paper is organized as follows. In the preliminary section 2 we provide background on the cubic Hecke algebras, and start by a definition of the quotient Q_n over the field $\mathbb{Q}(a, b, c)$ – that we denote $Q_n \otimes K$ for coherence with the sequence. This definition does not involve any braid formula, and is enough to justify that a few quantum quotients of the braid group factorize through it. With the exception of the BMW algebra, which is quite well known (see e.g. [1, 35, 30]), we describe them for small n . We also prove here that the other ‘maximal cubic quotient’ mentioned above is actually a ‘tripled’ version of the ordinary (quadratic) Hecke algebra. Finally, we introduce Vogel’s algebra and explain the connection with our quotient Q_n .

Section 3 investigates the structure of Q_3 on 3 strands, notably from a computer algebra point of view. It also establishes some tools which are useful for the sequel.

The module structure of Q_4 is determined in section 4, as well as its structure as a Q_3 -bimodule. Finally, we prove in section 5 that Q_5 is a finitely generated module, and investigate it as a Q_4 -bimodule.

2. PRELIMINARIES

2.1. The cubic Hecke algebras. Let us denote $R = \mathbb{Z}[a, a^{-1}, b, b^{-1}, c, c^{-1}] = \mathbb{Z}[a, b, c, (abc)^{-1}]$. We let H_n denote the R -algebra defined as the quotient of the group algebra RB_n of the braid group on n strands by the relations $(s_i - a)(s_i - b)(s_i - c) = 0$ for $1 \leq i \leq n - 1$ or, equivalently – since each s_i is conjugated to s_1 – by the relation $(s_1 - a)(s_1 - b)(s_1 - c) = 0$. It is known that H_n is a free R -module of finite rank for $n \leq 5$ (see [25]) and that the specialization of H_n at $\{a, b, c\} = \mu_3(\mathbb{C})$ – that is, the group algebra of B_n/s_1^3 – is infinite-dimensional for $n \geq 6$ by a theorem of Coxeter (see [9]). In the present state of knowledge it remains however possible

that H_n is finite dimensional for other values of n when extended over $\mathbb{Q}(a, b, c)$, although there is no evidence in this direction so far.

2.1.1. *The cubic Hecke algebra for $n = 3$.* More precisely, for $n = 3$, one may excerpt from [25] the following result (see also [3, ?, 21] for related statements).

Proposition 2.1.

(1) *The algebra H_3 is a free H_2 -module of rank 8, with basis the elements $1, s_2, s_2^{-1}, s_1^\alpha s_2^\beta$ for $\alpha, \beta \in \{1, -1\}$, $s_2 s_1^{-1} s_2$.*

(2) *The algebra H_3 is a free R -module of rank 24, with basis the elements*

$$\begin{aligned} \mathcal{B}_1 = & (1, s_1, s_1^{-1}, s_2, s_2^{-1}, s_1 s_2, s_1 s_2^{-1}, s_1^{-1} s_2, s_1^{-1} s_2^{-1}, s_1 s_2 s_1, s_1 s_2 s_1^{-1}, s_1^{-1} s_2 s_1, s_1^{-1} s_2 s_1^{-1}, \\ & s_1 s_2^{-1} s_1, s_1^{-1} s_2^{-1} s_1, s_2 s_1, s_2^{-1} s_1, s_2 s_1^{-1}, s_2^{-1} s_1^{-1}, s_1 s_2^{-1} s_1^{-1}, s_1^{-1} s_2^{-1} s_1^{-1}, s_2 s_1^{-1} s_2, \\ & s_1 s_2 s_1^{-1} s_2, s_1^{-1} s_2 s_1^{-1} s_2). \end{aligned}$$

Proof. From [25] theorem 3.2 we know that H_3 is generated as a H_2 -module by the 8 elements on the first statement. Since H_2 is spanned by $1, s_1, s_1^{-1}$ it follows that H_3 is generated as a H_3 -module by the 24 elements of the second statement. Since Γ_3 has 24 elements and by an argument of [4] (see also [27], proposition 2.4 (1)) it follows that these 24 elements are a basis over R of H_3 . It readily follows that the 8 original elements provide a basis of H_3 as a H_2 -module. \square

A consequence is that H_3 is a free deformation of the group algebra $R\Gamma_3$, where Γ_n denotes the quotient of the braid group by the relations $s_i^3 = 1$, and H_3 becomes isomorphic to it after extension of scalars to the algebraic closure \overline{K} of the field of fractions K of R . Actually, one has the stronger result $H_3 \otimes_R K \simeq K\Gamma_3$, because the irreducible representations of KH_3 are absolutely irreducible.

We will use the following explicit matrix models for the representations, which are basically the same which were obtained in [3], §5B. We endow $\{a, b, c\}$ with the total ordering $a < b < c$. We denote

- (1) S_α for $\alpha \in \{a, b, c\}$ the 1-dimensional representation $s_1, s_2 \mapsto \alpha$
- (2) $U_{\alpha, \beta}$ for $\alpha, \beta \in \{a, b, c\}$ with $\alpha < \beta$ the 2-dimensional representation

$$U_{\alpha, \beta} : s_1 \mapsto \begin{pmatrix} \alpha & 0 \\ -\alpha & \beta \end{pmatrix} s_2 \mapsto \begin{pmatrix} \beta & \beta \\ 0 & \alpha \end{pmatrix}$$

- (3) V the 3-dimensional irreducible representation

$$s_1 \mapsto \begin{pmatrix} c & 0 & 0 \\ ac + b^2 & b & 0 \\ b & 1 & a \end{pmatrix} s_2 \mapsto \begin{pmatrix} a & -1 & b \\ 0 & b & -ac - b^2 \\ 0 & 0 & c \end{pmatrix}$$

We note the important feature that these representations are actually defined over R . As a consequence, these formulas provide an explicit embedding

$$\Phi_{H_3} : H_3 \hookrightarrow R^3 \oplus M_2(R)^3 \oplus M_3(R)$$

with the RHS being a free R -module of rank 24.

Another interesting property that we have in H_3 is the following relation (see [25], lemma 3.6).

$$(2.1) \quad s_2^{-1} s_1 s_2^{-1} s_1 - s_1 s_2^{-1} s_1 s_2^{-1} \in u_1 u_2 + u_2 u_1$$

Finally we recall from e.g. [30] section 2 that specializations of H_3 remain semisimple as soon as $(x - y)(x^2 - xy + y^2)(xy + z^2) \neq 0$ for $\{x, y, z\} = \{a, b, c\}$.

2.1.2. *The cubic Hecke algebra on 4 strands.* A description of the irreducible representations of H_4 can be found in [21] and [29]. We use the same notation here. There are

- three 1-dimensional representations S_x for $x \in \{a, b, c\}$, defined by $s_i \mapsto x$.
- three 2-dimensional representations $T_{x,y}$ indexed by the subsets $\{x, y\} \subset \{a, b, c\}$ of cardinality 2, which factorize through the special morphism $B_4 \rightarrow B_3$ (hence through H_3).
- one 3-dimensional representation V , factorizing through B_3 .
- six 3-dimensional representations $U_{x,y}$ for each tuple (x, y) with $x \neq y$ and $x, y \in \{a, b, c\}$.
- six 6-dimensional representations $V_{x,y,z}$ for each tuple (x, y, z) with $\{x, y, z\} = \{a, b, c\}$
- three 8-dimensional representations W_x for $x \in \{a, b, c\}$
- two 9-dimensional representations X, X' .

Except for X, X' , they are uniquely determined by their restriction to B_3 .

$$\begin{aligned} \text{Res}U_{x,y} &= S_x + U_{x,y} \\ \text{Res}V_{x,y,z} &= S_x + U_{x,y} + V \\ \text{Res}W_x &= S_x + U_{x,y} + U_{x,z} + V \\ \text{Res}X &= U_{x,y} + U_{x,z} + U_{y,z} + V \end{aligned}$$

The representations $U_{x,y}$ of B_3 are well-determined by their restriction to B_2 : the restriction to B_2 of $U_{x,y}$ is the sum of two 1-dimensional representations on which s_1 acts by x and y , respectively.

A complete set of matrices for these representations was first found by Broué and Malle in [3]. Other constructions were subsequently given, in [21] and [19]. The latter ones have been included in the development version of the CHEVIE package for GAP3, and the order in which they are stored in this package at the present time is $S_a, S_c, S_b, T_{b,c}, T_{a,b}, T_{a,c}, V, U_{b,a}, U_{a,c}, U_{c,b}, U_{c,a}, U_{a,b}, U_{b,c}, V_{c,a,b}, V_{b,c,a}, V_{a,b,c}, V_{b,a,c}, V_{c,b,a}, V_{a,c,b}, W_a, W_c, W_b, X, X'$.

For a printed version of these models, we refer the reader to the tables of [29].

A consequence of the trace conjecture of [5] for the complex reflection group G_{25} would be that there exists a symmetrizing trace on this algebra with a unicity condition enabling one to compute the corresponding Schur elements. Under this conjecture, these Schur elements have been computed (see [18, 7]). Of interest for us will be the following fact: none of these Schur elements vanish in $R/(a^3 + b^2c)$ (while some do inside $R/(a^3 - b^2c)$, though!). Therefore this conjecture implies that $H_4 \otimes L$ is semisimple for L the fraction field of $R/(a^3 - b^2c)$. We check this as follows. In CHEVIE there are matrix models of all irreducible representations of H_4 . We check on these by direct (computer) computation that for the specialization $a = -4, b = 8, c = 1$ all these representations remain irreducible, and pairwise non isomorphic.

2.2. **A first definition of Q_n , over K .** Our quotient Q_n can be defined over K as follows. Consider the ideal of $H_n \otimes K$ generated by the (images inside $H_n \otimes K$) of the ideal of H_3 associated to the irreducible representation $U_{b,c}$. Then this version of Q_n over K , that we denote $Q_n \otimes K$ for compatibility reasons with the sequel, is the quotient of $H_n \otimes K$ by this ideal.

From the description of $H_4 \otimes K$ given here, it follows that $Q_4 \otimes K$ is semisimple, and that its irreducible representations are (see [29]) the S_x for $x \in \{a, b, c\}$ and $T_{a,b}, T_{a,c}, V, U_{a,b}, U_{b,a}, U_{a,c}, U_{c,a}, V_{a,b,c}, V_{b,a,c}, V_{a,c,b}, V_{c,a,b}, W_a$.

The above definition can already be slightly generalized to the case where L is the field and we consider a specialization $R \rightarrow L$ where the image of $(x - y)(x^2 - xy + y^2)(xy + z^2)$ is nonzero for any $\{x, y, z\} = \{a, b, c\}$. In this case $\mathbb{Q}_4 \otimes L$ is again the quotient of the semisimple algebra $H_4 \otimes L$ by the ideal corresponding to $U_{b,c}$.

2.3. Quantum cubic quotients. Let \mathfrak{g} denote a (finite dimensional) semisimple Lie algebra over the complex numbers, endowed with a \mathfrak{g} -invariant non-degenerate bilinear form $\langle \cdot, \cdot \rangle$ (usually the Killing form). We fix an arbitrary basis e_1, \dots, e_m of \mathfrak{g} , denote e^1, \dots, e^m its dual basis with respect to the given form. Let $C = \sum_{i=1}^m e_m e^m \in Z(\mathbf{U}\mathfrak{g})$. If our form is the Killing form, then C is the Casimir operator.

Let us fix an finite dimensional \mathfrak{g} -module U , and denote $\tau \in \text{End}(U \otimes U)$ denote the action of $\sum_{i=1}^m e_i \otimes e_i$ on $U \otimes U$. It commutes with the flip $x \otimes y \mapsto y \otimes x$. Defining $\tau_{ij} \in \text{End}(U^{\otimes n})$ for $1 \leq i \neq j \leq n$ as in [17] we get a linear representation $\mathfrak{B}_n \rightarrow \text{End}(U^{\otimes n})$, induced by $t_{ij} \mapsto \tau_{ij}$ and extending the natural action of \mathfrak{S}_n on $U^{\otimes n}$. This action commutes with the action of the envelopping algebra $\mathbf{U}\mathfrak{g}$. By the Drinfeld-Kohno theorem, for generic values of q , this $\mathfrak{B}_n \otimes \mathbf{U}\mathfrak{g}$ -module structure provides through the monodromy of the associated KZ-system

$$\frac{h}{i\pi} \sum_{1 \leq i < j \leq n} \tau_{ij} d \log(z_i - z_j)$$

the $\mathbb{C}B_n \otimes \mathbf{U}_q \mathfrak{g}$ -action on the quantized module also denoted $U^{\otimes n}$ (see e.g. [17]), where $q = e^h$.

2.3.1. Action on $U^{\otimes 2}$. In particular the action of the braid generator s_1 is conjugate (for generic values of h) to $(1 \ 2) \exp(h\tau)$. Since

$$2 \sum_{i=1}^m e_i \otimes e^i = \Delta(C) - C \otimes 1 - 1 \otimes C$$

where $\Delta : \mathbf{U}\mathfrak{g} \rightarrow \mathbf{U}\mathfrak{g} \otimes \mathbf{U}\mathfrak{g}$ is the coproduct, we know that τ acts by a scalar on any simple component of $U^{\otimes 2}$. Also recall that τ commutes with $(1 \ 2)$ and that the value of the Casimir element $C \in Z(\mathbf{U}\mathfrak{g})$ on $V(\lambda)$ is equal to $\langle \lambda, \lambda + 2\rho \rangle$, where ρ is equal to the half-sum of the positive roots ([11], (25.14)).

2.3.2. Commutant algebra. We set $\mathcal{C}_n = \text{End}_{\mathbf{U}\mathfrak{g}}(U^{\otimes n})$ and assume $U^{\otimes n}$ is semisimple as a $\mathbf{U}\mathfrak{g}$ -module for n smaller than some $n_\infty \in \mathbb{N}$. Let P_+ denote the lattice of dominant weights for \mathfrak{g} , and $V(\lambda)$ the (irreducible) highest weight module associated to it. We set $P_+(n)$ the set of all $\lambda \in P_+$ such that $U^{\otimes n}$ contains an irreducible component isomorphic to $V(\lambda)$. As a $\mathcal{C}_n \otimes \mathbf{U}\mathfrak{g}$ -module there is a canonical multiplicity-free decomposition of $U^{\otimes n}$ of the form

$$U^{\otimes n} = \bigoplus_{\lambda \in P_+(n)} \hat{M}_n(\lambda), \quad \hat{M}_n(\lambda) \simeq V_n(\lambda) \otimes M_n(\lambda)$$

with $M_n(\lambda) = \text{Hom}_{\mathbf{U}\mathfrak{g}}(V_n(\lambda), U^{\otimes n}) = \text{Hom}_{\mathbf{U}\mathfrak{g}}(V_n(\lambda), \hat{M}_n(\lambda))$ and the isomorphism $V_n(\lambda) \otimes M_n(\lambda) \rightarrow \hat{M}_n(\lambda)$ being given by the evaluation map. Since, as a $\mathbf{U}\mathfrak{g}$ -module, $U^{\otimes n}$ is the direct sum of the $\hat{M}_{n-1}(\mu) \otimes U$ and this decomposition is stable under \mathfrak{B}_{n-1} , we get a canonical decomposition

$$\text{Res}_{\mathcal{C}_{n-1}} M_n(\lambda) = \bigoplus_{\mu \in P_{n-1}(\lambda)} \text{Hom}_{\mathbf{U}\mathfrak{g}}(V(\lambda), \hat{M}_{n-1}(\mu) \otimes U) = \bigoplus_{\mu \in P_{n-1}(\lambda)} c(V(\lambda), V(\mu) \otimes U) M_{n-1}(\mu)$$

where $c(V(\lambda), V(\mu) \otimes U)$ denotes the multiplicity of the simple $\mathbf{U}\mathfrak{g}$ -module $V(\lambda)$ inside $V(\mu) \otimes U$.

When every such restriction is multiplicity free, up to the restriction to \mathfrak{B}_2 , we get a canonical decomposition of $M_n(\lambda)$ as direct sum of lines. A basis obtained by choosing one nonzero vector per line will be called a *suitable basis*. It is naturally indexed by paths in the following levelled graph. If $U = V(\lambda_0)$, then there is only one vertex of level 0, corresponding to λ_0 . In general, the vertices of level k are 1-1 correspondence with the $\mu \in P_+(k)$, and there is an edge between the level k vertex attached to $\lambda \in P_+(k)$ and $\mu \in P_+(k+1)$ if and only if $V(\lambda) \otimes U$ contains an irreducible component isomorphic to $V(\mu)$ (and there will be only one such component under our multiplicity free assumption, for the λ, μ we will consider). When needed, we denote $\mu^{(k)}$ the vertex of level k attached to $\mu \in P_+(k)$.

The indexing paths are the following ones. Consider paths from $\lambda_0^{(1)}$ to $\lambda^{(n)}$ (always passing from one level to the next). Every such path $\lambda_0^{(1)} = \mu_1^{(1)} \rightarrow \mu_2^{(2)} \rightarrow \cdots \rightarrow \mu_n^{(n)} = \lambda^{(n)}$ is canonically associated to the only line corresponding to the inclusions $M_1(\mu_1) \subset M_2(\mu_2) \subset \cdots \subset M_n(\mu_n) = M_n(\lambda)$ in the direct sum decomposition above.

2.3.3. Action of infinitesimal braids. From now on we assume that the multiplicity free assumption is satisfied by $M_n(\lambda)$, and that we have pick a suitable basis.

There is a natural morphism $\mathfrak{B}_n \rightarrow \mathcal{C}_n$, and we want to know whether the restriction to \mathfrak{B}_n of the modules $M_n(\lambda)$ is irreducible. We introduce the elements $Y_r = \sum_{i < r} t_{ir} \in \mathfrak{B}_r$. They commute to $\mathfrak{B}_{r-1} \subset \mathfrak{B}_r$ and in particular they commute to each other. Under our assumption it is readily checked that they act diagonally on our suitable basis. The scalar by which Y_r acts on the (basis element indexed by the) path $\mu_1^{(1)} \rightarrow \mu_2^{(2)} \rightarrow \cdots \rightarrow \mu_n^{(n)}$ is equal to $\frac{1}{2}(C(\mu_r) - C(\mu_{r-1}) - C(\lambda_0))$ (see e.g. [21] ch. 4 or [29] lemma 2.1). Setting $s_r = (r, r+1)$ we get $s_r Y_r s_r = Y_{r+1} - t_{r,r+1}$ hence $s_r W_r s_r + W_r = 2t_{r,r+1}$ where $W_r = Y_{r+1} - Y_r$. Notice that W_r, s_r and $t_{r,r+1}$ all commute with \mathfrak{B}_{r-1} and that their action on a given basis element only depends on the section of the path given by $\mu_{(r-1)}^{(r-1)} \rightarrow \mu_{(r)}^{(r)} \rightarrow \mu_{(r+1)}^{(r+1)}$. We call *brick* between $\mu_{(r-1)}^{(r-1)}$ and $\mu_{(r+1)}^{(r+1)}$ the vector space indexed by the paths of the form $\mu_{(r-1)}^{(r-1)} \rightarrow \mu_{(r)}^{(r)} \rightarrow \mu_{(r+1)}^{(r+1)}$, endowed with the \mathcal{A} -module structure $W \mapsto W_r, s \mapsto s_r, u \mapsto t_{r,r+1}$, where \mathcal{A} is the unital algebra defined by generators W, s, u and relations

$$W + sWs = 2u, su = us, s^2 = 1$$

Lemma 2.2. *Let E be a \mathcal{A} -module on which s or $-s$ acts as a reflection, and on which W, u act diagonally with a disjoint set of distinct eigenvalues. Then E is irreducible.*

Proof. Since \mathcal{A} admits an algebra automorphism exchanging $s \leftrightarrow -s$, we can assume s acts by a reflection. We choose a basis e_1, \dots, e_{m+1} on which s, u both act diagonally, $s = \text{diag}(-1, 1, \dots, 1)$, $u = \text{diag}(\alpha, \beta_1, \dots, \beta_m)$. Then the equation $W + sWs = 2u$ implies that $W = u + F$ with F mapping $\text{Vect}(e_2, \dots, e_{m+1})$ to $\text{Vect}(e_1)$ and $\text{Vect}(e_1)$ to $\text{Vect}(e_2, \dots, e_{m+1})$. Since u has distinct eigenvalues, E is irreducible unless one of the lines $\text{Vect}(e_i)$ is stable. But this is possible only if $F.e_i = 0$, and then $\text{Vect}(e_i)$ would be a common eigenspace for u, W and for the same eigenvalue for both u and W . This is excluded by assumption, hence E is irreducible. \square

In dimension 2, we have the following much stronger form.

Lemma 2.3. *Let E be a 2-dimensional \mathcal{A} -module on which s or $-s$ acts as a reflection, and on which W, u act diagonally. If u has two distinct eigenvalues and $Sp(u) \neq Sp(W)$ then E is irreducible.*

Proof. The proof starts in the same way as in the previous lemma, but then $W + sWs = 2u$ implies that, either W and u have the same spectrum, or none of the two eigenspaces of u is stable under W . This implies that E is irreducible. \square

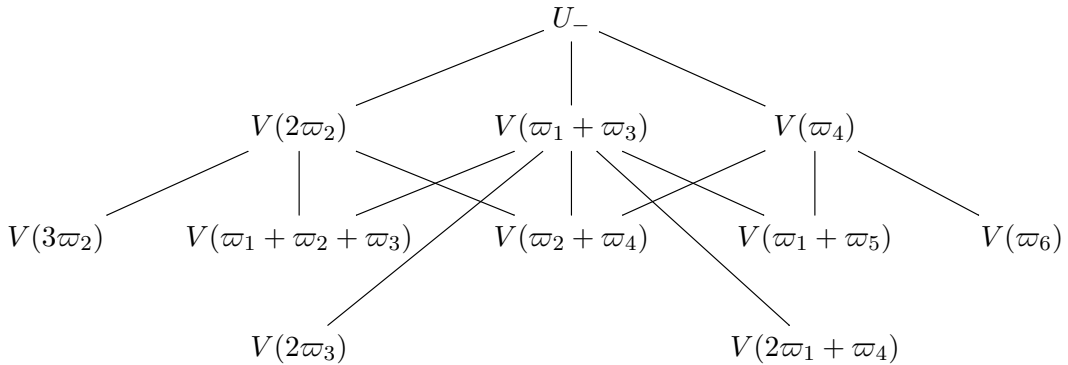
Assume that the $M_{n-1}(\mu)$ are all pairwise non-isomorphic irreducible \mathfrak{B}_{n-1} -modules for all the $\mu \in P_+(n-1)$ for which $V(\lambda)$ is an irreducible component of $V(\mu) \otimes U$. Then, the *irreducibility graph* is the unoriented graph whose vertices are all such μ , and there is an edge between μ_1 and μ_2 if there is an irreducible brick between some $\nu^{(n-2)}$ and $\lambda^{(n)}$ and paths $\nu^{(n-2)} \rightarrow \mu_1^{(n-1)} \rightarrow \lambda^{(n)}$ and $\nu^{(n-2)} \rightarrow \mu_2^{(n-1)} \rightarrow \lambda^{(n)}$. It is easily proved (see e.g. [21], proposition 17, p. 51) that, if the irreducibility graph is connected, then $M_n(\lambda)$ is irreducible.

2.4. Quantum cubic quotients : the $\mathfrak{sl}(V)$ modules $\Lambda^2 V$ and $S^2 V$. Let $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ for $n \geq 7$, $V = \mathbb{C}^n$ and $U_- = \Lambda^2 V$. We denote E_{ij} the elementary matrix containing 1 in row i and column j and 0 otherwise. We consider the standard generators $X_i = E_{i,i+1}$, $Y_i = E_{i+1,i}$, $H_i = E_{ii} - E_{i+1,i+1}$. A highest weight vector v is characterised by $X_i.v = 0$ for all i .

We use the notation $V(\alpha)$ for the highest weight \mathfrak{g} -module of highest weight α . Denoting $\varpi_1, \dots, \varpi_{n-1}$ the fundamental weight of \mathfrak{g} we have $U_- = V(\varpi_2)$. Let (e_1, \dots, e_n) denote the standard basis of $\mathbb{C}^n = V$. We have $X_i.e_j = \delta_{i+1,j}e_{j-1}$ hence e_1 is a highest weight vector for V . Then a h.w. vector for $U_- = \Lambda^2 V$ is $e_1 \wedge e_2$.

A related module is $U_+ = S^2 V$. All the results below concerning the action of B_k or \mathfrak{B}_k are equally valid with U_- replaced by U_+ , after exchanging n with $-n$. This is because these two modules can be considered as modules under the Lie superalgebra $\mathfrak{sl}_n = \mathfrak{sl}(n|0) = \mathfrak{sl}(0|n)$ of the form $S^2 E$ where E is a super vector space of type $n|0$ for U_+ or $0|n$ for U_- , and of superdimension n or $-n$. We provide the details in the case of U_- , the corresponding details for U_+ being left to the reader.

2.4.1. Action of B_k , $k \leq 3$. From the fairly easy multiplication rule by $U_- = V(\varpi_2)$ (see [11] proposition 15.25) one gets that the Bratteli diagram of the tensor powers $U_-, U_-^{\otimes 2}, U_-^{\otimes 3}$ is as follows.



We have $S^2 U_- = V(2\varpi_2) + V(\varpi_4)$ ([11], ex. 15.34) hence $\Lambda^2 U_- = V(\varpi_1 + \varpi_3)$. Also, $\Lambda^3 U_- = V(2\varpi_1 + \varpi_4) + V(2\varpi_3)$ and $S^3 U_- = V(3\varpi_2) + V(\varpi_2 + \varpi_4) + V(\varpi_6)$.

The value of the Casimir element $C \in Z(\mathbf{Ug})$ on $V(\lambda)$ is equal to $(\lambda, \lambda + 2\rho)$, where ρ is equal to the half-sum of the positive roots ([11], (25.14)). We refer to Bourbaki ([2], planche 1) for the basic datas involved in this case. Straightforward calculations show $(\varpi_i, 2\rho) = i(n-i)$ and $(\varpi_i, \varpi_j) = \min(i, j) - ij/n$. From this the value of the Casimir is readily computed.

The value of τ on each of the irreducible components of $U_-^{\otimes 2}$ is then :

$$V(\varpi_1 + \varpi_3) : \frac{-4}{n} \quad V(2\varpi_2) : \frac{2(n-2)}{n} \quad V(\varpi_4) : \frac{-4(n+1)}{n}$$

Therefore, the action of the braid group factorises through the cubic Hecke algebras, with

$$a = -\exp\left(\frac{-4}{n}h\right), b = \exp\left(\frac{2(n-2)}{n}h\right), c = \exp\left(\frac{-4(n+1)}{n}h\right),$$

We have $a^3 + b^2c = 0$, so in this section we replace K by the fraction field K_Λ of $R_\Lambda = R/(a^3 + b^2c)$. Note that, inside K_Λ , $(x-y)(x^2 - xy + y^2)(xy + z^2) \neq 0$ for $\{x, y, z\} = \{a, b, c\}$, so $\mathbb{Q}_4 \otimes K_\Lambda$ is well-defined, and semisimple by section 2.1.2. Moreover, $R/(a^3 + b^2c) = \mathbb{Z}[a^{\pm 1}, b^{\pm 1}, c^{\pm 1}]/(a^3 + b^2c)$ is equal to

$$\mathbb{Z}\left[a^{\pm 1}, \left(\frac{b}{-a}\right)^{\pm 1}, \left(\frac{c}{-a}\right)^{\pm 1}\right] / \left(1 - \left(\left(\frac{b}{a}\right)^2 \left(\frac{c}{-a}\right)\right)\right) \simeq \mathbb{Z}[a^{\pm 1}, b^{\pm 1}, c^{\pm 1}] / (1 - b^2c)$$

which is isomorphic to $(\mathbb{Z}[a^{\pm 1}, b^{\pm 1}])[c^{\pm 1}]/(c - b^{-2}) \simeq \mathbb{Z}[a^{\pm 1}, b^{\pm 1}]$.

In order to check that the specialization we are considering is generic inside $R_\Lambda = R/(a^3 + b^2c)$, we need to check that the discriminant of the corresponding algebra is nonzero. Since it is homogeneous we can renormalize $s_i \mapsto -a^{-1}s_i$, and assume $a = -1$, $b = e^{2h}$, $c = e^{4h}$. Then, under the above isomorphism, this discriminant becomes a Laurent polynomial in b specialized at $b = e^{2h}$, which is nonzero for generic h since the Laurent polynomial itself is nonzero by section 2.1.2. Therefore, we will loosely work in the sequel as though we were working inside $\mathbb{Q}_4 \otimes K_\Lambda$.

Similarly, the value of Y_3 can be computed easily, and one checks that the irreducibility criteria above are satisfied (for n large enough), so that all the $M_3(\lambda)$ are irreducible as \mathfrak{B}_3 -modules. They are obviously pairwise non-isomorphic, except for the 1-dimensional representations. Actually, a 1-dimensional representation of \mathfrak{B}_n has the form $t_{ij} \mapsto \alpha$, $(i \ j) \mapsto \varepsilon$ for some $\varepsilon = \pm 1$ and $\alpha \in \mathbb{C}$. But since here $(1 \ 2)$ acts as a polynomial of t_{12} the value of α determines ε . And there can be at most 4 values of α , which are the values of τ on $U_-^{\otimes 2}$. This shows a priori that there cannot be 4 non-isomorphic 1-dimensional components for the action of \mathfrak{B}_3 . Actually, the ones which are isomorphic are the ones whose restriction to \mathfrak{B}_2 are isomorphic, and these are $M_3(2\varpi_1 + \varpi_4)$ and $M_3(2\varpi_3)$. Another reason for this is that $\Lambda^3 U_- = V(2\varpi_1 + \varpi_4) + V(2\varpi_3)$ and that, for all $x \in U_-^{\otimes n}$, we have $(i \ j).x = -x \Rightarrow t_{ij}.x = (-4/n)x$. Indeed, this property is true for $t_{12} = \tau$ on $U^{\otimes 2}$, therefore also on $U^{\otimes n}$ and through \mathfrak{S}_n -conjugation one gets it holds for arbitrary i, j .

From this one gets the following decomposition of the quantized module $U_-^{\otimes 3}$,

$$U_-^{\otimes 3} = V(3\varpi_2) \otimes S_b + V(\varpi_1 + \varpi_2 + \varpi_3) \otimes U_{a,b} + V(\varpi_2 + \varpi_4) \otimes V + V(\varpi_1 + \varpi_5) \otimes U_{a,c} + V(\varpi_6) \otimes S_c + (V(2\varpi_3) + V(2\varpi_1 + \varpi_4)) \otimes S_a$$

This implies the following.

Proposition 2.4. *The action of B_k on $U_-^{\otimes n}$ factorizes through $\mathbb{Q}_k \otimes K$. Moreover, the morphism $\mathbb{Q}_3 \otimes K \rightarrow \text{End}(U_-^{\otimes 3})$ is injective and not surjective.*

2.4.2. *The bimodule $U_-^{\otimes 4}$.*

The goal of this section is to prove the following.

$V(3\varpi_2) \otimes V(\varpi_2)$	$= V(4\varpi_2) + V(\varpi_1 + 2\varpi_2 + \varpi_3) + V(2\varpi_2 + \varpi_4)$
$V(\varpi_1 + \varpi_2 + \varpi_3) \otimes V(\varpi_2)$	$= V(\varpi_1 + 2\varpi_2 + \varpi_3) + V(2\varpi_1 + 2\varpi_3) + V(2\varpi_1 + \varpi_2 + \varpi_4)$ $+ V(\varpi_2 + 2\varpi_3) + V(2\varpi_2 + \varpi_4) + V(\varpi_1 + \varpi_3 + \varpi_4)$ $+ V(\varpi_1 + \varpi_2 + \varpi_5)$
$V(\varpi_2 + \varpi_4) \otimes V(\varpi_2)$	$= V(2\varpi_2 + \varpi_4) + V(\varpi_1 + \varpi_3 + \varpi_4) + V(\varpi_1 + \varpi_2 + \varpi_5)$ $+ V(2\varpi_4) + V(\varpi_3 + \varpi_5) + V(\varpi_2 + \varpi_6)$
$V(\varpi_1 + \varpi_5) \otimes V(\varpi_2)$	$= V(\varpi_1 + \varpi_2 + \varpi_5) + V(\varpi_3 + \varpi_5) + V(2\varpi_1 + \varpi_6)$ $+ V(\varpi_2 + \varpi_6) + V(\varpi_1 + \varpi_7)$
$V(\varpi_6) \otimes V(\varpi_2)$	$= V(\varpi_2 + \varpi_6) + V(\varpi_1 + \varpi_7) + V(\varpi_8)$
$V(2\varpi_3) \otimes V(\varpi_2)$	$= V(\varpi_2 + 2\varpi_3) + V(\varpi_1 + \varpi_3 + \varpi_4) + V(\varpi_3 + \varpi_5)$
$V(2\varpi_1 + \varpi_4) \otimes V(\varpi_2)$	$= V(2\varpi_1 + \varpi_2 + \varpi_4) + V(\varpi_1 + \varpi_3 + \varpi_4) + V(3\varpi_1 + \varpi_5)$ $+ V(\varpi_1 + \varpi_2 + \varpi_5) + V(2\varpi_1 + \varpi_6)$

TABLE 1. Decomposition of $V(\lambda) \otimes V(\varpi_2)$

Proposition 2.5. *For generic values of q , the $(U_q\mathfrak{g}, B_4)$ -bimodule $U_-^{\otimes 4}$ admits the following decomposition*

$$\begin{aligned}
U_-^{\otimes 4} = & V(4\varpi_2) \otimes S_b + V(\varpi_1 + 2\varpi_2 + \varpi_3) \otimes U_{b,a} + V(2\varpi_1 + 2\varpi_3) \otimes T_{a,b} \\
& + V(2\varpi_2 + \varpi_4) \otimes V_{b,a,c} + V(\varpi_1 + \varpi_3 + \varpi_4) \otimes (S_a + V_{a,b,c}) + V(2\varpi_4) \otimes V \\
& + V(2\varpi_1 + \varpi_2 + \varpi_4) \otimes U_{a,b} + V(\varpi_2 + 2\varpi_3) \otimes U_{a,b} \\
& + V(\varpi_1 + \varpi_2 + \varpi_5) \otimes W_a + V(\varpi_3 + \varpi_5) \otimes V_{a,c,b} + V(\varpi_2 + \varpi_6) \otimes V_{c,a,b} \\
& + V(3\varpi_1 + \varpi_5) \otimes S_a + V(2\varpi_1 + \varpi_6) \otimes U_{a,c} + V(\varpi_1 + \varpi_7) \otimes U_{c,a} + V(\varpi_8) \otimes S_c
\end{aligned}$$

Corollary 2.6. *The image of $Q_4 \otimes K$ inside $\text{End}(U_-^{\otimes 4})$ has dimension 260, and its irreducible representations are $\text{Irr}(Q_4 \otimes K) \setminus \{T_{a,c}\}$.*

In order to prove this decomposition, we need to determine the restriction to \mathfrak{B}_4 of the \mathcal{C}_4 -modules $M_4(\lambda)$ for $\lambda \in P_+(4)$. For this we will, in most cases, first prove irreducibility, and then determine the isomorphism type by looking at the restriction on 3 strands. Notice that all the representations of \mathfrak{B}_4 we are considering here are semisimple, as they provide monodromy representations of $H_4 \otimes K_\Lambda$ and $H_4 \otimes K_\Lambda$ is semisimple (see [26] on a detail account on these monodromy properties).

From the Littlewood-Richardson rule (see e.g. [11], (15.23) and (A.8)) we get the decompositions of the $V(\lambda) \otimes V(\varpi_2)$ for $\lambda \in P_3(U_-)$ (see table 1).

From these we readily get that $M_4(4\varpi_2) \simeq S_b$, $\text{Res}M_4(2\varpi_1 + 2\varpi_3) = M_3(\varpi_1 + \varpi_2 + \varpi_3) \simeq U_{a,b}$ hence $M_4(2\varpi_1 + 2\varpi_3) \simeq T_{a,b}$, $\text{Res}M_4(2\varpi_4) = M_3(\varpi_2 + \varpi_4) \simeq V$ hence $M_4(2\varpi_4) \simeq V$, $\text{Res}M_4(3\varpi_1 + \varpi_5) = M_3(2\varpi_1 + \varpi_4) \simeq S_a$ hence $M_4(3\varpi_1 + \varpi_5) \simeq S_a$, $\text{Res}M_4(\varpi_8) = M_3(\varpi_6) \simeq S_c$ hence $M_4(\varpi_8) \simeq S_c$.

Again by using the Littlewood-Richardson rule we get the decomposition of the $V(\lambda) \otimes V(\mu)$ for $\lambda, \mu \in P_2(U_-)$ (see table 2). These decompositions determine the spectrum of t_{34} (hence of (3 4)) on any given brick.

We first deal with $2\varpi_1 + \varpi_2 + \varpi_4, \varpi_2 + 2\varpi_3, 2\varpi_1 + \varpi_6, \varpi_1 + \varpi_7, \varpi_1 + 2\varpi_2 + \varpi_3$. In all these cases, the restriction to \mathfrak{B}_3 has 2 distinct irreducible components, and there are two bricks, one being 1-dimensional and the other being 2-dimensional. In order to prove irreducibility we thus only need to prove irreducibility of the 2-dimensional brick under the action of u and W , since the irreducibility graph is then obviously connected. By lemma 2.3 this only depends on the spectra of u and W , provided s is a reflection, which means here

$V(2\varpi_2) \otimes V(2\varpi_2)$	$=$	$V(4\varpi_2) + V(\varpi_1 + 2\varpi_2 + \varpi_3) + V(2\varpi_1 + 2\varpi_3) + V(2\varpi_2 + \varpi_4)$ $+V(\varpi_1 + \varpi_3 + \varpi_4) + V(2\varpi_4)$
$V(2\varpi_2) \otimes V(\varpi_1 + \varpi_3)$	$=$	$V(\varpi_1 + 2\varpi_2 + \varpi_3) + V(2\varpi_1 + \varpi_2 + \varpi_4) + V(\varpi_2 + 2\varpi_3)$ $+V(2\varpi_2 + \varpi_4) + V(\varpi_1 + \varpi_3 + \varpi_4) + V(\varpi_1 + \varpi_2 + \varpi_5)$ $+V(\varpi_3 + \varpi_5)$
$V(2\varpi_2) \otimes V(\varpi_4)$	$=$	$V(2\varpi_2 + \varpi_4) + V(\varpi_1 + \varpi_2 + \varpi_5) + V(\varpi_2 + \varpi_6)$
$V(\varpi_1 + \varpi_3) \otimes V(\varpi_1 + \varpi_3)$	$=$	$V(2\varpi_1 + 2\varpi_3) + V(2\varpi_1 + \varpi_2 + \varpi_4) + V(\varpi_2 + 2\varpi_3)$ $+V(2\varpi_2 + \varpi_4) + 2V(\varpi_1 + \varpi_3 + \varpi_4) + V(3\varpi_1 + \varpi_5)$ $+2V(\varpi_1 + \varpi_2 + \varpi_5) + V(2\varpi_4) + V(\varpi_3 + \varpi_5) + V(2\varpi_1 + \varpi_6)$ $+V(\varpi_2 + \varpi_6)$
$V(\varpi_1 + \varpi_3) \otimes V(\varpi_4)$	$=$	$V(\varpi_1 + \varpi_3 + \varpi_4) + V(\varpi_1 + \varpi_2 + \varpi_5) + V(\varpi_3 + \varpi_5)$ $+V(2\varpi_1 + \varpi_6) + V(\varpi_2 + \varpi_6) + V(\varpi_1 + \varpi_7)$
$V(\varpi_4) \otimes V(\varpi_4)$	$=$	$V(2\varpi_4) + V(\varpi_3 + \varpi_5) + V(\varpi_2 + \varpi_6) + V(\varpi_1 + \varpi_7) + V(\varpi_8)$

TABLE 2. Decomposition of the $V(\lambda) \otimes V(\mu)$ for $V(\lambda), V(\mu) \hookrightarrow (\Lambda^2 V)^{\otimes 2}$

$U_-^{\otimes 2}$	$\varpi_1 + \varpi_3$	$\varpi_1 + \varpi_3$	$\varpi_1 + \varpi_3$
$U_-^{\otimes 3}$	$\varpi_1 + \varpi_2 + \varpi_3; 2\varpi_1 + \varpi_4$	$\varpi_1 + \varpi_2 + \varpi_3; 2\varpi_3$	$2\varpi_1 + \varpi_4; \varpi_1 + \varpi_5$
$U_-^{\otimes 4}$	$2\varpi_1 + \varpi_2 + \varpi_4$	$\varpi_2 + 2\varpi_3$	$2\varpi_1 + \varpi_6$
$Sp(u')$	$-4; 2(n-2)$	$-4; 2(n-2)$	$-4; -4(n+1)$
$Sp(W')$	$-2(n+2); 4(n-1)$	$-2(n+2); 4(n-1)$	$-4(2n+1); 4(n-1)$
$U_-^{\otimes 2}$	ϖ_4	$2\varpi_2$	
$U_-^{\otimes 3}$	$\varpi_1 + \varpi_5; \varpi_6$	$3\varpi_2; \varpi_1 + \varpi_2 + \varpi_3$	
$U_-^{\otimes 4}$	$\varpi_1 + \varpi_7$	$\varpi_1 + 2\varpi_2 + \varpi_3$	
$Sp(u')$	$-4; -4(n+1)$	$-4; 2(n-2)$	
$Sp(W')$	$-4(2n+1); 4(n-1)$	$-4(n+1); 2(n-2)$	

TABLE 3. Irreducibility of the 2-dimensional bricks.

that $\{-4/n\} \not\subseteq Sp(u)$. In table 3 we give the vertices of each brick as well as the spectra, where $u' = u$, $W' = W$. This proves irreducibility in all these cases. Since the irreducible representations of H_4 are uniquely determined by their restriction to H_3 and since we know the restrictions of the $M_4(\lambda)$, we get the conclusion for these weights.

We now consider the weights $2\varpi_2 + \varpi_4$, $\varpi_3 + \varpi_5$, $\varpi_2 + \varpi_6$. In all the cases the restriction of $M_4(\lambda)$ admits 3 pairwise non-isomorphic irreducible components, and there is 3-dimensional brick. Therefore, it remains to prove that s acts as a reflection and that $Sp(u') \cap Sp(W') = \emptyset$. From table 2 we get that the spectrum of u has 3 elements, equal to the 3 eigenvalues of t_{12} . In particular, s acts as a reflection, and it thus sufficient to compute the spectrum of W to prove irreducibility, by lemma 2.2. This is done in table 4. Then, by considering the restriction, we similarly get the conclusion for these weights.

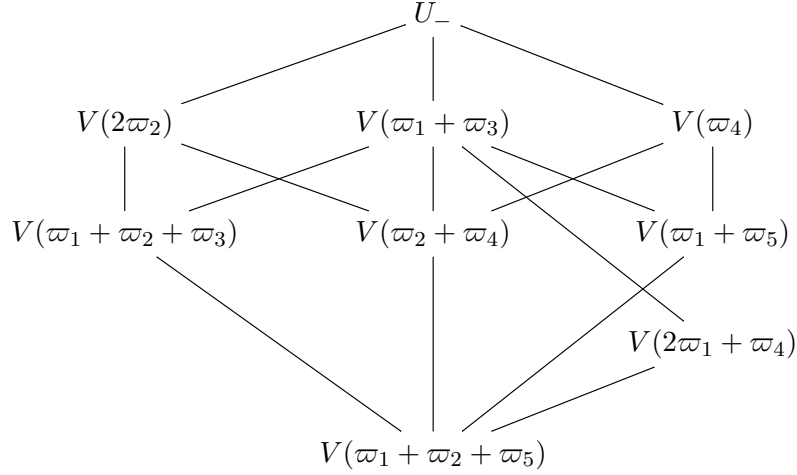
$U_-^{\otimes 2}$	$2\varpi_2$	$\varpi_1 + \varpi_3$
$U_-^{\otimes 3}$	$3\varpi_2; \varpi_1 + \varpi_2 + \varpi_3; \varpi_2 + \varpi_4$	$2\varpi_3; \varpi_2 + \varpi_4; \varpi_1 + \varpi_5$
$U_-^{\otimes 4}$	$2\varpi_2 + \varpi_4$	$\varpi_3 + \varpi_5$
$Sp(W')$	$-4(2n+1); -2(n+2); 4(2n-1)$	$-2(3n+2); -2(n+2); 2(3n-2)$
$U_-^{\otimes 2}$	ϖ_4	
$U_-^{\otimes 3}$	$\varpi_2 + \varpi_4; \varpi_1 + \varpi_5; \varpi_6$	
$U_-^{\otimes 4}$	$\varpi_2 + \varpi_6$	
$Sp(W')$	$-2(5n+2); -2(n+2); 2(5n-2)$	

TABLE 4. Irreducibility of some 3-dimensional bricks.

$S^4 U_-$	$= F_{[4]}(U_-)$	$= V(4\varpi_2) + V(2\varpi_2 + \varpi_4) + V(2\varpi_4) + V(\varpi_2 + \varpi_6) + V(\varpi_8)$
$F_{[3,1]}(U_-)$	$= V(\varpi_1 + 2\varpi_2 + \varpi_3) + V(2\varpi_2 + \varpi_4) + V(\varpi_1 + \varpi_3 + \varpi_4)$	
		$+ V(\varpi_1 + \varpi_2 + \varpi_5) + V(\varpi_3 + \varpi_5) + V(\varpi_2 + \varpi_6) + V(\varpi_1 + \varpi_7)$
$F_{[2,2]}$	$= V(2\varpi_1 + 2\varpi_3) + V(2\varpi_2 + \varpi_4) + V(\varpi_1 + \varpi_2 + \varpi_5)$	
		$+ V(2\varpi_4) + V(\varpi_2 + \varpi_6)$
$F_{[2,1,1]}(U_-)$	$= V(2\varpi_1 + \varpi_2 + \varpi_4) + V(\varpi_2 + 2\varpi_3) + V(\varpi_1 + \varpi_3 + \varpi_4)$	
		$+ V(\varpi_1 + \varpi_2 + \varpi_5) + V(\varpi_3 + \varpi_5) + V(2\varpi_1 + \varpi_6)$
$\Lambda^4 U_-$	$= F_{[1,1,1,1]}(U_-)$	$= V(\varpi_1 + \varpi_3 + \varpi_4) + V(3\varpi_1 + \varpi_5)$

TABLE 5. Plethysm of $V(\varpi_2)^{\otimes 4}$ for $\mathfrak{sl}_9(\mathbb{C})$.

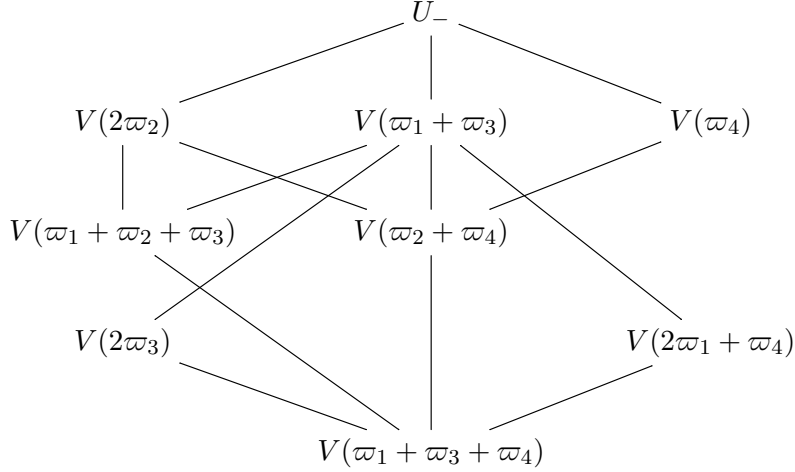
$M_4(\varpi_1 + \varpi_2 + \varpi_5)$ is irreducible .



We consider the two 2-dimensional bricks. For the one based at $V(2\varpi_2) \subset U_-^{\otimes 2}$, t_{34} has eigenvalues $-4/n$ and $\frac{-4(n+1)}{n}$. For the one based at $V(\varpi_4) \subset U_-^{\otimes 2}$, t_{34} has eigenvalues $-4/n$ and $\frac{2(n-2)}{n}$. Hence on both bricks s acts as a reflection. On the first one the eigenvalues of W are $-\frac{(7n+4)}{n}$ and $\frac{(3n-4)}{n}$, while on the second one the eigenvalues are $-\frac{(3n+4)}{n}$ and $\frac{(5n-4)}{n}$. In the irreducibility graph, we thus see that $\varpi_1 + \varpi_2 + \varpi_3$, $\varpi_2 + \varpi_4$ and $\varpi_1 + \varpi_5$ belong to the same connected component. It follows that either $M_4(\varpi_1 + \varpi_2 + \varpi_3)$

is irreducible, or it is the direct sum of two irreducible representations, one 1-dimensional and one 7-dimensional. Since there is no 7-dimensional irreducible representation of H_4 the conclusion follows. The restriction to B_3 is isomorphic to $S_a + U_{a,c} + U_{a,b} + V$ hence $M_4(\varpi_1 + \varpi_2 + \varpi_5) \simeq W_a$.

The case $M_4(\varpi_1 + \varpi_3 + \varpi_4)$.



Here the restriction to \mathfrak{B}_3 is *not* multiplicity free, hence our criterion cannot be applied directly. Consider however the 2-dimensional brick based at $V(2\varpi_2)$. The eigenvalues of t_{34} are $-4/n$ and $2(n-2)/n$ (hence s acts by a reflection). The eigenvalues of W are $-4(n+1)/n$ and $2(3n-2)/n$, hence the brick is irreducible by our criterion. Since the action of \mathfrak{B}_4 is semisimple, it easily follows that our 7-dimensional representation is a sum of an irreducible component of dimension at least 5, and of another component on which t_{12} has for only eigenvalue $-4/n$. Since there are no 5-dimensional or 7-dimensional irreducible representation for H_4 , the only possibility is that we have the sum of a 6-dimensional irreducible representations and a 1-dimensional one. The restriction to B_3 is isomorphic to $2S_a + U_{a,b} + V$ hence $M_4(\varpi_1 + \varpi_3 + \varpi_4) \simeq S_a + V_{a,b,c}$.

Remark 2.7. *We are not able to follow the same method to elucidate the bimodule structure on $U_-^{\otimes 5}$ because some of the putative Schur elements of the cubic Hecke algebra on 5 strands vanish inside R_Λ , suggesting that its representation-theoretic behavior is not generic. As a matter of fact, it can be checked that the restriction to B_4 of $M_5(\varpi_3 + \varpi_7)$ cannot be obtained by restriction of a representation of the generic Hecke algebra, thus proving that the discriminant of H_5 admits $a^3 + b^2c$ as a factor. This has for consequence that we are not able to find the dimension by easy representation-theoretic arguments based on the generic cubic Hecke algebra. The question of whether the action of B_5 is semisimple over K_Λ remains open, though.*

2.5. Quantum cubic quotients : the Links-Gould invariant. In [29] we investigated the quotient of the braid algebra involved in the Links-Gould polynomial invariant. Recall that this invariant arises through the consideration of a 1-parameter family of 4-dimensional representations of the Lie superalgebra $\mathfrak{sl}(2|1)$. This invariant is stronger than the Alexander polynomial, and yet shares a number of properties with it (for instance, it vanishes on split links).

We proved in [29] that the centralizer algebra LG_n involved in this construction is a quotient of the cubic Hecke algebra H_n , and even of Q_n . We defined a quotient (denoted A_n in [29]) of Q_n , proper when $n \geq 4$, as the quotient of KB_n by the ideal generated by $\text{Ker}(KB_4 \rightarrow LG_4)$. Let us denote it by LG'_n . We conjectured $LG_n \simeq LG'_n$. The dimensions for $n = 2, 3, 4, 5$ are 3, 20, 175, 1764, and conjecturally $\dim LG_{n+1} = (2n)!(2n+1)!/(n!(n+1)!)^2$. The description of the defining ideal $\text{Ker}(KB_4 \rightarrow LG_4)$ given in [29] was representation-theoretic at first. In particular we got that

$$\text{Irr}(LG_4) = \text{Irr}(GQ_4 \otimes K) \setminus \{T_{a,b}, T_{a,c}, V, V_{a,b,c}, V_{a,c,b}\}$$

and from this we get that the quotient map $Q_4 \otimes K \rightarrow LG_4$ factorizes through the algebra $\Lambda S_4 = \text{Im}(KB_4 \rightarrow \text{End}(U_-^{\otimes 4}))$ described above.

From this representation-theoretic description we got in particular some remarkable properties that we recall here :

$$\begin{aligned} s_1^{-1}(s_3^{-1}s_2s_3^{-1}) &\equiv (s_3^{-1}s_2s_3^{-1})s_1^{-1} \pmod{LG_3s_3LG_3 + LG_3s_3^{-1}LG_3 + LG_3} \\ s_1(s_3^{-1}s_2s_3^{-1}) &\equiv (s_3^{-1}s_2s_3^{-1})s_1 \pmod{LG_3s_3LG_3 + LG_3s_3^{-1}LG_3 + LG_3} \end{aligned}$$

2.6. The tripled quadratic Hecke algebra. In this section R denotes an arbitrary domain with $x, y \in R^\times$. We let $H_n(x, y)$ denote the (ordinary) Hecke algebra with these parameters, namely the quotient of the group algebra RB_n by the quadratic relations $(s_i - x)(s_i - y) = 0$ for $1 \leq i \leq n-1$, or equivalently by the relation $(s_1 - x)(s_1 - y) = 0$. We denote $J_n(x, y) \subset RB_n$ the (twosided) ideal generated by these relations, so that $H_n(x, y) = RB_n/J_n(x, y)$.

Now assume $a, b, c \in R^\times$ and assume the additional condition that $(a-b)(a-c)(b-c) \in R^\times$. Then, we define the *tripled quadratic algebra* as

$$\mathcal{H}_n = \mathcal{H}_n(a, b, c) = \frac{RB_n}{J_n(a, b) \cap J_n(a, c) \cap J_n(b, c)}$$

We studied this algebra in detail in [6], in the special case where R was a field of characteristic 2 containing \mathbb{F}_4 , and $\{a, b, c\} = \mathbb{F}_4 \setminus \{0\}$. It turns out that most results of [6] are also valid in the present setting, as we explain now.

Notice first that \mathcal{H}_n obviously projects onto $H_n(x, y)$ for all $x \neq y$ with $\{x, y\} \subset \{a, b, c\}$, and in particular there is a natural morphism $\mathcal{H}_n \rightarrow H_n(a, b) \oplus H_n(a, c) \oplus H_n(b, c)$. We denote $q_x : H_n(x, y) \rightarrow R$ characterized by $s_i \mapsto x$.

$$\begin{array}{ccccc} H_n(b, c) & & H_n(a, c) & & H_n(a, b) \\ q_c \downarrow & \swarrow q_b & & \swarrow q_b & \downarrow q_a \\ & R & & R & R \\ & \nwarrow q_c & & \nwarrow q_a & \\ & & & & \end{array}$$

Proposition 2.8. *The natural morphism $\mathcal{H}_n \rightarrow H_n(a, b) \oplus H_n(a, c) \oplus H_n(b, c)$ is injective, and its image is made of the triples (z_c, z_b, z_a) such that $q_\alpha(z_{\alpha'}) = q_\alpha(z_{\alpha''})$ whenever $\{\alpha, \alpha', \alpha''\} = \{a, b, c\}$. If R is a field, then \mathcal{H}_n has dimension $3(n! - 1)$.*

Proof. This proposition follows from general arguments as in [6], proposition 5.7, as soon as we know that, for $\{x, y, z\} = \{a, b, c\}$, the (twosided) ideal $J_n(x, y) + J_n(x, z)$ is generated by $s_1 - x$. This follows from the fact that $(s_1 - x)(s_1 - y) - (s_1 - x)(s_1 - z) = (z - y)(s_1 - x)$ and that $(z - y) \in R^\times$. Indeed, this imply immediately that $J_n(a, b) + J_n(b, c) + J_n(a, c) = RB_n$ again because $(a - b)(a - c)(b - c) \in R^\times$. \square

Lemma 2.9. *The following equalities hold inside \mathcal{H}_3 , where $\Sigma_1 = a + b + c$, $\Sigma_2 = ab + bc + ac$, $\Sigma_3 = abc$.*

- (1) $(s_i - a)(s_i - b)(s_i - c) = 0, i \in \{1, 2\}$
- (2) $[s_2^2, s_1] = [s_2, s_1^2]$, that is $s_2^2 s_1 = s_2 s_1^2 - s_1^2 s_2 + s_1 s_2^2$
- (3) $s_2 s_1^2 s_2 = -\Sigma_3 s_1 + \Sigma_2 s_1 s_2 + s_1 s_2 s_1^2 - \Sigma_1 s_1 s_2^2 + s_1^2 s_2^2$

The following equalities hold inside \mathcal{H}_4 ,

- (1) $s_2 s_3^2 = -s_1^2 s_3 + s_1^2 s_2 + s_2^2 s_3 - s_1 s_2^2 + s_1 s_3^2$
- (2) $s_2^2 s_3 s_1 = s_2 s_1^2 s_3 - s_1^2 s_2 s_3 + s_1 s_2^2 s_3$
- (3) $s_2^2 s_3^2 = -\Sigma_3 s_1 + \Sigma_2 s_1 s_3 - \Sigma_1 s_1^2 s_3 + \Sigma_3 s_2 - \Sigma_2 s_2 s_3 + \Sigma_1 s_1^2 s_2 - s_1^2 s_2 s_3 + \Sigma_1 s_2^2 s_3 - \Sigma_1 s_1 s_2^2 + s_1 s_2^2 s_3 + s_1^2 s_3^2$
- (4) $s_2^2 s_3^2 = -\Sigma_3 s_1 + \Sigma_2 s_1 s_3 - \Sigma_1 s_1^2 s_3 + \Sigma_3 s_2 - \Sigma_2 s_2 s_3 + \Sigma_1 s_1^2 s_2 + \Sigma_1 s_2^2 s_3 - \Sigma_1 s_1 s_2^2 + s_1^2 s_3^2 + s_2^2 s_1 s_3 - s_2 s_1^2 s_3$

Proof. By the previous proposition it is enough to prove these equalities inside each of the $H_n(x, y)$ for $\{x, y\} \subset \{a, b, c\}$. Depending on taste, this can be done (by hand or by computer) either by using the natural bases of the Hecke algebras or models over $\mathbb{Z}[a, b, c]$ of their simple modules. \square

From now on we assume that $R = K$ is a field of characteristic 0, with $a, b, c \in K$ being generic (e.g., algebraic independent over $\mathbb{Q} \subset K$). Actually, the genericity condition under which the remaining part of the section is valid is precisely the following. One needs a, b, c as well as $(a - b)(a - c)(b - c)$ to be nonzero, and also that the algebras $H_3(a, b, c)$ and $H_4(a, b, c)$ are split semisimple. This last condition can be made quite explicit under the trace conjecture of Broué, Malle and Michel (see [5] §2), which implies that the algebras H_3 and H_4 should be symmetric algebras over $\mathbb{Z}[a^\pm, b^\pm, c^\pm]$, this conjecture being known to be true for H_3 (see [19, 29]). Under this conjecture the condition of being semisimple amounts to the nonvanishing of a collection known as the Schur elements of these algebras, and they have been determined in [30] for H_3 . For H_3 this condition implies in particular $ac + b^2 \neq 0$.

We consider the element

$$\mathbf{b} = [s_2^2, s_1] - [s_2, s_1^2] = s_2^2 s_1 - s_1 s_2^2 - s_2 s_1^2 + s_1^2 s_2$$

and define $\mathcal{K}_n = \mathcal{K}_n(a, b, c) = H_n(a, b, c)/(\mathbf{b})$. Note that, because of the cubic relation on s_1 and s_2 one can express inside $H_n(a, b, c)$ each s_i^2 as a linear combination of s_i^{-1} , s_i and 1. By direct calculation one gets that \mathcal{K}_n is equivalently defined as the quotient of $H_n(a, b, c)$ by the relation $[s_2^{-1}, s_1] = [s_2, s_1^{-1}]$, that is

The image of $\mathbf{b} = [s_2^2, s_1] - [s_2, s_1^2]$ inside the 3-dimensional representation V of $H_3(a, b, c)$ (using the matrix model of section 2.1.1) is

$$\begin{pmatrix} -(a-c)(ac+b^2) & 2c(a+b) - ab - c^2 & (a-c)(ac+b^2) \\ -(ac+b^2)(a^2+bc-2a(b+c)) & 2(a-c)(ac+b^2) & (ac+b^2)(a^2+bc-2a(b+c)) \\ (a-c)(ac+b^2) & ab+c^2-2c(a+b) & -(a-c)(ac+b^2) \end{pmatrix}$$

Proposition 2.10. *Assume that a, b, c are generic. The twosided ideal of H_3 generated by \mathbf{b} is the indecomposable ideal attached to the representation V . Moreover, the natural morphism $\mathcal{K}_n \rightarrow \mathcal{H}_n$ is an isomorphism for $n \leq 4$. In particular the relations of lemma 2.9 hold true inside \mathcal{K}_3 and \mathcal{K}_4 , respectively.*

Proof. Under the genericity assumption, the first assertion is equivalent to the non-vanishing of \mathbf{b} under V , which has been established above, and its vanishing under the other irreducible representations of H_3 . But each one of these factorizes through one of the (quadratic) Hecke algebras $H_3(x, y)$, and \mathbf{b} maps to 0 in each one of them. This proves the first claim. Since this quotient has dimension $24 - 3^2 = 15 = 3(3! - 1)$ this also proves that the surjective morphism $\mathcal{K}_3 \twoheadrightarrow \mathcal{H}_3$ is an isomorphism. The ideal generated by \mathbf{b} inside the semisimple algebra $H_4(a, b, c)$ is attached, up to possibly extending the scalars to an algebraic closure of K , to the irreducible representations of $H_4(a, b, c)$ whose restriction to $H_3(a, b, c)$ contains V as a constituent. From the restriction rules recalled in section 2.1.2 we get that these irreducible representations are exactly the ones which do *not* factorize through one of the $H_4(x, y)$ and we get $\dim \mathcal{K}_4 = 3 \times 1^2 + 6 \times 3^2 + 3 \times 2^2 = 69 = 3(4! - 1) = \dim \mathcal{H}_4$ which proves the claim. \square

The next propositions are similar to the ones established in [6]. Moreover, the proofs are most of the time exactly the same. Therefore, we only provide precise references to them as well as the small changes to make when they are needed. Notice that the results below do not use the genericity condition (except for corollary 2.14).

Proposition 2.11. *For $n \geq 2$ one has*

- (1) $\mathcal{K}_{n+1} = \mathcal{K}_n + \mathcal{K}_n s_n \mathcal{K}_n + \mathcal{K}_n s_n^2 \mathcal{K}_n$
- (2) $\mathcal{K}_{n+1} = \mathcal{K}_n + \mathcal{K}_n s_n \mathcal{K}_n + \mathcal{K}_n s_n^2$
- (3) *If $k < n$, $r, t \in \{0, 1, 2\}$ we have $s_k^r s_1^t s_n^2 \equiv s_1^{r+t} s_n^2 \pmod{\mathcal{K}_n + \mathcal{K}_n s_n}$*
- (4) $\mathcal{K}_{n+1} = \mathcal{K}_n + \mathcal{K}_n s_n \mathcal{K}_n + \mathcal{K}_2 s_n^2$

Proof. Inside \mathcal{K}_3 , from $\mathbf{b}s_2 = 0$ we get $s_2 s_1^2 s_2 = s_2^2 s_1 - s_1 s_2^3 + s_1^2 s_2^2$. Using the cubic relation to expand s_2^3 one gets $s_2 s_1^2 s_2 \in \mathcal{K}_2 + \mathcal{K}_2 s_2 \mathcal{K}_2 + \mathcal{K}_2 s_2^2 \mathcal{K}_2$ hence $s_n s_{n-1}^2 s_n \in \mathcal{K}_n + \mathcal{K}_n s_n \mathcal{K}_n + \mathcal{K}_n s_n^2 \mathcal{K}_n$. From this one gets by induction (1) following the proof of [6], proposition 4.2. Then (2) follows from (1) and $\mathbf{b} = 0$ with the same proof as in [6], lemma 5.11. When $n = 2$ (3) is trivial so we can assume $n \geq 3$. By proposition 2.10 we know that the identities of lemma 2.9 are valid inside \mathcal{K}_{n+1} hence in particular $s_2 s_3^2 \equiv s_1 s_3$ and $s_2^2 s_3^2 \equiv s_1^2 s_3^2$ modulo $\mathcal{K}_3 + \mathcal{K}_3 s_3$. This implies that $s_{n-1} s_n^2 \equiv s_{n-2} s_n$ and $s_{n-1}^2 s_n^2 \equiv s_{n-2}^2 s_n$ modulo $\mathcal{K}_n + \mathcal{K}_n s_n$. From this the arguments for [6] lemma 5.11 can be applied directly and to prove (3) and then (4). \square

For $0 \leq k \leq n$, we let $s_{n,k} = s_n s_{n-1} \dots s_{n-k+1}$ with the convention that $s_{n,0} = 1$ and $s_{n,1} = s_n$. We let $\mathcal{K}_n^k = \mathcal{K}_n s_{n,k}$ (hence $\mathcal{K}_n^0 = \mathcal{K}_n$). Similarly, we let $x_{n,k} = s_n s_{n-1} \dots s_{n-k+2} s_{n-k+1}^2$ for $1 \leq k \leq n$, with the convention $x_{n,1} = s_n^2$.

Lemma 2.12.

- (1) *If $r \leq n - 1$, $1 \leq k \leq n$ and $c \in \{0, 1, 2\}$, then $s_r s_1^c x_{n,k} \in s_1^{c+1} x_{n,k} + \mathcal{K}_n^0 + \dots + \mathcal{K}_n^k$.*
- (2) $\mathcal{K}_n x_{n,k} \subset \mathcal{K}_2 x_{n,k} + \mathcal{K}_n^0 + \dots + \mathcal{K}_n^n$.

Proof. The proof of lemma 5.12 in [6] can be applied directly, as it only uses the previous result as well as $\mathbf{b} = 0$. \square

Finally, all these partial results imply the following one.

Proposition 2.13. *Let $n \geq 2$. Then $\dim \mathcal{K}_n = 3(n! - 1)$ and*

$$\mathcal{K}_{n+1} = \mathcal{K}_n \oplus \mathcal{K}_n^1 \oplus \dots \oplus \mathcal{K}_n^n \oplus \mathcal{K}_2 x_{n,1} \oplus \dots \oplus \mathcal{K}_2 x_{n,n}$$

Proof. The proof is the same as the one of proposition 5.13 in [6]. \square

Corollary 2.14. *For generic a, b, c , and all $n \geq 3$, the natural morphism $\mathcal{K}_n \rightarrow \mathcal{H}_n$ is an isomorphism.*

Corollary 2.15. *Let $x'_{n,k} = s_n s_{n-1} \cdots s_{n-k+2} s_{n-k+1}^{-1}$. Then*

$$\mathcal{K}_{n+1} = \mathcal{K}_n \oplus \mathcal{K}_n^1 \oplus \cdots \oplus \mathcal{K}_n^n \oplus \mathcal{K}_2 x'_{n,1} \oplus \cdots \oplus \mathcal{K}_2 x'_{n,n}$$

Proof. From $s_i^3 - \Sigma_1 s_i^2 + \Sigma_2 s_i - \Sigma_3 = 0$ one gets $s_i^2 - \Sigma_1 s_i + \Sigma_2 = \Sigma_3 s_i^{-1}$ hence $x_{n,k} \in \Sigma_3 x'_{n,k} + \mathcal{K}_n \oplus \mathcal{K}_n^1 \oplus \cdots \oplus \mathcal{K}_n^n$. Since $\Sigma_3 = abc$ is invertible this proves the claim. \square

Consider the shift morphism $\mathcal{K}_n \rightarrow \mathcal{K}_{n+1}$ given by $s_i \mapsto s_{i+1}$. We define a basis of \mathcal{K}_n inductively by choosing $\mathcal{B}_1 = \{1\}$ as a basis of \mathcal{K}_1 , $\mathcal{B}_2 = \{1, s_1, s_1^{-1}\}$ as a basis of \mathcal{K}_2 , and

$$\mathcal{B}_{n+1} = \left(\bigcup_{k=0}^n \mathcal{B}_n s_{n,k} \right) \cup \left(\bigcup_{k=1}^n \mathcal{B}_2 x'_{n,k} \right)$$

2.7. Vogel's algebra : definition.

2.7.1. Trivalent diagrams. We recall here basic material from the theory of Vassiliev invariants of knots and links. We let \mathbf{D} denote the category whose objects are the $[n] = \{1, \dots, n\}$ for $n \in \mathbb{N} = \mathbb{Z}_{\geq 0}$ with the convention $[0] = \emptyset$, and $\text{Hom}_{\mathbf{D}}([p], [q])$ is made of the linear combinations of the set of couples (Γ, α) where Γ is a graph with vertices of degrees either 1 or 3, together with a cyclic ordering α of the neighbours of any vertex of degree 3, such that the set of vertices of degree 1 is $[p] \sqcup [q]$, modulo the relations that

- changing α to α' where the cyclic ordering of one single vertex has been changed yields $(\Gamma, \alpha') \equiv -(\Gamma, \alpha)$
- the following local 'IHX' relation, where the orientation of the plane determines the chosen cyclic orderings

If \mathfrak{g} is a 'quadratic' Lie algebra in the sense of [31], for example a semisimple Lie algebra endowed with its Killing form, there is a well-defined functor $\Phi_{\mathfrak{g}} : \mathbf{D} \rightarrow \mathfrak{g}\text{-mod}$, where $\mathfrak{g}\text{-mod}$ is the category of (finite-dimensional) \mathfrak{g} -modules, such that $\Phi_{\mathfrak{g}}([n]) = \mathfrak{g}^{\otimes n}$ is the n -th tensor power of the adjoint representation of \mathfrak{g} . Moreover, in case it is a simple Lie algebra endowed with its Killing form,

$$\Phi_{\mathfrak{g}} : \begin{array}{c} \bullet \text{---} \bullet \\ | \\ \bullet \text{---} \bullet \end{array} \mapsto \psi_{\mathfrak{g}} = \sum_{i=1}^n \varepsilon_i \otimes \varepsilon^i \in \text{End}(\mathfrak{g})^{\otimes 2} = \text{End}(\mathfrak{g}^{\otimes 2})$$

where e_1, \dots, e_n is a fixed arbitrary basis of \mathfrak{g} , e^1, \dots, e^n is its dual basis with respect to the Killing form, and $\varepsilon_i, \varepsilon^i$ denote the images of $e_i, e^i \in \mathfrak{g} \subset \mathbf{U}\mathfrak{g}$ under the natural map $\mathbf{U}\mathfrak{g} \rightarrow \text{End}(\mathfrak{g})$. Note that

$$2 \sum_{i=1}^n e_i \otimes e^i = \Delta(C) - C \otimes 1 - 1 \otimes C$$

where $C = \sum_{i=1}^n e_i e^i \in Z(\mathfrak{U}\mathfrak{g})$ is the Casimir operator. Therefore $\psi_{\mathfrak{g}}$ acts by a scalar on any simple component of $\mathfrak{g}^{\otimes 2}$.

The action of $\psi_{\mathfrak{g}}$ on $\Lambda^2 \mathfrak{g}$ has 2 eigenvalues, 0 and $t \neq 0$. Moreover, $K = \text{Ker}(\psi_{\mathfrak{g}})_{\Lambda^2 \mathfrak{g}} = \text{Ker}([\ , \] : \mathfrak{g}^{\otimes 2} \rightarrow \mathfrak{g})$ and $\Lambda^2 \mathfrak{g}/K \simeq \mathfrak{g}$. For most Cartan types, $\psi_{\mathfrak{g}}$ acts on $S^2 \mathfrak{g} \subset \mathfrak{g} \otimes \mathfrak{g}$ with 4 eigenvalues $2t, \alpha, \beta, \gamma$.

2.7.2. The case of exceptional Lie algebras. Assume now that \mathfrak{g} is a (complex) simple Lie algebra of exceptional Cartan type E_6, E_7, E_8, F_4 or G_2 . Then $\psi_{\mathfrak{g}}$ acts on $S^2 \mathfrak{g} \subset \mathfrak{g} \otimes \mathfrak{g}$ with only 3 eigenvalues $2t, \alpha, \beta$ with the relation $\alpha + \beta = t/3$. With the purpose of uniformizing the results in the spirit of [32], another parameter $\gamma = 2t/3$ is introduced, so that $\alpha + \beta + \gamma = t$. The preferred choice of ordering for α, β is as follows (see [32]).

	α	β	γ
E_6	3	-1	4
E_7	4	-1	6
E_8	6	-1	10
F_4	5	-2	6
G_2	5	-3	4

Assume that such a Lie algebra \mathfrak{g} is fixed, together with the corresponding choice of α, β . Then the functor $\Phi_{\mathfrak{g}}$ factors through the category \mathbf{D}^{exc} , quotient of \mathbf{D} by the local relation

$$\begin{aligned}
 & \left(\text{Diagram 1} \right) = (\alpha + \beta) \left(\text{Diagram 2} + \text{Diagram 3} \right) \\
 & - \frac{\alpha\beta}{2} \left(\text{Diagram 4} + \text{Diagram 5} + \text{Diagram 6} \right)
 \end{aligned}$$

Let us now consider the algebra

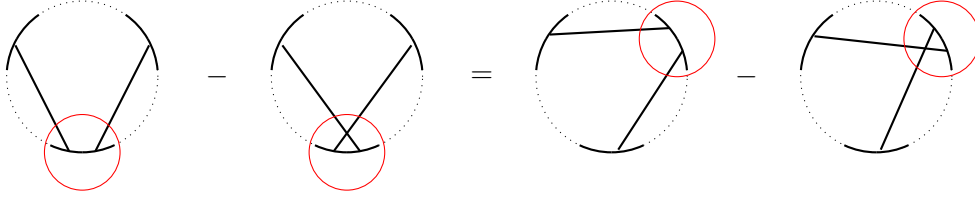
$$D^{exc}(n) = \frac{\mathbf{D}^{exc}([n], [n])}{\sum_{k < n} \mathbf{D}^{exc}([n], [k]) \circ \mathbf{D}^{exc}([k], [n])}$$

Inside this quotient, the following local relation obviously holds.

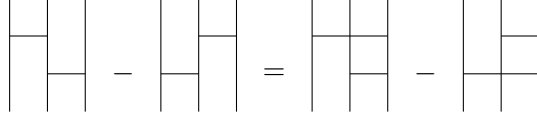
$$\left(\text{Diagram 1} \right) = (\alpha + \beta) \left(\text{Diagram 2} \right) - \frac{\alpha\beta}{2} \left(\text{Diagram 3} + \text{Diagram 4} \right)$$

2.7.3. Trivalent diagrams and infinitesimal braids.

$$\left(\text{Diagram 1} \right) - \left(\text{Diagram 2} \right) = \left(\text{Diagram 3} \right) - \left(\text{Diagram 4} \right)$$



The IHX relations imply the above so-called 4T relations (apply the IHX relation inside the red circles). These can be rewritten ‘horizontally’ as



and can be viewed as a relation inside $\mathbf{D}([3], [3])$. Let $n \geq 2$. For $1 \leq i < j \leq n$ we let $d_{ij} \in \mathbf{D}([n], [n])$ denote the diagram that differs from the identity diagram only in that the i -th and j -th strands are connected by an additional straight arc. Then the above relation reads $[d_{ij}, d_{ik} + d_{kj}] = 0$, with the convention that $d_{ij} = d_{ji}$ when $i > j$. Obviously one also has the relations $[d_{ij}, d_{rs}] = 0$ whenever $\#\{i, j, r, s\} = 4$.

Moreover, there exists a natural embedding $\mathfrak{S}_n \subset \mathbf{D}([n], [n])$. We denote s_{ij} the (image of) the transposition $(i j)$. Clearly, for $w \in \mathfrak{S}_n$, one has $wd_{ij}w^{-1} = d_{w(i), w(j)}$.

Recall that the holonomy Lie algebra \mathcal{T}_n of the ordered configuration space of n points in the plane admits a presentation by generators $t_{ij} = t_{ji}$ for $1 \leq i \neq j \leq n$ and relations $[t_{ij}, t_{rs}] = 0$ whenever $\#\{i, j, r, s\} = 4$ and $[t_{ij}, t_{ik} + t_{kj}] = 0$. It is acted upon by \mathfrak{S}_n such that $w.t_{ij} = t_{w(i), w(j)}$. It follows that there is a natural algebra morphism $\mathfrak{B}_n \rightarrow \mathbf{D}([n], [n])$ inducing the identity on \mathfrak{S}_n and mapping t_{ij} to d_{ij} , where $\mathfrak{B}_n = \mathbb{k}\mathfrak{S}_n \ltimes \mathcal{U}\mathcal{T}_n$.

2.7.4. Vogel’s algebra. The image of \mathfrak{B}_n under the composite map $\mathfrak{B}_n \rightarrow \mathbf{D}([n], [n]) \rightarrow \mathbf{D}^{exc}([n], [n]) \rightarrow D^{exc}(n)$ is an algebra generated by \mathfrak{S}_n and the (images of the) t_{ij} ’s, that satisfies the defining relations of \mathfrak{B}_n together with the following additional ones.

- (1) For all i, j , $t_{ij}(i, j) = (i, j)t_{ij} = t_{ij}$
- (2) For all i, j ,

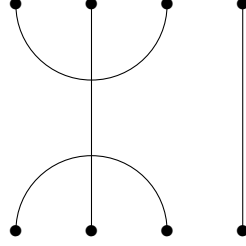
$$t_{ij}^2 - (\alpha + \beta)t_{ij} + \frac{\alpha\beta}{2}(1 + (i, j)) = 0$$

Indeed, the first one is a consequence of the image of the IHX relation inside $D^{exc}(n)$, and the second one is the image of the ‘exceptional’ relation of \mathbf{D}^{exc} . In this paper we call the corresponding quotient of \mathfrak{B}_n by these abstract relations *Vogel’s algebra* and denote it by \mathfrak{V}_n . P. Vogel communicated to me ([34]) that he computed the dimension over $\mathbb{Q}(\alpha, \beta)$ of these algebras for small n under the assumption that they are finite-dimensional, and that he got that they are semisimple of dimensions 3, 20, 264, 6490, 141824, 6799151 for $n = 2, 3, 4, 5, 6, 7$. He conjectured this algebra to be finite-dimensional and semisimple in general.

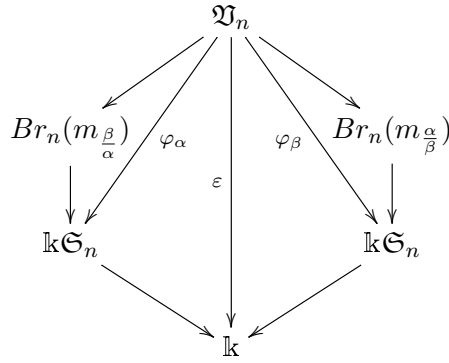
2.8. Vogel’s algebra : representations.

2.8.1. Quotients of Vogel’s algebra. We now assume that \mathbb{k} is a field of characteristic not 2. For all $u, v \in \mathbb{k}$ there exists a well-defined (surjective) morphism $\varphi_{u,v} : \mathfrak{B}_n \rightarrow \mathbb{k}\mathfrak{S}_n$ restricting to the identity on $\mathbb{k}\mathfrak{S}_n$ and mapping t_{ij} to $u.1 + v.(i j)$. It is immediately checked that it factorizes through \mathfrak{V}_n iff $u = v \in \{\alpha/2, \beta/2\}$. It follows that the composite of natural maps $\mathbb{k}\mathfrak{S}_n \rightarrow \mathfrak{B}_n \rightarrow \mathfrak{V}_n$ is injective. We also note that the morphism $\mathbb{k}\mathfrak{B}_n \rightarrow \mathbb{k}$ mapping each $w \in \mathfrak{S}_n$ to its sign and t_{ij} to 0 also factorizes through \mathfrak{V}_n . We denote it by $\varepsilon : \mathfrak{V}_n \rightarrow \mathbb{k}$.

We now introduce the algebra $Br_n(m)$ of Brauer diagrams. Its natural basis is made of Brauer diagrams, a Brauer diagram being a collection of matchings between $2n$ points which is depicted with n points on the top and n points on the bottom, so that they can be composed to the expense of possibly making a circle appear. In this case this circle is converted into the scalar m . We refer to e.g. [12] for more details on this object, and recall that it admits a natural subalgebra isomorphic to $\mathbb{k}\mathfrak{S}_n$. By abuse of notations we identify the transposition (i, j) with its image. We denote p_{ij} the diagram matching i with j , $n + i$ with $n + j$ and k with $n + k$ for $k \geq n$ and $k \notin \{i, j\}$. For instance the following picture depicts $p_{13} \in Br_4(m)$.



There exists a 2-parameters family of morphisms $\psi_{u,v} : \mathfrak{B}_n \rightarrow Br_n(m)$ restricting to the identity on $\mathbb{k}\mathfrak{S}_n$ and mapping t_{ij} to $u \cdot 1 + v \cdot ((i, j) - p_{ij})$ (see [?]). It factors through the relation $t_{ij}(i, j) = t_{ij}$ only if $u = v$, and finally factors through \mathfrak{V}_n iff $u \in \{\alpha/2, \beta/2\}$ and $u(m-4) = -(\alpha+\beta)$. We thus get the following commutative diagram, with $m_x = 4 - 2(1+x^{-1})$:



2.8.2. *Vogel's algebra for $n = 2, 3$.* First assume $n = 2$, $\alpha\beta \neq 0$ and $\alpha \neq \beta$. Also assume that \mathbb{k} is a field. From the relations it is clear that \mathfrak{V}_2 is spanned by $1, (1, 2), t_{12}$ and also by $1, t_{12}, t_{12}^2$. Moreover, we easily get from the relations that $t_{ij}^3 = (\alpha + \beta)t_{ij}^2 - \alpha\beta t_{ij}$ which can be rewritten $t_{ij}(t_{ij} - \alpha)(t_{ij} - \beta) = 0$. Therefore

$$\mathfrak{V}_2 \simeq \mathbb{k}[X]/X(X - \alpha)(X - \beta) \simeq \frac{\mathbb{k}[X]}{X} \oplus \frac{\mathbb{k}[X]}{X - \alpha} \oplus \frac{\mathbb{k}[X]}{X - \beta} \simeq \mathbb{k} \oplus \mathbb{k} \oplus \mathbb{k}$$

and, under this isomorphism, we have $t_{12} \mapsto (0, \alpha, \beta)$ while $(1, 2) \mapsto (-1, 1, 1)$.

We now consider the case $n = 3$, and assume that $\text{char.}\mathbb{k} \notin \{2, 3\}$. In particular the algebra $\mathbb{k}\mathfrak{S}_3$ is split semisimple. The two surjective morphisms φ_α and φ_β together with the irreducible representations of \mathfrak{S}_3 provide irreducible representations of dimension 1 and 2 of \mathfrak{V}_3 . They can be distinguished up to isomorphism from the spectrum of $t_{1,2}$, which is $\{\alpha, 0\}, \{\beta, 0\}, \{\alpha\}, \{\beta\}, \{0\}$. Moreover, since the Brauer algebra admits an irreducible representation of dimension 3, it provides generically at least one 3-dimensional generically irreducible representation for \mathfrak{V}_3 . Actually, the irreducible representations of dimension 3 of \mathfrak{B}_3

have been classified in [21] (see proposition 10). There is up to isomorphism only one such irreducible representation such that the spectrum of t_{12} is (contained in) $\{0, \alpha, \beta\}$, provided that $(\alpha\beta)/2 \neq ((\alpha + \beta)/3)^2$. One matrix model for it is

$$(1, 2) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (2, 3) = \frac{1}{2} \begin{pmatrix} 1 & 1 & 0 \\ 3 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad t_{12} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2d & b \\ 0 & c & d \end{pmatrix}$$

with $d = (\alpha + \beta)/3$ and $bc = 2d^2 - \alpha\beta$. It is straightforward to check that it defines indeed a representation of \mathfrak{B}_3 , which is irreducible under the above condition, and that the defining relations of \mathfrak{Y}_3 are indeed satisfied. This proves that \mathfrak{Y}_3 has dimension at least $3 \times 1^2 + 2 \times 2^2 + 3^2 = 20$.

2.8.3. Braids and infinitesimal braids. In this section, we let $R = \mathbb{C}[[h]]$, and assume $\alpha, \beta \in \mathbb{k} \setminus \{0\} \subset \mathbb{C}$. We also assume $\alpha \neq \beta$. We identify B_n with the fundamental group of $\mathbb{C}_*^n / \mathfrak{S}_n$ with $\mathbb{C}_*^n = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j\}$ with respect with some arbitrarily chosen base point (alternatively, we could choose an arbitrary associator). From the $\mathfrak{Y}_n \otimes R$ -valued 1-form

$$\frac{h}{i\pi} \sum_{1 \leq i < j \leq n} t_{ij} d \log(z_i - z_j)$$

we get by monodromy a morphism $RB_n \rightarrow \mathfrak{Y}_n \otimes R$ that can be extended to $KB_n \rightarrow \mathfrak{Y}_n \otimes K$ with $K = \mathbb{C}((h))$. The image of an arbitrary Artin generator s_i is then conjugated to $(i, i+1)e^{ht_{i,i+1}}$ and therefore to $(1, 2)e^{ht_{1,2}} \in \mathfrak{Y}_2 \otimes R$. It follows that it is annihilated by the polynomial $(X+1)(X-e^{h\alpha})(X-e^{h\beta})$. Therefore the morphism $RB_n \rightarrow \mathfrak{Y}_n \otimes R$ factorizes through the cubic Hecke algebra.

We now prove that the morphism $KB_n \rightarrow \mathfrak{Y}_n \otimes K$ is surjective, following arguments of [26]. Indeed, The image of s_i is congruent to $(i, i+1)$ modulo h , and the image of $h^{-1}(s_i^2 - 1)$ is congruent to $2t_{i,i+1}$ modulo h . Let C be the R -subalgebra of KB_n generated by the σ_i and the $h^{-1}(s_i^2 - 1)$ for $1 \leq i \leq n-1$. The monodromy morphism $RB_n \rightarrow \mathfrak{Y}_n \otimes R$ can be extended to a morphism $C \rightarrow \mathfrak{Y}_n \otimes R$. Since \mathfrak{S}_n is generated by the $(i, i+1)$ and since \mathfrak{Y}_n is generated by \mathfrak{S}_n and t_{12} , we get that that reduction mod h of this morphism is surjective, and therefore the morphism $C \rightarrow \mathfrak{Y}_n \otimes R$ is surjective by Nakayama's lemma. It follows immediately that the morphism $KB_n \rightarrow \mathfrak{Y}_n \otimes K$ is surjective.

In the cases $n \leq 3, 4, 5$, since the cubic Hecke algebras are finite dimensional and semisimple, it follows that $K\mathfrak{Y}_n$ is also finite-dimensional and semisimple.

We now restrict to the case $n = 3$. We know that $K\mathfrak{Y}_3$ is isomorphic to a quotient of the cubic Hecke algebra of dimension at least 20, with at least 3 irreducible 1-dimensional representations, 2 of dimension 2, 1 of dimension 3. It follows that either $K\mathfrak{Y}_3 \simeq \mathbb{Q}_3$, or $K\mathfrak{Y}_3 \simeq H_3$ and there exists a 2-dimensional irreducible representation of \mathfrak{Y}_3 such that the spectrum of σ_i under the monodromy morphism is $\{e^{h\alpha}, e^{h\beta}\}$. This is possible only if there exists a 2-dimensional irreducible representation of \mathfrak{Y}_3 such that $Sp(t_{ij}) = \{\alpha, \beta\}$. But since $\alpha\beta \neq 0$ this forces t_{ij} to be invertible, whence by the relation $t_{ij}(i, j) = t_{ij}$ we would have $(i, j) = 1$ for all i, j . But then the action of \mathfrak{S}_n would be trivial, and $\text{Ker}(t_{12} - \alpha)$ would be proper stable subspace of the representation. This contradiction proves that the irreducible representations of \mathfrak{Y}_3 determined above are the only ones, that $\dim \mathfrak{Y}_3 = 20$ and that $\mathbb{Q}_3 \simeq K\mathfrak{Y}_3$.

2.8.4. *Representations.* If E be a representation of \mathfrak{S}_n it can be expanded as a representation of \mathfrak{B}_n by letting $t_{ij} = \frac{\alpha}{2}(1 + (i\ j))$ or $t_{ij} = \frac{\beta}{2}(1 + (i\ j))$.

For E associated to $[3, 1]$ we denote these representations $\mathcal{U}_{b,a}$ and $\mathcal{U}_{c,a}$, while for E associated to $[2, 1, 1]$ we denote these representations $\mathcal{U}_{a,b}$ and $\mathcal{U}_{a,c}$. One readily checks that the monodromy representation associated to $\mathcal{U}_{x,y}$ is $U_{x,y}$.

There is an algebra morphism $\mathfrak{B}_4 \rightarrow \mathfrak{B}_3$ induced by the special morphism $\mathfrak{S}_4 \rightarrow \mathfrak{S}_3$ and $t_{ij} \mapsto t_{rs}$ with $\{r, s\} = \{i, j\}$ if $1 \leq i, j \leq 3$ and $\{r, s, i, j\} = \{1, 2, 3, 4\}$ otherwise. From this the irreducible representations of \mathfrak{B}_3 can be expanded to \mathfrak{B}_4 . The monodromy of these representations yields the representations $V, T_{a,b}, T_{a,c}, S_a, S_b, S_c$.

Let $E = \mathbb{C}\mathcal{R}$ be a vector space spanned by the set \mathcal{R} of transpositions of \mathfrak{S}_n . We denote $v_s = v_{ij}$ the basis vector corresponding to $s = (i\ j) \in \mathcal{R}$. We endow it with the \mathfrak{S}_n -permutation module structure associated to the action by conjugation on the transpositions, $w.v_s = v_{wsw^{-1}}$ and we fix $\lambda \in \mathbb{C}^\times, m, x \in \mathbb{C}$,

We set $t_s.v_s = \lambda(m+x)v_s$, $t_s.v_u = x\lambda v_u + \lambda v_{sus}$ if $s \neq u$, $su = us$, and $t_s.v_u = x\lambda v_u + \lambda v_{sus} - \lambda v_s$ otherwise. It is known (see [22], [24]) that this defines a representation of \mathfrak{B}_n for $x = 0$, which is irreducible for generic values of m . Since $t_{ij} \mapsto \lambda(t_{ij} + x)$ defines an automorphism of \mathfrak{B}_n these formulas provide a representation of \mathfrak{B}_n for arbitrary values of x , irreducible for generic values of m . Moreover, $st_s.v_s = \lambda(m+x)v_s = t_s.v_s$; if $s \neq u$, $su = us$, then $st_s.v_u = x\lambda v_{sus} + \lambda v_u = t_s.v_u$ iff $x = 1$; otherwise $st_s.v_u = x\lambda v_{sus} + \lambda v_u - \lambda v_s = t_s.v_u$ iff $x = 1$.

Consider $s = (1\ 2)$. The eigenspaces for t_{12} are $\mathbb{C}v_s$ with eigenvalue $\lambda(m+1)$; the subspace spanned by the v_u for $su = us$, $s \neq u$ and the $v_{1k} + v_{2,k} + (2/(m-1))v_s$ for $3 \leq k \leq n$ with eigenvalue 2λ , which has dimension $(n-1)(n-2)/2$; the subspace spanned by the $v_{1k} - v_{2k}$ with eigenvalue 0 has dimension 0, which has dimension $n-2$. This proves that t_{12} is diagonalizable and satisfies the relation $t_{12}(t_{12} - \lambda(m+1))(t_{12} - 2\lambda) = 0$. Choosing $2\lambda \in \{\alpha, \beta\}$ and $m = (\alpha + \beta)/\lambda - 3$ we get that t_{12} satisfies the relation $t_{12}(t_{12}^2 - (\alpha + \beta)t_{12} + \alpha\beta) = 0$ whence we get two irreducible representations of \mathfrak{B}_n .

We now construct matrix models for the remaining representations. From the study of the quantum construction in section 2.4 we get an hint about their restriction to \mathfrak{S}_4 . Computing the plethysm of $V(\varpi_2)^{\otimes 4}$ for e.g. type A_8 we get table 5 which implies that the restriction of $M(\varpi_3 + \varpi_5)$ to \mathfrak{S}_4 has isomorphism type $[3, 1] + [2, 1, 1]$ while the restriction of $M(\varpi_1 + \varpi_2 + \varpi_5)$ has type $[3, 1] + [2, 2] + [2, 1, 1]$ (at least for $n = 9$).

Let us then introduce the matrices

$$A_1 = \begin{pmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} A_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

and

$$B_1 = \text{diag}(-1, 1) = B_3, B_2 = \frac{1}{2} \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}$$

Then $s_i \mapsto A_i$ defines the representation of \mathfrak{S}_4 associated with the partition $[3, 1]$, while $s_i \mapsto B_i$ defines the representation associated with the partition $[2, 2]$,

We first consider the representation of \mathfrak{S}_4 associated to $[3, 1] + [2, 1, 1]$, given by $s_i \mapsto \text{diag}(A_i, -A_i)$. We set $t_{12} \mapsto \frac{\alpha}{2}(s_1 + 1) + p_{12}$ with

$$p_{12} = \begin{pmatrix} 0 & 0 & \frac{2\alpha-\beta}{4} & \frac{\beta(\beta-2\alpha)}{16} & 0 & 0 \\ 0 & 0 & \frac{\beta-2\alpha}{2} & \frac{\beta(2\alpha-\beta)}{8} & 0 & 0 \\ 0 & 0 & -2 & \beta/2 & 0 & 0 \\ 0 & 0 & 1 & -\beta/4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The endomorphism p_{12} has rank 1 and nontrivial eigenvalue $\beta - \alpha$. This provides an irreducible representation of dimension 6 of \mathfrak{A}_4 , for which t_{12} has eigenvalues $0, 0, 0, \alpha, \alpha, \beta$. Another one is obtained by exchanging α and β .¹

We now construct a 8-dimensional representation. We set $s_i \mapsto \text{diag}(A_i, B_i, -A_i)$ and

$$t_{12} \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2c & 4c & c & 0 & 2c^2 + 2a & a & 0 & 0 \\ 0 & 0 & 2c & 0 & 0 & -2a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 4c & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 6c & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -3c & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

It is readily checked that, $t_{ij} = wt_{12}w^{-1}$ does not depend on the choice of $w \in \mathfrak{S}_4$ such that $w(\{1, 2\}) = \{i, j\}$, and that this provides a representation of \mathfrak{B}_4 . One then checks that the other relations of \mathfrak{A}_4 are satisfied for $\alpha + \beta = 8c$ and $\alpha\beta = 12c^2 - 4a$, that is $\alpha = 4c + 2\sqrt{c^2 + a}$ and $\beta = 4c - 2\sqrt{c^2 + a}$. The representation is clearly irreducible (look at the restriction to \mathfrak{S}_4 and check that the isotopic components are mapped to each other by t_{12}). The spectrum is easily checked to be $0, 0, 0, 0, \alpha, \alpha, \beta, \beta$, and the representation is clearly irreducible for generic values of $a, c \neq 0$.

2.9. Freeness for the case $n = 3$. We set $x_i = t_{jk}$ and $z_i = (j, k)$ whenever $\{i, j, k\} = \{1, 2, 3\}$. From the defining relation $x_1(x_2 + x_3) = (x_2 + x_3)x_1$ we get $x_1(x_2 + x_3)z_3 = (x_2 + x_3)x_1z_3$ hence

$$\begin{aligned} x_1x_2z_3 + x_1(x_3z_3) &= x_2x_1z_3 + x_3(x_1z_3) \\ \Leftrightarrow x_1x_2z_3 + x_1x_3 &= x_2x_1z_3 + (x_3z_3)x_2 \\ \Leftrightarrow x_1x_2z_3 + x_1x_3 &= x_2x_1z_3 + x_3x_2 \\ \Leftrightarrow (x_1x_2 - x_2x_1)z_3 &= x_3x_2 - x_1x_3 \end{aligned}$$

that is $[x_1, x_2]z_3 = x_3x_2 - x_1x_3$. Multiplying on the right by x_3 yields $[x_1, x_2]x_3 = x_3x_2x_3 - x_1x_3^2$ hence $x_1x_2x_3 - x_2x_1x_3 = x_3x_2x_3 - x_1x_3^2$. and

$$[x_1, x_2]x_3 = x_3x_2x_3 - x_1x_3^2 = x_3(x_2 + x_1)x_3 - x_3x_1x_3 - x_1x_3^2 = (x_2 + x_1)x_3^2 - x_3x_1x_3 - x_1x_3^2 = x_2x_3^2 - x_3x_1x_3 = (\alpha + \beta)x_2x_3 - \frac{\alpha\beta}{2}x_2(1 + z_3) - x_3x_1x_3$$

Also note that $x_1x_2x_1 = x_1x_2z_1x_1 = x_1z_1x_3x_1 = x_1x_3x_1$.

Let

$$\mathcal{B}(3) = \{1, x_1, x_2, x_3, z_1, z_2, z_3, x_1x_2, x_1x_3, x_1z_2, x_1z_3, x_2x_1, x_2x_3, x_2z_1, x_2z_3, x_3z_1, x_3z_2, z_1z_2, z_1z_3, x_1x_2x_3\}$$

¹This representation of \mathfrak{S}_4 is its hyperoctahedral representation as a Coxeter group of type A_3 , see [23]. However, on the corresponding easy matrix model we were unable to find a nicer description of t_{12} , so we prefer this one where irreducibility is obvious.

We let $\mathcal{B}_0(3) = \mathcal{B}(3) \setminus \{x_1x_2x_3\}$. We need to prove $bx_i \in RB(3)$ for $i \in \{1, 2, 3\}$ and $b \in \mathcal{B}(3)$. For $b = 1$ this is clear. Let us consider the case $b = x_j$. The case $i = j$ is clear, so we assume $i \neq j$. We have $bx_i \in \mathcal{B}(3)$ for all i , except for $j = 3$. From $(x_3 + x_2)x_1 = x_1(x_2 + x_3)$ we get $x_3x_1 = x_1x_2 + x_1x_3 - x_2x_1 \in RB_0(3)$ and similarly from $(x_3 + x_1)x_2 = x_2(x_1 + x_3)$ we get $x_3x_2 \in RB_0(3)$. The case $b = z_j$ is clear since $\mathfrak{S}_3 \in \mathcal{B}(3)$ and from the defining relations $z_jx_i = x_kz_j$ and $z_jx_j = x_jz_j = x_j$ for $\{i, j, k\} = \{1, 2, 3\}$. Before the next case we first prove the following lemma.

Lemma 2.16.

- (1) For all i, j , $x_ix_j \in RB_0(3)$, $x_iz_j \in RB_0(3)$ and $\mathfrak{S}_3 \subset RB_0(3)$.
- (2) For all i, j we have $2x_ix_jx_i \in RB_0(3)$
- (3) For all $\sigma \in \mathfrak{S}_3$ we have $2x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)} \equiv 2x_1x_2x_3 \pmod{RB_0(3)}$
- (4) $(\alpha + \beta)x_1x_2x_3 \in 2x_1x_2x_3 + RB_0(3)$

Proof. (1) is a direct consequence of the definition of $\mathcal{B}_0(3)$ and of the defining relations, except for $x_3x_i \in RB_0(3)$ for $i = 1, 2$. From $(x_3 + x_2)x_1 = x_1(x_2 + x_3)$ we get $x_3x_1 = x_1x_2 + x_1x_3 - x_2x_1 \in RB_0(3)$ and similarly from $(x_3 + x_1)x_2 = x_2(x_1 + x_3)$ we get $x_3x_2 \in RB_0(3)$. In order to prove (2) we establish the identity $2x_1x_2x_1 + (\alpha\beta/2)x_3z_1 + (\alpha\beta/2)x_2z_1 - (\alpha + \beta)x_1x_3 - (\alpha + \beta)x_1x_2 + (\alpha\beta/2)x_3 + (\alpha\beta/2)x_2$. From $x_1(x_2 - x_3)x_1 = 0$ we get $0 = 2x_1x_2x_1 - x_1(x_2 + x_3)x_1 = 2x_1x_2x_1 - x_1^2(x_2 + x_3)$ and the identity follows by expanding x_1^2 and easy defining relations. Therefore $2x_1x_2x_1 \in \mathcal{B}_0(3)$. Since the statement is \mathfrak{S}_3 -symmetric this proves (2). Now notice $x_2x_1x_3 = x_2(x_1 + x_3)x_3 - x_2x_3^2 = (x_1 + x_3)x_2x_3 - x_2x_3^2 = x_1x_2x_3 + x_3x_2x_3 - x_2x_3^2$ hence $2x_2x_1x_3 \equiv 2x_1x_2x_3 + RB_0(3)$. Similarly we get $2x_1x_3x_2 \equiv 2x_1x_2x_3 + RB_0(3)$ and this proves (3). From the defining relations we get $(\alpha + \beta)x_1x_2z_3 = x_1^2x_2z_3 + (\alpha\beta/2)(1 + z_1)x_2z_3$. Since $z_1x_2z_3 = z_1z_3x_2 = z_3z_2x_2 = z_3x_2 = x_1z_3RB_0(3)$ we only need to prove $x_1^2x_2z_3 \in 2x_1x_2x_3 + RB_0(3)$. Now $x_1^2x_2z_3 = x_1^2(x_2 + x_3)z_3 - x_1^2x_3z_3 = x_1(x_2 + x_3)x_1z_3 - x_1^2x_3$. But we get $x_1^2x_3 \in RB_0(3)$ by expanding x_1^2 and we have $x_1(x_2 + x_3)x_1 = x_1x_2x_1 + x_1x_3x_1 = 2x_1x_3x_1$ hence $x_1(x_2 + x_3)x_1z_3 = 2x_1x_3x_1z_3 = 2x_1x_3z_3x_2 = 2x_1x_3x_2 \equiv 1x_1x_2x_3 \pmod{RB_0(3)}$ and this proves (4). \square

Now consider the case $b = x_jx_k$ with $j \neq k$. If $i \in \{j, k\}$ then we get $2bx_i \in RB_0(3)$ either by expanding x_k^2 or by part (2) of the lemma. If not, then there exists $\sigma \in \mathfrak{S}_3$ such that $x_jx_kx_i = x_{\sigma(1)}x_{\sigma(2)}x_{\sigma(3)}$ hence $2bx_i \in 2x_1x_2x_3 + RB_0(3)$ by part (3) of the lemma. Now consider the case $b = x_jz_k$. If $i = k$ it is clear, and if $i \notin \{k, j\}$ we get $bx_i = x_jz_kx_i = x_j^2z_k \in RB_0(3)$ by expanding x_j^2 . Therefore we can assume $i = j$. Then $bx_i = x_iz_kx_i = x_{\sigma(1)}x_{\sigma(2)}z_{\sigma(3)}$ for some $\sigma \in \mathfrak{S}_3$. Therefore $(\alpha + \beta)bx_i \in 2x_1x_2x_3 + RB_0(3)$ by the lemma. Finally, if $b \in \mathfrak{S}_3$ is a 3-cycle, then b can be written as z_jz_i for some j hence $bx_i = z_jz_ix_i = z_jx_i \in RB_0(3)$. This proves $RB_0(3)x_i \subset RB(3)$ for all i .

Assume now that 2 and $(\alpha + \beta)$ are invertible. We need to prove $x_1x_2x_3x_i \in RB(3)$ for all i . By part (3) of the lemma we get that $x_1x_2x_3 \equiv x_kx_jx_i \pmod{RB_0(3)}$ for some k, j . Since $RB_0(3)x_i \subset RB(3)$ this implies $x_1x_2x_3x_i \in x_kx_jx_i^2 + RB(3)$. But expanding x_i^2 we get that $x_kx_jx_i^2 \in RB(3)$ and this proves the claim.

3. STRUCTURE ON 3 STRANDS AND GENERAL PROPERTIES

In this section we first define Q_n over $R = \mathbb{Z}[a^{\pm 1}, b^{\pm 1}, c^{\pm 1}]$, as in the introduction. By definition, Q_n is the quotient of H_n by one of the following two relations, that we will prove to be equivalent.

$$\begin{aligned}
(1) \quad s_1^{-1}s_2s_1 &= s_1s_2s_1^{-1} - a^{-1}s_1s_2 + as_1s_2^{-1} + as_1^{-1}s_2 - a^3s_1^{-1}s_2^{-1} + a^{-1}s_2s_1 - as_2s_1^{-1} - as_2^{-1}s_1 \\
&\quad + a^3s_2^{-1}s_1^{-1} + a^2s_1^{-1}s_2^{-1}s_1 - a^2s_1s_2^{-1}s_1^{-1} \\
(2) \quad s_1s_2s_1^2s_2 &= s_1^2s_2s_1^2 + as_2s_1^2s_2 - as_1s_2^2s_1 + a^2s_2^2s_1 - a^2s_2s_1^2 + a^2s_1s_2^2 - a^2s_1^2s_2 \\
&\quad - a^3s_2^2 + a^3s_1^2 + a^4s_2 - a^4s_1
\end{aligned}$$

In other words, it is defined as the quotient of RB_n for $n \geq 3$ by the cubic relation $(s_1 - a)(s_1 - b)(s_1 - c)$ and either (1) or (2). The proof that these two relations are equivalent will be given below.

In this section, we prove the following.

Theorem 3.1. *The natural morphism $Q_2 \rightarrow Q_3$ is injective, and*

- (1) Q_3 is a free R -module of rank 20.
- (2) We have $Q_3 = Q_2 + Q_2s_2Q_2 + Q_2s_2^{-1}Q_2 + Rs_2s_1^{-1}s_2$
- (3) We have $Q_3 = Q_2 + Q_2s_2Q_2 + Q_2s_2^{-1}Q_2 + Rs_2^{-1}s_1s_2^{-1}$

where Q_2 is identified with the subalgebra of Q_3 generated by s_1 .

Moreover we provide two R -bases, one of them being made of positive braids, and both of them originating from explicit Gröbner bases of the defining ideal that we determine.

Notice that it is known (see [29]) that $Q_3 \otimes_R K$ has dimension 20, where K is the field of fractions of R . Therefore, in order for a collection of elements spanning Q_3 to be a R -basis, it is necessary and sufficient that it has exactly 20 elements.

3.1. The defining relation(s). In order to compare the two possible defining relations, notice that $s_1s_2s_1^2s_2 = s_2s_1s_2s_1s_2$.

We let $\Phi^a : RB_3 \rightarrow M_2(R)$ denote the representation $U_{b,c}$ of section 2.1.1, and compute the image of each of these relations under Φ^a , and also the RB_3 -module they generate inside $M_2(R)$. Note that R is noetherian, and therefore every R -submodule of $M_2(R) \simeq R^4$ has finite type.

We check that all the other components of $\Phi_{H_3}(r)$ are 0 for $r \in \{r_1, r_2\}$. We get

$$\begin{aligned}
\Phi^a(r_1) &= \frac{(a-c)(a-b)(a^2+bc)}{abc} \begin{pmatrix} 1 & \frac{b-c}{b} \\ \frac{b-c}{c} & -1 \end{pmatrix} \\
\Phi^a(r_2) &= (a-b)(a-c)(a^2+bc) \begin{pmatrix} b-c & -c \\ -b & c-b \end{pmatrix}
\end{aligned}$$

We consider the natural projection $p : H_3 \rightarrow \mathcal{H}_3(b, c)$, where $\mathcal{H}_3(b, c)$ is the Iwahori-Hecke algebra of type A_2 defined over R with parameters b, c , that is the quotient of the group algebra RB_3 by the relations $(s_i - b)(s_i - c) = 0$. A straightforward computation in the standard basis of $\mathcal{H}_3(b, c)$ shows that

$$p(r_1) = \frac{(a-c)(a-b)(a^2+bc)}{ab^2c^2}(s_1s_2 - s_2s_1)$$

$$p(r_2) = (a-c)(a-b)(a^2+bc)(s_1 - s_2)$$

Now, note that $s_2(s_1s_2 - s_2s_1)s_1^{-1}s_2^{-1} = (s_2s_1s_2)s_1^{-1}s_2^{-1} - s_2s_2s_1s_1^{-1}s_2^{-1} = s_1s_2s_1s_1^{-1}s_2^{-1} - s_2 = s_1 - s_2$. This proves that $p(r_1)$ and $p(r_2)$ generate the same ideal inside $\mathcal{H}_3(b, c)$. From the

following commutative diagram and the fact that $\Phi_{H_3}(r_1)$ and $\Phi_{H_3}(r_2)$ both belong to the ideal $Mat_2(R) \times R^2$ we get that r_1 and r_2 generate the same ideal.

$$\begin{array}{ccc} H_3 & \hookrightarrow & Mat_3(R) \times Mat_2(R)^3 \times R^3 \\ \downarrow & & \updownarrow \\ \mathcal{H}_3(b, c) & \hookrightarrow & Mat_2(R) \times R^2 \end{array}$$

Actually, by the same argument we get that the relation between $p(r_1)$ and $p(r_2)$ inside $\mathcal{H}_3(b, c)$ has to lift to the equality

$$s_2 r_1 s_1^{-1} s_2^{-1} = ab^2 c^2 r_2.$$

Once this equality is guessed, the proof of its validity can also be obtained computationally by applying directly Φ_{H_3} to it.

More generally, the membership problem for an element to belong to the defining ideal of \mathbb{Q}_3 is easily reduced, by this method to

- first checking that its image under Φ_{H_3} belongs to $(a-c)(a-b)(a^2+bc)M_2(R)$
- and, if yes, whether its image inside $\mathcal{H}_3(b, c)$ divided by $(a-c)(a-b)(a^2+bc)$ belongs to the ideal of $\mathcal{H}_3(b, c)$ generated by $s_1 - s_2$.

For this last step, since $\mathcal{H}_3(b, c) = RB_3/(s_1 - b)(s_1 - c)$ and the quotient of B_3 by the relation $s_1 s_2^{-1}$ is $B_3^{ab} \simeq \mathbb{Z}$, we get $\mathcal{H}_3(b, c)/(s_1 - s_2) = R[s]/(s - b)(s - c)$, and this last membership problem is very easy to solve.

3.2. Automorphisms of \mathbb{Q}_n . The braid group B_n admits a group automorphism characterized by the property that each generator s_i is mapped to its inverse s_i^{-1} . The image of a given braid is usually called its mirror image. It induces an automorphism of H_n as a \mathbb{Z} -algebra through via $s_i \mapsto s_i^{-1}$, $a \mapsto a^{-1}$, $b \mapsto b^{-1}$, $c \mapsto c^{-1}$. We denote this automorphism by ϕ .

Now, the braid group B_n , as any other group, admits a group *antiautomorphism* mapping every element to its inverse. It induces an *antiautomorphism* of H_n as a \mathbb{Z} -algebra that maps s_i, a, b, c to the same images as ϕ . We denote this antiautomorphism by ψ . Note that $\phi \circ \psi = \psi \circ \phi$ is a *R-algebra antiautomorphism* of H_n which maps s_i to s_i .

Direct computation (either by hand or using a computer implementation of the injective morphism Φ_{H_3}) proves that $\phi(r_1) = a^{-2}r_1$ and $\psi(r_1) = -a^{-2}r_1$. Therefore $\phi \circ \psi(r_1) = \phi(-a^{-2}r_1) = -a^2\phi(r_1) = -r_1$.

This proves that ϕ induces a \mathbb{Z} -algebra automorphism of \mathbb{Q}_n , that ψ induces a \mathbb{Z} -algebra anti-automorphism of \mathbb{Q}_n , and that $\phi \circ \psi = \psi \circ \phi$ induces a *R-algebra anti-automorphism* of \mathbb{Q}_n .

3.3. A Gröbner basis with positive words. Using the GBNP package of GAP4 (see [8]) on specialized rational values of a, b, c , we guess a (noncommutative) Gröbner basis for \mathbb{Q}_3 . The rewriting system corresponding to it is the following one. Here and later on, we have used for concision the convention that the empty word is denoted \emptyset and the Artin generators s_i, s_i^{-1} are denoted i and \bar{i} , respectively.

- (1) $111 \rightsquigarrow \Sigma_1.11 - \Sigma_2.1 + \Sigma_3.\emptyset$
- (2) $222 \rightsquigarrow \Sigma_1.22 - \Sigma_2.2 + \Sigma_3.\emptyset$
- (3) $212 \rightsquigarrow 121$
- (4) $12112 \rightsquigarrow 11211 + a.2112 - a.1221 + a^2.221 - a^2.211$
 $+ a^2.122 - a^2.112 - a^3.22 + a^3.11 + a^4.2 - a^4.1$
- (5) $21121 \rightsquigarrow 11211 + a.2112 - a.1221 + a^2.221 - a^2.211$
 $+ a^2.122 - a^2.112 - a^3.22 + a^3.11 + a^4.2 - a^4.1$
- (6) $12211 \rightsquigarrow 11221 + a.2211 + (a^2 + \Sigma_2)a^{-1}.1211 - a.1122 - (a^2 + \Sigma_2)a^{-1}.1121$
 $- a^2.221 - (a^2 + \Sigma_2).211 + a^2.122 + (a^2 + \Sigma_2).112$
 $+ (a^2 + \Sigma_2)a.21 - (a^2 + \Sigma_2)a.12$
- (7) $21122 \rightsquigarrow 11221 + \Sigma_1.2112 - \Sigma_2.1221 + \Sigma_2.221 - \Sigma_2.211$
- (8) $22112 \rightsquigarrow 12211 + \Sigma_1.2112 - \Sigma_1.1221 + \Sigma_2.122 - \Sigma_2.112$

Notice that, inside B_3 , we have $21121 = 21212 = 12112$, whence the validity of (5) is equivalent to the validity of (4). Using Φ_{H_3} we check that the relations (7) and (8) are actually true inside H_3 . Relation (6) is mapped inside $(a^2 + bc)(a - b)(a - c)M_2(R)$, and its image inside $\mathcal{H}_3(b, c)$ is equal to $(a^2 + bc)(a - b)(a - c)a^{-1}(s_1s_2 - s_2s_1)$. Therefore it is valid inside Q_3 , and actually could be taken as a defining relation, too.

We then check (by computer) that the set of (positive) words avoiding the patterns

$$111, 222, 212, 12112, 21121, 12211, 21122, 22112$$

is finite, and has exactly 20 elements. We also check that it provides a generating set of Q_3 (by using the rewriting system described above). Actually, this proves that Q_3 is spanned by these 20 elements even if it is defined over the smaller ring $\mathbb{Z}[b, c, a, a^{-1}]$ using the relation r_2 .

This is remarkable because H_3 is *not* finitely generated if defined over $\mathbb{Z}[a, b, c]$ (see [27]).

\emptyset	1	2	11	12
21	22	112	121	122
211	221	1121	1122	1211
1221	2112	2211	11211	11221

3.4. Two Gröbner bases with signed words. We now construct two rewriting systems on *signed words*. The procedure is similar, as we use GBNP to find Gröbner basis on specialisations, and then the previously described algorithm to check the validity of the relations inside Q_3 . We use two different orderings on the signed generators to find the Gröbner basis. The first one is $1 < 2 < \bar{1} < \bar{2}$ and is described in table 6, the second one is $1 < \bar{1} < 2 < \bar{2}$ and is described in table 7. In these tables, we used the conventions $u = a + b + c$, $v = ab + ac + bc$ and $w = abc$.

For the first ordering, we avoid the patterns

$$1\bar{1}, 2\bar{2}, 2\bar{2}, \bar{1}1, \bar{2}2, 21\bar{2}, \bar{1}\bar{2}, \bar{1}\bar{2}, \bar{1}\bar{2}, \bar{1}\bar{2}, 11, \bar{1}\bar{1}, \bar{2}\bar{2}, \bar{1}\bar{2}\bar{1}, \bar{2}\bar{1}\bar{2}, 121\bar{2}, 12\bar{1}\bar{2}, 21\bar{2}\bar{1}, 2\bar{1}\bar{2}\bar{1}, \bar{1}\bar{2}\bar{1}\bar{2}$$

and we get the following basis

\emptyset	1	2	$\bar{1}$	$\bar{2}$
12	$\bar{1}\bar{2}$	21	$2\bar{1}$	$\bar{1}\bar{2}$
$\bar{1}\bar{2}$	$\bar{2}\bar{1}$	$\bar{2}\bar{1}$	121	$12\bar{1}$
$\bar{1}\bar{2}\bar{1}$	$\bar{1}\bar{2}\bar{1}$	$2\bar{1}\bar{2}$	$2\bar{1}\bar{2}$	$\bar{1}\bar{2}\bar{1}$

For the second ordering, we avoid the patterns

$$\bar{1}\bar{1}, \bar{1}1, 22, \bar{2}\bar{2}, \bar{2}2, \bar{1}\bar{2}\bar{1}, 212, 21\bar{2}, 2\bar{1}\bar{2}, \bar{2}1\bar{2}, \bar{2}\bar{1}\bar{2}, 11, \bar{1}\bar{1}, \bar{2}\bar{2}, \bar{1}\bar{2}\bar{1}, \bar{2}1\bar{2}, 12\bar{1}\bar{2}, \bar{1}\bar{2}\bar{1}\bar{2}, 2\bar{1}\bar{2}1, 2\bar{1}\bar{2}\bar{1}$$

and we get the following basis

\emptyset	1	$\bar{1}$	2	$\bar{2}$
12	$\bar{1}\bar{2}$	$\bar{1}2$	$\bar{1}\bar{2}$	21
$2\bar{1}$	$\bar{2}1$	$\bar{2}\bar{1}$	121	$12\bar{1}$
$1\bar{2}\bar{1}$	$\bar{1}\bar{2}\bar{1}$	$\bar{1}2\bar{1}$	$\bar{1}\bar{2}\bar{1}$	$2\bar{1}\bar{2}$

We notice that these two distinct collections of signed words represent exactly the same collection of elements inside the braid group B_3 , since $\bar{1}21 = 21\bar{2}$ in B_3 . In particular they provide the same basis of Q_3 .

3.5. Q_3 as a Q_2 -bimodule. An immediate consequence of the basis found above is that

$$Q_3 = \sum_{k=0}^2 Q_2 s_2^k Q_2 + R s_2 s_1^{-1} s_2$$

as a R -module. Let $M_1 = \sum_{k=0}^2 Q_2 s_2^k Q_2 \subset Q_3$. From the basis with signed words described above, it is clear that the bimodule Q_3/M_1 is a free R -module of rank 1 spanned over R by the element $s_2 s_1^{-1} s_2$, and that M_1 is a free R -module of rank 19. Since $s_1 \cdot s_2 s_1^{-1} s_2 = (s_1 s_2 s_1^{-1}) s_2 = s_2^{-1} s_1 s_2 s_2 = s_2^{-1} s_1 (s_2^2) = w \cdot s_2^{-1} s_1 s_2^{-1} + u \cdot (s_2^{-1} s_1 s_2) - v \cdot s_2^{-1} s_1 = w \cdot s_2^{-1} s_1 s_2^{-1} + u \cdot s_1 s_2 s_1^{-1} - v \cdot s_2^{-1} s_1 \equiv w \cdot s_2^{-1} s_1 s_2^{-1} \pmod{M_1}$. Now relation (21) of table 6 states that $s_2^{-1} s_1 s_2^{-1} \equiv (bc)^{-1} s_2 s_1^{-1} s_2 \pmod{M_1}$ hence $s_1 \cdot s_2 s_1^{-1} s_2 \equiv a \cdot s_2 s_1^{-1} s_2 \pmod{M_1}$. Similarly, $s_2 s_1^{-1} s_2 \cdot s_1 = s_2 (s_1^{-1} s_2 s_1) = s_2^2 s_1 s_2^{-1} = w \cdot s_2^{-1} s_1 s_2^{-1} + u \cdot s_2 s_1 s_2^{-1} - v \cdot s_1 s_2^{-1} \equiv w \cdot s_2^{-1} s_1 s_2^{-1} \equiv w(bc)^{-1} \cdot s_2 s_1^{-1} s_2 \equiv a \cdot s_2 s_1^{-1} s_2 \pmod{M_1}$. If S_x denotes the Q_2 -module defined by $s_1 \mapsto x$, this proves that $Q_3/M_1 \simeq S_a \otimes S_a$ as a Q_2 -bimodule. As a by-product we get the following alternative descriptions of Q_3 as a R -module :

$$Q_3 = \sum_{k=0}^2 Q_2 s_2^k Q_2 + R s_2^{-1} s_1 s_2^{-1}$$

Let $M_2 = Q_2 \subset M_1$. We know from the basis description that M_2 is a free R -module of rank 3 such that M_1/M_2 is a free R -module of rank 16. We consider $M_+ \subset M_1' = M_1/M_2$ the bimodule generated by s_2 . It is spanned as a R -module by $2, 12, \bar{1}\bar{2}, 2\bar{1}, 21, 121, 12\bar{1}, \bar{1}\bar{2}\bar{1}, \bar{1}\bar{2}\bar{1}$, and since they belong to the basis of Q_3 and are disjoint from the part of it providing a basis of M_2 , we get that they provide a basis of M_+ . Therefore, $M_+ \simeq Q_2 \otimes Q_2$ as a Q_2 -bimodule. We then consider M_1'/M_+ . It is a free R -module of rank 7, and is spanned by $\bar{2}, \bar{1}\bar{2}, \bar{1}\bar{2}, 2\bar{1}, 21, 121, 12\bar{1}$. Clearly we have a natural surjective map $Q_2 \otimes Q_2 \rightarrow M_1'/M_+$, defined by $x \otimes y \mapsto x\bar{2}y$. Inside the kernel we find $\bar{1}\bar{2}\bar{1} - 12\bar{1} + a \cdot \bar{2}\bar{1} - a^{-1} \cdot 21 - a \cdot \bar{1}\bar{2} + a^{-1} \cdot 1\bar{2}$ by relation (16) of table 7 and

$\bar{1}\bar{2}\bar{1} + (b+c)w^{-1} \cdot 1\bar{2}\bar{1} + (wa)^{-1} \cdot 1\bar{2}\bar{1} - (v/w) \cdot \bar{2}\bar{1} - w^{-1} \cdot 21 - a^{-1} \cdot \bar{1}\bar{2} - (u/(wa)) \cdot 1\bar{2} + ((a^2+v)/(wa)) \cdot \bar{2}$ by relation (17) of table 7.

3.6. Another basis and some special computations.

\emptyset	$\bar{2}$	$2\bar{1}$	21	$\bar{1}$
$\bar{1}\bar{2}$	$\bar{1}\bar{2}$	1	$\bar{1}\bar{2}$	12
2	$2\bar{1}$	21	$\bar{2}\bar{1}\bar{2}$	$2\bar{1}\bar{2}$
$21\bar{2}$	$\bar{2}\bar{1}\bar{2}$	212	$\bar{2}\bar{1}\bar{2}$	$\bar{1}\bar{2}\bar{1}$

(1)	$1\bar{1}$	$\rightsquigarrow \emptyset$	(2)	$2\bar{2}$	$\rightsquigarrow w.\bar{2} + u.2 - v.\emptyset$
(3)	$2\bar{2}$	$\rightsquigarrow \emptyset$	(4)	$\bar{1}1$	$\rightsquigarrow \emptyset$
(5)	$2\bar{2}$	$\rightsquigarrow \emptyset$	(6)	$21\bar{2}$	$\rightsquigarrow 12\bar{1}$
(7)	$\bar{1}21$	$\rightsquigarrow 21\bar{2}$	(8)	$\bar{1}\bar{2}1$	$\rightsquigarrow 2\bar{1}\bar{2}$
(9)	$\bar{2}1\bar{2}$	$\rightsquigarrow 1\bar{2}\bar{1}$	(10)	$\bar{2}\bar{1}\bar{2}$	$\rightsquigarrow 1\bar{2}\bar{1}$
(11)	$\bar{2}\bar{1}\bar{2}$	$\rightsquigarrow \bar{1}\bar{2}\bar{1}$	(12)	11	$\rightsquigarrow w.\bar{1} + u.1 - v.\emptyset$
(13)	$\bar{1}\bar{1}$	$\rightsquigarrow v/w.\bar{1} + w^{-1}.1 - u/w.\emptyset$	(14)	$\bar{2}\bar{2}$	$\rightsquigarrow v/w.\bar{2} + w^{-1}.2 - u/w.\emptyset$
(15)	$121\bar{2}$	$\rightsquigarrow 21$	(16)	$12\bar{1}\bar{2}$	$\rightsquigarrow w.\bar{2}1\bar{2} + u.1\bar{2}\bar{1} - v.\bar{2}1$
(17)	$21\bar{2}1$	$\rightsquigarrow w.\bar{1}\bar{2}\bar{1} + u.21\bar{2} - v.\bar{1}\bar{2}$	(18)	$21\bar{2}\bar{1}$	$\rightsquigarrow \bar{1}\bar{2}$

(19)	$21\bar{2}$	$\rightsquigarrow a^{-2}.21\bar{2} + 1\bar{2}\bar{1} - a^{-2}.1\bar{2}\bar{1} + (-a).\bar{2}\bar{1} + a^{-1}.\bar{2}\bar{1} + a.\bar{1}\bar{2} - a^{-1}.\bar{1}\bar{2} + a^{-1}.\bar{2}\bar{1} - a^{-3}.21$ $- a^{-1}.\bar{1}\bar{2} + a^{-3}.1\bar{2}$
(20)	$\bar{1}\bar{2}\bar{1}$	$\rightsquigarrow a^{-2}.\bar{1}\bar{2}\bar{1} - (b+c)/w.1\bar{2}\bar{1} - (1/(wa)).1\bar{2}\bar{1} + (b+c)/(wa^2).1\bar{2}\bar{1}$ $+ (1/(wa^3)).121 + v/w.\bar{2}\bar{1} + (1/w).\bar{2}\bar{1} + a^{-1}.\bar{1}\bar{2} - a^{-3}.\bar{1}\bar{2}$ $- v/(wa^2).\bar{2}\bar{1} - (wa^2)^{-1}.21 + u/(wa).1\bar{2}$ $- u/(wa^3).1\bar{2} - (a^2+v)/(wa).\bar{2}$ $+ (a^2+v)/(wa^3).2$
(21)	$\bar{2}1\bar{2}$	$\rightsquigarrow a^2.\bar{1}\bar{2}\bar{1} + (bc)^{-1}.2\bar{1}\bar{2} + (a(b+c))/(bc).1\bar{2}\bar{1} - (b+c)/w.1\bar{2}\bar{1} - (wa)^{-1}.121$ $- (va)/(bc).\bar{2}\bar{1} + v/w.\bar{2}\bar{1} - a.\bar{1}\bar{2} - u/(bc).\bar{1}\bar{2}$ $- a/(bc).\bar{2}\bar{1} + w^{-1}.21 + a^{-1}.\bar{1}\bar{2} + (u/(wa)).1\bar{2}$ $+ ((a^2+v)/bc).\bar{1} - (a^2+v)/(wa).1$
(22)	$2\bar{1}\bar{2}\bar{1}$	$\rightsquigarrow (bc)^{-1}.21\bar{2}\bar{1} + a^2.1\bar{2}\bar{1}\bar{1} + (-1).1\bar{2}\bar{1}\bar{1} - a^3.\bar{2}\bar{1}\bar{1} + (a(a^2+bc)).\bar{1}\bar{2}\bar{1} + (-(a^2+bc)/a).\bar{1}\bar{2}\bar{1}$ $+ a.2\bar{1}\bar{1} + a^{-1}.2\bar{1}\bar{2} + (-(a^2+v)/w).21\bar{2} + (b+c-2a).1\bar{2}\bar{1} + a^{-1}.1\bar{2}\bar{1}$ $- (b+c-a)/a^2.1\bar{2}\bar{1} - a^{-3}.121 + a^2.\bar{2}\bar{1} + (-1).\bar{2}\bar{1} + (-(a^2+bc)).\bar{1}\bar{2}$ $+ ((a^3(b+c) + 2a^2bc + b^2c^2)/(wa)).\bar{1}\bar{2}$ $+ (((b+c)a^3 + (b+c)^2a^2 + 2w(b+c) + (bc)^2)/(wa)).2\bar{1}$ $+ ((a^2+bc+v)/(wa)).21$ $- (b+c)a^{-1}.\bar{1}\bar{2} + (u/a^3).1\bar{2} + a.\bar{2} - (a+b)(a+c)v/w.\bar{1}$ $- (a^4 + 2a^3(b+c) + a^2(b^2 + 5bc + c^2) + 2w(b+c) + b^2c^2)/(wa^2).2$ $+ (-(a^2+v)/w).1 + ((a+b)^2(a+c)^2/(wa)).\emptyset$
(23)	$\bar{1}\bar{2}\bar{1}\bar{2}$	$\rightsquigarrow a.\bar{1}\bar{2}\bar{1} + a^{-1}.2\bar{1}\bar{2} - a^{-1}.21\bar{2} - a.1\bar{2}\bar{1} + a^2.\bar{2}\bar{1}$ $+ v.\bar{1}\bar{2} + (uv/w).\bar{1}\bar{2} + (-1).\bar{2}\bar{1} + a^{-2}.21 + 1\bar{2}$ $+ (u/w).1\bar{2} + (-(a^2+v)/a).\bar{2} + (-(a+b)(a+c)v/w).\bar{1}$ $+ (-u(a^2+v)/(wa)).2$ $+ (-(a^2+v)/w).1 + ((a+b)^2(a+c)^2/(wa)).\emptyset$

TABLE 6. Rewriting system for \mathbb{Q}_3 from $1 < 2 < \bar{1} < \bar{2}$

We claim that the above list of 20 elements provides an alternative basis for \mathbb{Q}_3 . This can be checked using Φ_{H_3} and express each element of this list as a linear combination over K of one of the previously obtained R -basis. Then, one checks that both the corresponding K -valued matrix and its inverse have all their coefficients inside R , which proves our claim.

An interest of this basis is that the 19 first elements of the basis belong to the R -submodule $u_1u_2 + \bar{2}u_1u_2 + u_2u_1\bar{2} + R.121$, which will have some importance in our computations. In

(1)	$\bar{1}\bar{1}$	\rightsquigarrow	\emptyset	(2)	$\bar{1}1$	\rightsquigarrow	\emptyset
(3)	22	\rightsquigarrow	$(w).\bar{2} + (u).2 + (-v).\emptyset$	(4)	$2\bar{2}$	\rightsquigarrow	\emptyset
(5)	22	\rightsquigarrow	\emptyset	(6)	212	\rightsquigarrow	121
(7)	$21\bar{2}$	\rightsquigarrow	$\bar{1}21$	(8)	$2\bar{1}\bar{2}$	\rightsquigarrow	$\bar{1}\bar{2}1$
(9)	$\bar{2}12$	\rightsquigarrow	$12\bar{1}$	(10)	$\bar{2}\bar{1}\bar{2}$	\rightsquigarrow	$1\bar{2}\bar{1}$
(11)	$\bar{2}\bar{1}\bar{2}$	\rightsquigarrow	$\bar{1}\bar{2}\bar{1}$	(12)	11	\rightsquigarrow	$(w).\bar{1} + (u).1 + (-v).\emptyset$
(13)	$\bar{1}\bar{1}$	\rightsquigarrow	$(v/w).\bar{1} + (1/w).1 + (-u/w).\emptyset$	(14)	$\bar{2}\bar{2}$	\rightsquigarrow	$(v/w).\bar{2} + (1/w).2 + (-u/w).\emptyset$
(15)	$12\bar{1}2$	\rightsquigarrow	$(w).\bar{2}1\bar{2} + (u).12\bar{1} + (-v).\bar{2}1$				

(16)	$\bar{1}\bar{2}1$	\rightsquigarrow	$a^{-2}.\bar{1}21 + 1\bar{2}\bar{1} - a^{-2}.12\bar{1} - a.\bar{2}\bar{1} + a^{-1}.\bar{2}1 + a^{-1}.2\bar{1} - a^{-3}.21 + a.\bar{1}\bar{2}$ $- a^{-1}.\bar{1}\bar{2} - a^{-1}.1\bar{2} + a^{-3}.12$
(17)	$\bar{1}\bar{2}\bar{1}$	\rightsquigarrow	$a^{-2}.\bar{1}2\bar{1} - (b+c)w^{-1}.1\bar{2}\bar{1} - (wa)^{-1}.1\bar{2}1 + ((b+c)/(wa^2)).12\bar{1}$ $+ (1/(wa^3)).121 + (v/w).\bar{2}\bar{1} + w^{-1}.\bar{2}1 - (v/(wa^2)).2\bar{1} - (1/(wa^2)).21 + a^{-1}.\bar{1}\bar{2}$ $- a^{-3}.\bar{1}2 + (u/(wa)).1\bar{2} - (u/(wa^3)).12 - ((a^2+v)/(wa)).\bar{2} + ((a^2+v)/(wa^3)).2$
(18)	$\bar{2}1\bar{2}$	\rightsquigarrow	$(bc)^{-1}.\bar{2}1\bar{2} + \bar{1}2\bar{1} - (bc)^{-1}.1\bar{2}1 + ((a^2+v)/w).\bar{2}1 - ((a^2+v)/w).2\bar{1}$ $- ((a^2+v)/w).\bar{1}2 + ((a^2+v)/w).1\bar{2} - ((a^2+v)/(bc)).\bar{2} + ((a^2+v)/(wa)).2$ $+ ((a^2+v)/(bc)).\bar{1} - ((a^2+v)/(wa)).1$
(19)	$\bar{1}2\bar{1}\bar{2}$	\rightsquigarrow	$-(bc).\bar{1}2\bar{1}\bar{1} + ((bc)/a^2).12\bar{1}\bar{1} + a^{-1}.\bar{2}1\bar{2} + ((a^2+v)/a).2\bar{1}\bar{1} + ((a^2+v)/a).\bar{1}2\bar{1}$ $- a.1\bar{2}\bar{1} - (v/a^3).12\bar{1} - a^{-3}.121 + a^2.\bar{2}\bar{1}$ $- ((a^3(b+c) + a^2(b^2 + c^2 + 4bc) + 2w(b+c) + (bc)^2)/(wa)).2\bar{1} - (u/w).21 + v.\bar{1}\bar{2}$ $+ ((u(b+c))/(bc)).\bar{1}2 - ((a^2+v)).\bar{1}\bar{1} + \bar{1}2 + (u(a^2+bc)/(wa^2)).12 - ((a^2+v)/a).\bar{2}$ $+ ((a^2+v)/a^2).\emptyset$
(20)	$2\bar{1}21$	\rightsquigarrow	$1\bar{2}\bar{1}2 + u.\bar{1}21 - u.12\bar{1} + v.\bar{2}1 - v.1\bar{2}$
(21)	$2\bar{1}\bar{2}\bar{1}$	\rightsquigarrow	$\bar{1}2\bar{1}2 + v.\bar{2}\bar{1} + ((uv)/w).2\bar{1} + (u/w).21 - v.\bar{1}\bar{2} - ((uv)/w).\bar{1}2 - (u/w).12$

TABLE 7. Rewriting system for \mathbb{Q}_3 from $1 < \bar{1} < 2 < \bar{2}$

particular, by expressing $\bar{2}1\bar{2}1\bar{2}$ in this basis one gets the following identities

$$(3.1) \quad \bar{2}1\bar{2}1\bar{2} \equiv \frac{bc}{a^2}.\bar{1}2\bar{1} + \frac{bc-a^2}{a^4bc}.121 \quad \text{mod } u_1u_2 + u_2u_1 + \bar{2}u_1u_2 + u_2u_1\bar{2}$$

$$(3.2) \quad 2\bar{1}2\bar{1}2 \equiv a^2.\bar{1}2\bar{1} \quad \text{mod } u_1u_2 + u_2u_1\bar{2}u_1u_2 + u_2u_1\bar{2} + R.121$$

3.7. Spanning sets for specific \mathbb{Q}_3 -modules.

Lemma 3.2. *The quotient of \mathbb{Q}_3 by the left ideal generated by $(2-a.\emptyset)1$ and $(2-a.\emptyset)(\emptyset-a.\bar{1})$ is spanned as a R -module by $\emptyset, 1, \bar{1}, 2, \bar{2}$.*

Proof. Let us consider the left ideal I generated by $(2-a.\emptyset)1$ and $(2-a.\emptyset)(\emptyset-a.\bar{1})$, and V the R -submodule spanned by $\emptyset, 1, \bar{1}, 2, \bar{2}$. Since \mathbb{Q}_3 is generated as a unital algebra by 1 and 2 it is sufficient to prove $2.V \subset V + I$ and $1.V \subset V + I$. We start with the former. We have $21 = (2-a.\emptyset)1 + a.1 \in a.1 + I \subset V + I$. Moreover

$$2\bar{1} = (2-a.\emptyset)\bar{1} + a.\bar{1} = -a^{-1}(2-a.\emptyset)(\emptyset-a.\bar{1}) + a^{-1}.2 - \emptyset + a.\bar{1} \in V + I$$

hence $2.V \subset V + I$. Now,

$$12 = \bar{2}212 = \bar{2}121 \in \bar{2}1(a.1 + I) \subset \bar{2}(a.11 + I) \subset \bar{2}(V + I) \subset V + I.$$

Finally, $\bar{1}\bar{2} = \bar{2}\bar{1}\bar{2} = \bar{2}\bar{1}2\bar{1} \in \bar{2}\bar{1}(a.1 + I) = a.\bar{2} + I \subset V + I$ hence $1.V \subset V + I$ and this proves the claim. \square

4. STRUCTURE ON 4 STRANDS

It was determined in [29] that $\dim Q_4 \otimes_R K = 264$. Moreover, there are models of all the irreducible representations of H_4 which are defined over $R' = \mathbb{Z}[j][a^{\pm 1}, b^{\pm 1}, c^{\pm 1}]$, and this provides an embedding Φ_{H_4} of H_4 inside a product of matrix algebras over R' . These models can be found in [29], too.

4.1. Q_4 as a Q_3 -bimodule. We denote $u_i = R + Rs_i + Rs_i^2 = R + Rs_i + Rs_i^2$ the R -subalgebra of Q_4 generated by s_i , and Q_3, Q_2 the subalgebra of Q_4 generated by s_1, s_2 and s_1 . Obviously $Q_2 = u_1$. We first basically use the decompositions

$$Q_3 = u_1u_2u_1 + u_2u_1u_2 = u_1u_2u_1 + Rs_2s_1^{-1}s_2 = u_1u_2u_1 + Rs_2^{-1}s_1s_2^{-1}.$$

which follow from theorem 3.1.

$$\text{We set } Q_4^{(1)} = Q_3u_3Q_3 \text{ and } Q_4^{(i+1)} = Q_4^{(i)}u_3Q_3.$$

Lemma 4.1.

- (1) $u_3u_2u_3Q_3u_3 \subset Q_4^{(2)}$
- (2) $u_3Q_3u_3u_2u_3 \subset Q_4^{(2)}$

Proof. We first show (1). We have $Q_3 = u_1u_2u_1 + u_2u_1u_2$ hence $u_3u_2u_3Q_3u_3 \subset u_3u_2u_3u_1u_2u_1u_3 + u_3u_2u_3u_2u_1u_2u_3$. From $u_3u_2u_3u_2 = u_3u_2u_3 + u_2u_3u_2$ we get $(u_3u_2u_3u_2)u_1u_2u_3 \subset u_3u_2u_3u_1u_2u_3 + Q_4^{(2)}$. Since $u_3u_2u_3u_1u_2u_1u_3 = u_3u_2u_3u_1u_2u_3u_1 \subset u_3u_2u_3u_1u_2u_3Q_3$ it is sufficient to show that $u_3u_2u_3u_1u_2u_3 \subset Q_4^{(2)}$. From $u_3 = R + Rs_3 + Rs_3^{-1}$ we deduce $u_3u_2u_3u_1u_2u_3 \subset Q_4^{(2)} + \sum_{\alpha \in \{-1,1\}} s_3^\alpha u_2u_3u_1u_2u_3$. But $s_3^\alpha u_2u_3u_1u_2u_3 = s_3^\alpha u_2u_1u_3u_2u_3 \subset s_3^\alpha u_2u_1s_3^{-\alpha}s_2^\alpha s_3^{-\alpha} + s_3^\alpha u_2u_1u_2u_3u_2 \subset (s_3^\alpha u_2s_3^{-\alpha})u_1s_2^\alpha s_3^{-\alpha} + Q_4^{(2)} \subset Q_4^{(2)}$. This proves (1), and (2) can be deduced from it, or can be proven similarly. \square

Proposition 4.2. $Q_4 = Q_4^{(2)}$.

Proof. It is sufficient to show $u_3Q_3u_3Q_3u_3 \subset Q_4^{(2)}$, hence that $s_3^\alpha Q_3s_3^\beta Q_3u_3 \subset Q_4^{(2)}$ pour $\alpha, \beta \in \{-1,1\}$. We have $Q_3 = Rs_2^\alpha s_1^{-\alpha} s_2^\alpha + u_1u_2u_1 = Rs_2^{-\alpha} s_1^\alpha s_2^{-\alpha} + u_1u_2u_1$, hence

$$\begin{aligned} s_3^\alpha Q_3s_3^\beta Q_3u_3 &\subset s_3^\alpha s_2^\alpha s_1^{-\alpha} s_2^\beta s_3^\beta Q_3u_3 + s_3^\alpha u_1u_2u_1s_3^\beta Q_3u_3 \\ &\subset s_3^\alpha s_2^\alpha s_1^{-\alpha} s_2^\beta s_3^\beta Q_3u_3 + Q_3s_3^\alpha u_2s_3^\beta Q_3u_3 \\ &\subset s_3^\alpha s_2^\alpha s_1^{-\alpha} s_2^\beta s_3^\beta Q_3u_3 + Q_4^{(2)} && \text{by lemma 4.1} \\ &\subset s_3^\alpha s_2^\alpha s_1^{-\alpha} s_2^\beta s_3^\beta (s_2^{-\alpha} s_1^\alpha s_2^{-\alpha})u_3 + Q_4^{(2)} && \text{again after lemma 4.1} \\ &\subset s_3^\alpha s_2^\alpha s_1^{-\alpha} (s_2^\alpha s_3^\beta s_2^{-\alpha})s_1^\alpha s_2^{-\alpha} u_3 + Q_4^{(2)} \\ &\subset s_3^\alpha s_2^\alpha s_1^{-\alpha} s_3^{-\alpha} s_2^\beta s_3^\alpha s_1^\alpha s_2^{-\alpha} u_3 + Q_4^{(2)} \\ &\subset (s_3^\alpha s_2^\alpha s_3^{-\alpha})s_1^{-\alpha} s_2^\beta s_1^\alpha s_3^\alpha s_2^{-\alpha} u_3 + Q_4^{(2)} \\ &\subset s_2^{-\alpha} s_3^\alpha (s_2^\alpha s_1^{-\alpha} s_2^\beta s_1^\alpha) s_3^\alpha s_2^{-\alpha} u_3 + Q_4^{(2)} \\ &\subset Q_3u_3Q_3u_3u_2u_3 + Q_4^{(2)} \\ &\subset Q_4^{(2)} && \text{by lemma 4.1} \end{aligned}$$

and this proves the claim. \square

Lemme 4.3.

- (1) $s_3 Q_3 s_3^{-1} \subset Q_4^{(1)} + Q_3 s_3 s_2^{-1} s_3 Q_3$.
(2) $s_3^{-1} Q_3 s_3 \subset Q_4^{(1)} + Q_3 s_3 s_2^{-1} s_3 Q_3$.

Proof. We prove (1). From $Q_3 = u_1 u_2 u_1 + R s_2 s_1^{-1} s_2$ we get $s_3 Q_3 s_3^{-1} \subset u_1 s_3 u_2 s_3^{-1} u_1 + R s_3 s_2 s_1^{-1} s_2 s_3^{-1} \subset Q_4^{(1)} + u_1 s_3 s_2^{-1} s_3 u_1 + R s_3 s_2 s_1^{-1} s_2 s_3^{-1}$, hence it is sufficient to prove $s_3 s_2 s_1^{-1} s_2 s_3^{-1} \in Q_4^{(1)} + Q_3 s_3 s_2^{-1} s_3 Q_3$. But

$$\begin{aligned} s_3 s_2 s_1^{-1} s_2 s_3^{-1} &= s_3 s_2 s_1^{-1} (s_2 s_3^{-1} s_2^{-1}) s_2 = s_3 s_2 s_1^{-1} s_3^{-1} s_2^{-1} s_3 s_2 \\ &= (s_3 s_2 s_3^{-1}) s_1^{-1} s_2^{-1} s_3 s_2 = s_2^{-1} s_3 (s_2 s_1^{-1} s_2^{-1}) s_3 s_2 = s_2^{-1} s_3 s_1^{-1} s_2^{-1} s_1 s_3 s_2 \\ &= s_2^{-1} s_1^{-1} s_3 s_2^{-1} s_3 s_1 s_2 \in Q_3 (s_3 s_2^{-1} s_3) Q_3 \end{aligned}$$

and this proves (1). The proof of (2) is similar. \square

We set $w_0 = s_3 s_2 s_1^2 s_2 s_3$, $w_+ = s_3 s_2 s_1^{-1} s_2 s_3$, $w_- = s_3^{-1} s_2^{-1} s_1 s_2^{-1} s_3^{-1}$.

Lemme 4.4. (1) $w_0 \in R^\times w_+ + u_1 u_3 u_2 u_3 u_1$

- (2) $w_0^{-1} \in R^\times w_- + u_1 u_3 u_2 u_3 u_1$
(3) $\forall x \in Q_3 \quad w_+ x \in x w_+ + Q_3 u_3 u_2 u_3 Q_3$.
(4) $\forall x \in Q_3 \quad w_- x \in x w_- + Q_3 u_3 u_2 u_3 Q_3$.
(5) $w_+ s_3, w_+ s_1 \in R w_+ + Q_3 u_3 u_2 u_3 Q_3$.
(6) $w_- s_3, w_- s_1 \in R w_- + Q_3 u_3 u_2 u_3 Q_3$.

Proof. (1) is a consequence of $s_1^2 \in R^\times s_1^{-1} + R s_1 + R$ and of the braid relations. (2) is similar. (3) is deduced from (1) and (2), and of the fact that w_0 commutes with s_1 et s_2 . (4) is similar. From $s_3^2 \in u_3 = R s_3 + R s_3^{-1} + R$ we deduce $w_+ s_3 \in R w_+ + Q_4^{(1)} + R s_3 s_2 s_1^{-1} s_2 s_3^{-1}$ and (5) is a consequence of lemma 4.3 (1). (6) is similar. \square

We now use another aspect of the defining relation of Q_3 , under the form

$$\begin{aligned} s_1^{-1} s_2 s_1 &= \frac{-1}{a} s_1 s_2 + a s_1 s_2^{-1} + a s_1^{-1} s_2 - a^3 s_1^{-1} s_2^{-1} + a^{-1} s_2 s_1 - a s_2 s_1^{-1} - a s_2^{-1} s_1 + a^3 s_2^{-1} s_1^{-1} \\ &\quad + a^2 s_1^{-1} s_2^{-1} s_1 - a^2 s_1 s_2^{-1} s_1^{-1} + s_1 s_2 s_1^{-1} \end{aligned}$$

In particular, $s_1^{-1} s_2 s_1 \equiv s_1 s_2 s_1^{-1} + a^2 (s_1^{-1} s_2^{-1} s_1 - s_1 s_2^{-1} s_1^{-1}) \pmod{u_1 u_2 + u_2 u_1}$.

Lemme 4.5. We have $w_+ u_2 \subset R w_+ + Q_3 u_3 u_2 u_3 Q_3$, and $w_- u_2 \subset R w_- + Q_3 u_3 u_2 u_3 Q_3$.

Proof. The second claim is deduced from the first one through the natural automorphisms, hence we can limit ourselves to considering the first one. Since u_2 is generated as a R -algebra by s_2^{-1} , it is sufficient to prove $w_+ s_2^{-1} \in R w_+ + Q_3 u_3 u_2 u_3 Q_3$. Using the obvious shift morphism $u_1 u_2 u_1 \mapsto u_2 u_3 u_2$ characterized by $s_1 \mapsto s_2$, $s_2 \mapsto s_3$, we deduce from the preceding relation (inside $u_1 u_2 u_1$) that $s_2^{-1} s_3 s_2 \equiv s_2 s_3 s_2^{-1} + a^2 (s_2^{-1} s_3^{-1} s_2 - s_2 s_3^{-1} s_2^{-1}) \pmod{u_2 u_3 + u_3 u_2}$. hence $s_2 s_3 s_2^{-1} \in s_2^{-1} s_3 s_2 - a^2 (s_2^{-1} s_3^{-1} s_2 - s_2 s_3^{-1} s_2^{-1}) + u_2 u_3 + u_3 u_2$. We deduce from this $w_+ s_2^{-1} = s_3 s_2 s_1^{-1} s_2 s_3 s_2^{-1} \in s_3 s_2 s_1^{-1} s_2^{-1} s_3 s_2 - a^2 (s_3 s_2 s_1^{-1} s_2^{-1} s_3^{-1} s_2 - s_3 s_2 s_1^{-1} s_2 s_3^{-1} s_2^{-1}) + s_3 s_2 s_1^{-1} u_2 u_3 + s_3 s_2 s_1^{-1} u_3 u_2$. We have $s_3 s_2 s_1^{-1} u_2 u_3 \subset R s_3 s_2 s_1^{-1} s_2 s_3 + u_3 u_1 u_2 u_1 u_3 + s_3 Q_3 s_3^{-1} + Q_4^{(1)} \subset R w_+ + Q_3 u_3 u_2 u_3 Q_3$ after lemma 4.3; Clearly $s_3 s_2 s_1^{-1} u_3 u_2 \subset Q_3 u_3 u_2 u_3 Q_3$ and $s_3 (s_2 s_1^{-1} s_2^{-1}) s_3 s_2 = s_3 s_1^{-1} s_2^{-1} s_1 s_3 s_2 = s_1^{-1} s_3 s_2^{-1} s_3 s_1 s_2 \in Q_3 u_3 u_2 u_3 Q_3$. Finally, $s_3 s_2 s_1^{-1} s_2^{-1} s_3^{-1} s_2 \in Q_3 u_3 u_2 u_3 Q_3$ after lemma 4.3. \square

Proposition 4.6. *We have*

$$Q_4 = Q_3 + Q_3 s_3 Q_3 + Q_3 s_3^{-1} Q_3 + Q_3 s_3 s_2^{-1} s_3 Q_3 + R w_0 + R w_0^{-1}$$

Proof. By proposition 4.2 we know that $Q_4 = Q_4^{(2)} = Q_3 u_3 Q_3 u_3 Q_3$. But $Q_3 = u_2 u_1 u_2 + u_1 u_2 u_1$, hence $Q_4 = Q_3 u_3 u_2 u_1 u_2 u_3 Q_3 + Q_3 u_3 u_1 u_2 u_1 u_3 Q_3 = Q_3 u_3 u_2 u_1 u_2 u_3 Q_3 + Q_3 u_3 u_2 u_3 Q_3$. Since $u_3 u_2 u_3 \subset R s_3 s_2^{-1} s_3 + u_2 u_3 u_2$ we have $Q_3 u_3 u_2 u_3 Q_3 = Q_3 + Q_3 s_3 Q_3 + Q_3 s_3^{-1} Q_3 + Q_3 s_3 s_2^{-1} s_3 Q_3$. On the other hand,

$$u_3(u_2 u_1 u_2) u_3 \subset Q_4^{(1)} + \sum_{\alpha, \beta \in \{-1, 1\}} s_3^\alpha u_2 u_1 u_2 s_3^\beta$$

hence, after lemma 4.3, we get

$$u_3(u_2 u_1 u_2) u_3 \subset Q_3 u_3 u_2 u_3 Q_3 + \sum_{\alpha \in \{-1, 1\}} s_3^\alpha u_2 u_1 u_2 s_3^\alpha.$$

From $u_2 u_1 u_2 \subset R s_2 s_1^{-1} s_2 + u_1 u_2 u_1$ and $u_2 u_1 u_2 \subset R s_2^{-1} s_1 s_2^{-1} + u_1 u_2 u_1$ we deduce that $u_3 u_2 u_1 u_2 u_3 \subset Q_3 u_3 u_2 u_3 Q_3 + R w_+ + R w_-$. From lemma 4.4 (3) and (4) we get $Q_3 u_3 u_2 u_1 u_2 u_3 Q_3 \subset Q_3 u_3 u_2 u_3 Q_3 + w_+ Q_3 + w_- Q_3$. Finally, from lemma 4.4 (5), (6) and lemma 4.5 we get $Q_3 u_3 u_2 u_1 u_2 u_3 Q_3 \subset Q_3 u_3 u_2 u_3 Q_3 + R w_+ + R w_-$, and this proves the claim. \square

4.2. $Q_3 u_3 Q_3$ as a R -module.

Proposition 4.7. $Q_3 s_3^{\pm 1} Q_3 = Q_3 \cdot s_3^{\pm 1} \cdot F_1 + R \cdot E \cdot \{s_3^{\pm 1} s_2^{-1} s_1^{-1}, s_3^{\pm 1} s_2 s_1^{-1} s_2\}$ where

$$F_1 = \{1, s_2, s_2 s_1, s_2 s_1^{-1}, s_2^{-1}, s_2^{-1} s_1\}$$

and $E = \{\emptyset, 2, \bar{2}, 12, \bar{12}, \bar{1}\bar{2}, 2\bar{1}\bar{2}\}$. In particular, $Q_3 s_3^{\pm 1} Q_3$ is spanned as a R -module by 136 elements.

From $Q_3 = u_1 u_2 u_1 + R s_2 s_1^{-1} s_2$ we get $Q_3 s_3 Q_3 = Q_3 s_3 u_1 u_2 u_1 + Q_3 s_2 s_1^{-1} s_2 = Q_3 s_3 u_2 u_1 + Q_3 s_2 s_1^{-1} s_2$. Now, $s_3 u_2 u_1 = R \cdot s_3 \cdot F_1 + R s_3 s_2^{-1} s_1^{-1}$ hence $Q_3 s_3 u_2 u_1 \subset Q_3 \cdot s_3 \cdot F_1 + Q_3 s_3 s_2^{-1} s_1^{-1}$.

We use that the defining relation can be rewritten

$$a^2(s_1 - a)s_2^{-1}s_1^{-1} = (s_1 - a)s_2 s_1^{-1} + a(as_1^{-1} - 1)s_2^{-1}s_1 + (a^{-1} - s_1^{-1})s_2 s_1 + (as_1^{-1} - a^{-1}s_1)s_2 + a(s_1 - a^2 s_1^{-1})s_2^{-1}$$

whence

$$a^2(1 - a \cdot \emptyset) \cdot 3\bar{2}\bar{1} \in Q_3 \cdot 321 + Q_3 \cdot 3\bar{2}\bar{1} + Q_3 \cdot 32 + Q_3 \cdot 3\bar{2} + Q_3 \cdot 3\bar{2}\bar{1} \subset Q_3 \cdot 3 \cdot F_1$$

From the R -bases for Q_3 obtained above, we know that Q_3 is spanned as a right u_1 -module by $E = \{\emptyset, 2, \bar{2}, 12, \bar{12}, \bar{1}\bar{2}, 2\bar{1}\bar{2}\}$. Therefore by the relation above we get that $Q_3 s_3 s_2^{-1} s_1^{-1} \subset R \cdot E \cdot 3\bar{2}\bar{1} + Q_3 \cdot 3 \cdot F_1$ hence $Q_3 s_3 u_2 u_1 \subset Q_3 \cdot s_3 \cdot F_1 + R \cdot E \cdot 3\bar{2}\bar{1}$.

We now notice that $s_1 s_3 s_2 s_1^{-1} s_2 = s_3 (s_1 s_2 s_1^{-1}) s_2 = s_3 s_2^{-1} s_1 s_2^2$. Since $s_2^2 = u s_2 - v + w s_2^{-1}$ we get $s_1 s_3 s_2 s_1^{-1} s_2 = u s_3 (s_2^{-1} s_1 s_2) - v s_3 s_2^{-1} s_1 + w s_3 s_2^{-1} s_1 s_2^{-1} = u s_3 s_1 s_2 s_1^{-1} - v s_3 s_2^{-1} s_1 + w s_3 s_2^{-1} s_1 s_2^{-1} = u s_1 s_3 s_2 s_1^{-1} - v s_3 s_2^{-1} s_1 + w s_3 s_2^{-1} s_1 s_2^{-1} \in Q_3 \cdot s_3 s_2 s_1^{-1} + Q_3 s_3 s_2^{-1} s_1 + w s_3 s_2^{-1} s_1 s_2^{-1}$ and this proves $(s_1 - w) \cdot s_3 s_2 s_1^{-1} s_2 \in Q_3 \cdot s_3 s_2 s_1^{-1} + Q_3 s_3 s_2^{-1} s_1 \subset Q_3 \cdot F_1$. Therefore, $Q_3 \cdot s_3 s_2 s_1^{-1} s_2 \subset R \cdot E \cdot s_3 s_2 s_1^{-1} s_2 + Q_3 \cdot F_1$ and this proves the claim for $Q_3 s_3 Q_3$. The proof for $Q_3 s_3^{-1} Q_3$ is the same.

Lemma 4.8. $Q_3 \cdot s_3 \cdot F + Q_3 s_3^{-1} \cdot F = Q_3 \cdot F_2 + R \cdot E' \cdot \{\bar{3}\bar{2}, \bar{3}\bar{2}\bar{1}\} + R \cdot E_0 \cdot \bar{3}\bar{2}\bar{1}$ with

$$F_2 = \{s_3, s_3^{-1}, s_3 s_2, s_3 s_2^{-1}, s_3^{-1} s_2, s_3 s_2 s_1, s_3^{-1} s_2 s_1, s_3 s_2 s_1^{-1}, s_3 s_2^{-1} s_1\}$$

and $E' = (s_1 s_2 s_1) E (s_1 s_2 s_1)^{-1} = \{\emptyset, 1, \bar{1}, 21, 2\bar{1}, \bar{2}\bar{1}, \bar{2}\bar{1}\bar{1}, 1\bar{2}\bar{1}\}$, $E_0 = \{\emptyset, 1, \bar{1}, 2, \bar{2}\}$. In particular, $Q_3 \cdot s_3 \cdot F + Q_3 s_3^{-1} \cdot F$ is spanned as a R -module by 201 elements.

Proof. By definition, $Q_3.s_3.F + Q_3.s_3^{-1}.F$ is the (left) Q_3 -module generated by

$$\begin{aligned} & \{s_3, s_3s_2, s_3s_2s_1, s_3s_2s_1^{-1}, s_3s_2^{-1}, s_3s_2^{-1}s_1, s_3^{-1}, s_3^{-1}s_2, s_3^{-1}s_2s_1, s_3^{-1}s_2s_1^{-1}, s_3^{-1}s_2^{-1}, s_3^{-1}s_2^{-1}s_1\} \\ & = F_2 \cup \{s_3^{-1}s_2s_1^{-1}, s_3^{-1}s_2^{-1}, s_3^{-1}s_2^{-1}s_1\} \end{aligned}$$

Notice that $Q_3.s_3.F_1 \subset Q_3.F_2$. Using the defining relation as above (and taking its image by conjugation under $s_1s_2s_3$), we have

$$a^2(s_2-a)s_3^{-1}s_2^{-1} = (s_2-a)s_3s_2^{-1} + a(as_2^{-1}-\emptyset)s_3^{-1}s_2 + (a^{-1}-s_2^{-1})s_3s_2 + (as_2^{-1}-a^{-1}s_2)s_3 + a(s_2-a^2s_2^{-1})s_3^{-1}.$$

Since Q_3 is spanned as a right u_1 -module by E , it is spanned as a right u_2 -module by $E' = (s_1s_2s_1)E(s_1s_2s_1)^{-1}$ hence $Q_3.s_3^{-1}s_2^{-1} \subset Q_3.F_2 + R.E'.s_3^{-1}s_2^{-1}$. Similarly, $Q_3.s_3^{-1}s_2^{-1}s_1 \subset Q_3.F_2 + R.E'.s_3^{-1}s_2^{-1}s_1$.

Now, $1.\bar{3}2\bar{1} = \bar{3}(1\bar{2}\bar{1}) = (\bar{3}\bar{2})12$. Using the defining relation we get

$$\begin{aligned} a^2(2-a.\emptyset)1.\bar{3}2\bar{1} &= a^2(2-a.\emptyset)(\bar{3}\bar{2})12 \\ &= (2-a.\emptyset)\bar{3}\bar{2}12 + a(a\bar{2}-\emptyset)\bar{3}\bar{2}12 + (a^{-1}.\emptyset-\bar{2})\bar{3}\bar{2}12 + (a\bar{2}-a^{-1}2)3\bar{1}2 \\ &\quad + a(2-a^2\bar{2})\bar{3}12 \\ &= (2-a.\emptyset)\bar{3}\bar{2}12 + a(a\bar{2}-\emptyset)\bar{3}121 + (a^{-1}.\emptyset-\bar{2})3\bar{1}21 + (a\bar{2}-a^{-1}2)132 \\ &\quad + a(2-a^2\bar{2})1\bar{3}2 \\ &\in (2-a.\emptyset)\bar{3}\bar{2}12 + Q_3.F_2 \\ &\subset Q_3.F_2 \end{aligned}$$

We now use that $1\bar{2}\bar{1} \in \bar{1}21 + u_1.2 + u_1.\bar{2} + R.21 + a.2\bar{1} + u_1.\bar{2}1 - a^3.\bar{2}\bar{1}$ to get

$$\bar{3}2\bar{1} = \bar{1}\bar{3}(1\bar{2}\bar{1}) \in u_1.\bar{3}21 + u_1.\bar{3}2 + u_1.\bar{3}\bar{2} + a.\bar{1}\bar{3}2\bar{1} + u_1.\bar{3}\bar{2}1 - a^3.\bar{3}\bar{1}\bar{2}\bar{1}$$

that is

$$(\emptyset - a.\bar{1})\bar{3}2\bar{1} \in u_1.\bar{3}21 + u_1.\bar{3}2 + u_1.\bar{3}\bar{2} + u_1.\bar{3}\bar{2}1 - a^3.\bar{3}\bar{1}\bar{2}\bar{1}$$

and $(\emptyset - a.\bar{1})\bar{3}2\bar{1} \in Q_3.F_2 + Q_3.\{\bar{3}\bar{2}, \bar{3}\bar{2}1\} - a^3.\bar{3}\bar{1}\bar{2}\bar{1}$. Now, $\bar{3}(1\bar{2}\bar{1}) = \bar{3}\bar{2}\bar{1}\bar{2}$ and, by the defining relation,

$$a^2(2-a.\emptyset)\bar{3}\bar{1}\bar{2}\bar{1} = a^2(2-a.\emptyset)\bar{3}\bar{2}\bar{1}\bar{2} \in Q_3.3.Q_3 + u_2.\bar{3}\bar{2}\bar{1}\bar{2} + u_2.\bar{3}\bar{2}$$

and, since $\bar{3}\bar{2}\bar{1}\bar{2} = \bar{3}\bar{1}\bar{2}\bar{1} = \bar{1}\bar{3}\bar{2}\bar{1} \in Q_3.\bar{3}\bar{2}\bar{1}$ we get $a^2(2-a.\emptyset)\bar{3}\bar{1}\bar{2}\bar{1} \in Q_3.F_2 + R.E'.\{\bar{3}\bar{2}, \bar{3}\bar{2}1\}$. Therefore, $(2-a.\emptyset)(\emptyset - a.\bar{1})\bar{3}2\bar{1} \in Q_3.F_2 + Q_3.\{\bar{3}\bar{2}, \bar{3}\bar{2}1\} = Q_3.F_2 + R.E'.\{\bar{3}\bar{2}, \bar{3}\bar{2}1\}$. From lemma 3.2 we deduce that

$$Q_3.\bar{3}2\bar{1} \subset Q_3.F_2 + R.E'.\{\bar{3}\bar{2}, \bar{3}\bar{2}1\} + R.E_0.\bar{3}2\bar{1}$$

and this proves the claim. \square

Lemma 4.9. $Q_3s_3.Q_3 + Q_3s_3^{-1}.Q_3 = Q_3.s_3.F + Q_3.s_3^{-1}.F + R.E.3\bar{2}\bar{1} + R.E.3\bar{2}\bar{1}2 + R.\bar{3}\bar{2}\bar{1} + R.\bar{3}\bar{2}\bar{1}2$. In particular, it is spanned by 219 elements, and $Q_3u_3Q_3 = Q_3 + Q_3s_3.Q_3 + Q_3s_3^{-1}.Q_3$ is spanned by 239 elements.

Proof. Note that, according to proposition 4.7, we have $Q_3.s_3.Q_3 = Q_3.s_3.F + R.E.3\bar{2}\bar{1} + R.E.3\bar{2}\bar{1}2 \subset Q_3.s_3.F + Q_3.s_3^{-1}.F + R.E.3\bar{2}\bar{1} + R.E.3\bar{2}\bar{1}2$. In particular $I = Q_3.s_3.F + Q_3.s_3^{-1}.F + R.E.3\bar{2}\bar{1} + R.E.3\bar{2}\bar{1}2$ is a left Q_3 -module. From the defining relation we get

$$a^2(2-a.\emptyset)\bar{3}\bar{2}\bar{1} = (2-a.\emptyset)\bar{3}\bar{2}\bar{1} + a(a.\bar{2}-1)\bar{3}\bar{2}\bar{1} + (a^{-1}.\emptyset-\bar{2})\bar{3}\bar{2}\bar{1} + (a\bar{2}-a^{-1}2)3\bar{1} + a(2-a^2\bar{2})\bar{3}\bar{1}$$

hence $a^2(2-a.\emptyset)\bar{3}\bar{2}\bar{1} \in Q_3.3.Q_3 + Q_3.\bar{3}.F \subset I$. Now, in the proof of proposition 4.7 we proved that $(1-a.\emptyset)\bar{3}\bar{2}\bar{1} \in Q_3.\bar{3}.F_1 \subset I$. Since the quotient of Q_3 by its left ideal generated by $1-a.\emptyset$ and $2-a.\emptyset$ is obviously spanned by \emptyset we get that $Q_3.\bar{3}\bar{2}\bar{1} \subset R.\bar{3}\bar{2}\bar{1} + I$. In particular $J = R.\bar{3}\bar{2}\bar{1} + I$ is a Q_3 -submodule.

From the defining relation we get $a(a.\bar{2} - \emptyset)\bar{3}\bar{2} \in a^2(2 - a.\emptyset).\bar{3}\bar{2} + Q_3.3.Q_3 + Q_3.\bar{3}$ hence $a(a.\bar{2} - \emptyset)\bar{3}\bar{2}\bar{1}\bar{2} \in a^2(2 - a.\emptyset).\bar{3}\bar{2}\bar{1}\bar{2} + Q_3.3.Q_3 + Q_3.\bar{3}\bar{1}\bar{2}$ that is

$$a(a.\bar{2} - \emptyset)\bar{3}\bar{2}\bar{1}\bar{2} \in a^2(2 - a.\emptyset).\bar{3}\bar{1}\bar{2}\bar{1} + Q_3.3.Q_3 + Q_3.\bar{3}\bar{2} \subset Q_3.\bar{3}\bar{2}\bar{1} + I = J.$$

Similarly, from $(1 - a.\emptyset)2\bar{1} \in a^2(1 - a.\emptyset)\bar{2}\bar{1} - a(a.\bar{1} - \emptyset)\bar{2}\bar{1} - (a^{-1}.\emptyset - \bar{1})2\bar{1} + u_1u_2$ we get

$$\begin{aligned} (1 - a.\emptyset)\bar{3}\bar{2}\bar{1}\bar{2} &\in a^2(1 - a.\emptyset)\bar{3}\bar{2}\bar{1}\bar{2} - a(a.\bar{1} - \emptyset)\bar{3}\bar{2}\bar{1}\bar{2} - (a^{-1}.\emptyset - \bar{1})\bar{3}\bar{2}\bar{1}\bar{2} + u_1\bar{3}u_2 \\ &\in a^2(1 - a.\emptyset)\bar{3}\bar{1}\bar{2}\bar{1} - a(a.\bar{1} - \emptyset)\bar{3}\bar{1}\bar{2}\bar{1} - (a^{-1}.\emptyset - \bar{1})\bar{3}\bar{1}\bar{2}\bar{1} + u_1\bar{3}u_2 \\ &\in u_1\bar{3}\bar{2}\bar{1} - u_1\bar{3}\bar{2}\bar{1} - u_1\bar{3}\bar{2}\bar{1} + u_1\bar{3}u_2 \\ &\in Q_3\bar{3}\bar{2}\bar{1} + Q_3.\bar{3}.F_1 \subset J \end{aligned}$$

Since the quotient of Q_3 by its left ideal generated by $(1 - a.\emptyset)$ and $a(a.\bar{2} - \emptyset)$ is spanned by the image of \emptyset , we get $Q_3.\bar{3}\bar{2}\bar{1}\bar{2} \in R.\bar{3}\bar{2}\bar{1}\bar{2} + J$ and this proves the claim. \square

4.3. Spanning $Q_4/Q_3u_3Q_3$ as a R -module - preliminaries. By proposition 4.6 we know that $Q_4 = Q_3u_3Q_3 + Rw_0 + Rw_0^{-1} + Q_3.3\bar{2}\bar{3}.Q_3$.

4.3.1. *Step 1 :* $Q_3u_3Q_3 + u_2\bar{3}\bar{2}\bar{3}u_2 = Q_3u_3Q_3 + R.3\bar{2}\bar{3}$. First note that $2.3\bar{2}\bar{3} = (2\bar{3}\bar{2})3 = \bar{3}\bar{2}\bar{3}\bar{3} \in u_2u_3u_2 + R.3\bar{2}\bar{3}$ by theorem 3.1. Since Q_3 is spanned as a right u_2 -module by E' , we get that $Q_3u_3Q_3 + Q_3.3\bar{2}\bar{3} = Q_3u_3Q_3 + R.E'.3\bar{2}\bar{3}$. By a similar argument we get $\bar{3}\bar{2}\bar{3}.2 \in u_2u_3u_2 + R.3\bar{2}\bar{3}$ hence $Q_3u_3Q_3 + Q_3.3\bar{2}\bar{3}.u_2 = Q_3u_3Q_3 + Q_3.3\bar{2}\bar{3} = Q_3u_3Q_3 + R.E'.3\bar{2}\bar{3}$.
suite. ble

4.3.2. *Step 2 :* $Q_3u_3Q_3 + u_1\bar{3}\bar{2}\bar{3}u_1 = Q_3u_3Q_3 + u_1.x + x.u_1 + R.1x1 + R.1x\bar{1}$, $x = \bar{3}\bar{2}\bar{3}$.

While studying Q_3 as a Q_2 -module in section 3.5, the quotient module of $Q_2u_2Q_2$ by its submodule $Q_2 + Q_2s_2Q_2$ has been determined under the name M'_1/M_+ , and it was proven to be generated by (images of) the seven elements $\bar{2}$, $\bar{1}\bar{2}$, $\bar{1}\bar{2}$, $\bar{2}\bar{1}$, $\bar{2}\bar{1}$, $\bar{1}\bar{2}\bar{1}$, $\bar{1}\bar{2}\bar{1}$. Therefore, $Q_3u_3Q_3 + u_1\bar{3}\bar{2}\bar{3}u_1$ is spanned by the already determined $219 + 20 = 239$ elements spanning $Q_3u_3Q_3$ plus the 7 elements $\bar{3}\bar{2}\bar{3}$, $1\bar{3}\bar{2}\bar{3}$, $\bar{1}\bar{3}\bar{2}\bar{3}$, $\bar{3}\bar{2}\bar{3}\bar{1}$, $\bar{3}\bar{2}\bar{3}\bar{1}$, $1\bar{3}\bar{2}\bar{3}\bar{1}$, $1\bar{3}\bar{2}\bar{3}\bar{1}$

4.3.3. *Step 3 :* $u_2\bar{3}\bar{1}\bar{2}\bar{1}\bar{3} \subset Q_3u_3Q_3 + R.3\bar{1}\bar{2}\bar{1}\bar{3} + u_2.3\bar{1}\bar{2}\bar{1}\bar{3} + u_2.1\bar{3}\bar{2}\bar{3}$.

By the study at the end of section 3.5 we know that $\bar{1}\bar{2}\bar{1} \equiv -(b+c)w^{-1}.\bar{1}\bar{2}\bar{1} - (wa)^{-1}.\bar{1}\bar{2}\bar{1} + (v/w).\bar{2}\bar{1} + w^{-1}\bar{2}\bar{1} + a^{-1}\bar{1}\bar{2} + (u/(wa)).\bar{1}\bar{2} - ((a^2+v)/(wa))\bar{2}$ modulo $Q_2 + Q_2s_2Q_2$. Therefore $\bar{3}\bar{1}\bar{2}\bar{1}\bar{3} \equiv -(b+c)w^{-1}.\bar{3}\bar{1}\bar{2}\bar{1}\bar{3} - (wa)^{-1}.\bar{3}\bar{1}\bar{2}\bar{1}\bar{3} + (v/w).\bar{3}\bar{2}\bar{3}\bar{1} + w^{-1}\bar{3}\bar{2}\bar{3}\bar{1} + a^{-1}\bar{1}\bar{3}\bar{2}\bar{3} + (u/(wa)).\bar{3}\bar{1}\bar{2}\bar{3} - ((a^2+v)/(wa))\bar{3}\bar{2}\bar{3}$ modulo $Q_3u_3Q_3$ and $1\bar{2}\bar{3}\bar{1}\bar{2}\bar{1}\bar{3} \equiv -(b+c)w^{-1}.\bar{1}\bar{2}\bar{3}\bar{1}\bar{2}\bar{1}\bar{3} - (wa)^{-1}.\bar{1}\bar{2}\bar{3}\bar{1}\bar{2}\bar{1}\bar{3} + (v/w).\bar{1}\bar{2}\bar{3}\bar{2}\bar{3}\bar{1} + w^{-1}.\bar{1}\bar{2}\bar{3}\bar{2}\bar{3}\bar{1} + a^{-1}.\bar{1}\bar{2}\bar{3}\bar{1}\bar{2}\bar{3} + (u/(wa)).\bar{1}\bar{2}\bar{3}\bar{1}\bar{2}\bar{3} - ((a^2+v)/(wa))\bar{1}\bar{2}\bar{3}\bar{2}\bar{3}$ modulo $Q_3u_3Q_3$.

Since $2\bar{3}\bar{2}\bar{3} \equiv a.3\bar{2}\bar{3} \pmod{Q_3u_3Q_3}$, we have $1\bar{2}\bar{3}\bar{1}\bar{2}\bar{1}\bar{3} = 1\bar{2}\bar{3}\bar{2}\bar{3}\bar{1}\bar{3} = 2\bar{1}\bar{2}\bar{3}\bar{2}\bar{3}\bar{1} \equiv a.2\bar{1}\bar{3}\bar{2}\bar{3}\bar{1}$, $1\bar{2}\bar{3}\bar{1}\bar{2}\bar{1}\bar{3} = 1\bar{2}\bar{1}\bar{3}\bar{2}\bar{3}\bar{1} = 2\bar{1}\bar{2}\bar{3}\bar{2}\bar{3}\bar{1} \equiv a.2\bar{1}\bar{3}\bar{2}\bar{3}\bar{1}$, $1\bar{2}\bar{3}\bar{2}\bar{3}\bar{1} \equiv a.1\bar{3}\bar{2}\bar{3}\bar{1}$, $1\bar{2}\bar{3}\bar{2}\bar{3}\bar{1} \equiv a.1\bar{3}\bar{2}\bar{3}\bar{1}$, $1\bar{2}\bar{1}\bar{3}\bar{2}\bar{3} = \bar{2}\bar{1}\bar{2}\bar{3}\bar{2}\bar{3} \equiv a.\bar{2}\bar{1}\bar{3}\bar{2}\bar{3}$ and $1\bar{2}\bar{3}\bar{1}\bar{2}\bar{3} = 1\bar{2}\bar{1}\bar{3}\bar{2}\bar{3} = 2\bar{1}\bar{2}\bar{3}\bar{2}\bar{3} \equiv a.2\bar{1}\bar{3}\bar{2}\bar{3}$ modulo $Q_3u_3Q_3$.

It follows that $1\bar{2}\bar{3}\bar{1}\bar{2}\bar{1}\bar{3} \equiv -(b+c)w^{-1}.a.2\bar{1}\bar{3}\bar{2}\bar{3}\bar{1} - w^{-1}.2\bar{1}\bar{3}\bar{2}\bar{3}\bar{1} + (v/w).a.1\bar{3}\bar{2}\bar{3}\bar{1} + w^{-1}.a.1\bar{3}\bar{2}\bar{3}\bar{1} + \bar{2}\bar{1}\bar{3}\bar{2}\bar{3} + (u/w).2\bar{1}\bar{3}\bar{2}\bar{3} - ((a^2+v)/w)1\bar{3}\bar{2}\bar{3}$ modulo $Q_3u_3Q_3$. On the other hand, we have $1\bar{2}\bar{3}\bar{1}\bar{2}\bar{1}\bar{3} = 1\bar{2}\bar{1}\bar{3}\bar{2}\bar{3}\bar{1} = \bar{2}\bar{1}\bar{2}\bar{3}\bar{2}\bar{3}\bar{1} \equiv a.\bar{2}\bar{1}\bar{3}\bar{2}\bar{3}\bar{1}$ modulo $Q_3u_3Q_3$. This yields to $\bar{2}\bar{1}\bar{3}\bar{2}\bar{3}\bar{1} \equiv -(b+c)w^{-1}.2\bar{1}\bar{3}\bar{2}\bar{3}\bar{1} - (aw)^{-1}.2\bar{1}\bar{3}\bar{2}\bar{3}\bar{1} + (v/w).1\bar{3}\bar{2}\bar{3}\bar{1} + w^{-1}.1\bar{3}\bar{2}\bar{3}\bar{1} + a^{-1}.\bar{2}\bar{1}\bar{3}\bar{2}\bar{3} + (u/(wa)).2\bar{1}\bar{3}\bar{2}\bar{3} - ((a^2+v)/(aw))1\bar{3}\bar{2}\bar{3}$ modulo $Q_3u_3Q_3$, which can be rephrased as $(\bar{2} + (b+c)w^{-1}.2 - (v/w).\emptyset)1\bar{3}\bar{2}\bar{3}\bar{1}$

being congruent to $-(aw)^{-1}.213\bar{2}31 + w^{-1}.13\bar{2}31 + a^{-1}.\bar{2}13\bar{2}3 + (u/(wa)).213\bar{2}3 - ((a^2 + v)/(aw)).13\bar{2}3$ modulo $\mathbb{Q}_3u_3\mathbb{Q}_3$. Now, $\bar{2} + (b+c)w^{-1}.2 - (v/w).\emptyset = w^{-1}.2.(2-a.\emptyset)$ hence

$$2(2-a.\emptyset)31\bar{2}\bar{1}3 \equiv -a^{-1}.213\bar{2}31 + 13\bar{2}31 + wa^{-1}.\bar{2}13\bar{2}3 + (u/a).213\bar{2}3 - ((a^2+v)/a).13\bar{2}3$$

modulo $\mathbb{Q}_3u_3\mathbb{Q}_3$ and

$$(4.1) \quad (2-a.\emptyset)31\bar{2}\bar{1}3 \equiv -a^{-1}.13\bar{2}31 + \bar{2}13\bar{2}31 + wa^{-1}.\bar{2}\bar{2}13\bar{2}3 + (u/a).13\bar{2}3 - ((a^2+v)/a).\bar{2}13\bar{2}3$$

modulo $\mathbb{Q}_3u_3\mathbb{Q}_3$ and this proves in particular that $u_231\bar{2}\bar{1}3 \subset \mathbb{Q}_3u_3\mathbb{Q}_3 + R.31\bar{2}\bar{1}3 + u_2.31\bar{2}13 + u_2.13\bar{2}3$. Applying the usual automorphisms and the previous reductions we deduce that $31\bar{2}\bar{1}3u_2 \subset \mathbb{Q}_3u_3\mathbb{Q}_3 + R.31\bar{2}\bar{1}3 + 31\bar{2}13.u_2 + 3\bar{2}31.u_2 + u_1.3\bar{2}3.u_1$

4.4. $\mathbb{Q}_4/\mathbb{Q}_3u_3\mathbb{Q}_3$ as a R -module - computational description. We concentrate our attention on the \mathbb{Q}_3 -bimodule $\tilde{A}_4 = \mathbb{Q}_4/\mathbb{Q}_3u_3\mathbb{Q}_3$.

4.4.1. A convenient basis. We introduce the following list of 25 elements of \tilde{A}_4 . They will turn out to provide a R -basis. We set $x = 3\bar{2}3$ and $y = 1\bar{2}1$.

$e_1 = 3\bar{2}3$	$e_2 = 13\bar{2}3$	$e_3 = \bar{1}3\bar{2}3$	$e_4 = 213\bar{2}3$
$e_5 = 2\bar{1}3\bar{2}3$	$e_6 = \bar{2}13\bar{2}3$	$e_7 = \bar{2}\bar{1}3\bar{2}3$	$e_8 = 3\bar{2}31$
$e_9 = 3\bar{2}3\bar{1}$	$e_{10} = 3\bar{2}312$	$e_{11} = 3\bar{2}31\bar{2}$	$e_{12} = 3\bar{2}3\bar{1}\bar{2}$
$e_{13} = 3\bar{2}3\bar{1}\bar{2}$	$e_{14} = 213\bar{2}31$	$e_{15} = \bar{2}13\bar{2}3\bar{1}$	$e_{16} = 13\bar{2}312$
$e_{17} = 13\bar{2}31\bar{2}$	$e_{18} = 13\bar{2}3\bar{1}$	$e_{19} = 13\bar{2}31$	$e_{20} = 213\bar{2}31\bar{2}$
$e_{21} = 213\bar{2}31\bar{2}$	$e_{22} = 3\bar{2}\bar{1}23$	$e_{23} = \bar{3}\bar{2}\bar{1}\bar{2}\bar{3}$	$e_{24} = 3\bar{2}31\bar{2}1$
$e_{25} = 1\bar{2}13\bar{2}3$			

4.4.2. Description of $f = \Phi \circ \Psi$ on \tilde{A}_4 . We let f denote the R -module automorphism induced by $\Phi \circ \Psi = \Psi \circ \Phi$ on \tilde{A}_4 . We have $f(3\bar{2}3) = 3\bar{2}3$, and an immediate verification shows that $f(e_i) = e_{\sigma_f(i)}$ for all $i \notin \{18, 21\}$, with

$$\sigma_f = (2, 8)(3, 9)(4, 10)(5, 12)(6, 11)(7, 13)(14, 16)(15, 17)(24, 25) \in \mathfrak{S}_{25}.$$

It remains to compute $f(e_{18})$ and $f(e_{21})$.

We have $f(e_{18}) = \bar{1}x1 = 3\bar{1}\bar{2}13$. By section 3.5 (or by relation (16) of table 7) we know that $\bar{1}\bar{2}1 \equiv 1\bar{2}\bar{1} - a.\bar{2}\bar{1} + a^{-1}.\bar{2}1 + a.\bar{1}\bar{2} - a^{-1}.\bar{1}\bar{2}$ modulo M_+ . This implies, as in step 2, that $\bar{1}x1 \equiv 1x\bar{1} - a.x\bar{1} + a^{-1}.x1 + a.\bar{1}x - a^{-1}.1x$ (mod. $\mathbb{Q}_3u_3\mathbb{Q}_3$). This proves that

$$f(e_{18}) = e_{18} + a.(e_3 - e_9) + a^{-1}.(e_8 - e_2).$$

which completes the explicit determination of f , except for $f(e_{21}) = \bar{2}1x12$. We will determine in section 4.4.4 that

$$(4.2) \quad f(e_{21}) = e_{21} + a.(e_{15} - e_{17}) + a^{-1}.(e_{16} - e_{14})$$

Note that this description provides a square matrix of size 25 with coefficients in the subring $\mathbb{Z}[a^{\pm 1}]$ of R .

4.4.3. *Description of $u_2u_1xu_1 + u_1xu_1u_2$.* We have $\bar{1}x12 = f(21x\bar{1})$ and, from equation (4.1) we get $(2-a.\emptyset).1x\bar{1} = -a^{-1}.e_{19} + e_{15} + wa^{-1}.221x + (u/a).e_2 - (a^2+v)a^{-1}.e_6$. Since $221x = w^{-1}21.x - uw^{-1}.1x + vw^{-1}.21x = w^{-1}.e_4 - uw^{-1}.e_2 + vw^{-1}.e_6$ we get $(2-a.\emptyset)1x\bar{1} = -a^{-1}e_{19} + e_{15} + a^{-1}.e_4 - a.e_6$, that is

$$(4.3) \quad 21x\bar{1} = a.e_{18} - a^{-1}e_{19} + e_{15} + a^{-1}.e_4 - a.e_6$$

Then $\bar{1}x12 = f(21x\bar{1}) = a.f(e_{18}) - a^{-1}e_{19} + e_{17} + a^{-1}.e_{10} - a.e_{11}$

We want to compute $\bar{1}x\bar{1}$. Following the indications of step 2, we use the results of section 3.5 (in particular relation (17) of table 7) to expand $\bar{1}x\bar{1} = 3(\bar{1}\bar{2}\bar{1})2$ and get

$$(4.4) \quad \bar{1}x\bar{1} = -\frac{b+c}{w}.e_{18} - \frac{1}{wa}.e_{19} + \frac{v}{w}.e_9 + w^{-1}.e_8 + a^{-1}.e_3 + \frac{u}{wa}.e_2 - \frac{a^2+v}{wa}.e_1$$

From this and the identity $2x = x2 = a.x$ one readily gets

$$(4.5) \quad \bar{1}x\bar{1}2 = -\frac{b+c}{w}.1x\bar{1}2 - \frac{1}{wa}.e_{16} + \frac{v}{w}.e_{12} + w^{-1}.e_{10} + e_3 + \frac{u}{w}.e_2 - \frac{a^2+v}{w}.e_1$$

From the identity $\bar{1}x1 \equiv 1x\bar{1} - a.x\bar{1} + a^{-1}.x1 + a.\bar{1}x - a^{-1}.1x$ obtained above, we get $2\bar{1}x1 = 21x\bar{1} - a.2x\bar{1} + a^{-1}.2x1 + a.2\bar{1}x - a^{-1}.21x = 21x\bar{1} - a^2.x\bar{1} + x1 + a.2\bar{1}x - a^{-1}.21x$ hence

$$(4.6) \quad 2\bar{1}x1 = 21x\bar{1} - a^2e_9 + e_8 + a.e_5 - a^{-1}.e_4$$

and $21x\bar{1}$ is known by (4.3). Similarly, we get

$$(4.7) \quad \bar{2}1x\bar{1} = \bar{2}1x\bar{1} - e_9 + a^{-2}.e_8 + a.e_7 - a^{-1}.e_6$$

But $\bar{2}1x\bar{1}$ is not known yet. We get it as follows. From the description of $21x\bar{1}$ in (4.3) we get, after using the cubic relation a couple of times, that $221x\bar{1} = a.21x\bar{1} - a^{-1}.e_{14} + e_{19} + (u/a).e_4 - \frac{v+a^2}{a}.e_2 + (w/a).e_6$. Expanding $22 = u.2 - v.\emptyset + w.\bar{2}$ we get from this that

$$(4.8) \quad \bar{2}1x\bar{1} = a^{-1}.e_{18} + \frac{u}{wa}.e_{19} + \frac{a-u}{w}.e_{15} + w^{-1}.e_4 + \frac{v}{w}.e_6 - \frac{a^2+v}{aw}.e_2 - \frac{1}{aw}.e_{14}$$

4.4.4. *Description of $u_2u_1xu_1u_2$.* Let us denote $e' = \bar{2}1x\bar{1}2$. We postpone for now the determination of its value. Note that $f(e') = e'$. Multiplying (4.3) on the right by 2 and using expansion of $2\bar{2}$ by the cubic relation as well as $x2 = a.x$, we get that

$$(4.9) \quad 21x\bar{1}2 = a.f(2\bar{1}x1) + f(e_{21}) + e_4 - a^2.e_6 - a^{-1}.e_{16}$$

Similarly, multiplying (4.3) on the right by $\bar{2}$, one gets

$$(4.10) \quad 21x\bar{1}\bar{2} = a.f(\bar{2}\bar{1}x1) - a^{-1}.e_{17} + e' + a^{-2}.e_4 - e_6$$

We start over the same computations, this time from (4.8), multiplying first by 2 and then by $\bar{2}$ on the right. One gets

$$(4.11) \quad \bar{2}1x\bar{1}2 = \frac{bc}{w}.f(2\bar{1}x1) + \frac{u}{wa}.e_{16} + \frac{a-u}{w}.f(e_{21}) - \frac{1}{wa}.e_{20} + \frac{va}{w}.e_6 - \frac{v+a^2}{w}.e_2 + \frac{1}{bc}.e_4$$

and

$$(4.12) \quad \bar{2}1x\bar{1}\bar{2} = \frac{bc}{w}.f(\bar{2}\bar{1}x1) + \frac{u}{wa}.e_{17} + \frac{a-u}{w}.e' + w^{-1}a^{-1}.e_4 + \frac{v}{wa}.e_6 - \frac{v+a^2}{a^2w}.e_2 - \frac{1}{aw}.e_{21}$$

Note that the above four equations need an expression of $f(e_{21})$ and e' to be expressible as a linear combination of the $(e_i)_{1 \leq i \leq 25}$. We first compute $f(e_{21}) = 21x1\bar{2}$. Its expression as

been given in (4.2), but is not yet justified. We do it now. Following rule (18) of table 3.5, we can expand $1\bar{2}1$ inside $21x1\bar{2} = 231\bar{2}13\bar{2}$. This yields

$$(4.13) \quad 21x1\bar{2} \equiv 232\bar{1}23\bar{2} - (bc).\bar{3}213\bar{2}3\bar{2} + (a^2 + v).x1\bar{2} + \frac{a^2 + v}{a^2}.21x - (a^2 + v).e_1$$

Note that $232\bar{1}23\bar{2} = 2.w_+.\bar{2} = w_+$. We now use rule (23) of table 6 (after applying the shift morphism $1 \mapsto 2, 2 \mapsto 3$) to expand $\bar{2}3\bar{2}3$ inside $\bar{2}3\bar{2}31\bar{2}3 = f(\bar{3}213\bar{2}3\bar{2})$. Using in addition a few easy braid relations we get from this expansion that

$$\bar{2}3\bar{2}31\bar{2}3 \equiv a^{-1}.\bar{3}\bar{2}31\bar{2}3 - \frac{a}{bc}.21x\bar{1} + \frac{a^2}{bc}.1x\bar{1} + \frac{v}{bc}.\bar{2}1x + (bc)^{-1}.21x - \frac{a^2 + v}{w}.1x$$

modulo $Q_3u_3Q_3$. Plugging this into (4.13) we get that $21x1\bar{2}$ is equal to

$$w_+ - \frac{bc}{a}.\bar{3}\bar{2}13\bar{2}3 + a.\bar{1}x12 - a^2.\bar{1}x1 + a^2.x1\bar{2} - x12 + \frac{a^2 + v}{a}.x1 + \frac{a^2 + v}{a^2}.21x - (a^2 + v).x$$

modulo $Q_3u_3Q_3$. Now, $\bar{3}\bar{2}13\bar{2}3 = \bar{3}\bar{2}31\bar{2}3 = 23\bar{2}1\bar{2}3$ and expanding $\bar{2}1\bar{2}$ by rule (18) of table 3.5, we get that

$$(bc).23\bar{2}1\bar{2}3 = 2.w_+ - 21x1 + (a^2 + v).x1 + \frac{a^2 + v}{a}.21x - a(a^2 + v).x$$

Altogether, this yields

$$(4.14) \quad 21x1\bar{2} = (\emptyset - a^{-1}.2).w_+ + a^{-1}.21x1 + a.\bar{1}x12 - a^2.\bar{1}x1 + a^2.x1\bar{2} - x12$$

Applying f we get

$$(4.15) \quad \bar{2}1x12 = (\emptyset - a^{-1}.2).w_+ + a^{-1}.1x12 + a.21x\bar{1} - a^2.1x\bar{1} + a^2.\bar{2}1x - 21x$$

since $w_+2 = 2w_+$. Therefore, we have

$$(4.16)$$

$$21x1\bar{2} - \bar{2}1x12 = a^{-1}.(21x1 - 1x12) + a.(\bar{1}x12 - 21x\bar{1}) + a^2.(1x\bar{1} - \bar{1}x1) + a^2.(x1\bar{2} - \bar{2}1x) + (21x - x12)$$

and from this we get the expression of $f(e_{21})$ obtained above (4.2).

We now compute $e' = \bar{2}1x1\bar{2}$. Multiplying (4.15) on the right by 2 yields (after expanding $2\bar{2}$ inside $1x12$)

$$\bar{2}1x122 = w_+.((\emptyset - a^{-1}.2)2) + a^{-1}u.1x12 - a^{-1}v.1x1 + a^{-1}w.1x1\bar{2} + a.21x\bar{1}2 - a^2.1x\bar{1}2 + a^3.\bar{2}1x - a.21x$$

Using that $\bar{2} = w^{-1}.22 - uw^{-1}.2 + vw^{-1}.\emptyset$ we deduce from this that $\bar{2}1x1\bar{2}$ is equal to

$$w^{-1}.w_+.((\emptyset - a^{-1}.2)2) + \frac{u}{aw}.1x12 - \frac{v}{aw}.1x1 + \frac{1}{a}.1x1\bar{2} + \frac{a}{w}.21x\bar{1}2 - \frac{a^2}{w}.1x\bar{1}2 + \frac{a^3}{w}.\bar{2}1x - \frac{a}{w}.21x - \frac{u}{w}.\bar{2}1x12 + \frac{v}{w}.\bar{2}1x1$$

Now, we use that $1x12 = 2\bar{2}1x12$. From (4.15) this yields $1x12 = 2.w_+.((\emptyset - a^{-1}.2) + a^{-1}.21x12 + a.221x\bar{1} - a^2.21x\bar{1} + a^2.1x - 221x)$. Expanding $2\bar{2}$ twice, we get that $w_+.((\emptyset - a^{-1}.2)2) = 2.w_+.((\emptyset - a^{-1}.2)$ is equal to

$$1x12 - a^{-1}.21x12 - au.21x\bar{1} - av.1x\bar{1} - aw.\bar{2}1x\bar{1} + a^2.21x\bar{1} - (a^2 + v).1x + u.21x + w.\bar{2}1x$$

Plugging this into the former equation provides an expression for $\bar{2}1x1\bar{2}$, as

$$\begin{aligned} \bar{2}1x1\bar{2} = & \frac{a + u}{aw}.e_{16} - \frac{1}{aw}.e_{20} + \frac{a(a - u)}{w}.21x\bar{1} + \frac{av}{w}.e_{18} - a.\bar{2}1x\bar{1} - \frac{a^2 + v}{w}.e_2 + \frac{u - a}{w}.e_4 - \frac{v}{aw}.e_{19} \\ & + \frac{a}{w}.21x\bar{1}2 + a^{-1}.e_{17} - \frac{a^2}{w}.1x\bar{1}2 + \left(\frac{a^3}{w} + 1\right).e_6 - \frac{u}{w}.\bar{2}1x12 + \frac{v}{w}.e_{15} \end{aligned}$$

We shall need to compute $x\bar{1}2\bar{1}$. Using relation (18) of table 7 to expand $\bar{1}2\bar{1}$, we get after an easy calculation, that

$$(4.17) \quad x\bar{1}2\bar{1} = \frac{v}{w}.e_{12} + (bc)^{-1}.e_{24} - \frac{au}{w}.e_{11}$$

4.5. $\mathbb{Q}_4/\mathbb{Q}_3u_3\mathbb{Q}_3$ as a \mathbb{Q}_3 -bimodule - computational description.

4.5.1. Left multiplication by s_1 inside \tilde{A}_4 .

$e_1 \mapsto e_2$	$e_6 \mapsto e_{25}$	$e_{11} \mapsto e_{17}$
$e_2 \mapsto -v.e_1 + u.e_2 + w.e_3$	$e_7 \mapsto a.e_7$	$e_{12} \mapsto f(2\bar{1}x\bar{1})$
$e_3 \mapsto e_1$	$e_8 \mapsto e_{19}$	$e_{13} \mapsto f(\bar{2}\bar{1}x\bar{1})$
$e_4 \mapsto a.e_4$	$e_9 \mapsto e_{18}$	$e_{14} \mapsto a.e_{14}$
$e_5 \mapsto a.e_6$	$e_{10} \mapsto e_{16}$	$e_{15} \mapsto f(1.e_{24})$

and

$e_{16} \mapsto u.e_{16} - v.e_{10} + w.f(2\bar{1}x\bar{1})$	$e_{21} \mapsto a.e_{21}$
$e_{17} \mapsto u.e_{17} - v.e_{11} + w.f(\bar{2}\bar{1}x\bar{1})$	$e_{22} \mapsto (4.19)$
$e_{23} \mapsto (4.22)\&(4.4)$	$e_{23} \mapsto (4.22)$
$e_{18} \mapsto u.e_{18} - v.e_9 + w.\bar{1}x\bar{1}$	$e_{24} \mapsto (4.23)$
$e_{19} \mapsto u.e_{19} - v.e_8 + w.f(e_{18})$	$e_{25} \mapsto u.e_{25} - v.e_6 + wa^{-1}.e_5$
$e_{20} \mapsto a.e_{20}$	

We now consider $1w_+ = 3(12\bar{1}2)3$. Using rule (15) in table 7 we get that $1w_+ = w.3\bar{2}\bar{1}\bar{2}3 + u.3\bar{1}2\bar{1}3 - v.e_8$. We have $3\bar{1}2\bar{1}3 = 132\bar{3}\bar{1} = 1232\bar{1} \equiv 0$, hence $1w_+ \equiv w.3\bar{2}\bar{1}\bar{2}3 - v.e_8$. Now, using rule (18) in table 7 we get

$$(4.18) \quad 3\bar{2}\bar{1}\bar{2}3 \equiv (bc)^{-1}.e_{22} - (bc)^{-1}.e_{19} + \frac{a^2+v}{w}.e_8 + \frac{a^2+v}{w}.e_2 - \frac{a^2+v}{bc}.e_1$$

hence

$$(4.19) \quad 1w_+ \equiv a.e_{22} - a.e_{19} + a^2.e_8 + (a^2+v)e_2 - a(a^2+v).e_1.$$

In particular we get the following potentially useful property

$$(4.20) \quad 1.w_+ \equiv a.w_+ \pmod{\mathbb{Q}_4^{(1)} + u_1xu_1}.$$

We now want to compute $1w_- = w_-1 = 1.e_{23}$. For this we first compute $\bar{1}w_- = \bar{1}3\bar{2}\bar{1}\bar{2}\bar{3} = \bar{3}(\bar{1}2\bar{1})2\bar{3} = \bar{3}\bar{2}\bar{1}(22)\bar{3} = w^{-1}\bar{3}\bar{2}\bar{1}2\bar{3} - uw^{-1}.\bar{3}\bar{2}\bar{1}\bar{3} + vw^{-1}.\bar{3}\bar{2}\bar{1}2\bar{3}$. Since $\bar{3}\bar{2}\bar{1}2\bar{3} = \bar{3}\bar{1}2\bar{1}\bar{3} = \bar{1}\bar{3}\bar{2}\bar{3}\bar{1}\bar{1}\bar{2}\bar{3}\bar{3}\bar{1} \equiv 0$ we get $\bar{1}w_- = w^{-1}.\bar{3}\bar{2}\bar{1}2\bar{3} - uw^{-1}(bc)^{-1}.x\bar{1}$.

Then using relation (18) of table 7 to replace $2\bar{1}2$, we get after a straightforward computation that

$$(4.21) \quad \bar{3}\bar{2}\bar{1}2\bar{3} \equiv (bc)e_{23} - \bar{1}x\bar{1} + \frac{a^2+v}{w}e_9 + \frac{a^2+v}{w}e_3 - \frac{a^2+v}{wa}e_1$$

Now, $\bar{1}\bar{1}.w_- = w^{-1}.\bar{1}.\bar{3}\bar{2}\bar{1}2\bar{3} - uw^{-1}(bc)^{-1}.\bar{1}x\bar{1}$. From the expression of $\bar{3}\bar{2}\bar{1}2\bar{3}$ given above, this yields after a again straightforward computation that

$$\bar{1}\bar{1}.w_- = w^{-2}(bc)^2.e_{23} + \frac{a^2+v}{w^2a}.e_9 + \frac{v(a^2+v)}{w^3}.e_3 - \frac{a^2+v}{w^3a}(bc+ua).e_1 - w^{-2}.e_{18} - \frac{v}{w^2}.\bar{1}x\bar{1} + \frac{a^2+v}{w^3}.e_2$$

Now, $1.w_- = u.w_- - v.\bar{1}.w_- + w.\bar{1}\bar{1}.w_-$. From this we easily get

$$(4.22) \quad 1.w_- = a.e_{23} + \frac{a}{w}.e_9 - \frac{a(a^2+v)}{w^2}.e_1 - w^{-1}.e_{18} + \frac{a^2+v}{w^2}.e_2$$

We now want to compute $1.e_{24} = 1xy$. We use the identity $1x\bar{1} = \bar{1}x1 + a.x\bar{1} - a^{-1}.x1 - a.\bar{1}x + a^{-1}.1x \pmod{Q_3u_3Q_3}$. Multiplying on the right by $2\bar{1}$ we get (through a couple of braid relations and $x2 = ax$) that

$$1x\bar{1}2\bar{1} = a^{-1}.f(21x\bar{1}) + a.x\bar{1}2\bar{1} - a^{-2}.e_{10} - a^2.\bar{1}x\bar{1} + e_{18}$$

which provides an explicit description of $1x\bar{1}2\bar{1}$ thanks to (4.17).

Using relation (18) of table 7 to replace $y = 1\bar{2}1$, we get after a straightforward computation making use of $x2 = ax$ and $x\bar{2} = a^{-1}x$ that

$$(4.23) \quad 1.xy = u.e_{17} - \frac{v}{a}.f(2\bar{1}x1) + (bc).1x\bar{1}2\bar{1}$$

Now, $1.e_{15} = 1\bar{2}13\bar{2}31 = yx1 = f(1xy)$ and this completes the table of left multiplication by s_1 .

4.5.2. Left multiplication by s_2 inside \tilde{A}_4 .

$e_1 \mapsto a.e_1$	$e_6 \mapsto e_2$	$e_{11} \mapsto a.e_{11}$
$e_2 \mapsto e_4$	$e_7 \mapsto e_3$	$e_{12} \mapsto a.e_{12}$
$e_3 \mapsto e_5$	$e_8 \mapsto a.e_8$	$e_{13} \mapsto a.e_{13}$
$e_4 \mapsto u.e_4 - v.e_2 + w.e_6$	$e_9 \mapsto a.e_9$	$e_{14} \mapsto u.e_{14} - v.e_{19} + w.e_{15}$
$e_5 \mapsto u.e_5 - v.e_3 + w.e_7$	$e_{10} \mapsto a.e_{10}$	$e_{15} \mapsto e_{19}$

and

$e_{16} \mapsto e_{20}$	$e_{21} \mapsto u.e_{21} - v.e_{17} + w.e'$
$e_{17} \mapsto e_{21}$	$e_{22} \mapsto (4.25)$
$e_{18} \mapsto 21x\bar{1} \quad (4.3)$	$e_{23} \mapsto (4.26)$
$e_{19} \mapsto e_{14}$	$e_{24} \mapsto a.e_{24}$
$e_{20} \mapsto u.e_{20} - v.e_{16} + w.f(e_{21})$	$e_{25} \mapsto (4.24)$

where $e' = \bar{2}1x1\bar{2}$ has been computed in section 4.4.4.

Every entry in the table is straightforward to compute, except for 3 of them. We need to compute $2.e_{22} = 2.w_+$, $2.e_{23} = 2.w_-$ and $2.e_{25}$. We start with $2.e_{25} = 2yx = 21\bar{2}1x = \bar{1}211x = u.\bar{1}21x - v.\bar{1}2.x + w.\bar{1}2\bar{1}x = u.21\bar{2}x - av.\bar{1}.x + w.f(x\bar{1}2\bar{1})$ hence

$$(4.24) \quad 2.e_{25} = \frac{u}{a}.e_4 - av.e_3 + w.f(x\bar{1}2\bar{1})$$

Now, from (4.14) one easily gets that

$$(4.25) \quad 2.e_{22} = 2.w_+ = -a.e_{21} + a.e_{22} + e_{14} + a^2.\bar{1}x12 - a^3.f(e_{18}) + a^3.e_{11} - a.e_{10}$$

We now compute $2.w_- = 2\bar{3}\bar{2}1\bar{2}\bar{3}$. Using rule (16) of table 7, we get $2\bar{3}\bar{2} = \bar{2}\bar{3}2 - a^{-2}.\bar{2}32 + a^{-2}.\bar{2}3\bar{2} + a.\bar{3}\bar{2} - a^{-1}.\bar{3}2 - a^{-1}.\bar{3}\bar{2} + a^{-3}.\bar{3}2 - a.\bar{2}\bar{3} + a^{-1}.\bar{2}\bar{3} + a^{-1}.\bar{2}\bar{3} - a^{-3}.\bar{2}3$. Multiplying on the right by $1\bar{2}\bar{3}$, we get after a straightforward computation that

$$2.w_- \equiv (bc)^{-1}.\bar{2}\bar{1}x1 + a^{-2}.\bar{2}3\bar{2}1\bar{2}\bar{3} + a.e_{23} - w^{-1}.f(e_{18}) - a^{-1}.\bar{3}\bar{2}1\bar{2}\bar{3}$$

Since $\bar{2}1\bar{2} \equiv (bc)^{-1}.\bar{2}\bar{1}2 \pmod{u_1u_2u_1}$ (see section 3.5), we have $2\bar{3}\bar{2}1\bar{2}\bar{3} \equiv (bc)^{-1}.\bar{2}3\bar{2}1\bar{2}\bar{3}$ and $\bar{3}\bar{2}1\bar{2}\bar{3} \equiv (bc)^{-1}.\bar{3}\bar{2}1\bar{2}\bar{3}$ modulo $Q_3u_3Q_3$. Now, in the proof of lemma 4.3 we checked that $\bar{3}\bar{2}1\bar{2}\bar{3} = \bar{2}\bar{1}x12$, whence $2\bar{3}\bar{2}1\bar{2}\bar{3} = \bar{1}x12$. Altogether, this yields

$$(4.26) \quad 2.e_{23} = a.e_{23} + (bc)^{-1}.\bar{2}\bar{1}x1 + \frac{1}{aw}.f(21x\bar{1}) - w^{-1}.f(e_{18}) - w^{-1}.f(21x\bar{1}2)$$

4.6. Freeness of Q_4 as a R -module. Let $\mathcal{B}_{(0)}$ be a basis of Q_3 . We recall from lemmas 4.8 and 4.9 that $Q_3u_3Q_3 = Q_3 + Q_3s_3Q_3 + Q_3s_3^{-1}Q_3$ is spanned by

$$\mathcal{B}_{(1)} = \mathcal{B}_{(0)} \sqcup \mathcal{B}_{(0)} \times F_2 \sqcup E' \times \{\bar{3}\bar{2}, \bar{3}\bar{2}\bar{1}\} \sqcup E_0 \times \{\bar{3}\bar{2}\bar{1}\} \sqcup E \times \{\bar{3}\bar{2}\bar{1}, \bar{3}\bar{2}\bar{1}\bar{2}\} \sqcup \{\bar{3}\bar{2}\bar{1}, \bar{3}\bar{2}\bar{1}\bar{2}\}$$

where

- $E = \{\emptyset, 2, \bar{2}, 12, \bar{1}\bar{2}, \bar{1}\bar{2}, \bar{1}\bar{2}, 2\bar{1}\bar{2}\}$
- $F = \{\emptyset, 2, 2\bar{1}, \bar{2}\bar{1}, \bar{2}, \bar{2}\bar{1}\}$.
- $F_2 = \{\bar{3}, \bar{3}, \bar{3}\bar{2}, \bar{3}\bar{2}, \bar{3}\bar{2}, \bar{3}\bar{2}\bar{1}, \bar{3}\bar{2}\bar{1}, \bar{3}\bar{2}\bar{1}\}$
- $E_0 = \{\emptyset, 1, \bar{1}, 2, \bar{2}\}$
- $E' = \{\emptyset, 1, \bar{1}, 2\bar{1}, \bar{2}\bar{1}, \bar{2}\bar{1}, \bar{2}\bar{1}, \bar{2}\bar{1}\}$

Since $\mathcal{B}_{(0)}$ has 20 elements, $\mathcal{B}_{(1)}$ has 239 elements. We now add to $\mathcal{B}_{(1)}$ the 25 elements described by the words $(e_i)_{1 \leq i \leq 25}$ of section 4.4.1 to get a collection $\mathcal{B}_{(2)}$ of 264 elements.

Theorem 4.10. Q_4 is a free R -module of rank 264, and $\mathcal{B}_{(2)}$ is a basis.

Proof. We know that $Q_4 = Q_3u_3Q_3 + Q_3xQ_3 + Rw_+ + Rw_-$, where $x = \bar{3}\bar{2}\bar{3}$, by proposition 4.6 and lemma 4.4,. We want to prove that $\mathcal{B}_{(2)}$ spans Q_4 as a R -module. We recalled that $\mathcal{B}_{(1)}$ spans $Q_3u_3Q_3$, so it is sufficient to prove that the image of $\mathcal{B}_{(2)} \setminus \mathcal{B}_{(1)}$ span $Q_4/Q_3u_3Q_3$. We know that $Q_4/Q_3u_3Q_3$ is generated as a Q_3 -bimodule by x, w_+, w_- . Let M denote the R -submodule of $Q_4/Q_3u_3Q_3$ spanned by the image of $\mathcal{B}_{(2)} \setminus \mathcal{B}_{(1)}$. It contains x, w_+, w_- , and it is stable by left multiplication by Q_3 , by section 4.5. Moreover, it is also stable by the antimorphism of Q_3 -module f , hence it is stable by left and right multiplication by Q_3 . This proves $M = Q_4/Q_3u_3Q_3$, hence Q_4 is spanned by $\mathcal{B}_{(2)}$. Since $Q_4 \otimes K$ has dimension 264, this proves that Q_4 is a free R -module with basis $\mathcal{B}_{(2)}$. \square

An immediate corollary is the following one.

Corollary 4.11. *The natural morphism $Q_3 \rightarrow Q_4$ is injective.*

The matrix of left and right multiplication by s_1, s_2 on the basis $\mathcal{B}_{(2)} \setminus \mathcal{B}_{(1)}$ of \tilde{A}_4 can be found in the file `A4tilde.gap` at <http://www.lamfa.u-picardie.fr/marin/data/A4tilde.gap>.

In [29] we described an explicit isomorphism

$$\Phi_4^K : Q_4 \otimes \tilde{K} \rightarrow \tilde{K}^3 \times M_2(\tilde{K})^2 \times M_3(\tilde{K})^5 \times M_6(\tilde{K})^4 \times M_8(\tilde{K})$$

from the explicit matrix models of the irreducible representations of the semisimple algebra $H_4 \otimes \tilde{K}$, where $\tilde{K} = \mathbb{Q}(\zeta_3, a, b, c)$. We denote Φ_4 the composite of Φ_4^K by the natural R -algebra morphism $Q_4 \rightarrow Q_4 \otimes \tilde{K}$. By theorem 4.10 we know that Φ_4 is injective, and therefore Φ_4 can be used for explicit computations inside Q_4 .

Computing the images of the elements of the relevant bases, together with their images by left and right multiplications by the generators and their inverses, we could in principle get in this way the structure constants of Q_4 on the basis $\mathcal{B}_{(2)}$. However, because the coefficients of the equations belong to the field $\mathbb{Q}(a, b, c)$, this linear algebra matter is computationally nontrivial (even after having reduced the problem to $a = 1$).

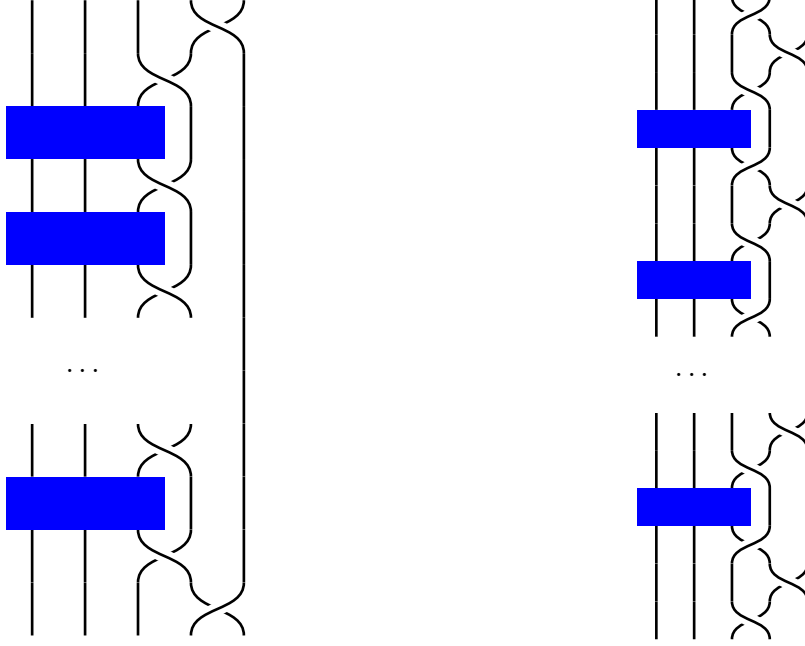


FIGURE 1. Handle reduction

5. STRUCTURE ON 5 STRANDS

5.1. **General properties.** We denote $Q_{n+1}^{(1)} = Q_n \cdot 1$ the image of Q_n inside Q_{n+1} under the natural map. A collection of Q_n -subbimodules of Q_{n+1} is defined inductively by the formula $Q_{n+1}^{(k+1)} = Q_{n+1}^{(k)} u_n Q_n$. In other terms,

$$Q_{n+1}^{(k)} = \underbrace{Q_n u_n Q_n u_n \dots u_n Q_n u_n Q_n}_{k+1 \text{ terms}}$$

We know that $Q_3^{(2)} = Q_3$ and $Q_4^{(2)} = Q_4$.

Every element x of the braid group B_{n+1} either belongs to B_n (that is, the image of B_n under the usual map $B_n \rightarrow B_{n+1}$ of adding one strand ‘on the right’), or can be written either as $x_1 s_n x_2 s_n \dots s_n x_{k+1}$ for some k , or as $x_1 s_n^{-1} x_2 s_n^{-1} \dots s_n^{-1} x_{k+1}$ for some k . The process to convert any given braid to one of these forms is called by Dehornoy ‘handle reduction’, and is at the origin of his ordering on the braid group B_{n+1} . A nice reference for this is [10], ch. 3. The basic ‘handle reduction’ has the following form. If $a \in B_{n+1}$ is written

$$a = s_n s_{n-1} a_1 s_{n-1} a_2 \dots s_{n-1} a_k s_{n-1} s_n^{-1}$$

with $a_i \in B_{n-1}$, then it can be rewritten as

$$a = s_n^{-1} (s_n (s_{n-1} a_1) s_n (s_{n-1} a_2) \dots s_n (s_{n-1} a_k) s_n) s_{n-1}$$

An iterated application of handle reduction proves the following identities

$$(5.1) \quad s_n^{-1} (s_{n-1}^{-1} \dots s_2^{-1} s_1 s_2^{-1} \dots s_{n-1}^{-1}) s_n = (s_{n-1} \dots s_1) (s_n^{-1} \dots s_3^{-1} s_2 s_3^{-1} \dots s_n^{-1}) (s_{n-1} \dots s_1)^{-1}$$

and its mirror image

$$(5.2) \quad s_n(s_{n-1} \dots s_2 s_1^{-1} s_2 \dots s_{n-1}) s_n^{-1} = (s_{n-1}^{-1} \dots s_1^{-1})(s_n \dots s_3 s_2^{-1} s_3 \dots s_n)(s_{n-1}^{-1} \dots s_1^{-1})^{-1}$$

Let us define $Q_{n+1}^{(1)+} = Q_{n+1}^{(1)-} = Q_n \cdot 1$ and $Q_{n+1}^{(k+1)+} = Q_{n+1}^{(k)+} s_n Q_n$, $Q_{n+1}^{(k+1)-} = Q_{n+1}^{(k)-} s_n^{-1} Q_n$. In other terms,

$$Q_{n+1}^{(k)+} = \underbrace{Q_n s_n Q_n s_n \dots s_n Q_n s_n Q_n}_{k+1 \text{ terms}} \quad Q_{n+1}^{(k)-} = \underbrace{Q_n s_n^{-1} Q_n s_n^{-1} \dots s_n^{-1} Q_n s_n^{-1} Q_n}_{k+1 \text{ terms}}$$

Let

$$Q_{n+1}^+ = \sum_{k \geq 1} Q_{n+1}^{(k)+} \quad Q_{n+1}^- = \sum_{k \geq 1} Q_{n+1}^{(k)-}$$

The immediate consequence of Handle reduction in our context can be stated in the following form, although it is the explicit process recalled above (and illustrated in figure 1) that will be useful to us.

Proposition 5.1. *For all n we have $Q_n = Q_n^+ + Q_n^-$.*

5.2. $Q_5^{(2)}$ as a Q_4 -bimodule. We know by proposition 4.6 that $Q_4 = Q_4^{(2)} = Q_3 u_3 Q_3 + Q_3 \cdot 3\bar{2}3 \cdot Q_3 + Q_3 \cdot 3\bar{2}1\bar{2}3 + Q_3 \cdot \bar{3}2\bar{1}2\bar{3}$. Since $\bar{4}3\bar{4} \in R^\times \cdot 4\bar{3}4 + u_3 u_4 u_3$ and, by handle reduction, $4X\bar{4}$ for $X \in \{3\bar{2}3, 3\bar{2}1\bar{2}3, \bar{3}2\bar{1}2\bar{3}\}$ can be written as $X4Y4Z$ or $X\bar{4}Y\bar{4}Z$ for $X, Z \in B_4$ and $Y \in \{3, \bar{3}, 3\bar{2}3, \bar{3}2\bar{3}\}$, this implies that $Q_5^{(2)}$ is generated as a Q_4 -bimodule by $\emptyset, 4, \bar{4}, 4\bar{3}4, 4\bar{3}2\bar{3}4, \bar{4}3\bar{2}3\bar{4}, 4\bar{3}2\bar{1}2\bar{3}4, \bar{4}3\bar{2}1\bar{2}3\bar{4}, 4\bar{3}2\bar{1}2\bar{3}4, \bar{4}3\bar{2}1\bar{2}3\bar{4}$.

Let us introduce $Q_5^{(1.5)}$ the Q_4 -subbimodule of Q_5 generated by $\emptyset, 4, \bar{4}, 4\bar{3}4, 4\bar{3}2\bar{3}4, \bar{4}3\bar{2}3\bar{4}$, and $Q_5^{(1.3)}$ the Q_4 -subbimodule of Q_5 generated by $\emptyset, 4, \bar{4}, 4\bar{3}4$. Notice that both are stable under $F = \Phi \circ \Psi$.

We use the defining relation under the form

$$(5.3) \quad (3 - a \cdot \emptyset) \cdot 4\bar{3} = a^2(3 - a \cdot \emptyset) \cdot \bar{4}3 + a(\emptyset - a \cdot \bar{3}) \cdot \bar{4}3 + (\bar{3} - a^{-1} \cdot \emptyset)43 + (a^{-1} \cdot 3 - a \cdot \bar{3}) \cdot 4 + a(a^2 \cdot \bar{3} - 3) \cdot \bar{4}$$

and in particular $(3 - a \cdot \emptyset) \cdot 4\bar{3} \in u_3 \bar{4} u_3 + u_3 \cdot 4 + u_3 \cdot 4\bar{3}$. Since $4\bar{3}2\bar{1}2\bar{3}4 = \bar{2}1 \cdot 4\bar{3}2\bar{3}4 \cdot 12$ this implies $(3 - a \cdot \emptyset) \cdot 4w_-4 \in u_3 \cdot \bar{4} \cdot u_3 \bar{2}1\bar{2}3\bar{4} + Q_4 \cdot 4\bar{3}4 + u_3 u_2 u_1 \cdot 4\bar{3}2\bar{3}4 \cdot u_1 u_2 \subset u_3 \cdot \bar{4} \cdot Q_4 \cdot 4 + Q_4 \cdot 4\bar{3}4 + Q_4 \cdot 4\bar{3}2\bar{3}4 \cdot Q_4$

Proposition 5.2.

- (1) $4 \cdot Q_4 \cdot \bar{4} \subset Q_5^{(1.5)}$ and $\bar{4} \cdot Q_4 \cdot 4 \subset Q_5^{(1.5)}$
- (2) $Q_5^{(2)} = Q_5^{(1.5)} + R \cdot 4w_+4 + R \cdot \bar{4}w_-4 + R \cdot 4w_-4 + R \cdot \bar{4}w_+4$. Moreover, for all $X \in \{4w_+4, \bar{4}w_-4, 4w_-4, \bar{4}w_+4\}$ and $\lambda \in Q_4$, we have $\lambda \cdot X \equiv X \cdot \lambda \equiv \varepsilon(\lambda)X \pmod{Q_5^{(1.5)}}$, where $\varepsilon : Q_4 \rightarrow R$ is induced by $s_i \mapsto a$.

Proof. We first prove (1). Since $Q_5^{(1.5)}$ is stable under F we only need to prove $4 \cdot Q_4 \cdot \bar{4} \subset Q_5^{(1.5)}$. Since Q_4 is generated as a Q_3 -bimodule by $S = \{\emptyset, 3, \bar{3}, 3\bar{2}3, 3\bar{2}1\bar{2}3, \bar{3}2\bar{1}2\bar{3}\}$, and 4 commutes with Q_3 , we need to prove $4 \cdot X \cdot \bar{4} \in Q_5^{(1.5)}$ for all $X \in S$. Clearly $4X\bar{4} \in Q_5^{(1)} \subset Q_5^{(1.5)}$ for $X \in \{\emptyset, 3, \bar{3}\}$. By the proof of lemma 4.3, or handle reduction, we know that $3\bar{2}1\bar{2}3 \in Q_3 \cdot 3\bar{2}3 \cdot Q_3$, whence $4 \cdot 3\bar{2}3 \cdot \bar{4} = s(3\bar{2}1\bar{2}3) \in s(Q_3 \cdot 3\bar{2}3 \cdot Q_3) \subset Q_4 \cdot 4\bar{3}3 \cdot Q_4 \subset Q_5^{(1.3)}$. It remains to consider $X \in \{w_+, w_-\}$. By handle reduction (see (5.2)) we have $4w_+\bar{4} = \bar{3}2\bar{1} \cdot 4\bar{3}2\bar{3}4 \cdot 123 \in Q_5^{(1.5)}$. Similarly, using (5.1) we get $\bar{4}w_-4 = 321 \cdot \bar{4}3\bar{2}3\bar{4} \cdot \bar{1}2\bar{3} \in Q_5^{(1.5)}$. Applying F we get $4w_-4 \in Q_5^{(1.5)}$ and this concludes (1).

We now prove (2). We compute modulo the \mathbb{Q}_4 -bimodule $\mathbb{Q}_5^{(1.5)}$. We proved that $(3 - a.\emptyset).4w_{-4} \in u_3.\bar{4}.\mathbb{Q}_4.4 + \mathbb{Q}_4.4\bar{3}4$ hence $(3 - a.\emptyset).4w_{-4} \equiv 0$ by (1). By the computations of section 4.5 we know that $(2 - a.\emptyset).w_{\pm} \in \mathbb{Q}_3.3\bar{2}3.\mathbb{Q}_3$ and $(1 - a.\emptyset).w_{\pm} \in \mathbb{Q}_3.3\bar{2}3.\mathbb{Q}_3$. It follows that $\lambda.4w_{-4} \equiv \varepsilon(\lambda).4w_{-4}$ for all $\lambda \in \mathbb{Q}_4$.

From relation (5.3) we get similarly that $a.(\emptyset - a.\bar{3}).\bar{4}3 \in u_34u_4 + a^2.(a.\emptyset - 3).\bar{4}\bar{3} + u_3.\bar{4}$ whence $a.(\emptyset - a.\bar{3}).\bar{4}w_{+}\bar{4} \in \mathbb{Q}_5^{(1.5)}$. This can be rewritten as $\bar{3}.\bar{4}w_{+}\bar{4} \equiv \varepsilon(\bar{3})$. Since \mathbb{Q}_4 is generated by $1, 2, \bar{3}$ this yields $\lambda.\bar{4}w_{+}\bar{4} \equiv \varepsilon(\lambda).\bar{4}w_{+}\bar{4}$ for all $\lambda \in \mathbb{Q}_4$.

Again by (5.3) we get that $(\bar{3} - a^{-1}.\emptyset).\bar{4}3 \in (3 - a.\emptyset).\bar{4}\bar{3} + u_3.\bar{4}.u_3 + u_3u_4$ hence $(\bar{3} - a^{-1}.\emptyset).4w_{+}4 \in (3 - a.\emptyset).\bar{4}\bar{3}2\bar{2}234 + u_3.\bar{4}.\mathbb{Q}_4.4.u_3 + u_3u_4.\mathbb{Q}_4.u_4$ and this yields $(\bar{3} - a^{-1}.\emptyset).4w_{+}4 \in \mathbb{Q}_5^{(1.5)}$ since $4\bar{3}2\bar{1}234 = 21.4\bar{3}2\bar{3}4.\bar{1}\bar{2}$ by handle reduction. Since we already know $\lambda.4w_{+}4 = 4(\lambda.w_{+})4 \equiv \varepsilon(\lambda)4w_{+}4$ for $\lambda \in \mathbb{Q}_3$, this implies $\lambda.4w_{+}4 \equiv \varepsilon(\lambda)4w_{+}4$. The case $\lambda.\bar{4}w_{-}\bar{4} \equiv \varepsilon(\lambda).\bar{4}w_{-}\bar{4}$ is similar and left to the reader.

We have now proved that $\lambda.X \equiv \varepsilon(\lambda)X$ for all $\lambda \in \mathbb{Q}_4$ and $X \in \{4w_{+}4, \bar{4}w_{-}\bar{4}, 4w_{-}4, \bar{4}w_{+}\bar{4}\}$. Since $F(X) = X$ this implies $X.\lambda \equiv \varepsilon(\lambda)X$ for all λ, X , and (2). \square

We now claim that $\mathbb{Q}_5 \neq \mathbb{Q}_5^{(2)}$, that is $\mathbb{Q}_5^{(2)} \subsetneq \mathbb{Q}_5^{(3)}$. In order to prove this, one just need to check this on one specialization over a field \mathbb{k} , by comparing the dimensions of the two. Let us consider the specialization at $\{a, b, c\} = \mu_3(\mathbb{k})$ with $|\mu_3(\mathbb{k})| = 3$, in which case \mathbb{Q}_5 is a quotient of the group algebra $\mathbb{k}\Gamma_5$, with $\Gamma_5 = B_5/s_1^3$. If $\text{car.}\mathbb{k} \notin \{2, 3, 5\}$ then the algebra $\mathbb{k}\Gamma_5$ is split semisimple (of dimension 155520), and one has a description of its irreducible representation. Therefore, one can identify (this specialization of) \mathbb{Q}_5 to a sum of matrix algebras. By computer means, we get that $\mathbb{Q}_5^{(2)}$ has dimension 6489 (over $\mathbb{k} = \mathbb{F}_{103}$) while $\mathbb{Q}_5 = \mathbb{Q}_5^{(3)}$ has dimension 6490. Similarly, we check that $\mathbb{Q}_5^{(1.5)}$ has dimension 6485 = 6489 - 4, thus the equality $\mathbb{Q}_5^{(2)} = \mathbb{Q}_5^{(1.5)} + R.4w_{+}4 + R.\bar{4}w_{-}\bar{4} + R.4w_{-}4 + R.\bar{4}w_{+}\bar{4}$ is sharp.

We now consider $\mathbb{Q}_5^{(3)}$.

5.3. $\mathbb{Q}_5^{(3)}/\mathbb{Q}_5^{(2)}$ as a \mathbb{Q}_4 -bimodule. We first need to prove the following lemma.

Lemma 5.3.

- (1) $sh(\mathbb{Q}_4) \subset \mathbb{Q}_5^{(2)}$
- (2) $u_4\mathbb{Q}_4u_4u_3u_4 \subset \mathbb{Q}_5^{(2)}$.
- (3) $u_4u_3u_4\mathbb{Q}_4u_4 \subset \mathbb{Q}_5^{(2)}$.
- (4) $u_4(u_3u_2u_3)(u_2u_1u_2)u_4(u_3u_2u_3)u_4 = u_4(u_3u_2u_3)u_4(u_2u_1u_2)(u_3u_2u_3)u_4 \subset \mathbb{Q}_5^{(2)}$

Actually part (4) easily implies the first items, but the first items will be easier to use in the sequel.

Proof. From $\mathbb{Q}_4 = \mathbb{Q}_4^{(2)}$ we get (1). For (2) we need to prove that $s_4^\alpha\mathbb{Q}_4u_4u_3u_4 \subset \mathbb{Q}_5^{(2)}$ for $\alpha \in \{-1, 1\}$. We have $u_4u_3u_4 \subset sh^2(\mathbb{Q}_3) = R.s_4^\alpha s_3^{-\alpha} s_4^\alpha + u_3u_4u_3$ hence $s_4^\alpha\mathbb{Q}_4u_4u_3u_4 \subset \mathbb{Q}_5^{(2)}$ iff $s_4^\alpha\mathbb{Q}_4s_4^\alpha s_3^{-\alpha} s_4^\alpha \subset \mathbb{Q}_5^{(2)}$.

Let $A \in \mathbb{Q}_3u_3\mathbb{Q}_3$. We prove that $s_4^\alpha A s_4^\alpha s_3^{-\alpha} s_4^\alpha \in \mathbb{Q}_5^{(2)}$. We have $s_4^\alpha A s_4^\alpha s_3^{-\alpha} s_4^\alpha \in s_4^\alpha\mathbb{Q}_3u_3\mathbb{Q}_3s_4^\alpha s_3^{-\alpha} s_4^\alpha = \mathbb{Q}_3s_4^\alpha u_3\mathbb{Q}_3s_4^\alpha s_3^{-\alpha} s_4^\alpha$. Now $s_4^\alpha s_3^{-\alpha} s_4^\alpha \in R.s_4^{-\alpha} s_3^\alpha s_4^{-\alpha} + u_3u_4u_3$ hence $s_4^\alpha u_3\mathbb{Q}_3s_4^\alpha s_3^{-\alpha} s_4^\alpha \subset s_4^\alpha u_3\mathbb{Q}_3s_4^{-\alpha} s_3^\alpha s_4^{-\alpha} + \mathbb{Q}_5^{(2)}$. Since $s_4^\alpha u_3\mathbb{Q}_3s_4^{-\alpha} s_3^\alpha s_4^{-\alpha} = (s_4^\alpha u_3 s_4^{-\alpha})\mathbb{Q}_3s_3^\alpha s_4^{-\alpha} \subset \mathbb{Q}_5^{(2)}$ we get $s_4^\alpha A s_4^\alpha s_3^{-\alpha} s_4^\alpha \in \mathbb{Q}_5^{(2)}$ for $A \in \mathbb{Q}_3u_3\mathbb{Q}_3$.

Now, we note that we can assume $\alpha = 1$, up to applying Φ . Let us assume $A \in Q_3.3\bar{2}3.Q_3$. We need to prove that $43\bar{2}3Q_34\bar{3}4 \in Q_5^{(2)}$. We have $Q_3 = u_1u_2u_1 + u_2u_1u_2$. Then, $43\bar{2}3(u_2u_1u_2)4\bar{3}4 = 4(3\bar{2}3u_2)u_1u_24\bar{3}4$ and we know that $(3\bar{2}3u_2) \subset R.3\bar{2}3 + u_2u_3u_2$ hence $4(3\bar{2}3u_2)u_1u_24\bar{3}4 \subset 43\bar{2}3u_1u_24\bar{3}4 + u_24u_3u_2u_1u_24\bar{3}4$. Since we already proved $u_24u_3u_2u_1u_24\bar{3}4 \subset 4Q_3u_3Q_34\bar{3}4 \subset Q_5^{(2)}$ it thus remains to prove that $43\bar{2}3u_1u_2u_14\bar{3}4 \subset Q_5^{(2)}$. It follows from $4\bar{3}4 \subset R.\bar{4}\bar{3}\bar{4} + u_3u_4u_3$, as depicted in figure 2. There we depict words in the generators as music notes on the staff : bullets correspond to Artin generators, with white/black coloring corresponds to \pm power signs, grey coloring corresponds to indeterminate power signs, and the height of the bullet determines the position of the strand.

We now assume $A \in R.w_+$. Then $4A4\bar{3}4 \in R.4w_+4\bar{3}4 = R.\bar{3}4w_+44 \subset Q_5^{(2)}$. Now assume $A \in R.w_-$. Then $4A4\bar{3}4 \in R.4w_-4\bar{3}4$ and $4w_-4\bar{3}4 = 4\bar{3}\bar{2}1\bar{2}(\bar{3}4\bar{3}4) \in a^{-1}.4\bar{3}\bar{2}1\bar{2}4\bar{3}4 + Q_5^{(2)}$. But $4\bar{3}\bar{2}1\bar{2}4\bar{3}4 \in 4u_3Q_34\bar{3}4 \subset Q_5^{(2)}$ hence $4A4\bar{3}4 \in Q_5^{(2)}$.

Since $Q_4 = Q_3u_3Q_3 + Q_3.3\bar{2}3.Q_3 + R.w_+ + R.w_-$ this proves (2). (3) follows by applying F .

We now prove (4). First note that $u_3u_2u_3u_2 = sh(Q_3) = u_2u_3u_2u_3$, whence

$$\begin{aligned} u_4(u_3u_2u_3)(u_2u_1u_2)u_4(u_3u_2u_3)u_4 &= u_4(u_3u_2u_3u_2)u_1u_4(u_2u_3u_2u_3)u_4 \\ &= u_2u_4(u_3u_2u_3)u_1u_4(u_3u_2u_3)u_4u_2 \end{aligned}$$

and we need to prove $u_4(u_3u_2u_3)u_1u_4(u_3u_2u_3)u_4 \subset Q_5^{(2)}$. Now, since

$$s_4^\alpha u_3u_2u_3s_4^{-\alpha} = sh(s_3^\alpha u_2u_1u_2s_3^{-\alpha}) \subset Q_5^{(1)} + Q_4u_4u_3u_4$$

by lemma 4.3, we only need to prove $s_4^\alpha(u_3u_2u_3)u_1s_4^\alpha(u_3u_2u_3)s_4^\alpha \subset Q_5^{(2)}$ for $\alpha \in \{-1, 1\}$. Using Φ , we can assume $\alpha = 1$. We then use that $u_3u_2u_3 \subset R.3\bar{2}3 + u_2u_3u_2$ and that $s_4(u_2u_3u_2)u_1s_4(u_3u_2u_3)s_4 = u_2s_4u_3s_4u_2u_1(u_3u_2u_3)s_4 \subset Q_5^{(1)}$ by (2), so we only need to prove $43\bar{2}3.u_1.43\bar{2}34 \subset Q_5^{(2)}$. But $43\bar{2}3.u_1.43\bar{2}34 = 43\bar{2}.u_1.343\bar{2}34 = 43\bar{2}.u_1.434\bar{2}34 = 434\bar{2}.u_1.3\bar{2}434 = 343\bar{2}.u_1.3\bar{2}3434 \subset Q_5^{(2)}$ as in figure 3, and this concludes the proof of (4). \square

We then claim that $4.3\bar{2}3.Q_3.4.3\bar{2}3.4 \subset R.4.3\bar{2}3.1\bar{2}1.4.3\bar{2}3.4 + Q_5^{(2)}$. This is depicted in figure 3, as well as the fact that $4xu_2u_14u_2x4 \subset Q_5^{(2)}$.

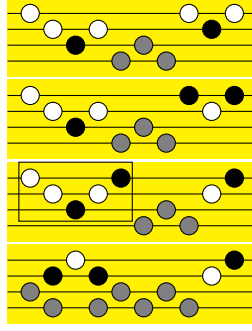
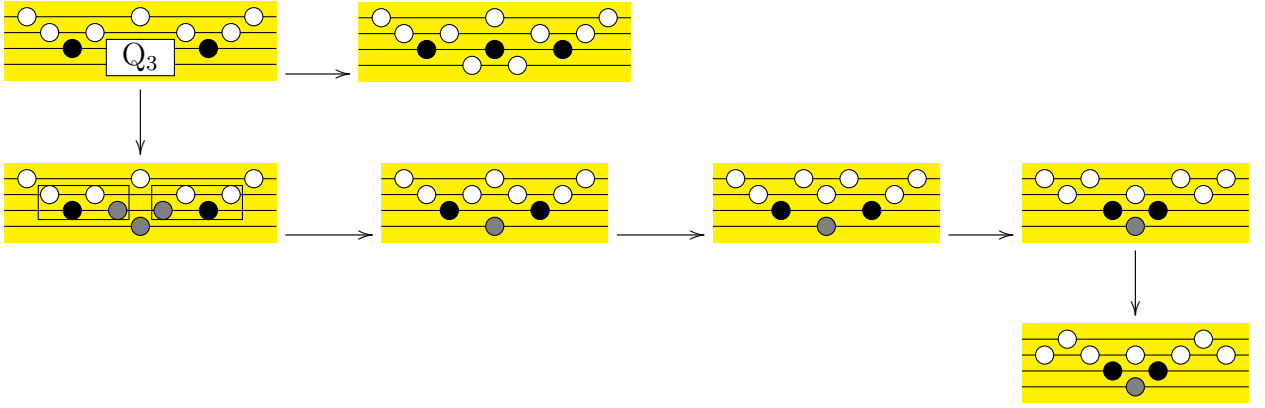
Lemma 5.4. *For all $i \in \{1, 2, 3\}$ we have $s_i.4xy4x4 \equiv 4xy4x4.s_i \equiv a.4xy4x4 \pmod{Q_5^{(2)}}$.*

Proof. From the computations in section 4.4 we get that $s_i.xy \equiv a.xy \pmod{Q_3u_3Q_3 + u_2u_1xu_1u_2}$ for $i \in \{1, 2\}$ (notice that the image of $u_2u_1xu_1u_2$ in \tilde{A}_4 is spanned by the e_i for $i < 22$). Since $4u_2u_1xu_1u_24x4 = u_2u_14xu_1u_24x4 \subset Q_5^{(2)}$ this implies $s_i.4xy4x4 \equiv a.4xy4x4 \pmod{Q_5^{(2)}}$ for $i \in \{1, 2\}$.

Since $Q_5^{(2)}$ is stable under $F = \Phi \circ \Psi$ and $F(4xy4x4) = 4x4yx4 = 4xy4x4$ this implies $4xy4x4.s_i \equiv a.4xy4x4 \pmod{Q_5^{(2)}}$ for $i \in \{1, 2\}$. For the same reason, the statement $3.4xy4x4 \equiv a.4xy4x4$ is equivalent to the statement $4xy4x4.3 \equiv a.4xy4x4$.

We use equation (5.3) under the form $(\bar{3} - a^{-1}.\emptyset).43 \in (\bar{3} - a.\emptyset).4\bar{3} + u_3\bar{4}u_3 + u_3u_4$ to get $(\bar{3} - a^{-1}.\emptyset).4xy4x4 = (\bar{3} - a^{-1}.\emptyset).43\bar{2}3y4x4 \in Q_5^{(2)}$ since

- $(\bar{3} - a.\emptyset).4\bar{3}\bar{2}3y4x4 = (\bar{3} - a.\emptyset).42\bar{3}\bar{2}y4x4 \in (\bar{3} - a.\emptyset)2u_3.4\bar{3}4\bar{2}yx4 \subset Q_5^{(2)}$

FIGURE 2. $4.3\bar{2}3.(u_1u_2u_1).4.\bar{3}.4 \subset Q_5^{(2)}$ FIGURE 3. $4.3\bar{2}3.Q_3.4.3\bar{2}3.4 \subset R.4.3\bar{2}3.1\bar{2}1.4.3\bar{2}3.4 + Q_5^{(2)}$

- $u_3\bar{4}u_3\bar{2}3y4x4 \subset u_3\bar{4}u_3\bar{2}34yx4$ and $\bar{4}u_3\bar{2}34 = R.\bar{4}\bar{2}34 + R.\bar{4}3\bar{2}34 + R.\bar{4}\bar{3}\bar{2}34 = R.\bar{2}(\bar{4}34) + R.(\bar{4}34)\bar{2}(\bar{4}34) + R.\bar{4}\bar{2}\bar{3}\bar{2}4 \subset Q_5^{(1)} + R.34\bar{3}\bar{2}3)4\bar{3} \subset Q_5^{(1)} + u_3u_24u_34u_2u_3$ whence $\bar{4}u_3\bar{2}34yx4 \subset Q_5^{(1)}Q_4u_4 + u_3u_24u_34u_2u_3yx4 \subset Q_5^{(2)}$.
- $u_3u_4\bar{2}3y4x4 = u_3\bar{2}u_434yx4 \subset Q_5^{(2)}$

Therefore $s_3^{-1}.4xy4x4 \equiv a^{-1}.4xy4x4$ whence $s_3.4xy4x4 \equiv a.4xy4x4$ and this completes the proof of the lemma. \square

Note that $4x\bar{4} = \bar{3}\bar{2}.4\bar{3}4.23$ hence $s_4^\alpha x y s_4^\beta x s_4^\gamma \in Q_5^{(2)}$ whenever $\alpha, \beta, \gamma \in \{-1, 1\}$ with $\#\{\alpha, \beta, \gamma\} > 1$.

Lemma 5.5.

- (1) $\bar{4}w_+\bar{4}w_+\bar{4} \equiv -(a^2/w)^3.4xy4x4 \pmod{Q_5^{(2)}}$.
- (2) $4w_-\bar{4}w_-\bar{4} \equiv -(a^6/w^5).4xy4x4 \pmod{Q_5^{(2)}}$.

Proof. We first prove (1). We use that $4\bar{3}4 \equiv (bc).\bar{4}3\bar{4} + 3\bar{4}3 \pmod{u_34\bar{3} + u_4u_3 + u_3u_4}$. Since

- $\bar{4}3\bar{2}\bar{1}2(u_34\bar{3})2\bar{1}23\bar{4} = \bar{4}3\bar{2}\bar{1}2u_34(\bar{3}\bar{2}\bar{1}23)\bar{4} = \bar{4}3\bar{2}\bar{1}2u_3421.3\bar{2}3.\bar{1}\bar{2}.\bar{4} = \bar{4}3\bar{2}\bar{1}2u_321.(43\bar{2}3\bar{4}).\bar{1}\bar{2} = \bar{4}3\bar{2}\bar{1}2u_321.\bar{3}\bar{2}.4\bar{3}4.23.\bar{1}\bar{2} \subset Q_5^{(2)}$

- $\bar{4}3\bar{2}\bar{1}2u_3u_4\bar{2}\bar{1}2\bar{3}\bar{4} = \bar{4}3\bar{2}\bar{1}2u_3\bar{2}\bar{1}2u_4\bar{3}\bar{4} \subset Q_5^{(2)}$ and $\bar{4}3\bar{2}\bar{1}2u_4u_3\bar{2}\bar{1}2\bar{3}\bar{4} = \bar{4}3u_4\bar{2}\bar{1}2u_3\bar{2}\bar{1}2\bar{3}\bar{4} \subset Q_5^{(2)}$
- $\bar{4}3\bar{2}\bar{1}2(\bar{4}\bar{3}\bar{4})\bar{2}\bar{1}2\bar{3}\bar{4} = \bar{4}3\bar{4}\bar{2}\bar{1}2\bar{3}\bar{2}\bar{1}2\bar{4}\bar{3}\bar{4} = 3\bar{4}\bar{3}\bar{2}\bar{1}2\bar{3}\bar{2}\bar{1}2\bar{3}\bar{4}\bar{3} \in Q_5^{(2)}$

we get that $\bar{4}3\bar{2}\bar{1}2(3\bar{4}3)\bar{2}\bar{1}2\bar{3}\bar{4} \equiv -(bc).\bar{4}3\bar{2}\bar{1}2\bar{4}3\bar{4}\bar{2}\bar{1}2\bar{3}\bar{4} \pmod{Q_5^{(2)}}$. Now $\bar{4}3\bar{2}\bar{1}2\bar{4}3\bar{4}\bar{2}\bar{1}2\bar{3}\bar{4} = (\bar{4}\bar{3}\bar{4})\bar{2}\bar{1}2\bar{3}\bar{2}\bar{1}2\bar{4}\bar{3}\bar{4} \equiv (bc)^{-1}\bar{4}\bar{3}\bar{4}\bar{2}\bar{1}2\bar{3}\bar{2}\bar{1}2(\bar{4}\bar{3}\bar{4}) \equiv (bc)^{-2}\bar{4}\bar{3}\bar{4}\bar{2}\bar{1}2\bar{3}\bar{2}\bar{1}2\bar{4}\bar{3}\bar{4} \pmod{Q_5^{(2)}}$. Now $\bar{4}\bar{3}\bar{4}\bar{2}\bar{1}2\bar{3}\bar{2}\bar{1}2\bar{4}\bar{3}\bar{4} = \bar{4}\bar{3}\bar{2}\bar{1}2(4\bar{3}\bar{4})\bar{2}\bar{1}2\bar{3}\bar{4} = 4(\bar{3}\bar{2}\bar{1}2\bar{3})4(\bar{3}\bar{2}\bar{1}2\bar{3})4 = 4.2\bar{1}.x.\bar{1}\bar{2}.4.\bar{2}\bar{1}.x.12.4 = 2\bar{1}.4.x.\bar{1}\bar{2}\bar{2}\bar{1}.4.x.4.12$. We use that $\bar{1}\bar{2}\bar{2}\bar{1} \equiv (wbc)^{-1}.\bar{1}\bar{2}\bar{1} \pmod{u_2u_1u_2}$ to get $2\bar{1}.4.x.\bar{1}\bar{2}\bar{2}\bar{1}.4.x.4.12 \equiv (wbc)^{-1}.2\bar{1}.4xy4x4.12 \pmod{Q_5^{(2)}}$. Now, by lemma 5.4 we know $2\bar{1}.4xy4x4.12 \equiv a^4.4xy4x4 \pmod{Q_5^{(2)}}$. Altogether this proves (1).

We now prove (2), We note that $\bar{4}w_+\bar{4}w_+\bar{4} = \bar{4}3\bar{2}\bar{1}2\bar{3}\bar{4}\bar{3}\bar{2}\bar{1}2\bar{3}\bar{4} = 3(\bar{3}\bar{4}\bar{3})\bar{2}\bar{1}2\bar{3}\bar{4}\bar{3}\bar{2}\bar{1}2(\bar{3}\bar{4}\bar{3})\bar{3} = 3\bar{4}\bar{3}\bar{4}\bar{2}\bar{1}2\bar{3}\bar{4}\bar{3}\bar{2}\bar{1}2\bar{4}\bar{3}\bar{4} = 3\bar{4}\bar{3}\bar{2}\bar{1}2(\bar{4}\bar{3}\bar{4}\bar{3}\bar{4})\bar{2}\bar{1}2\bar{3}\bar{4}\bar{3}$. Now,

- $\bar{4}\bar{3}\bar{2}\bar{1}2(u_3u_4)\bar{2}\bar{1}2\bar{3}\bar{4} = \bar{4}\bar{3}\bar{2}\bar{1}2u_3\bar{2}\bar{1}2u_4\bar{3}\bar{4} \subset Q_5^{(2)}$
- $\bar{4}\bar{3}\bar{2}\bar{1}2(u_4u_3)\bar{2}\bar{1}2\bar{3}\bar{4} = \bar{4}\bar{3}u_4\bar{2}\bar{1}2u_3\bar{2}\bar{1}2\bar{3}\bar{4} \subset Q_5^{(2)}$
- $\bar{4}\bar{3}\bar{2}\bar{1}2(\bar{4}u_3u_4)\bar{2}\bar{1}2\bar{3}\bar{4} = (\bar{4}\bar{3}\bar{4})\bar{2}\bar{1}2u_3\bar{2}\bar{1}2u_4\bar{3}\bar{4} = \bar{3}\bar{4}\bar{3}\bar{2}\bar{1}2u_3\bar{2}\bar{1}2u_4\bar{3}\bar{4} \subset Q_5^{(2)}$
- $\bar{4}\bar{3}\bar{2}\bar{1}2(u_4u_3\bar{4})\bar{2}\bar{1}2\bar{3}\bar{4} = \bar{4}\bar{3}u_4\bar{2}\bar{1}2u_3\bar{2}\bar{1}2(\bar{4}\bar{3}\bar{4}) = \bar{4}\bar{3}u_4\bar{2}\bar{1}2u_3\bar{2}\bar{1}2\bar{3}\bar{4}\bar{3} \subset Q_5^{(2)}$
- $\bar{4}\bar{3}\bar{2}\bar{1}2(3\bar{4}\bar{3})\bar{2}\bar{1}2\bar{3}\bar{4} = \bar{4}\bar{3}\bar{2}\bar{1}2\bar{3}\bar{4}(\bar{3}\bar{2}\bar{1}2\bar{3})\bar{4} = \bar{4}\bar{3}\bar{2}\bar{1}2\bar{3}\bar{2}\bar{1}4\bar{3}\bar{2}\bar{3}\bar{4}\bar{1}2$ by handle reduction, and this is equal to $4(\bar{3}\bar{2}\bar{1}2\bar{3})\bar{2}\bar{1}4x4\bar{1}2 = 2\bar{1}4\bar{3}\bar{2}\bar{3}\bar{1}\bar{2}\bar{2}\bar{1}4x4\bar{1}2 = 2\bar{1}4x\bar{1}\bar{2}\bar{2}\bar{1}4x4\bar{1}2$. Expanding $\bar{2}\bar{2}$ by the cubic relation, we get $\bar{1}\bar{2}\bar{2}\bar{1} \equiv w^{-1}.\bar{1}\bar{2}\bar{1} \pmod{u_2u_1u_2}$, hence $2\bar{1}4x\bar{1}\bar{2}\bar{2}\bar{1}4x4\bar{1}2 \equiv w^{-1}.2\bar{1}4x\bar{1}\bar{2}\bar{1}4x4\bar{1}2 \pmod{Q_5^{(2)}}$. Now $\bar{1}\bar{2}\bar{1} \equiv (bc)^{-1} \pmod{u_2u_1u_2}$ hence $2\bar{1}4x\bar{1}\bar{2}\bar{2}\bar{1}4x4\bar{1}2 \equiv (wbc)^{-1}.2\bar{1}4x\bar{1}\bar{2}\bar{1}4x4\bar{1}2 \equiv (wbc)^{-1}.2\bar{1}.4xy4x4.12 \equiv (a^4/wbc).4xy4x4 \pmod{Q_5^{(2)}}$.
- $\bar{4}\bar{3}\bar{2}\bar{1}2(\bar{3}\bar{4}\bar{3})\bar{2}\bar{1}2\bar{3}\bar{4} = \bar{4}\bar{3}(\bar{2}\bar{1}2)\bar{3}\bar{4}\bar{3}(\bar{2}\bar{1}2)\bar{3}\bar{4}$. We have $\bar{2}\bar{1}2 \equiv (bc).\bar{2}\bar{1}\bar{2} \pmod{u_1u_2u_1}$ hence $\bar{4}\bar{3}(\bar{2}\bar{1}2)\bar{3}\bar{4} \equiv (bc).\bar{4}\bar{3}\bar{2}\bar{1}2\bar{3}\bar{4} \pmod{u_1\bar{4}\bar{3}u_2\bar{3}\bar{4}u_1}$. Now $u_1\bar{4}\bar{3}u_2\bar{3}\bar{4}u_1 \subset u_1\bar{4}u_2u_3u_2\bar{4}u_1 + u_1\bar{4}\bar{3}\bar{2}\bar{3}\bar{4}u_1$ whence $\bar{4}\bar{3}(\bar{2}\bar{1}2)\bar{3}\bar{4}\bar{3}(\bar{2}\bar{1}2)\bar{3}\bar{4} \equiv (bc).\bar{4}\bar{3}(\bar{2}\bar{1}2)\bar{3}\bar{4}\bar{3}\bar{2}\bar{1}2\bar{3}\bar{4} \pmod{\bar{4}\bar{3}(\bar{2}\bar{1}2)\bar{3}u_1\bar{4}u_2u_3u_2\bar{4}u_1 + \bar{4}\bar{3}(\bar{2}\bar{1}2)\bar{3}u_1\bar{4}\bar{3}\bar{2}\bar{3}\bar{4}u_1}$. Since $\bar{4}\bar{3}(\bar{2}\bar{1}2)\bar{3}u_1\bar{4}u_2u_3u_2\bar{4}u_1 = \bar{4}\bar{3}(\bar{2}\bar{1}2)\bar{3}u_1u_2\bar{4}u_3\bar{4}u_2u_1 \subset Q_5^{(2)}$ and $\bar{4}\bar{3}(\bar{2}\bar{1}2)\bar{3}u_1\bar{4}\bar{3}\bar{2}\bar{3}\bar{4}u_1 = \bar{4}\bar{3}(\bar{2}\bar{1}2)u_1(\bar{3}\bar{4}\bar{3})\bar{2}\bar{3}\bar{4}u_1 = \bar{4}\bar{3}(\bar{2}\bar{1}2)u_1\bar{4}\bar{3}\bar{4}\bar{2}\bar{3}\bar{4}u_1 = \bar{4}\bar{3}(\bar{2}\bar{1}2)u_1\bar{4}\bar{3}\bar{2}(\bar{4}\bar{3}\bar{4})u_1 \subset Q_5^{(2)}$ this implies $\bar{4}\bar{3}(\bar{2}\bar{1}2)\bar{3}\bar{4}\bar{3}(\bar{2}\bar{1}2)\bar{3}\bar{4} \equiv (bc).\bar{4}\bar{3}(\bar{2}\bar{1}2)\bar{3}\bar{4}\bar{3}\bar{2}\bar{1}2\bar{3}\bar{4} \pmod{Q_5^{(2)}}$. Now $\bar{4}\bar{3}(\bar{2}\bar{1}2)\bar{3}\bar{4}\bar{3}\bar{2}\bar{1}2\bar{3}\bar{4} \equiv (bc).\bar{4}\bar{3}\bar{2}\bar{1}2\bar{3}\bar{4}\bar{3}\bar{2}\bar{1}2\bar{3}\bar{4} \pmod{\bar{4}\bar{3}u_1u_2u_1\bar{3}\bar{4}\bar{3}\bar{2}\bar{1}2\bar{3}\bar{4}}$ and $\bar{4}\bar{3}u_1u_2u_1\bar{3}\bar{4}\bar{3}\bar{2}\bar{1}2\bar{3}\bar{4} = u_1\bar{4}\bar{3}u_2\bar{3}\bar{4}\bar{3}u_1\bar{2}\bar{1}2\bar{3}\bar{4}$ with $\bar{4}\bar{3}u_2\bar{3}\bar{4}\bar{3}u_1\bar{2}\bar{1}2\bar{3}\bar{4} = \bar{4}\bar{3}\bar{3}\bar{4}\bar{3}u_1\bar{2}\bar{1}2\bar{3}\bar{4} + \bar{4}\bar{3}\bar{2}\bar{3}\bar{4}\bar{3}u_1\bar{2}\bar{1}2\bar{3}\bar{4} + \bar{4}\bar{3}\bar{2}\bar{3}\bar{4}\bar{3}u_1\bar{2}\bar{1}2\bar{3}\bar{4} \subset Q_5^{(2)} + \bar{4}\bar{3}\bar{2}\bar{3}\bar{4}\bar{3}u_1\bar{2}\bar{1}2\bar{3}\bar{4}$. But $4(\bar{3}\bar{2}\bar{3})\bar{4}\bar{3}u_1\bar{2}\bar{1}2\bar{3}\bar{4} \equiv (bc)^{-1}\bar{4}\bar{3}\bar{2}\bar{3}\bar{4}\bar{3}u_1\bar{2}\bar{1}2\bar{3}\bar{4} \pmod{Q_5^{(2)}}$ and $\bar{4}\bar{3}\bar{2}\bar{3}\bar{4}\bar{3}u_1\bar{2}\bar{1}2\bar{3}\bar{4} = \bar{4}\bar{3}\bar{2}\bar{4}\bar{3}\bar{4}u_1\bar{2}\bar{1}2\bar{3}\bar{4} = (\bar{4}\bar{3}\bar{4})\bar{2}\bar{3}u_1\bar{2}\bar{1}2\bar{4}\bar{3}\bar{4} = \bar{3}\bar{4}\bar{3}\bar{2}\bar{3}u_1\bar{2}\bar{1}2\bar{4}\bar{3}\bar{4} \in Q_5^{(2)}$. This proves $\bar{4}\bar{3}(\bar{2}\bar{1}2)\bar{3}\bar{4}\bar{3}\bar{2}\bar{1}2\bar{3}\bar{4} \equiv (bc).4w_-4w_-4 \pmod{Q_5^{(2)}}$ hence $\bar{4}\bar{3}(\bar{2}\bar{1}2)\bar{3}\bar{4}\bar{3}(\bar{2}\bar{1}2)\bar{3}\bar{4} \equiv (bc)^2.4w_-4w_-4$.

Therefore, by (3.1) we get $\bar{4}w_+\bar{4}w_+\bar{4} \equiv \frac{bc-a^2}{a^4bc}.(a^4/wbc).3\bar{4}xy4x4\bar{3} + \frac{bc}{a^2}.(bc)^2.3\bar{4}w_-4w_-4\bar{3} \pmod{Q_5^{(2)}}$ that is $\bar{4}w_+\bar{4}w_+\bar{4} \equiv \frac{bc-a^2}{bc}.(a^2/wbc).4xy4x4 + (bc)^3.4w_-4w_-4 \pmod{Q_5^{(2)}}$ hence

$$4w_-4w_-4 \equiv -(a^6/w^5).4xy4x4 \pmod{Q_5^{(2)}}$$

and this proves (2). \square

Proposition 5.6.

- (1) $Q_5^{(3)} = Q_5^{(2)} + R.4xy4x4$.
- (2) $Q_5^{(3)} = Q_5^{(2)} + R.4w_-4w_-4 = Q_5^{(2)} + R.\bar{4}w_+\bar{4}w_+\bar{4}$
- (3) Modulo $Q_5^{(2)}$, $\lambda.4w_-4w_-4 \equiv 4w_-4w_-4.\lambda \equiv \varepsilon(\lambda).4w_-4w_-4$ and $\lambda.\bar{4}w_+\bar{4}w_+\bar{4} \equiv \bar{4}w_+\bar{4}w_+\bar{4}.\lambda \equiv \varepsilon(\lambda).\bar{4}w_+\bar{4}w_+\bar{4}$ for all $\lambda \in Q_4$.

Proof. By lemma 5.5 we know that $Q_5^{(2)} + R.4xy4x4 = Q_5^{(2)} + R.4w_-4w_-4 = Q_5^{(2)} + R.\bar{4}w_+\bar{4}w_+\bar{4}$. We denote H this R -module, and aim to show that $H = Q_5^{(3)}$. Clearly $H \subset Q_5^{(3)}$, and $\Phi(H) = H$ since $\Phi(w_\pm) = w_\mp$. By handle reduction it is thus sufficient to check that $s_4\alpha s_4\beta s_4 \in H$ for all $\alpha, \beta \in Q_4$. Recall that $Q_4 = Q_3u_3Q_3 + Q_3xQ_3 + R.w_+ + R.w_-$. It is thus sufficient to check $s_4\alpha s_4\beta s_4 \in H$ for all $\alpha, \beta \in \{Q_3u_3Q_3, Q_3xQ_3, w_+, w_-\}$. We first consider the following cases.

- if α or β belong to $Q_3u_3Q_3$ we have $s_4\alpha s_4\beta s_4 \in Q_5^{(2)} \subset H$.
- if α and β belong to Q_3xQ_3 we have that $s_4\alpha s_4\beta s_4 \in Q_3s_4x\gamma s_4x s_4Q_3$ with $\gamma \in Q_3 = R.y + u_2u_1u_2$ and $s_4x\gamma s_4x s_4 \in R.s_4xys_4x s_4 + s_4xu_2u_1u_2 \subset R.s_4xys_4x s_4 + Q_5^{(2)} = H$.

We now notice that the cubic relation almost immediatly implies

$$\begin{cases} w_+ & \in w^{-1}.w_0 + Q_3u_3Q_3 + Q_3.x \\ w_- & \in w.w_0^{-1} + Q_3u_3Q_3 + Q_3.x \end{cases}$$

where $w_0 = 321123$ centralizes s_1 (and s_2). By the above cases we know this implies that $s_4xQ_3s_4w_\pm s_4 \subset s_4xs_4w_\pm s_4Q_3 + H$. Therefore,

- if $\alpha \in Q_3xQ_3$ and $\beta = w_+$, we have $s_4\alpha s_4\beta s_4 \in H$ iff $s_4xs_4w_+s_4 \in H$. But x commutes with $s_4w_+s_4$ whence $s_4xs_4w_+s_4 = s_4s_4w_+s_4x \in Q_5^{(2)} \subset H$ and this solves the case;
- if $\alpha \in Q_3xQ_3$ and $\beta = w_-$, we have $s_4\alpha s_4\beta s_4 \in H$ iff $s_4xs_4w_-s_4 \in H$. But $s_4xs_4w_-s_4 = 43\bar{2}(34\bar{3})\bar{2}1\bar{2}\bar{3}4 = 43\bar{2}434\bar{2}1\bar{2}\bar{3}4 = (43\bar{4})\bar{2}3\bar{2}1\bar{2}4\bar{3}4 = \bar{3}43\bar{2}3\bar{2}1\bar{2}4\bar{3}4 \in Q_5^{(2)} \subset H$ and this solves the case.

The cases where the roles of α and β are exchanged are deduced from these ones by applying F . Therefore, we are reduced to considering $\alpha, \beta \in \{w_+, w_-\}$. If $\alpha = \beta$ this is clear, so we can assume $\alpha \neq \beta$, and via F there is only one case to consider, namely $4w_+4w_-4 = 43\bar{2}1\bar{2}(34\bar{3})\bar{2}1\bar{2}\bar{3}4 = 43\bar{2}1\bar{2}434\bar{2}1\bar{2}\bar{3}4 = (43\bar{4})2\bar{1}23\bar{2}1\bar{2}4\bar{3}4 = \bar{3}43\bar{2}1\bar{2}3\bar{2}1\bar{2}4\bar{3}4 \in Q_5^{(2)} \subset H$, and this proves the claim that $H = Q_5^{(3)}$, which implies (1) and (2). Then (3) is an immediate consequence of lemmas 5.4 and 5.5. □

Theorem 5.7. Q_5 is a R -module of finite rank, and $Q_5 = Q_5^{(3)}$.

Proof. It is sufficient to show that $Q_5^{(4)} = Q_5^{(3)}$. But $Q_5^{(4)} = Q_5^{(3)}u_4Q_4 = Q_5^{(2)}u_4Q_4 + 4xy4x4u_4Q_4$. and $Q_5^{(2)}u_4Q_4 = Q_5^{(3)}$ while $4xy4x4u_4Q_4 = 4xy4xu_4Q_4 \subset Q_5^{(3)}$, whence the claim. □

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